Nonexistence results and estimates for some nonlinear elliptic problems

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Abstract

Here we study the local or global behaviour of the solutions of elliptic inequalities involving quasilinear operators, of the type

$$L_{\mathcal{A}}u = -div\,[\mathcal{A}(x,u,\nabla u)] \geq |x|^{\sigma}u^{Q},$$

or

$$\begin{cases} L_{\mathcal{A}}u = -div\,[\mathcal{A}(x,u,\nabla u)] \geq |x|^{a}u^{S}v^{R}, \\ L_{\mathcal{B}}v = -div\,[\mathcal{B}(x,v,\nabla v)] \geq |x|^{b}u^{Q}v^{T}. \end{cases}$$

We give integral estimates and nonexistence results. They depend on properties of the supersolutions of the equations $L_{\mathcal{A}}u = 0$, $L_{\mathcal{B}}v = 0$, which suppose weak coercivity conditions. Under stronger conditions, we give punctual estimates in case of equalities, using Harnack properties.

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1 Introduction

Here we study the existence and the behaviour of nonnegative solutions of elliptic problems in an open set Ω of $\mathbb{R}^N$ $(N \geq 2)$, involving quasilinear operators in divergential form. We discuss this question for *inequalities* of the type

$$-div\,[\mathcal{A}(x,u,\nabla u)] \geq |x|^{\sigma} u^{Q}, \tag{1.1}$$

where $Q, \sigma \in \mathbb{R}$, $Q > 0$, or for Hamiltonian systems of the form

$$\begin{cases} -div\,[\mathcal{A}(x,u,\nabla u)] \geq |x|^{a} v^{R}, \\ -div\,[\mathcal{B}(x,v,\nabla v)] \geq |x|^{b} u^{Q}, \end{cases} \tag{1.2}$$

where $Q, R, a, b \in \mathbb{R}$, with $Q, R > 0$. More generally we can reach multipower systems of the form

$$\begin{cases} -div\,[\mathcal{A}(x,u,\nabla u)] \geq |x|^{a} u^{S} v^{R}, \\ -div\,[\mathcal{B}(x,v,\nabla v)] \geq |x|^{b} u^{Q} v^{T}, \end{cases} \tag{1.3}$$

where $S, T \geq 0$.

In the sequel Ω will be either $\mathbb{R}^N$ or $\mathbb{R}^N \setminus \{0\}$, or an exterior or interior domain

$$\Omega_e = \left\{ x \in \mathbb{R}^N \mid |x| > 1 \right\}, \qquad \Omega_i = \left\{ x \in \mathbb{R}^N \mid 0 < |x| < 1 \right\},$$

or the halfspace $\mathbb{R}^{N+} = \left\{ x \in \mathbb{R}^N \mid x_N > 0 \right\}$, or

$$\Omega_e^+ = \Omega_e \cap \mathbb{R}^{N+}, \qquad \Omega_i^+ = \Omega_i \cap \mathbb{R}^{N+}.$$

Our aim is not only to give *nonexistence results*, but also *integral estimates* for the solutions in case of existence. The problem of the nonexistence, the so-called Liouville problem, has been the subject of several works. We can cite in the nonradial case the results of [20], [17], [33], [24] to [26], [23] in the case of $\mathbb{R}^N$; of [19], [3], [13] in the case of half-spaces or cones; and [4] for exterior domains. In the radial case the number of publications is so great that we cannot cite all of them, let us only mention [27], [28], [31], in the scalar case, and [34], [15], [18] in case of systems. Recall that such results can be used for finding a priori estimates in bounded domains via a blow-up technique; see [19]. Obtaining a priori estimates is most often difficult, even in the case of an equation; and many questions are still open. The main results can be found in [32], [20], [14], and also [2], [4], [7], [10].

Let us give an example showing the connections between local and global existence problems, and between equations and inequalities. Assume for simplicity that $N \geq 3$ and $Q > 1$. It is well known that the equation

$$-\Delta u = u^Q \tag{1.4}$$

has no positive C^2_{loc} solution in $\mathbb{R}^N$ if and only if $Q < (N+2)/(N-2)$, see [20], [29]. In fact in case $N(N-2) < Q < (N+2)/(N-2)$, it admits solutions in $\mathbb{R}^N \setminus \{0\}$, but they are singular at 0. Now the problem

$$-\Delta u \geq u^Q \tag{1.5}$$

has no positive solution in $\mathbb{R}^N$ if and only if $Q \leq N(N-2)$, see for example [3], [24]. An easy way to get the part "if" is as follows. The mean value of u on the sphere of center 0 and radius r also satisfies (1.5), by Jensen's inequality. Then we are reduced to the radial case. When $Q > N(N-2)$, the function $u(x) = c(1+|x|^2)^{-1/(Q-1)}$ is a solution of (1.5) if c is small enough, which gives the "only if" part. The problem (1.5) in Ω_e has no positive solution in Ω_e if and only if $Q \leq N(N-2)$, see for example [4], [7]. In fact the two problems in $\mathbb{R}^N$ and in Ω_e are equivalent, because under a supersolution one can construct a solution. There is a deep connection between the problems in Ω_i and Ω_e. The inequality

$$-\Delta u \geq |x|^\sigma u^Q \tag{1.6}$$

has no positive solution in Ω_e if and only if $Q \leq (N+\sigma)/(N-2)$, see [7]. Equivalently, by the Kelvin transform, it has no solution in Ω_i if and only if $\sigma \leq -2$, see also [20]. In the sequel we shall compare the problems in $\mathbb{R}^N, \mathbb{R}^N \setminus \{0\}$, Ω_e or Ω_i, according to the assumptions on the operators.

In **Section 2**, we give general properties of the supersolutions u of the operator $u \mapsto L_{\mathcal{A}} u = -div\,[\mathcal{A}(x,u,\nabla u)]$, that means

$$-div\,[\mathcal{A}(x,u,\nabla u)] = f \geq 0. \tag{1.7}$$

They are the key tool of our study. Here we combine two different approaches of the problem. First we precise a technique introduced in [24] and developped in [25], [26]. Under some weak assumptions on $\mathcal{A}$, it gives integral upper estimates of f with respect to u, in $\mathbb{R}^N, \Omega_i$ or Ω_e. Then using the method of [4] extended in [5], we also obtain estimates on f , independently of u. Under stronger assumptions on $\mathcal{A}$, we can complete them by integral estimates on u in Ω_i, Ω_e. Combining the two techniques, we get estimates of f in $\mathbb{R}^N \setminus \{0\}$. We also give lower estimates when $\mathcal{A}$ does not depend on x and u.

The **Section 3** deals with the inequality (1.1). We get a priori integral estimates which are new in the case of quasilinear operators. We also improve the nonexistence results of [26] in several directions: nonexistence in $\mathbb{R}^N$ for a larger class of operators,

nonexistence in $\mathbb{R}^N \setminus \{0\}$, or Ω_i, Ω_e, for any $Q > 0$ and any real σ. In case of the equation

$$-div\,[\mathcal{A}(x,u,\nabla u)] = |x|^\sigma u^Q, \tag{1.8}$$

we obtain pointwise a priori estimates in Ω_i via the Harnack inequality, in the first subcritical case. When $\mathcal{A}$ does not depend on x and u, we show that the problems of existence in $\mathbb{R}^N$ and in Ω_e are equivalent, and that the problems in Ω_i or Ω_e are of the same type, even if the Kelvin transform cannot be used.

In **Section 4**, we consider the problems in $\mathbb{R}^{N+}$, Ω_e^+ or Ω_i^+. The works of [19], [3], [13] in $\mathbb{R}^{N+}$ concern the case of the Laplacian $L_{\mathcal{A}} = -\Delta$. They use either its symmetry properties, or the first eigen function ϕ_1 of the Dirichlet problem in $\Omega \cap S^{N-1}$. Such methods cannot be used for quasilinear operators, and the question is more complex. We study the model case of the p-Laplace operator and show that the difficulties are due to the structure of p-harmonic functions in $\mathbb{R}^{N+}$ when $p \neq 2$. For some operators of order 2, we can overcome the difficulties by reporting a derivation on the test function, which recalls the use of ϕ_1 in [3].

In **Section 5** we extend the integral estimates to the multipower system (1.3). In case of a Hamiltonian system of equations ($S = T = 0$), this gives pointwise estimates, which are new for quasilinear operators. Then we get nonexistence results for the system (1.3) for any $Q, R > 0$, $S \in [0, p-1)$, $T \in [0, m-1)$. Thus we extend the results of [7] relative to the case $L_{\mathcal{A}} = L_{\mathcal{B}} = -\Delta$. and [25], [26] relative to the case of system (1.2) with $L_{\mathcal{A}} = -\Delta_p$, $L_{\mathcal{B}} = -\Delta_m$, and $S = T = 0$, $Q > p-1$ and $R > m-1$; and also many radial results, as [15] or [16].

2 General properties of supersolutions

Here we extend and compare some of the results of the first author [4], [5] and the second jointly with E. Mitidieri, [24] to [26], relative to the supersolutions of quasilinear equations.

2.1 Notations

For any $x \in \mathbb{R}^N$ and $r > 0$, we set $B(x,r) = \left\{ y \in \mathbb{R}^N \,||y-x| < r \right\}$ and $B_r = B(0,r)$. For any $\rho_2 > \rho_1 > 0$, let

$$\mathcal{C}_{\rho_1,\rho_2} = \left\{ y \in \mathbb{R}^N \mid \rho_1 < |y| < \rho_2 \right\}.$$

Let Ω be any open set of $\mathbb{R}^N$. For any function $f \in L^1(\Omega)$, and for any weight function $\varphi \in L^\infty(\Omega)$ such that $\varphi \geq 0, \varphi \neq 0$, we denote by

$$\oint_{\Omega,\varphi} f = \frac{1}{\int_\Omega \varphi} \int_\Omega f\varphi$$

the mean value of f with respect to φ and write

$$\oint_{\Omega} f = \oint_{\Omega,1} f.$$

When $\Omega \supset \mathcal{C}_{\rho_1,\rho_2}$, we also define on (ρ_1, ρ_2) the mean value,

$$\overline{f}(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r} f \, ds,$$

of f on the sphere ∂B_r of center 0 and radius r.

2.2 Assumptions on the operators

Let $\mathcal{A} : \Omega \times \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^N$ be a Caratheodory function, and

$$L_{\mathcal{A}} u = -div\,[\mathcal{A}(x, u, \nabla u)] \tag{2.1}$$

for any $u \in W^{1,1}_{loc}(\Omega)$ such that $\mathcal{A}(x, u, \nabla u) \in (L^1_{loc}(\Omega))^N$.

In this Section, we study the properties of the nonnegative supersolutions of equation $L_{\mathcal{A}} u = 0$, and more precisely the solutions of

$$L_{\mathcal{A}} u = f \geq 0 \tag{2.2}$$

where $f \in L^1_{loc}(\overline{\Omega})$. We shall say that a nonnegative function $u \in C^0(\Omega) \cap W^{1,1}_{loc}(\overline{\Omega})$ satisfies (2.2) if $\mathcal{A}(x, u, \nabla u) \in (L^1_{loc}(\overline{\Omega}))^N$, $L_{\mathcal{A}} u \in L^1_{loc}(\overline{\Omega})$ and

$$\int_{\Omega} \mathcal{A}(x, u, \nabla u).\nabla \phi \geq \int_{\Omega} f \phi, \tag{2.3}$$

for any nonnegative $\phi \in W^{1,\infty}(\Omega)$ with compact support in Ω.

Definition 1. Let $p > 1$. The function $\mathcal{A}$ is called **W-p-C (weakly-p-coercive)** if

$$\mathcal{A}(x, u, \eta).\eta \geq K \ |\mathcal{A}(x, u, \eta)|^{p'} \tag{2.4}$$

for some $K > 0$, and for all $(x, u, \eta) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^N$.

Definition 2. The function $\mathcal{A}$ is called **S-p-C (strongly-p-coercive)** if

$$\mathcal{A}(x, u, \eta).\eta \geq K_1 \ |\eta|^p \geq K_2 \ |\mathcal{A}(x, u, \eta)|^{p'} \tag{2.5}$$

for some $K_1, K_2 > 0$, and for all $(x, u, \eta) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^N$.

