An alternative proof of Wigner theorem on quantum transformations based on elementary complex analysis

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1. Preliminaries

Wigner theorem is a cornerstone of theoretical physics since it encapsulates all the linear structure of quantum transformations, among which the evolution of quantum systems (aside from the measurement process). More precisely, given a Hilbert space endowed with a Hermitian product $(\langle \chi | \psi \rangle)^*$, Wigner, in the early 30s [8, Appendix to Chap. 20, pp. 233–236, for the updated English translation], acknowledged that any transformation $T : |\psi\rangle \mapsto |\psi^\prime\rangle = T(|\psi\rangle)$ such that

\[ \forall (|\chi\rangle, |\psi\rangle) \in \mathcal{H}^2, \quad |T(|\chi\rangle |\psi\rangle)|^2 = |\langle \chi | \psi \rangle|^2 \]

is either

(a) linear $T(c_1 |\psi_1\rangle + c_2 |\psi_2\rangle) = (c_1 |\psi_1\rangle^T + c_2 |\psi_2\rangle^T)$ and unitary $T^{-1} = T^*$;

or

(b) antilinear $T(c_1 |\psi_1\rangle + c_2 |\psi_2\rangle) = c_1^* |\psi_2\rangle^T + c_2^* |\psi_2\rangle^T$ and unitary $T^{-1} = T^*$ (such a map is also called antiunitary).

We shall systematically use the usual Dirac bra–ket notation and, in the above definitions, $(c_1, c_2)$ stands for any pair of complex coefficients, $(|\psi_1\rangle, |\psi_2\rangle)$ is any pair of elements of $\mathcal{H}$. The Hermitian conjugate will be denoted by $^*$ and therefore, if $x$ is just one complex number, $x^*$ stands for its complex conjugate.

Since Wigner's original work that pertained to the representation of the rotation group, many proofs and generalizations have been proposed whose levels of rigor are not necessarily correlated to their length but, rather, vary depending on the concern of their author. See for example [6] and references therein to which I shall add [3, §1.3.2] and the concise and elegant proof given by [5, §XV-2]. The present work is further added to this list because it appears to be almost a back-of-the-envelope presentation while keeping a level of rigor that is acceptable by physicists (hopefully, little additional work on the main key ideas should meet the requirements of mathematicians as well). Moreover, the majority of the previous proofs, if not all of them, can hardly be transposed to non-separable Hilbert spaces, that is to spaces where a countable orthonormal basis does not exist. Yet, in quantum field theory, such non-separable Hilbert spaces are unavoidable: any continuous canonical transformation or rearrangement of the infinitely many degrees of freedom — that physically describe a renormalization of the bare particles into the dressed ones like, for instance, when a condensation occurs — requires that the Hilbert space is made of a continuous family of orthogonal Fock spaces and one cannot content one self with the unique Fock space that represents the physical (dressed) particles; see for example [2, §1.1] or [7, Chap. 3].

The proof presented in the second part of the present Letter will at first make use of a (at most) countable basis $(|\psi_\nu\rangle)_{\nu \in I}$ in $\mathcal{H}$,

\[ \langle \psi_\nu | \psi_\mu \rangle = \delta_{\nu\mu}, \]

but the discreteness of the set $I$ will be just a matter of convenience. The transposition to a continuous (multi-)index is straightforward (we will not use any induction arguments that would
in a transformation. Discontinuity nor singularity with respect to the quantum state induced by the measuring process—we have never observed any as condition (1) are lost by the transformation on the states induced by the measuring process, when the number of degrees of freedom involved in the interaction of the system with a measuring device becomes infinite—and we know that the superposition principle as well as the null vector, being a vector orthogonal to any \(v\). For instance, with an appropriate choice of normalization, the Hermitian product in \(H\) reads
\[
\langle w|z\rangle \equiv \langle w^*|z\rangle = \sum_v w_v^* z_v = \langle \chi|\psi \rangle
\]
with \(z = (\langle \varphi_v|\psi\rangle)_{v \in I}\) and \(w = (\langle \varphi_v|\chi\rangle)_{v \in I}\). Any application \(T\) in \(H\) can be seen as a either a function \(T(x, y)\) of the sequence of the real part \(x = \text{Re} z\) and imaginary part \(y = \text{Im} z\) of the components of the state, or rather as a function of two independent complex sequences \(z\) and \(\bar{z}\)
\[
T(z, \bar{z}) \equiv T((\bar{z} + z)/2, i(\bar{z} - z)/2)
\]
evaluated at \(\bar{z} = z^*\). When \(T\) can be differentiated, we can define the derivatives with respect to the complex variables by (see [1, §II.1] for instance):
\[
\partial_z T \equiv \frac{1}{2} (\partial_k - i\partial_y) T; \\
\partial_{\bar{z}} T \equiv \frac{1}{2} (\partial_k + i\partial_y) T.
\]
In particular, \(T\) is analytic if and only if \(\partial_z T = 0\). From the differential of \(T\), we get
\[
\partial_{z^*} T(z, z^*) = (\partial_{\bar{z}} T)(z, z^*).
\]
In the following we will drop the distinction between \(z^*\) and the variable \(\bar{z}\) that may vary independently of \(z\) because the continuation of any application \(T(z, \bar{z})\) defined by (8) in the domain where \(\bar{z} \neq z^*\) is unique.