Remark 2.1 The condition (2.5) is a classical frame for the study of quasilinear operators, see [32] and [36]. It implies that $L_{\mathcal{A}}$ satisfies the weak Harnack inequality, and hence the strong maximum principle. The condition (2.4) is clearly weaker . Let us give some examples.

i) Assume that $\mathcal{A} : \Omega \times \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^N$ is given by

$$\mathcal{A}_i(x,u,\eta) = \sum_{j=1}^{N} a_{i,j}(x,u,\eta)\eta_j. \tag{2.6}$$

Then $\mathcal{A}$ is **W-**p**-C** if

$$\sum_{i,j=1}^{N} a_{i,j}(x,u,\eta)\eta_i\eta_j \geq K \left[\sum_{i=1}^{N} \left(\sum_{j=1}^{N} a_{i,j}(x,u,\eta)\eta_j \right)^2 \right]^{p'/2}$$

for all $(x,u,\eta) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^N$.

ii) In particular, suppose that

$$\mathcal{A}_i(x,u,\eta) = A(x,u,|\eta|)\ \eta_i \tag{2.7}$$

with $A : \Omega \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$. Then $\mathcal{A}$ is **W-**p**-C** as soon as

$$0 \leq A(x,u,t) \leq M\ t^{\,p-2} \tag{2.8}$$

for some $M > 0$. Indeed $A(x,u,t)^{p'-1} \leq M^{p'-1}\ t^{\,(p-2)(p'-1)} = M^{p'-1}\ t^{\,2-p'}$, hence

$$|\mathcal{A}(x,u,\eta)|^{p'} \leq C(N,p) \sum_{i=1}^{N} A(x,u,|\eta|)^{p'}\ \ |\eta_i|^{p'} \leq C(N,p) M^{p'-1} A(x,u,|\eta|)\ |\eta|^2 .$$

Moreover $\mathcal{A}$ is **S-**p**-C** if and only if

$$M^{-1}\ t^{\,p-2} \leq A(x,u,t) \leq M\ t^{\,p-2} \tag{2.9}$$

for some $M > 1$.

iii) The same happens if (2.7) is replaced by

$$\mathcal{A}_i(x,u,\eta) = A_i(x,u,|\eta_i|)\ \eta_i \tag{2.10}$$

where A_i satisfy (2.8).

iv) Suppose that $\mathcal{A}$ is given by (2.6), with $a_{i,j} = a_{j,i}$, and

$$0 \leq \sum_{i,j=1}^{N} a_{i,j}(x,u,\eta)\xi_i\xi_j \leq M\ \ |\xi|^2$$

for some $M > 0$, and any $\xi \in \mathbb{R}^N$ and all $(x,u,\eta) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^N$. Then $\mathcal{A}$ is **W-2-C.**

In some cases we shall need to make more precise assumptions on $\mathcal{A}$, in particular that $\mathcal{A}$ does not depend on x and u.

Definition 3. We shall say that $\mathcal{A}$ satisfies the property $(\mathbf{H}_p)$ if $\mathcal{A} : \mathbb{R}^N \to \mathbb{R}^N$ is given by

$$\mathcal{A}_i(\eta) = A(|\eta|)\, \eta_i \tag{2.11}$$

where $A \in C([0,+\infty), \mathbb{R}) \cap C^1((0,+\infty), \mathbb{R})$, and if there exists $M > 0$ such that

$$\begin{cases} A(t) \leq M\, t^{p-2}, & \text{for any } t > 0, \\ A(t) \geq M^{-1}\, t^{p-2} & \text{for small } t > 0, \end{cases} \tag{2.12}$$

$$t \mapsto A(t)t \quad \text{non decreasing.} \tag{2.13}$$

Hence any operator satisfying $(\mathbf{H}_p)$ is **W-p-C.**

Remark 2.2 In particular the p-Laplace operator

$$Lu = -\Delta_p u = -div(|\nabla u|^{p-2}\, \nabla u), \tag{2.14}$$

is **S-p-C**, and satisfies $(\mathbf{H}_p)$. The mean curvature operator

$$Lu = -div(\nabla u / \sqrt{1 + |\nabla u|^2}), \tag{2.15}$$

$(p = 2)$ and more generally the operator

$$Lu = -div(|\nabla u|^{p-2}\, \nabla u / \sqrt{1 + |\nabla u|^p}),$$

$(p > 1)$ satisfy $(\mathbf{H}_p)$.

Remark 2.3 Under the assumption (2.12), $L_{\mathcal{A}}$ satisfies the strong maximum principle. Indeed we can find a function $\widetilde{A} \in C((0,+\infty), \mathbb{R})$, such that such that

$$M^{-1}\, t^{\,p-2} \leq \widetilde{A}(t) \leq M\, t^{\,p-2} \qquad \text{for any } t > 0,$$

and $\widetilde{A}(t) = A(t)$ for small $t > 0$. Then the associated operator $\widetilde{\mathcal{A}}$ is **S-p-C**, and $L_{\widetilde{\mathcal{A}}}$ is uniformly elliptic. Then it satisfies the strong maximum principle. This implies the same property for $L_{\mathcal{A}}$. If moreover $\mathcal{A}$ satisfies (2.13), then we can find $\widetilde{A}$ as above such that $t \mapsto \widetilde{A}(t)t$ is non decreasing.

Remark 2.4 For simplification we supposed that the rate of growth of $\mathcal{A}$ does not depend on $|x|$. Many of our results can be extended to the case where $\mathcal{A}(x, u, \eta)$ has a power growth in $|x|$, that is when (2.4) is replaced by

$$\mathcal{A}(x, u, \eta).\eta \geq K\ |x|^{\tau(1-p')}\, |\mathcal{A}(x, u, \eta)|^{p'}$$

for some $\tau \in \mathbb{R}$, and (2.5) is replaced by

$$\mathcal{A}(x, u, \eta).\eta \geq K_1\, |x|^{\tau}\, |\eta|^p \geq K_2\, |\mathcal{A}(x, u, \eta)|^{p'},$$

see [30].

2.3 First estimates on f *in* $\mathbb{R}^N$, Ω_i, Ω_e.

First we extend and axiomatise some results of [24] to [26]. For any solution u of equation (2.2), we give integral estimates of f *with respect to* u. The proof is very linked to the proof of the weak Harnack inequality for **S**-p-**C** operators given in [32] and [36]. It uses the same test function, a negative power of u. In the proof of [36], the greater coercivity allows to give estimates on the gradient of u, and in turn on u from the Sobolev injection and the Moser technique. Here we impose less coercivity, and we do not use the gradient term.

Proposition 2.1 *Let $\Omega = \mathbb{R}^N$ (resp. Ω_i, resp. Ω_e). Let $p > 1$. Let $\mathcal{A} : \Omega \times \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^N$ be a Caratheodory function, and* **W**-p-**C**. *Let $u \in C^0(\Omega) \cap W^{1,1}_{loc}(\overline{\Omega})$ be a nonnegative solution of equation 2.2.*

Let $\varphi_\rho = \xi_\rho^\lambda$ with $\lambda > 0$ large enough, and $\xi_\rho \in \mathcal{D}(\Omega)$ with values in $[0,1]$, such that $|\nabla \xi_\rho| \le C/\rho$, and

$$\begin{cases} \xi_\rho = 1 \quad \text{for } |x| \le \rho \quad (\text{resp. } \rho/2 \le |x| \le \rho), \\ \xi_\rho = 0 \quad \text{for } |x| \ge 2\rho \quad (\text{resp. } |x| \ge 2\rho \quad \text{and } |x| \le \rho/4). \end{cases}$$

Then for any $\rho > 0$ (resp. small $\rho > 0$, resp. large $\rho > 0$), any $\alpha \in [1-p, 0]$ and $\ell > p - 1 + \alpha$,

$$\begin{aligned} \oint_{\Omega,\varphi_\rho} f u^\alpha &\le C\, \rho^{-p} \left(\oint_{supp\ \nabla\varphi_\rho} u^\ell \varphi_\rho \right)^{(p-1+\alpha)/\ell} \\ &\le C\, \rho^{-p} \left(\oint_{\Omega,\varphi_\rho} u^\ell \right)^{(p-1+\alpha)/\ell} . \end{aligned} \tag{2.16}$$

In particular for any $\ell > p - 1$,

$$\oint_{\Omega,\varphi_\rho} f \le C\, \rho^{-p} \left(\oint_{supp\ \nabla\varphi_\rho} u^\ell \varphi_\rho \right)^{(p-1)/\ell} \le C\, \rho^{-p} \left(\oint_{\Omega,\varphi_\rho} u^\ell \right)^{(p-1)/\ell} . \tag{2.17}$$

Proof. Let $\alpha < 0$. We set $u_\varepsilon = u + \varepsilon$, for any $\varepsilon > 0$. Let $\zeta \in \mathcal{D}(\Omega)$ with values in $[0,1]$. Then we can take

$$\phi = u_\varepsilon^\alpha \zeta^\lambda$$

as a test function. Hence

$$\int_\Omega f u_\varepsilon^\alpha \zeta^\lambda + |\alpha| \int_\Omega u_\varepsilon^{\alpha-1} \zeta^\lambda \mathcal{A}(x,u,\nabla u)\nabla u \le \lambda \int_\Omega u_\varepsilon^\alpha \zeta^{\lambda-1} \mathcal{A}(x,u,\nabla u)\nabla\zeta$$

From (2.4), it follows that

$$\int_\Omega f u_\varepsilon^\alpha \zeta^\lambda + |\alpha|\, K^{-1} \int_\Omega u_\varepsilon^{\alpha-1} \zeta^\lambda\, |\mathcal{A}(x,u,\nabla u)|^{p'} \le \lambda \int_\Omega u_\varepsilon^\alpha \zeta^{\lambda-1} \mathcal{A}(x,u,\nabla u)\nabla\zeta$$

$$\leq \frac{|\alpha|\, K^{-1}}{2} \int_\Omega u_\varepsilon^{\alpha-1} \zeta^\lambda \, |\mathcal{A}(x,u,\nabla u)|^{p'} + C(\alpha) \int_\Omega u_\varepsilon^{\alpha+p-1} \zeta^{\lambda-p} \, |\nabla \zeta|^p \, .$$

Hence

$$\int_\Omega f u_\varepsilon^\alpha \zeta^\lambda + \int_\Omega u_\varepsilon^{\alpha-1} \zeta^\lambda \, |\mathcal{A}(x,u,\nabla u)|^{p'} \leq C(\alpha) \int_\Omega u_\varepsilon^{\alpha+p-1} \zeta^{\lambda-p} \, |\nabla \zeta|^p \tag{2.18}$$

Then we use Hölder's inequality and make ε tend to 0. Thus if $\alpha > 1-p$, for any $\ell > p-1+\alpha$, setting $\theta = \ell/(p-1+\alpha) > 1$,

$$\int_\Omega f u^\alpha \zeta^\lambda \leq C(\alpha) \left(\int_{\text{supp } \nabla\zeta} u^\ell \zeta^\lambda \right)^{1/\theta} \left(\int_\Omega \zeta^{\lambda - p\theta'} \, |\nabla \zeta|^{p\theta'} \right)^{1/\theta'} \tag{2.19}$$

with a new constant $C(\alpha)$, from the Hölder inequality. In particular, chosing $\zeta = \xi_\rho$ with λ large enough,

$$\int_\Omega f u^\alpha \xi_\rho^\lambda \leq C(\alpha) \rho^{N/\theta' - p} \left(\int_{\text{supp } \nabla \xi_\rho} u^\ell \xi_\rho^\lambda \right)^{1/\theta}$$

If $\alpha = 1-p$, we get directly from (2.18)

$$\int_\Omega f u^\alpha \xi_\rho^\lambda \leq C \int_\Omega \xi_\rho^{\lambda-p} \, |\nabla \xi_\rho|^p \leq C \rho^{N-p}.$$

Hence we obtain (2.16) for $\alpha \neq 0$. Now we suppose $\ell > p-1$, and take

$$\phi = \zeta^\lambda$$

as a test function. We get

$$\int_\Omega f \zeta^\lambda \leq \lambda \int_\Omega \zeta^{\lambda-1} \mathcal{A}(x,u,\nabla u).\nabla \zeta.$$

Hence for any $\alpha \in (1-p, 0)$,

$$\int_\Omega f \zeta^\lambda \leq \lambda \left(\int_\Omega u_\varepsilon^{\alpha-1} \zeta^\lambda \, |\mathcal{A}(x,u,\nabla u)|^{p'} \right)^{1/p'} \left(\int_\Omega u_\varepsilon^{(1-\alpha)(p-1)} \zeta^{\lambda-p} \, |\nabla \zeta|^p \right)^{1/p} .$$