In this language, the invariance condition (1) can be reformulated as follows,
\[
\forall (w, z) \in \mathbb{H}^2, \quad |(T(w, w^*))^* T(z, z^*)| = |w^* z|.
\]
Therefore we have two possibilities: there exists a real function \(\theta\) of the complex variables \((w, w^*, z, z^*)\) such that
(a) either \(T(w, w^*) = e^{i\theta(w, w^*, z, z^*)} w^* z\); (12a)
(b) or \(T(z, z^*) = e^{-i\theta(w, w^*, z, z^*)} w^* z\). (12b)
To understand where these two conditions come from, first divide the both sides of the equality in (11) by \(|w^* z|\) when non-vanishing (then eventually include these cases by continuity of the transformation \(T\)). Thus, we are led to an equation of the form \(|Z| = 1\) where \(Z\) is a complex number. Then, the argument \(\eta\) of \(Z\) can take the two solutions \(\pm \text{acccos}(\text{Re} Z)\) in \([-\pi, \pi]\) of the equation \(\cos \eta = \text{Re} Z\). The two options (a) and (b) correspond to the two forms \(e^{i\eta \text{acccos} (\text{Re} Z)}\) that is \(Z = e^{i\theta \text{acccos} (\text{Re} Z)}\) or \(Z^* = e^{i\theta \text{acccos} (\text{Re} Z)}\).

Permuting \((w, w^*)\) and \((z, z^*)\), then by conjugation we have necessarily
\[
\theta(z, z^*, w, w^*) = -\theta(w, w^*, z, z^*),
\]
which incidentally shows that \(\theta(0, 0, 0, 0) = 0\).

2.2. Adjusting the phase

The first step of our proof is to redefine the phase of the transformed states by considering
\[
\tilde{T}(z, z^*) = e^{i\alpha(z, z^*)} T(z, z^*),
\]
where \(\alpha\) is a real function. Conditions (12) can be written

2. The core of the proof

2.1. The invariance property formulated in terms of complex analysis

We shall take advantage of the isomorphism that allows us to identify the Hilbert space \(H\) to the set of sequences of complex numbers \(z = (z_v)_{v \in I}\). More precisely, once given an orthonormal basis \((|\varphi_v\rangle)_{v \in I}\), we can map any \(|\psi\rangle\) in \(H\) to \(z_v = \langle \varphi_v|\psi\rangle\). The Hermitian product in \(H\) reads
\[
\langle w|z\rangle \equiv w^* z = \sum_v w_v^* z_v = \langle \chi|\psi \rangle
\]
with \(z = (\langle \varphi_v|\psi\rangle)_{v \in I}\) and \(w = (\langle \varphi_v|\chi\rangle)_{v \in I}\). Any application \(T\) in \(H\) can be seen as a function \(T(x, y)\) of the sequence of the real part \(x = \text{Re} z\) and imaginary part \(y = \text{Im} z\) of the components of the state, or rather as a function of two independent complex sequences \(z\) and \(\bar{z}\)
\[
T(z, \bar{z}) \equiv T((\bar{z} + z)/2, i(\bar{z} - z)/2)
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evaluated at \(\bar{z} = z^*\). When \(T\) can be differentiated, we can define the derivatives with respect to the complex variables by (see [1, §II.1] for instance):
\[
\partial_z T \equiv \frac{1}{2} (\partial_k - i\partial_y) T; \\
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\]
In particular, \(T\) is analytic if and only if \(\partial_z T = 0\). From the differential of \(T\), we get
\[
\partial_{z^*} T(z, z^*) = (\partial_{\bar{z}} T)(z, z^*).
\]

\[
\forall (u, v) \in \mathbb{C}^2, \quad T(u) \cdot T(v) = u \cdot v.
\]
(a) either \( \left( \hat{T}(w, w^*) \right)^* \hat{T}(z, z^*) = e^{i\partial(\theta(w, w^*, z, z^*))} w^* z \); \hfill (15a)
(b) or \( \left( \hat{T}(z, z^*) \right)^* \hat{T}(w, w^*) = e^{-i\partial(\theta(w, w^*, z, z^*))} w^* z \), with
\[ \hat{\theta}(w, w^*, z, z^*) = \theta(w, w^*, z, z^*) + \alpha(z, z^*) - \alpha(w, w^*). \] \hfill (16)