Since $\ell > p-1$, we can fix an $\alpha \in (1-p, 0)$ such that $\tau = \ell/(1-\alpha)(p-1) > 1$. Then from (2.19) and Hölder's inequality, we get, as $\varepsilon \to 0$,

$$\begin{aligned} \int_\Omega f \zeta^\lambda \quad &\leq \quad C(\alpha) \left(\int_{\text{supp } \nabla\zeta} u^\ell \zeta^\lambda \right)^{1/\theta p' + 1/\tau p} \times \\ &\qquad \left(\int_\Omega \zeta^{\lambda - \theta' p} \, |\nabla \zeta|^{\theta' p} \right)^{1/\theta' p'} \left(\int_\Omega \zeta^{\lambda - \tau' p} \, |\nabla \zeta|^{\tau' p} \right)^{1/\tau' p} . \end{aligned} \tag{2.20}$$

But $1/\theta p' + 1/\tau p = (p-1)/\ell = 1 - (1/\theta' p' + 1/\tau' p)$. Hence with $\zeta = \xi_\rho$ as above

$$\int_\Omega f\xi_\rho^\lambda \le C(\alpha)\left(\int_{\text{supp }\nabla\xi_\rho} u^\ell \xi_\rho^\lambda\right)^{(p-1)/\ell} \rho^{N(1-((p-1)/\ell)-p}$$

and (2.17) follows.■

Remark 2.5 For the solutions of $L_{\mathcal{A}}u \ge 0$, this shows that any estimate on u in some $L^s_{loc}(\Omega)$ with $s > p-1$ implies an estimate of $L_{\mathcal{A}}u$ in $L^1_{loc}(\Omega)$. When $p = 2$ and $L = -\Delta$ it is a simple consequence of the fact that $-\Delta\overline{u} \ge \overline{f}$, and we have the result with $s = 1$, see for example [9]. In the general case the result is new so far as we know, all the more since the conditions on $L_{\mathcal{A}}$ are quite weak.

Remark 2.6 Under the assumptions of Proposition 2.1, we can also estimate the term $\mathcal{A}(x,u,\nabla u)\nabla u$: for any $\alpha \in [1-p, 0]$ and $\ell > p-1+\alpha$, and for any $k > 1+1/\ell$,

$$\left(\oint_{\Omega,\varphi_\rho} [\mathcal{A}(x,u,\nabla u)\nabla u]^{1/k}\right)^{k/p} \le \frac{C}{\rho}\left(\oint_{\text{supp }\nabla\varphi_\rho} u^\ell\right)^{1/\ell} \le \frac{C}{\rho}\left(\oint_{\Omega,\varphi_\rho} u^\ell\right)^{1/\ell}. \tag{2.21}$$

Indeed we have

$$\int_\Omega u_\varepsilon^{\alpha-1}\xi^\lambda \mathcal{A}(x,u,\nabla u)\nabla u \le C(\alpha)\left(\int_{\text{supp }\nabla\varphi_\rho} u_\varepsilon^\ell \xi^\lambda\right)^{1/\theta}\left(\int_\Omega \xi^{\lambda-p\theta'}\,|\nabla\xi|^{p\theta'}\right)^{1/\theta'}. \tag{2.22}$$

Then from Hölder's inequality, for any given $k > 1$,

$$\int_\Omega \xi^\lambda\,[\mathcal{A}(x,u,\nabla u)\nabla u]^{1/k} = \int_\Omega \xi^\lambda u_\varepsilon^{(1-\alpha)/k} u_\varepsilon^{(\alpha-1)/k}\,[\mathcal{A}(x,u,\nabla u)\nabla u]^{1/k}$$

$$\le \left(\int_\Omega \xi^\lambda u_\varepsilon^{\alpha-1}\mathcal{A}(x,u,\nabla u)\nabla u\right)^{1/k}\left(\int_{\text{supp }\nabla\varphi_\rho} \xi^\lambda u_\varepsilon^{(1-\alpha)/(k-1)}\right)^{1/k'}. \tag{2.23}$$

Fix α as above, such that $\omega = \ell(k-1)/(1-\alpha) > 1$. Then

$$\int_\Omega \xi^\lambda u_\varepsilon^{(1-\alpha)/(k-1)} \le \left(\int_\Omega u_\varepsilon^\ell \xi^\lambda\right)^{1/\omega}\left(\int_\Omega \xi^\lambda\right)^{1/\omega'}. \tag{2.24}$$

Consequently

$$\begin{aligned}\int_\Omega \xi^\lambda\,[\mathcal{A}(x,u,\nabla u)\nabla u]^{1/k} &\le C\rho^{N/k'\omega'+(N-p\theta')/\theta' k}\left(\int_{\text{supp }\nabla\varphi_\rho} u_\varepsilon^\ell\xi^\lambda\right)^{1/\theta k+1/\omega k'}\\ &\le C\rho^{N(1-p/k\ell)-p/k}\left(\int_{\text{supp }\nabla\varphi_\rho} u_\varepsilon^\ell\xi^\lambda\right)^{p/k\ell},\end{aligned}$$

hence (2.21) follows from (2.22), (2.23) and (2.24).

If $\mathcal{A}$ is **S-**p**-C**, the estimate (2.21) gives an estimate for the gradient, which leads to the weak Harnack inequality, see [32], [36]: for any $\ell > 1$and $k > 1 + 1/\ell$,

$$\left(\fint_{\Omega,\varphi_\rho} |\nabla u|^{p/k}\right)^{k/p} \leq \frac{C}{\rho} \left(\fint_{\text{supp } \nabla\varphi_\rho} u^\ell\right)^{1/\ell}. \tag{2.25}$$

In particular in the radial case, it reduces to

$$\left|u'(\rho)\right| \leq C\ u(\rho)/\rho.$$

Remark 2.7. We supposed $\alpha \in [1-p, 0]$ in Proposition 2.1. If $\alpha < 1-p$, then (2.16) still holds, for any negative $\ell < p-1+\alpha$. Indeed we still have $\theta = \ell/(p-1+\alpha) > 1$.

2.4 Other estimates on f and u

Here we develop another approach, introduced in [4], and in [5] for a **S-**p**-C** operator. We show that it works for a **W-**p**-C** operator. For any solution u of equation (2.2),we give integral estimates of f in Ω_i, which do not depend on u.

Proposition 2.2 *Let $\Omega = \Omega_i = B_1 \setminus \{0\}$. Assume $N > p > 1$, and $\mathcal{A}$ is* **W-**p**-C**. *Let u be a nonnegative solution of (2.2).*

Then $f \in L^1_{loc}(\overline{B_1})$, and for any $\zeta \in \mathcal{D}(B_1)$ with values in $[0,1]$, such that $\zeta = 1$ near 0,

$$\int_{B_1} f\zeta^p \leq \int_{B_1} \mathcal{A}(x,u,\nabla u)\nabla(\zeta^p). \tag{2.26}$$

Proof Here we chose a test function of [4]. Let $0 < \rho < R < 1$, and $\zeta \in \mathcal{D}(\Omega)$ and $\xi_\rho \in C^\infty(\Omega)$ with values in $[0,1]$, such that

$$\zeta = 1 \quad \text{on } B_R, \quad \xi_\rho = 0 \quad \text{for } |x| \leq \rho, \quad \xi_\rho = 1 \quad \text{for } |x| \geq 2\rho,$$

$|\nabla\xi_\rho| \leq C/\rho$. Let $n \in \mathbb{N}$. We take

$$\phi = (n+1-u)^+(\xi_\rho\zeta)^p$$

as a test function. We get

$$\begin{aligned} & \int_{\{u<n+1\}} f(n+1-u)(\xi_\rho\zeta)^p + \int_{\{u<n+1\}} (\xi_\rho\zeta)^p \mathcal{A}(x,u,\nabla u)\nabla u \\ \leq\ & p\int_{\{u<n+1\}} \xi_\rho^{p-1}\zeta^p(n+1-u)\mathcal{A}(x,u,\nabla u)\nabla\xi_\rho \\ & + \int_{\{u<n+1\}} \xi_\rho^p(n+1-u)\mathcal{A}(x,u,\nabla u)\nabla(\zeta^p). \end{aligned}$$

Now for any $h>0$, we have $n+1-u>(n+1)h/(h+1)$ on the set $\{u<(n+1)/(h+1)\}$. Hence dividing by $n+1$,

$$\begin{aligned}
&\frac{h}{h+1}\int_{\{u<(n+1)/(h+1)\}} f(\xi_\rho\zeta)^p+\frac{1}{n+1}\int_{\{u<n+1\}}(\xi_\rho\zeta)^p\mathcal{A}(x,u,\nabla u)\nabla u\\
\leq\ & p\int_{\{u<n+1\}}\xi_\rho^{p-1}\zeta^p(1-\frac{u}{n+1})\mathcal{A}(x,u,\nabla u)\nabla\xi\\
&+\int_{\{u<n+1\}}(1-\frac{u}{n+1})\mathcal{A}(x,u,\nabla u)\nabla(\zeta^p)\\
\leq\ & p\varepsilon\int_{\{u<n+1\}}(\xi_\rho\zeta)^p\left|\mathcal{A}(x,u,\nabla u)\right|^{p'}+p\varepsilon^{1-p}\int_{\{u<n+1\}}|\nabla\xi_\rho|^p\\
&+\int_{\{u<n+1\}}(1-\frac{u}{n+1})\mathcal{A}(x,u,\nabla u)\nabla(\zeta^p),
\end{aligned}$$

for any $\varepsilon>0$. Choosing $\varepsilon=K/2p(n+1)$ gives

$$\begin{aligned}
&\frac{h}{h+1}\int_{\{u<(n+1)/2\}} f(\xi_\rho\zeta)^p+\frac{1}{2(n+1)}\int_{\{u<n+1\}}(\xi_\rho\zeta)^p\mathcal{A}(x,u,\nabla u)\nabla u\\
\leq\ & C(n+1)^{p-1}\rho^{N-p}+\int_{\{u<n+1\}}(1-\frac{u}{n+1})\mathcal{A}(x,u,\nabla u)\nabla(\zeta^p),
\end{aligned}$$

with $C=C(K,p)$. Now we make successively $\rho\to 0$, $n\to+\infty$, and $h\to+\infty$. Thus we get (2.26), which proves that $f\in L^1_{loc}(\overline{B_1})$.■

Remark 2.8 The two techniques give complementary results. In fact they have a commun idea: to multiply the inequality (2.2) by a function $P(u)\ \varphi$, where φ has compact support in Ω, and P *is decreasing in* u, in order to obtain some coercivity. In the first case, $P(u)=u^\alpha$ with $\alpha<0$, and in the second one, $P(u)=(n+1-u)^+$.

In the case of a **S-p-C** operator, the second method gives optimal estimates for u and ∇u in L^r spaces or in Marcinkiewicz spaces. Let us recall the main results of [4] and [5].

Proposition 2.3 *Let $\Omega=\Omega_i$. Assume $N\geq p>1$, and $\mathcal{A}$ is* **S-p-C**. *Let u be a nonnegative solution of (2.2). Then*

i) For any $\ell\in(0,N(p-1)/(N-p))$, and for $\rho>0$ small enough,

$$\left(\fint_{B_\rho}u^\ell\,dx\right)^{1/\ell}\leq\begin{cases}\leq C\,\rho^{-(N-p)/(p-1)}, & \text{if } N>p\\ \leq C\,\left|\ln\rho\right|, & \text{if } N=p.\end{cases}\tag{2.27}$$

For any $s\in(0,N(p-1)/(N-1))$, and for $\rho>0$ small enough,

$$\left(\fint_{B_\rho}|\nabla u|^s\,dx\right)^{1/s}\leq\begin{cases}\leq C\,\rho^{-(N-1)/(p-1)}, & \text{if } N>p\\ \leq C\,\rho^{-1}\left|\ln\rho\right|, & \text{if } N=p.\end{cases}\tag{2.28}$$

ii) Moreover if $N > p$,

$$u \in M^{N/(N-p)}_{loc}(\overline{B_1}), \qquad |\nabla u|^{p-1} \in M^{N/(N-1)}_{loc}(\overline{B_1}).$$

iii) If $N > p$, or $N = p$ and $\lim_{x\to 0} u(x) = +\infty$, then there exists $\beta \geq 0$ such that

$$-div\,[\mathcal{A}(x, u, \nabla u)] = f + \beta\,\delta_0 \qquad in\ \mathcal{D}'(B_1), \tag{2.29}$$

where δ_0 is the Dirac mass at 0.

Remark 2.9 These results can be *false in case of a* **W-p-C** *operator*: consider the equation involving the mean curvature operator:

$$-div(\nabla u/\sqrt{1 + |\nabla u|^2}) = u^Q$$

with $Q > 0$. From [6] it admits a singular radial solution near 0, such that

$$\lim_{|x|\to 0} |x|^{1/Q}\, u(x) = (N-1)^{1/Q}.$$

Hence it does not satisfy (2.27) when $N > 2$ and $Q < 1/(N-2)$. Moreover, choosing Q small enough, we see that for any $m > 0$ we can find a function u such that

$$-div(\nabla u/\sqrt{1 + |\nabla u|^2}) \geq 0$$

in Ω_i, and $u(x) \geq |x|^{-m}$ near 0. Observe that $\lim_{|x|\to 0} |\nabla u| = +\infty$, and $\mathcal{A}(x, u, \eta) = \eta/\sqrt{1 + |\eta|^2}$, hence $\mathcal{A}(x, u, \eta).\eta/\,|\eta|^2$ is not bounded from below for large $|\eta|$.

Now we give a corresponding result in Ω_e.

Proposition 2.4 *Let $\Omega = \Omega_e$. Assume $N > p > 1$, and $\mathcal{A}$ is* **S-p-C**. *Let u be a nonnegative solution of (2.2). Then*

i) for any $\ell \in (0, N(p-1)/(N-p))$, and for $\rho > 0$ large enough,

$$\left(\oint_{\mathcal{C}_{\rho/2,\rho}} u^\ell\, dx \right)^{1/\ell} \leq C; \tag{2.30}$$

ii) for any $s \in (0, N(p-1)/(N-1))$, and for $\rho > 0$ large enough,

$$\left(\oint_{\mathcal{C}_{\rho/2,\rho}} |\nabla u|^s\, dx \right)^{1/s} \leq C\, \rho^{-1}. \tag{2.31}$$

Proof We just give the scheme of the proof, since it is very similar to the proof of Proposition 2.3, given in [4] . Let $C_1 = 2\max_{|x|=2} u(x)$ and $u_1 = u - C_1$. For any $\rho > 2$, let $m_1(\rho) = \min_{|x|=\rho} u_1(x)$. Since we are looking for an upper estimate of $m_1(\rho)$, we can assume that $m_1(\rho) > 0$, and define

$$v(x) = \begin{cases} 0 & \text{if } \ |x| > \rho \text{ and } \ u_1(x) \le 0, \text{ or if } |x| \le 2, \\ u_1(x) & \text{if } \ 2 < |x| < \rho \text{ and } \ 0 \le u_1(x) \le m_1(\rho), \\ m_1(\rho) & \text{if } \ 2 < |x| < \rho \text{ and } \ u_1(x) > m_1(\rho), \text{ or if } |x| \ge \rho. \end{cases}$$

Take as test function

$$\phi = v - m_1(\rho)\eta,$$

where η is radial, with values in $[0,1]$, such that

$$\eta = 0 \quad \text{for } |x| \le 2, \quad \text{and} \quad \eta = 1 \text{ near infinity.}$$

Then using the capacity of the annulus $\mathcal{C}_{\rho,2\rho}$, we get the estimate

$$\min_{|x|=\rho} u(x) \le C(1 + \rho^{(p-N)/(p-1)}) \le C$$

for large ρ. We deduce (2.30) from the weak Harnack inequality, after recovering the annulus by a finite number of balls. Then (2.31) follows from (2.25) and (2.30). ■

2.5 Estimates on f in $\mathbb{R}^N \setminus \{0\}$

Combining the two techniques, we can extend some estimates in $\mathbb{R}^N$ to $\mathbb{R}^N \setminus \{0\}$ in the case $\mathcal{A}$ is **S-p-C**.

Proposition 2.5 *Let $\Omega = \mathbb{R}^N \setminus \{0\}$. Assume that $N > p > 1$ and $\mathcal{A}$ is* **S-p-C**. *Let u be a nonnegative solution of equation (2.2).*
Let $\varphi_\rho = \xi_\rho^\lambda$ with $\lambda > 0$ large enough, and $\xi_\rho \in \mathcal{D}(\mathbb{R}^N)$ with values in $[0,1]$, such that

$$\xi_\rho = 1 \quad \text{for } |x| \le \rho, \quad \xi_\rho = 0 \quad \text{for } |x| \ge 2\rho, \tag{2.32}$$

and $|\nabla \xi_\rho| \le C/\rho$.

Then (2.16) still holds for any $\rho > 0$, any $\alpha \in [1-p, 0)$ and $\ell > p - 1 + \alpha$.

Proof Let $\alpha < 0$. Let $0 < \delta < \rho/2$. Now we take

$$\phi = u_\varepsilon^\alpha \xi^\lambda, \quad \text{where} \quad \xi = \xi_\rho(1 - \xi_\delta),$$

as a test function. As in the Proposition 2.1, we get

$$\int_\Omega f u_\varepsilon^\alpha \xi^\lambda + \int_\Omega u_\varepsilon^{\alpha-1} \xi^\lambda \left|\mathcal{A}(x,u,\nabla u)\right|^{p'} \le C(\alpha) \int_\Omega u_\varepsilon^{\alpha+p-1} \xi^{\lambda-p} \left|\nabla \xi\right|^p$$

if $\alpha \geq 1-p$. Then for any for any $k, \ell > p-1+\alpha$,

$$\begin{aligned}
&\int_\Omega f u_\varepsilon^\alpha \xi^\lambda + \int_\Omega u_\varepsilon^{\alpha-1} \xi^\lambda \, |\mathcal{A}(x,u,\nabla u)|^{p'} \\
\leq \;& C(\alpha) \left(\int_{\text{supp } \nabla \xi_\delta} u_\varepsilon^k \xi^\lambda \right)^{1/\tau} \left(\int_{\text{supp } \nabla \xi_\delta} \xi^{\lambda - p\tau'} \, |\nabla \xi|^{p\tau'} \right)^{1/\tau'} \\
&+ C(\alpha) \left(\int_{\text{supp } \nabla \xi_\rho} u_\varepsilon^\ell \xi^\lambda \right)^{1/\theta} \left(\int_{\text{supp } \nabla \xi_\rho} \xi^{\lambda - p\theta'} \, |\nabla \xi|^{p\theta'} \right)^{1/\theta'}
\end{aligned}$$

where $\theta = \ell/(p-1+\alpha)$ and $\tau = k/(p-1+\alpha)$. Now we choose $k < N(p-1)/(N-p)$. As $\mathcal{A}$ is **S**-p-**C**, we can use the estimate (2.27) in the ball B_δ for δ small enough. Hence

$$\begin{aligned}
\left(\int_{\text{supp } \nabla \xi_\delta} u_\varepsilon^k \xi^\lambda \right)^{1/\tau} \left(\int_{\text{supp } \nabla \xi_\delta} \xi^{\lambda - p\tau'} \, |\nabla \xi|^{p\tau'} \right)^{1/\tau'} &\leq C \, \delta^{(N-k(N-p)/(p-1))/\tau + N/\tau' - p} \\
&\leq C \, \delta^{(N-p)|\alpha|/(p-1))}.
\end{aligned}$$

Now we can pass to the limit as $\delta \to 0$, since $\delta^{(N-p)|\alpha|/(p-1))} \to 0$, and $\xi_\delta \to 1$ *a.e.*. Hence we deduce that (2.18) is still available, and we reach the desired conclusion as in Proposition 2.1.■

2.6 Lower estimates on u

In this paragraph we look for lower estimates for the supersolutions. Consider for example any superharmonic C^2 function $u \geq 0$ in a domain Ω. From the strong maximum principle, either $u \equiv 0$ or $u > 0$. Moreover if $\Omega = \Omega_i$, then there exists $C > 0$ such that

$$u(x) \geq C \quad \text{for } 0 < |x| \leq 1/2.$$

Indeed from the Brezis-Lions Lemma (or its extension (2.29)), the function $f = -\Delta u \in L^1(B_{1/2})$, and

$$-\Delta u = f + \beta \, \delta_0 \qquad \text{in } \mathcal{D}'(B_{1/2}),$$

for some $\beta \geq 0$. Denoting by μ the solution of

$$-\Delta \mu = \delta_0 \quad \text{in } \mathcal{D}'(B_{1/2}), \quad \mu = 0 \quad \text{for } |x| = 1/2,$$

we have $u - \beta\mu \geq 0$ and

$$-\Delta(u - \beta\mu) = f \qquad \text{in } L^1(B_{1/2}),$$

and the conclusion holds from [37]. By the Kelvin transform, if now $\Omega = \Omega_e$, then there exists $C > 0$ such that

$$u(x) \geq C \; |x|^{2-N} \quad \text{for } |x| \geq 2.$$

Now we give some extensions of these properties. The method is taken from [4], Theorem 1.3.

Proposition 2.6 *Assume that $\mathcal{A}$ satisfies* $(\mathbf{H}_p)$. *Let u be a nonnegative solution of (2.2), with $u \neq 0$.*

i) Assume $\Omega = \Omega_i$. Then there exists $C_\rho > 0$ such that

$$u(x) \geq C \qquad \text{for } \ 0 < |x| \leq 1/2. \tag{2.33}$$

ii) Assume that $\Omega = \Omega_e$. Then there exists $C > 0$ such that

$$\begin{cases} u(x) \geq C \ \ |x|^{(p-N)/(p-1)} & \text{for } \ |x| \geq 2, \quad \text{if } N > p, \\ u(x) \geq C & \text{for } \ |x| \geq 2, \quad \text{if } N \leq p. \end{cases} \tag{2.34}$$

Proof From Remark 2.3, $\mathcal{A}$ satisfies the strong maximum principle, hence $u > 0$ in Ω. Now we use the function $\widetilde{\mathcal{A}}$ associated to $\mathcal{A}$ in this remark.

i) Let $m = \min_{|x|=1/2} u(x)$ and $s \in (0, m]$. Let $n \in \mathbb{N}^*$ be fixed, such that $n > 2$. Then by minimisation we can construct a radial solution of

$$\begin{cases} L_{\widetilde{\mathcal{A}}} u_n = 0 & \text{for } 1/n < |x| < 1/2, \\ u_n = s & \text{for } \ |x| = 1/2, \\ u_n = 0 & \text{for } \ |x| = 1/n. \end{cases}$$

Sinced u_n is monotone, $u_n \leq s$ in $\mathcal{C}_{1/n,1/2}$. If s is small enough, we have

$$L_{\mathcal{A}} u_n = L_{\widetilde{\mathcal{A}}} u_n = 0 \leq L_{\mathcal{A}} u$$

in $\mathcal{C}_{1/n,1/2}$. Then $u_n \leq u$ in $\mathcal{C}_{1/n,1/2}$ from the comparison principle. For any $a \in (0, 1/2)$, the sequence (u_n) is bounded in $C^{1,\theta}(\mathcal{C}_{a,1/2})$ for some $\theta \in (0,1)$ from [35], and $u_n \leq u_{n+1}$ on $\mathcal{C}_{a,1/2}$. Then it converges strongly in $C^1_{loc}(\overline{B_{1/2}})$ to a nonnegative radial solution of

$$\begin{cases} L_{\mathcal{A}} w = L_{\widetilde{\mathcal{A}}} w = 0 & \text{for } 0 < |x| < 1/2, \\ w = s & \text{for } \ |x| = 1/2, \end{cases}$$

such that $w \leq \min(u, s)$. So there exists a real C such that

$$A(|w'(r)|)w'(r) = Cr^{1-N} \tag{2.35}$$

in $(0, 1/2)$. Then $|C|\, r^{1-N} \leq M \ \ |w'(r)|^{p-1}$, from (2.12), hence $C = 0$, and $w \equiv s$. Hence $u \geq s > 0$ in $\overline{B_{1/2}}$.

ii) In the same way, let $m = \min_{|x|=2} u(x)$ and $s \in (0, m]$ small enough. As above, replacing $\mathcal{C}_{1/n,1/2}$ by $\mathcal{C}_{2,n}$, we construct a nonnegative radial solution of

$$\begin{cases} L_{\mathcal{A}} w = L_{\widetilde{\mathcal{A}}} w = 0 & \text{for } \ |x| \geq 2, \\ w = s & \text{for } \ |x| = 2, \end{cases}$$

such that $w \leq \min(u, s)$. Then there exists a real $C \leq 0$ such that (2.35) holds in $(2, +\infty)$, and $|C|\, r^{1-N} \leq M\, |w'(r)|^{p-1}$. First suppose that $C \neq 0$. If $N \neq p$, there

exists $C_1 > 0$ such that the function $r \mapsto w(r) - C_1 r^{(p-N)/(p-1)}$ is nonincreasing and bounded below, hence it has a limit $C_2 \geq 0$. Thus $N > p$ and (2.34) follows. If $N = p$, the function $r \mapsto w(r) + (C/M)\ln r$ is nonincreasing, which is impossible. Now suppose $C = 0$, hence $w \equiv C_2$, and (2.34) follows again. ■

Now we need another result in order to cover some critical cases. Notice that the assumptions on $\mathcal{A}$ are different in Ω_i and Ω_e, because the minorizing functions which we construct can be unbounded in the first case.