The latter identity can be used to cancel \( \hat{\theta}(0, 0, z, z^*) \) with the choice
\[ \alpha(z, z^*) = -\theta(0, 0, z, z^*) = \theta(z, z^*, 0, 0). \] \hfill (17)

Therefore without loss of generality, in the following we can consider that for all \( z \)
\[ \theta(0, 0, z, z^*) = 0. \] \hfill (18)

2.3. Differentiation

The second step of the proof is to differentiate the conditions written in (12) with respect to \( w^* \) and \( z \). From (12a), using \( \hat{\partial}(T(w, w^*)) = (\hat{\partial} T(w, w^*))^* \), we get
\[ (\hat{\partial} T(w, w^*))^* \hat{\partial}_z T(z, z^*) \]
\[ = e^{i\partial(\theta(w, w^*, z, z^*))} \left[ 1 + i w^* \hat{\partial}_w \theta(w, w^*, z, z^*) \right. \]
\[ + i \partial_z \theta(w, w^*, z, z^*) z + \left[ i \partial^2_{ww} \theta(w, w^*, z, z^*) \right. \]
\[ - \partial_z \theta(w, w^*, z, z^*) \partial_w \theta(w, w^*, z, z^*) \] \hfill (19)

Because of this last computation, and in particular because of the presence of the Hessian of \( \theta \), we assume that the transformation is differentiable twice. When evaluated for \( w^* = 0, 0 \), we get
\[ (\hat{\partial}_z T(0, 0))^* \hat{\partial}_z T(z, z^*) = 1, \] \hfill (20)

since (18) implies \( \hat{\partial}_z \theta(0, 0, z, z^*) = 0 \). Then \( (\hat{\partial}_z T(z, z^*))^{-1} = (\hat{\partial}_z T(0, 0))^* \) appears to be independent of \( (z, z^*) \) and unitary. Moreover, because \( T(0, 0) = 0 \), we have, for all \( z \),
\[ T(z, z^*) = \hat{\partial}_z T(0, 0) z \] \hfill (21)

and therefore \( z \mapsto T^* = T(z, z^*) \) is linear. The linear operator \( \hat{U} \)
defined by its matrix elements \( \langle \phi_{\nu'} | \hat{U} | \phi_{\nu} \rangle \)
such that \( \hat{U}^{-1} = \hat{U}^* \) and the expression (21) can be written as
\[ \langle \phi_{\nu'} | \psi \rangle = \sum_{\nu'} \langle \phi_{\nu'} | \hat{U} | \phi_{\nu} \rangle \langle \phi_{\nu} | \psi \rangle = \langle \phi_{\nu'} | \hat{U} | \psi \rangle, \]
which is the first option offered by Wigner theorem.

The second option comes when differentiating (12b) which just modifies the left-hand side of (19) and the irrelevant sign in front of \( \theta \) in the right-hand side. Then we obtain
\[ (\hat{\partial}_w T(0, 0))^* \hat{\partial}_w T(z, z^*) = 1 \] \hfill (22)

and eventually
\[ T(z, z^*) = \hat{\partial}_w T(0, 0) z^*. \] \hfill (23)

The transformation \( z \mapsto T^* = T(z, z^*) \) is antilinear. An antilinear operator \( \hat{A} \) can be defined from its matrix elements \( \langle \phi_{\nu'} | \hat{A} | \phi_{\nu} \rangle \)
defined as \( (\hat{\partial}_w T(0, 0))^* \), and we have \( \langle \phi_{\nu'} | \psi \rangle^T = \sum_{\nu'} \langle \phi_{\nu'} | \hat{A} | \phi_{\nu} \rangle \langle \phi_{\nu} | \psi \rangle^* = \langle \phi_{\nu'} | \hat{A} \sum_{\nu'} \langle \phi_{\nu} | \phi_{\nu'} \rangle \langle \phi_{\nu'} | \psi \rangle \rangle = \langle \phi_{\nu'} | \hat{A} | \psi \rangle \rangle \) that is \( | \psi \rangle^T = \hat{A} | \psi \rangle \rangle \). The relation \( (\hat{\partial}_w T(0, 0))^* \) reads \( \hat{A}^{-1} = \hat{A}^* \). Therefore case (b) implies the second alternative of Wigner theorem.

3. Summary

Looking back the proof over one’s shoulder, once keeping in mind the simple line of thought used in the real case — the double differentiation of (3) that leads to (4) — the backbone of the argument in the Hilbert case do not appear much more complicated and can be captured in the following few lines : from the key equations (12), it is straightforward by a double differentiation to obtain (19). Then having redefined the phases of the transformed states in order to get (18), we are immediately led to (20) or (22) then to (21) or (23) respectively.

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References