Proposition 2.7 *Let $N \geq p > 1$. Assume that $\mathcal{A}$ satisfies* $(\mathbf{H}_p)$**.** *Let u be a nonnegative solution of*

$$L_{\mathcal{A}} u \geq C \ |x|^{\lambda} \tag{2.36}$$

for some $\lambda \in \mathbb{R}$, and $C > 0$, with $u \neq 0$.

i) Assume $\Omega = \Omega_i$, and $\mathcal{A}$ is **S-p-C**. *Then $\lambda + N > 0$, and there exists $C > 0$ such that*

$$\begin{cases} u(x) \geq C \ |x|^{(\lambda+p)/(p-1)} & \text{for } \ 0 < |x| \leq 1/2, \quad \text{if } \lambda \neq -p, \\ u(x) \geq C_\rho \ |\ln |x|| & \text{for } \ 0 < |x| \leq 1/2, \quad \text{if } \lambda = -p. \end{cases} \tag{2.37}$$

ii) Assume that $\Omega = \Omega_e$,and $\lambda < 0$. Then $\lambda + p < 0$, and there exists $C > 0$ such that

$$\begin{cases} u(x) \geq C \ |x|^{(\lambda+p)/(p-1)} & \text{for } \ |x| > 2, \quad \text{if } \lambda \neq -N, \\ u(x) \geq C\,|x|^{(p-N)/(p-1)} \ (\ln |x|)^{1/(p-1)} & \text{for } \ |x| > 2, \quad \text{if } \lambda = -N. \end{cases} \tag{2.38}$$

Proof i) We know that $|x|^{\lambda} \in L^1_{loc}(\overline{B_1})$ from Proposition 2.2, hence $\lambda + N > 0$. Let $n > 2$ be fixed. Here $\mathcal{A}$ is **S-p-C,** hence we can construct a radial solution of

$$\begin{cases} L_{\mathcal{A}} u_n = C \ |x|^{\lambda} & \text{for } 1/n < |x| < 1/2, \\ u_n = s & \text{for } \ |x| = 1/2, \\ u_n = 0 & \text{for } \ |x| = 1/n. \end{cases}$$

Then $L_{\mathcal{A}} u_n \leq L_{\mathcal{A}} u$ in $\mathcal{C}_{1/n,1/2}$. Then $u_n \leq u$ in $\mathcal{C}_{1/n,1/2}$, and u_n converges strongly in $C^1_{loc}(\overline{B_{1/2}})$ to a nonnegative radial solution of

$$\begin{cases} L_{\mathcal{A}} w = C \ |x|^{\lambda} & \text{for } 0 < |x| < 1/2, \\ w = s & \text{for } \ |x| = 1/2. \end{cases}$$

Let us compute w. There exists a real D such that

$$-A(|w'(r)|)w'(r) = C\, r^{\lambda+1}/(\lambda + N) + D\, r^{1-N}. \tag{2.39}$$

If $D \neq 0$, then $|D| \ r^{1-N}/2 \leq M \ |w'(r)|^{p-1}$ for small r, hence w is monotone. If $w' > 0$, then w has a limit, hence w' is integrable, which is impossible, since $N \geq p$. Now

$$w(r) - w(2r) = -\int_r^{2r} w'(s)ds,$$

so there exists $C_1 > 0$ such that

$$w(r) \geq C_1 r^{(p-N)/(p-1)} \quad \text{if} \quad N \neq p, \quad \text{and} \quad w(r) \geq C_1 \left|\ln r\right| \quad \text{if} \quad N = p.$$

If $D = 0$, then $w' < 0$ and $Cr^{\lambda+1}/2(\lambda+N) \leq M \ |w'(r)|^{p-1}$, hence there exists $C_2 > 0$ such that

$$w(r) \geq C_2 r^{(\lambda+p)/(p-1)} \quad \text{if} \quad \lambda \neq -p, \quad \text{and} \quad w(r) \geq C_2 \left|\ln r\right| \quad \text{if} \quad \lambda = -p.$$

In any case (2.37) follows.

i) Let $\rho > 2$, and $m_\rho = \min_{|x|=\rho} u(x)$ and $s_\rho \in (0, m]$. Here we construct a radial solution of

$$\begin{cases} L_{\widetilde{\mathcal{A}}} u_n = C \ |x|^\lambda & \text{for } \rho < |x| < n, \\ u_n = s_\rho & \text{for } |x| = \rho, \\ u_n = 0 & \text{for } |x| = n. \end{cases}$$

If $|x| \geq \rho$, then $|x|^\lambda \leq \rho^\lambda$, since $\lambda < 0$. Hence if ρ is large enough, u_n remains sufficiently small, so that $L_{\widetilde{\mathcal{A}}} u_n = L_{\mathcal{A}} u_n$, hence $u_n \leq u$ for $\rho < |x| < n$. Then u_n converges strongly in $C^1_{loc}(\mathbb{R}^N \backslash B_\rho)$ to a nonnegative radial solution of

$$\begin{cases} L_{\mathcal{A}} w = C \ |x|^\lambda & \text{for } |x| > \rho, \\ w = s_\rho & \text{for } |x| = \rho. \end{cases}$$

Hence w is given by

$$-A(|w'(r)|)w'(r) = \begin{cases} Cr^{\lambda+1}/(\lambda+N) + D\, r^{1-N} & \text{if } \lambda \neq -N, \\ Cr^{1-N} \ln r + D\, r^{1-N} & \text{if } \lambda = -N. \end{cases}$$

If $\lambda > -N$, then $w' < 0$ for large r, and $Cr^{\lambda+1} \ /2(\lambda+N) \leq M \ |w'(r)|^{p-1}$. Now w has a limit m, hence w' is integrable at infinity, hence $\lambda < -p$, (and $N > p$). And

$$w(r) - m = -\int_r^\infty w'(s) ds \geq C_1 r^{(\lambda+p)/(p-1)} \tag{2.40}$$

for some $C_1 > 0$. If $\lambda < -N$, and $D = 0$, then $w' > 0$ and (2.38) follows. If $D \neq 0$, then $r^{\lambda+1} \leq |D|\, r^{1-N}/2 \leq M \ |w'(r)|^{p-1}$ for large r, hence either $w' > 0$, or (2.40) holds. If $\lambda = -N$, then $w' < 0$ and $r^{1-N} \ln r \leq 2M \ |w'(r)|^{p-1}$, and similarly $N > p$ and

$$w(r) - m \geq C_2 r^{(p-N)/(p-1)} (\ln r)^{1/(p-1)}$$

for some $C_2 > 0$. ■

3 The scalar case in $\mathbb{R}^N, \mathbb{R}^N \setminus \{0\}, \Omega_i$, or Ω_e

3.1 Upper or lower estimates

As a corollary we get general estimates for the inequality (1.1). They extend the former results of [7], Lemma A.2, relative to the Laplacian $\mathcal{A} = -\Delta$. Let us define, for any $Q \neq p-1$,

$$\Gamma = \frac{p+\sigma}{Q-p+1}, \tag{3.1}$$

and denote by

$$Q_\sigma = (N+\sigma)(p-1)/(N-p). \tag{3.2}$$

Notice that the equation

$$-\Delta_p u = |x|^\sigma u^Q, \tag{3.3}$$

admits a particular solution u^* when $Q \neq p-1$, given by

$$u^*(x) = C^*|x|^{-\Gamma}, \qquad C^* = \left[\Gamma(N-p-\Gamma(p-1))\right]^{1/(Q-p+1)}, \tag{3.4}$$

whenever $N > p$ and $0 < \Gamma < (N-p)/(p-1)$, which means $Q > Q_\sigma > p-1$, or $Q < Q_\sigma < p-1$.

Theorem 3.1 *Assume that $N \geq p > 1$, and $\mathcal{A}$ is* **W-p-C**. *Let u be a nonnegative solution of*

$$-div\,[\mathcal{A}(x,u,\nabla u)] \geq |x|^\sigma u^Q \tag{3.5}$$

in $\Omega = \Omega_i$ (resp. Ω_e), with $\sigma \in \mathbb{R}$.

i) If $Q > p-1$, then for small $\rho > 0$, (resp. large $\rho > 0$),

$$\left(\oint_{\mathcal{C}_{\rho/2,\rho}} u^Q\right)^{1/Q} \leq C\,\rho^{-\Gamma}. \tag{3.6}$$

ii) If $Q < p-1$, and $u > 0$ in Ω, then for any $\ell > p-1-Q$,

$$\left(\oint_{\mathcal{C}_{\rho/2,\rho}} u^\ell\right)^{1/\ell} \geq C\,\rho^{-\Gamma}. \tag{3.7}$$

If moreover $\mathcal{A}$ is **S-p-C**, *then either $u \equiv 0$, or*

$$u(x) \geq C\ |x|^{-\Gamma} \qquad \text{near } 0 \text{ (resp. near infinity).} \tag{3.8}$$

Proof First suppose $Q > p-1$. We apply Proposition 2.1 with $f = |x|^\sigma u^Q$ and $\ell = Q$, and $\Omega = \Omega_i$ (resp. Ω_e), and the corresponding function φ_ρ :

$$\oint_{\Omega,\varphi_\rho} |x|^\sigma u^Q \leq C\,\rho^{-p}\left(\oint_{\text{supp}\,\nabla\varphi_\rho} u^Q\varphi_\rho\right)^{(p-1)/Q} \leq C\,\rho^{-p}\left(\oint_{\Omega,\varphi_\rho} u^Q\right)^{(p-1)/Q},$$

from (2.17). But $\rho/4 \le |x| \le 2\rho$ in the support of φ_ρ, so

$$\oint_{\Omega,\varphi_\rho} u^Q \le C\ \rho^{-(p+\sigma)} \left(\oint_{\Omega,\varphi_\rho} u^Q \right)^{(p-1)/Q},$$

and $\varphi_\rho = 1$ for $\rho/2 \le |x| \le \rho$. Since $Q > p-1$, (3.6) follows.
Now suppose $Q \le p-1$. Then from (2.16), for any $\alpha \in [1-p, 0]$ and $\ell > p-1+\alpha$,

$$\oint_{\Omega,\varphi_\rho} |x|^\sigma u^{Q+\alpha} \le C\ \rho^{-p} \left(\oint_{\Omega,\varphi_\rho} u^\ell \right)^{(p-1+\alpha)/\ell}.$$

We can take $\alpha = -Q$. Thus if $u > 0$ in Ω, then

$$1 \le C\ \rho^{-(p+\sigma)} \left(\oint_{\Omega,\varphi_\rho} u^\ell \right)^{(p-1-Q)/\ell}. \tag{3.9}$$

If $Q < p-1$, this implies (3.7). Now assume that $u \ne 0$, and $\mathcal{A}$ is **S-**p**-C**. Then $\mathcal{A}$ satisfies the weak Harnack inequality. Hence $u > 0$, and for any $k \in (0, N(p-1)/(N-p))$, there exists a constant C such that

$$\left(\oint_{\mathcal{C}_{3\rho/4,5\rho/4}} u^k \right)^{1/k} \le C \min_{|x|=\rho} u, \tag{3.10}$$

as in [4], Lemma 1.2. Then taking $k = \ell$ in (3.7), and changing sligthly the function φ_ρ, we deduce that

$$\min_{|x|=\rho} u \ge C\ \rho^\Gamma,$$

proving (3.8).■

3.2 Case of an equation

Using the estimates of Theorem 3.1 with $\Omega = \Omega_i$, we can find again the behaviour near 0 in the case of an equation, given in [4], [5] for $\sigma = 0$, and extend it to the case $\sigma \ne 0$. This result is new.

Theorem 3.2 *Assume that $N \ge p > 1$, $\mathcal{A}$ is* **S-**p**-C,** *and*

$$0 < Q < Q_0 = N(p-1)/(N-p).$$

Let u be a nonnegative solution of

$$-div\,[\mathcal{A}(x, u, \nabla u)] = |x|^\sigma\, u^Q \tag{3.11}$$

in Ω_i. Then u satisfies Harnack inequality, and consequently there exists $C > 0$ such that

$$\begin{cases} u(x) \leq C\,|x|^{(p-N)/(p-1)}\,, & \text{if} \quad N > p, \\ u(x) \leq C\ \ |\ln|x||\,, & \text{if} \quad N = p, \\ u(x) \leq C\ \ |x|^{-\Gamma}\,, & \text{if} \quad Q > p-1. \end{cases} \tag{3.12}$$

If moreover $Q < Q_\sigma$ and $Q > p-1$, then either the singularity is removable, or there exists $C > 0$ such that for small $|x|$,

$$\begin{cases} C^{-1}\,|x|^{(p-N)/(p-1)} \leq u(x) \leq C\ \ |x|^{(p-N)/(p-1)}\,, & \text{if} \quad N > p, \\ C^{-1}\,|\ln|x|| \leq u(x) \leq C\ \ |\ln|x||\,, & \text{if} \quad N = p. \end{cases} \tag{3.13}$$

Proof First suppose $Q > p-1$. We write the equation under the form

$$-div\,[\mathcal{A}(x,u,\nabla u)] = h\ u^{p-1},$$

with

$$h = |x|^\sigma\, u^{Q-p+1}.$$

If $\sigma = 0$, we can conclude rapidly: we have $u^Q \in L^1(B_{1/2})$ from Proposition 2.2. Hence $h^s \in L^1(B_{1/2})$ for

$$s = Q/(Q-p+1) > N/p,$$

since $Q < Q_0$. Then we can apply Serrin's results of [32], and conclude. In the general case $\sigma \in \mathbb{R}$, we use the estimate (3.6):

$$\int_{\mathcal{C}_{\rho/2,\rho}} h^s = \int_{\mathcal{C}_{\rho/2,\rho}} |x|^{\sigma s}\, u^Q \leq \rho^{\sigma s} \int_{\mathcal{C}_{\rho/2,\rho}} u^Q \leq C\ \rho^{N+\sigma s-\Gamma Q}.$$

That means

$$\int_{\mathcal{C}_{\rho/2,\rho}} h^s \leq C\ \rho^{N-ps}. \tag{3.14}$$

But (3.14) implies the Harnack inequality, see [21], [36]. Then (3.12) follows from (2.27) and (3.6). Assume moreover $Q < Q_\sigma$. Then

$$\eta = (Q-p+1)(N-p)/(p-1) < p+\sigma,$$

hence we can choose some $t \in (N/p, N/(\eta-\sigma)^+)$. Then $h^t \in L^1(B_{1/2})$, since

$$\int_{\mathcal{B}_{1/2}} h^t = \int_{\mathcal{B}_{1/2}} |x|^{\sigma t}\, u^{(Q-p+1)t} \leq C \int_0^{1/2} r^{N-1+(\sigma-\eta)t},$$

and we can again apply [32].

Now suppose $Q \leq p-1$. We observe that

$$h(x) = |x|^\sigma\, u^{Q-p+1}(x) \leq C\ \ |x|^{-p}$$

near 0, from (3.8) if $Q < p-1$, and from (3.9) if $Q = p-1$. Then h satisfies (3.14) for any $s > 1$, and the Harnack inequality still holds.■

Remark 3.1 Notice that the critical exponent is Q_0 and not Q_σ. In the case of the semilinear problem

$$-\Delta u = |x|^\sigma u^Q,$$

Gidas and Spruck [20] have shown the Harnack property and the poinwise estimate

$$u(x) \leq C \ |x|^{-\Gamma}$$

in Ω_i or Ω_e, up to the critical value $Q_2^* = (N+2)/(N-2)$ (with $Q \neq (N+2+2\sigma)/(N-2)$, when $\sigma < 2$). In the general case of equation 3.11, Q_2^* is replaced by $Q_p^* = (N(p-1)+p)/(N-p)$, and the question is opened for general operators in the range $Q \in (Q_0, Q^*)$.

3.3 Non existence results

We begin by the case $Q > p-1$. The following theorem extends the results of [24],[26] and [4], [7].

Theorem 3.3 *Assume that $N \geq p > 1$, and $Q > p-1$.*

i) If $Q \leq Q_\sigma$, and $\mathcal{A}$ is **W-p-C**, *then the problem (3.5) in $\mathbb{R}^N$ has only the solution $u \equiv 0$.*

ii) If $Q < Q_\sigma$, $N > p$ and $\mathcal{A}$ is **S-p-C**, *then the same result holds in $\mathbb{R}^N \setminus \{0\}$.*

iii) If $Q \leq Q_\sigma$ and $\mathcal{A}$ satisfies $(\mathbf{H}_p)$, *then the same result holds in Ω_e.*

iv) Assume $\sigma \leq -p$. If $\mathcal{A}$ satisfies $(\mathbf{H}_p)$ *and is* **S-p-C**, *then the problem (3.5) in Ω_i has only the solution $u \equiv 0$.*

Proof i) We apply Proposition 2.1, and Theorem 3.1. We get

$$\int_{B_\rho} |x|^\sigma u^Q \leq C \ \rho^{N-p-N(p-1)/Q} \left(\int_{\mathcal{C}_{\rho,2\rho}} u^Q \right)^{(p-1)/Q} \leq C \ \rho^\theta, \tag{3.15}$$

from (2.17) and (3.6), with

$$\theta = N - p - (p-1)\Gamma = (N-p)(Q-Q_\sigma)/(Q-p+1) \leq 0. \tag{3.16}$$

If $\theta < 0$, then as $\rho \to +\infty$, we deduce that $\int_{\mathbb{R}^N} |x|^\sigma u^Q = 0$, hence $u \equiv 0$. If $\theta = 0$, then $|x|^\sigma u^Q \in L^1(\mathbb{R}^N)$, hence

$$\lim \int_{\mathcal{C}_{2^n,2^{n+1}}} |x|^\sigma u^Q = 0.$$

And

$$\int_{B_{2^n}} |x|^\sigma u^Q \le C \left(\int_{\mathcal{C}_{2^n,2^{n+1}}} |x|^\sigma u^Q \right)^{(p-1)/Q}$$

from (3.15), hence again $u \equiv 0$.

ii) Here we apply Proposition 2.5: we have only (2.16) for $\alpha \in [1-p, 0)$. Hence

$$\int_{B_\rho} |x|^\sigma u^{Q+\alpha} \le C\ \rho^{N-p-N(p-1+\alpha)/Q} \left(\int_{\mathcal{C}_{\rho,2\rho}} u^Q \right)^{(p-1+\alpha)/Q} \le C\ \rho^{\theta-\alpha\Gamma}.$$

But here $\theta < 0$, since $Q < Q_\sigma$. Choosing $|\alpha|$ small enough, we get the conclusion.

iii) Here we use the lower bounds for u given by the Propositions 2.6 and 2.7. If the problem has a nontrivial solution u in Ω_e, then from (3.6) and (2.34), we get, for large ρ,

$$C_1 \rho^{-(N-p)/(p-1)} \le \left(\oint_{\mathcal{C}_{\rho/2,\rho}} u^Q \right)^{1/Q} \le C_2\ \rho^{-\Gamma},$$

but this contradicts (3.16), unless $Q = Q_\sigma$. In that case we observe that

$$L_{\mathcal{A}} u \ge |x|^\sigma\, u^Q \ge C\ \ |x|^{\sigma-(N-p)Q/(p-1)} = C\ \ |x|^{-N}\,,$$

hence

$$C_1 \rho^{-(N-p)/(p-1)} \left(\ln \rho\right)^{1/(p-1)} \le \left(\oint_{\mathcal{C}_{\rho/2,\rho}} u^Q \right)^{1/Q} \le C_2\ \rho^{-\Gamma},$$

from (2.38), which is a contradiction.

iv) In the same way, if the problem has a nontrivial solution u in Ω_i, then from (3.6) and (2.33),

$$C_1 \le \left(\oint_{\mathcal{C}_{\rho/2,\rho}} u^Q \right)^{1/Q} \le C_2\ \rho^{-\Gamma},$$

which implies $\sigma \ge -p$. If $\sigma = -p$, then

$$C_1 \left|\ln \rho\right| \le \left(\oint_{\mathcal{C}_{\rho/2,\rho}} u^Q \right)^{1/Q} \le C_2\ \rho^{-\Gamma},$$

from (2.37), which is also contradictory.■

Remark 3.2 More generally, as in ([26]), let $b \in C(\mathbb{R}^N \setminus \{0\})$, $b(x) > 0$ in $\mathbb{R}^N$, $b(x) \ge |x|^\sigma$ for large $|x|$. If $p-1 < Q \le Q_\sigma$, and $\mathcal{A}$ is **W-**p-**C**, then the problem

$$-div\left[\mathcal{A}(x,u,\nabla u)\right] \ge b(x) u^Q, \qquad \text{in } \mathbb{R}^N$$

has only the solution $u \equiv 0$. If $p-1 < Q < Q_\sigma$,and $\mathcal{A}$ is **S-**p-**C**, then the same result holds in $\mathbb{R}^N \setminus \{0\}$.

Now we study the case $Q \le p-1$:

Theorem 3.4 *Assume that $N \geq p > 1$, and $Q \leq p-1$.*

i) If $\sigma > -p$, and $\mathcal{A}$ is **W-p-C**, *then the problem (3.5) in $\mathbb{R}^N$ has only the solution $u \equiv 0$.*

ii) If $\sigma > -p$, and $\mathcal{A}$ is **S-p-C,** *the problem (3.5) in Ω_e has only the solution $u \equiv 0$. If moreover$\mathcal{A}$ satisfies $(\mathbf{H}_p)$, this is also true in case $\sigma = -p$, $Q \neq p-1$.*

iii) If $Q_\sigma < Q$,and $\mathcal{A}$ is **S-p-C,** *the problem (3.5) in Ω_i has only the solution $u \equiv 0$. If moreover $\mathcal{A}$ satisfies $(\mathbf{H}_p)$, this is also true in case $Q = Q_\sigma \neq p-1$.*

Proof Suppose that the problem has a nontrivial solution u.

i) Here we apply Proposition 2.1 with Remark 2.7, following an idea of [26]. We use (2.16) with $\ell < p-1+\alpha < 0$. Hence for any $\alpha < -Q$, choosing $\ell = Q+\alpha$, we have

$$\int_{B_\rho} |x|^\sigma u^{Q+\alpha} \leq C\ \rho^{N(Q-p+1)/(Q+\alpha)-p} \left(\int_{\mathcal{C}_{\rho,2\rho}} u^{Q+\alpha} \right)^{(p-1+\alpha)/(Q+\alpha)}.$$

Thus

$$\left(\int_{\mathcal{C}_{\rho,2\rho}} u^{Q+\alpha} \right)^{(Q-p+1)/(Q+\alpha)} \leq C\ \rho^{N(Q-p+1)/(Q+\alpha)-p-\sigma},$$

and consequently

$$\int_{B_\rho} |x|^\sigma u^{Q+\alpha} \leq C\ \rho^\vartheta,$$

with

$$\vartheta = N - p + (p+\sigma)(\alpha+p-1)/(p-1-Q).$$

If $\sigma > -p$, we can choose $\alpha < -Q$ such that $\vartheta < 0$, which yields a contradiction.

ii) If $Q < p-1$, then for any $\ell \in (0, N(p-1)/(N-p))$, and large ρ,

$$C_1 \rho^{-\Gamma} \leq \left(\oint_{\mathcal{C}_{\rho/2,\rho}} u^\ell\, dx \right)^{1/\ell} \leq C_2,$$

from (2.30) and (3.8). Then $\sigma \leq -p$. If $\mathcal{A}$ satisfies $(\mathbf{H}_p)$, then

$$L_{\mathcal{A}} u \geq |x|^\sigma u^Q \geq C\ |x|^{\sigma-\Gamma Q}. \tag{3.17}$$

If $\sigma = -p$, then $\sigma - \Gamma Q = -p$, but $\sigma - \Gamma Q + p < 0$, from Proposition 2.7, hence a contradiction. If $Q = p-1$, then again $\sigma \leq -p$ from (3.9).

ii) In the same way, if $Q < p-1$, then for small ρ,

$$C_1 \rho^{-\Gamma} \leq \left(\oint_{\mathcal{C}_{\rho/2,\rho}} u^\ell\, dx \right)^{1/\ell} \leq C_2\ \rho^{-(N-p)/(p-1)},$$

from (2.27) and (3.8), hence $Q \leq Q_\sigma$. If $Q = Q_\sigma$, and $\mathcal{A}$ satisfies $(\mathbf{H}_p)$, then we have (3.17) with $\sigma - \Gamma Q = -N$. This again contradicts Proposition 2.7. If $Q = p-1$, then $\sigma \geq -p$ from (3.9), hence again $Q \leq Q_\sigma$.■

4 The scalar case in half spaces

Here we consider the same problems in the case $\Omega = \mathbb{R}^{N+}$, or Ω_e^+, Ω_i^+. Let us show why some difficulties appear.

4.1 Upper estimates

First we adapt Proposition 2.1 to our new case:

Proposition 4.1 *Let $\Omega = \mathbb{R}^{N+}$ (resp. Ω_i^+, resp. Ω_e^+). Assume that $N \geq p > 1$, and $\mathcal{A}$ is* **W-**p-**C**. *Let u be a nonnegative solution of equation (2.2). Let*

$$\zeta_\rho = x_N \xi_\rho$$

with ξ_ρ as in Proposition 2.1.

Then for any $\rho > 0$ (resp. for small $\rho > 0$, resp. for large $\rho > 0$), any $\alpha \in [1-p, 0]$ and $\ell > p - 1 + \alpha$, and for $\lambda > 0$ large enough,

$$\int_\Omega f u^\alpha \zeta_\rho^\lambda \leq C\ \rho^{(N+\lambda)(1-((p-1+\alpha)/\ell))-p} \left(\int_{supp\ \nabla\zeta_\rho} u^\ell \zeta_\rho^\lambda \right)^{(p-1+\alpha)/\ell}. \tag{4.1}$$

Proof As in Proposition 2.1, for any function $\zeta \in \mathcal{D}(\Omega)$ with values in $[0,1]$ and any $\lambda > 0$, for any $\alpha \in [1-p, 0)$ and $\ell > p - 1 + \alpha$, we obtain (2.19) and then (2.20). Now let us take $\zeta = \zeta_\rho = x_N \xi_\rho$. Then for any $m \leq \lambda$, we have

$$\zeta^{\lambda - m} |\nabla \zeta|^m \leq C((x_N)^{\lambda-m} + (x_N)^\lambda \rho^{-m}) \leq C\rho^{\lambda-m}.$$

Taking λ large enough, we get (4.1), if $\alpha \neq 0$, and also for $\alpha = 0$. ■

Now we extend the estimates:

Theorem 4.2 *Assume that $\mathcal{A}$ is* **W-**p-**C**. *Let u be a nonnegative solution of (3.5) in $\Omega = \Omega_i^+$ (resp. Ω_e^+). Let $\mathcal{K}$ be any cone in $\mathbb{R}^{N+}$ with vertex 0, axis $0x_N$ and half-angle $< \pi/2$. Then for small $\rho > 0$, (resp. for large $\rho > 0$),*

i) If $Q > p - 1$, there exists $C_\mathcal{K} > 0$, such that

$$\left(\oint_{\mathcal{K} \cap \mathcal{C}_{\rho/2,\rho}} u^Q \right)^{1/Q} \leq C_\mathcal{K}\ \rho^{-\Gamma}. \tag{4.2}$$

ii) Suppose $Q < p - 1$ and $u > 0$. Then for any $\ell > p - 1 - Q$, there exists $C_{\mathcal{K},\ell} > 0$, such that

$$\left(\oint_{\mathcal{K} \cap \mathcal{C}_{\rho/2,\rho}} u^Q \right)^{1/Q} \geq C_{\mathcal{K},\ell}\ \rho^{-\Gamma}. \tag{4.3}$$

Proof i) Case $Q > p-1$. Let us apply (4.1) with $\alpha = 0$ and $\ell = Q$:

$$\int_\Omega |x|^\sigma u^Q \zeta_\rho^\lambda \leq C \left(\int_\Omega u^Q \zeta_\rho^\lambda \right)^{(p-1)/Q} \rho^{(N+\lambda)(1-((p-1)/Q)-p},$$

hence

$$\left(\int_\Omega u^Q \zeta_\rho^\lambda \right)^{1-(p-1)/Q} \leq C \rho^{(N+\lambda)(1-((p-1)/Q)-p-\sigma}, \tag{4.4}$$

which implies

$$\int_{\mathbb{R}^{N+} \cap \mathcal{C}_{\rho/2,\rho}} u^Q x_N^\lambda \leq C \rho^{N+\lambda-\Gamma Q}. \tag{4.5}$$

But there exists a constant $c_\mathcal{K} > 0$ such that $x_N \geq c_\mathcal{K} \, |x|$ on $\mathcal{K}$, so that 4.2 holds.

ii) Case $Q < p-1$. Here we apply (4.1) with $\alpha = -Q$ and $\ell > p-1-Q$, and get (4.3) in the same way.■

Remark 4.1 Now the question is to obtain nonexistence results. We shall restrict to the case $Q > p-1$ for simplicity. Here the results are not complete for general operators. Indeed suppose that u is a solution of (3.5) in $\mathbb{R}^{N+}$. Then from (4.4), we deduce

$$\int_{\mathbb{R}^{N+} \cap B_\rho} u^Q x_N^\lambda \leq C \rho^{N+\lambda-\Gamma Q}.$$

Then there is no solution except 0 if $N + \lambda - \Gamma Q < 0$. But this result is not optimal, because we had to chose λ large enough. In the case of the Laplacian, the optimal range is $N + 1 - \Gamma Q \leq 0$, from [3], [13]. But $\lambda = 1$ is not admissible in the proof of Proposition (4.1). We shall reduce to two different cases.

4.2 Nonexistence for the p-Laplacian

First we consider the model case of the p-Laplacian. Since this operator does not depend on x, u, we look for lower estimates. In Section 2.6, such estimates have been obtained by comparison with the radial elementary p-harmonic functions in $\mathbb{R}^N \setminus \{0\}$, that means the functions

$$\Phi_{1,p}(r) \equiv 1, \qquad \Phi_{2,p}(r) = \begin{cases} r^{(p-N)/(p-1)} & \text{if } \ N > p \\ \ln r & \text{if } \ N = p. \end{cases}$$

In $\mathbb{R}^{N+}$ the same role is played by the elementary p-harmonic functions which vanish on the set $x_N = 0$. In the case $p = 2$, they are given by

$$\Psi_{1,2}(x) = x_N, \qquad \Psi_{2,2}(x) = \frac{x_N}{|x|^N} = \frac{\sin(x/|x|))}{|x|^{N-1}}$$

(up to a constant, $\Psi_{2,2}$ is the Poisson kernel). In the general case, they are given by

$$\Psi_{1,p}(x) = x_N, \qquad \Psi_{2,p}(x) = \frac{\varpi(x/|x|)}{|x|^{\beta_{p,N}}} \tag{4.6}$$

for some unique $\beta_{p,N} > 0$ and some unique positive $\varpi \in C^1(S^{N-1})$ with maximum value 1, from [22], Theorem 4.3. The exact value of $\beta_{p,N}$ is unknown if $p \neq 2$, except in the case $N = 2$, where $\beta_{p,2}$ is given by

$$\beta_{p,2} = \frac{3 - p + \sqrt{(p-1)^2 + 2 - p}}{3(p-1)}.$$

Now we can give lower bounds as in Section 2.6.

Proposition 4.3 *Assume that $N \geq p > 1$.*

i) Let $u \in C^1(\overline{\Omega_e^+})$, $u \geq 0$, $u \neq 0$, and super-p-harmonic. Then there exists $C > 0$ such that

$$u \geq C\ \Psi_{2,p} \qquad \text{in } 2\Omega_e^+. \tag{4.7}$$

ii) Let $u \in C^1(\overline{\Omega_i^+} \setminus \{0\})$, $u \geq 0$, $u \neq 0$, and super-p-harmonic. Then there exists $C > 0$ such that

$$u \geq C\ x_N \qquad \text{in } (1/2)\Omega_i^+. \tag{4.8}$$

Proof i) Case of Ω_e^+. We have $u > 0$ in Ω_e^+ from the strong maximum principle, and $\min_{|x|=2}(u(x)/\ x_N) > 0$ from the Hopf Lemma. The function $\Psi_{2,p}$ be defined by (4.6) is also in $C^1(\overline{\Omega_e^+})$, so that we can find $C > 0$, such that $u(x) \geq C\Psi_{2,p}(x)$ for $|x| = 2$. Now for any $\delta > 0$, $u + \delta$ is also super-p-harmonic. Comparing $u + \delta$ and $C\Psi_{2,p}$ on $\mathcal{C}_{2,n} \cap \mathbb{R}^{N+}$ for sufficiently large n, we get $u + \delta \geq C\Psi_{2,p}$ in $2\Omega_e^+$ from the weak maximum principle. Then (4.7) follows as δ tends to 0.

ii) Case of Ω_i^+. Similarly we can find another $C > 0$ such that $u(x) \geq C\Psi_{1,p}(x)$ for $|x| = 1/2$. Comparing $u + \delta$ and $C\Psi_{1,p}$ on $\mathcal{C}_{1/n,1/2} \cap \mathbb{R}^{N+}$ for sufficiently large n, we get in the same way $u \geq C\Psi_{1,p}$ in $(1/2)\Omega_i^+$. ■

Theorem 4.4 *Assume that $N \geq p > 1$, and $Q > p - 1$.*

i) If $Q < Q_{\sigma,p}$, where $Q_{\sigma,p} = p - 1 + (p + \sigma)/\beta_{p,N}$, the problem

$$-\Delta_p u \geq |x|^\sigma u^Q \qquad \text{in } \Omega_e^+, \tag{4.9}$$

with unknown $u \in C^1(\overline{\Omega_e^+})$ has only the solution $u \equiv 0$.

ii) If $Q + \sigma + 1 < 0$, the problem

$$-\Delta_p u \geq |x|^\sigma u^Q \qquad \text{in } \Omega_i^+, \tag{4.10}$$

with $u \in C^1(\overline{\Omega_i^+} \setminus \{0\})$ has only the solution $u \equiv 0$.

Proof Consider for example the cone $\mathcal{K} = \left\{x \in \mathbb{R}^{N+} \mid x_N \geq |x|/2\right\}$ of half-angle $\pi/3$. Then

$$\int_{\mathcal{K} \cap \mathcal{C}_{\rho/2,\rho}} u^Q \leq C\rho^{N - \Gamma Q}$$

from Theorem 4.2. First suppose that $\Omega = \Omega_e^+$. Then with other constants $C > 0$,

$$\int_{\mathcal{K}\cap\mathcal{C}_{\rho/2,\rho}} u^Q \geq C \int_{\mathcal{K}\cap\mathcal{C}_{\rho/2,\rho}} \Psi_{2,p}^Q(x) \geq C\rho^{N-Q\beta_{p,N}}$$

from Proposition 4.3. And consequently $\beta_{p,N} \geq \Gamma$, which means $Q \geq Q_{\sigma,p}$. Now suppose that $\Omega = \Omega_i^+$. Then

$$\int_{\mathcal{K}\cap\mathcal{C}_{\rho/2,\rho}} u^Q \geq C \int_{\mathcal{K}\cap\mathcal{C}_{\rho/2,\rho}} \Psi_{1,p}^Q(x) \geq C\rho^{N+Q},$$

so that $\Gamma \geq -1$, which means $Q + \sigma + 1 \geq 0$. ■

Remark 4.2 Obviously, if $Q \leq Q_{\sigma,p}$ or $Q + \sigma + 1 < 0$, the problem has no solution in $\mathbb{R}^{N+}$. When $p = 2$, we find again the results of [3] or [13], since

$$\beta_{2,N} = N - 1, \qquad Q_{\sigma,2} = (N + 1 + \sigma)/(N - 1),$$

but for the critical case, which requires a finer study. Notice that the condition $Q \leq Q_{\sigma,2}$ is sharp: if $Q > Q_{\sigma,2}$, there exists a positive solution $u \in C^1(\overline{\mathbb{R}^{N+}})$ of the inequality $-\Delta u \geq u^Q$ in $\mathbb{R}^{N+}$, from [13]. Moreover if $Q \in (Q_{\sigma,2}, (N+1)/(N-3))$, there exists a positive solution $u \in C^1(\overline{\mathbb{R}^{N+}} \setminus \{0\})$ of the equation $-\Delta u = |x|^\sigma u^Q$ in $\mathbb{R}^{N+}$: the proof is given in [12] when $\sigma = 0$, and it works also if $\sigma \neq 0$.

Remark 4.3 In the case of the equation

$$-\Delta_p u = |x|^\sigma u^Q$$

in Ω_i^+, a question is to extend the results of Corollary 3.2. Suppose for simplification that $\sigma = 0$ and $1 < Q < Q_{0,p}$. Do we get the estimates

$$u(x) \leq C\ \Psi_{2,p}(x)$$

near 0 ? The result is true when $p = 2$, from [12], and moreover either u behaves like $\Psi_{2,p}$, or the singularity is removable. The proofs lie on precise properties of the Green function of the Laplacian. The question is open for $p \neq 2$.

4.3 Nonexistence for second order operators

Here we give a nonexistence result in $\mathbb{R}^{N+}$ in a case where the operator depends on x and u.

Theorem 4.5 *Assume that $N \geq 2$, $\Omega = \mathbb{R}^{N+}$ and that $\mathcal{A}$ is given by*

$$\mathcal{A}_i(x, u, \eta) = \sum_{j=1}^{N} a_i(x, u)\eta_i \tag{4.11}$$

where $a_i : \mathbb{R}^{N+} \times \mathbb{R}^+ \to \mathbb{R}^+$ are continuous and bounded, and a_N is C^1 with bounded derivatives, and

$$\partial a_N(x,u)/\partial x_N \geq 0.$$

If

$$1 < Q < Q_{\sigma,2} = (N+1+\sigma)/(N-1),$$

then the problem (3.5) in $\mathbb{R}^{N+}$, with $u \in W^{1,1}_{loc}(\mathbb{R}^{N+})$, has only the solution $u \equiv 0$.

Proof Notice that the assumptions on the coefficients imply that $\mathcal{A}$ is **W-p-C.** Let $u_\varepsilon = u + \varepsilon$, and $\alpha \in (-1,0)$. Here we take the test function

$$\phi = u_\varepsilon^\alpha x_N \xi_\rho^\lambda$$

with $\lambda > 0$ large enough, and $\xi_\rho \in \mathcal{D}(\mathbb{R}^N)$, radial, with values in $[0,1]$, satisfying (2.32), and $|\nabla \xi_\rho| \leq C/\rho$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^{N+}} |x|^\sigma x_N u^Q u_\varepsilon^\alpha \xi_\rho^\lambda + |\alpha| \sum_{i=1}^N \int_{\mathbb{R}^{N+}} u_\varepsilon^{\alpha-1} x_N \xi_\rho^\lambda a_i(x,u) \left|\frac{\partial u}{\partial x_i}\right|^2 \\
\leq\ & \lambda \sum_{i=1}^N \int_{\mathbb{R}^{N+}} u_\varepsilon^\alpha x_N \xi_\rho^{\lambda-1} a_i(x,u) \frac{\partial u}{\partial x_i}\frac{\partial \xi}{\partial x_i} + \int_{\mathbb{R}^{N+}} u_\varepsilon^\alpha \xi_\rho^\lambda a_N(x,u) \frac{\partial u}{\partial x_N} \\
\leq\ & \frac{|\alpha|}{2} \sum_{i=1}^N \int_{\mathbb{R}^{N+}} u_\varepsilon^{\alpha-1} x_N \xi_\rho^\lambda a_i(x,u) \left|\frac{\partial u}{\partial x_i}\right|^2 + C \int_{\mathbb{R}^{N+}} u_\varepsilon^{\alpha+1} x_N \xi_\rho^{\lambda-2} \left|\frac{\partial \xi_\rho}{\partial x_i}\right|^2 \\
&+ \int_{\mathbb{R}^{N+}} u_\varepsilon^\alpha \xi_\rho^\lambda a_N(x,u) \frac{\partial u}{\partial x_N},
\end{aligned}$$

since the a_i are bounded. Then

$$\begin{aligned}
&\int_{\mathbb{R}^{N+}} |x|^\sigma x_N u^Q u_\varepsilon^\alpha \xi_\rho^\lambda + \sum_{i=1}^N \int_{\mathbb{R}^{N+}} u_\varepsilon^{\alpha-1} x_N \xi_\rho^\lambda a_i(x,u) \left|\frac{\partial u}{\partial x_i}\right|^2 \\
\leq\ & C(\alpha) \left(\sum_{i=1}^N \int_{\mathbb{R}^{N+}} u_\varepsilon^{\alpha+1} x_N \xi_\rho^{\lambda-2} \left|\frac{\partial \xi_\rho}{\partial x_i}\right|^2 + \int_{\mathbb{R}^{N+}} u_\varepsilon^\alpha \xi_\rho^\lambda a_N(x,u) \frac{\partial u}{\partial x_N} \right).
\end{aligned}$$

Now

$$\int_{\mathbb{R}^{N+}} u_\varepsilon^\alpha \xi_\rho^\lambda a_N(x,u) \frac{\partial u}{\partial x_N} = X_\varepsilon + Y_\varepsilon,$$

with

$$X_\varepsilon = \int_{\mathbb{R}^{N+}} u_\varepsilon^\alpha \xi_\rho^\lambda a_N(x,u_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_N}, \qquad Y_\varepsilon = \int_{\mathbb{R}^{N+}} u_\varepsilon^\alpha \xi_\rho^\lambda (a_N(x,u) - a_N(x,u_\varepsilon)) \frac{\partial u}{\partial x_N}.$$

Then

$$Y_\varepsilon \leq C\varepsilon \int_{\mathbb{R}^{N+}} u_\varepsilon^\alpha \xi_\rho^\lambda \left|\frac{\partial u_\varepsilon}{\partial x_{Ni}}\right| \leq C\varepsilon^{1+\alpha} \int_{\mathbb{R}^{N+}} \xi_\rho^\lambda \left|\frac{\partial u}{\partial x_N}\right| \leq C\varepsilon^{1+\alpha} \int_{\mathbb{R}^{N+}\cap \mathcal{C}_{\rho,2\rho}} \left|\frac{\partial u}{\partial x_N}\right|$$

since $\partial u/\partial x_N \in L^1_{loc}(\mathbb{R}^{N+})$. Now let

$$F(x,t) = \int_0^t s^\alpha a_N(x,s)ds, \qquad \forall (x,t) \in \mathbb{R}^{N+} \times \mathbb{R}^+.$$

Then

$$\begin{aligned} X_\varepsilon &= \int_{\mathbb{R}^{N+}} \xi_\rho^\lambda \frac{\partial(F(x,u_\varepsilon(x))}{\partial x_N} - \int_{\mathbb{R}^{N+}} \xi_\rho^\lambda \int_0^{u_\varepsilon(x)} s^\alpha \frac{\partial a_N}{\partial x_N}(x,u_\varepsilon(x))dsdx \\ &\leq \int_{\mathbb{R}^{N+}} \xi_\rho^\lambda \frac{\partial(F(x,u_\varepsilon(x))}{\partial x_N}, \end{aligned}$$

since $\partial a_N(x,u)/\partial x_N \geq 0$. Then integrating by parts on $\mathbb{R}^{N+} \cap \mathcal{C}_{\rho,2\rho}$,

$$X_\varepsilon \leq \lambda \int_{\mathbb{R}^{N+}} F(x,u_\varepsilon)\xi_\rho^{\lambda-1} \left|\frac{\partial \xi_\rho}{\partial x_N}\right| \leq C \int_{\mathbb{R}^{N+}} u_\varepsilon^{1+\alpha}\xi_\rho^{\lambda-1}\left|\frac{\partial \xi_\rho}{\partial x_N}\right|,$$

since $\xi_\rho = 0$ for $|x| = 2\rho$ and $\xi_\rho \geq 0$ for $x_N = 0$, and a_N is bounded. Now we have

$$\begin{aligned} \int_{\mathbb{R}^{N+}} |x|^\sigma x_N u^Q u_\varepsilon^\alpha \xi_\rho^\lambda &\leq C\varepsilon^{1+\alpha} \int_{\mathbb{R}^{N+}\cap \mathcal{C}_{\rho,2\rho}} \left|\frac{\partial u}{\partial x_N}\right| + C\int_{\mathbb{R}^{N+}} u_\varepsilon^{1+\alpha}\xi_\rho^{\lambda-1}\left|\frac{\partial \xi_\rho}{\partial x_N}\right| \\ &\quad + \sum_{i=1}^N \int_{\mathbb{R}^{N+}} u_\varepsilon^{\alpha+1} x_N \xi_\rho^{\lambda-2}\left|\frac{\partial \xi_\rho}{\partial x_i}\right|^2 \end{aligned}$$

and we can pass to the limit from the Fatou Lemma as $\varepsilon \to 0$. It follows that

$$\int_{\mathbb{R}^{N+}} |x|^\sigma x_N u^{Q+\alpha}\xi_\rho^\lambda \leq C\int_{\mathbb{R}^{N+}} u^{1+\alpha}\xi_\rho^{\lambda-1}\left|\frac{\partial \xi_\rho}{\partial x_N}\right| + \sum_{i=1}^N \int_{\mathbb{R}^{N+}} u^{\alpha+1}x_N\xi_\rho^{\lambda-2}\left|\frac{\partial \xi_\rho}{\partial x_i}\right|^2.$$

But from the Hölder inequality, setting $\theta = (Q+\alpha)/(1+\alpha)$,

$$\int_{\mathbb{R}^{N+}} u^{1+\alpha}\xi_\rho^{\lambda-1}\left|\frac{\partial \xi_\rho}{\partial x_N}\right| \leq C\varepsilon \int_{\mathbb{R}^{N+}} |x|^\sigma x_N u^{Q+\alpha}\xi_\rho^\lambda + \frac{C}{\varepsilon}\int_{\mathbb{R}^{N+}\cap \mathrm{supp}|\nabla\xi_\rho|} |x|^{\sigma/(\theta-1)} x_N \rho^{-2\theta'}\xi_\rho^\lambda$$

since $|\partial\xi_\rho/\partial x_N| = x_N\left|\xi_\rho'(\rho)\right|/\rho \leq Cx_N/\rho^2$. And

$$\begin{aligned} \sum_{i=1}^N \int_{\mathbb{R}^{N+}} u^{\alpha+1}x_N\xi_\rho^{\lambda-2}\left|\frac{\partial \xi_\rho}{\partial x_i}\right|^2 &\leq C\varepsilon \int_{\mathbb{R}^{N+}} |x|^\sigma x_N u^{Q+\alpha}\xi_\rho^\lambda \\ &\quad + \frac{C}{\varepsilon}\int_{\mathbb{R}^{N+}\cap \mathrm{supp}|\nabla\xi_\rho|} |x|^{\sigma/(\theta-1)} x_N \xi_\rho^{\lambda-2\theta'}\left|\nabla\xi_\rho\right|^{2\theta'} \end{aligned}$$

Hence choosing ε small enough,

$$\int_{\mathbb{R}^{N+}} |x|^\sigma x_N u^{Q+\alpha}\xi_\rho^\lambda \leq C\int_{\mathbb{R}^{N+}\cap \mathrm{supp}|\nabla\xi_\rho|} |x|^{\sigma(1-\theta')}x_N\rho^{-2\theta'}\xi_\rho^\lambda + |x|^{\sigma(1-\theta')}x_N\xi_\rho^{\lambda-2\theta'}\left|\nabla\xi_\rho\right|^{2\theta'},$$

and thus

$$\int_{\mathbb{R}^{N+}} |x|^\sigma x_N u^{Q+\alpha} \xi_\rho^\lambda \leq C\rho^\beta,$$

where

$$\beta(Q-1) = (N-1)Q - (N+1+\sigma) - (\sigma+2)\alpha.$$

If $Q < Q_{\sigma,2}$, then we can chose α such that $\beta < 0$, hence we get $u \equiv 0$ by making ρ tend to 0.■

Remark 4.4 As in Remark 2.3, this result can be extended to the case where $a_i(x,u)$ has a power growth in $|x|$, after changing the value of $Q_{\sigma,2}$.

5 The case of systems

5.1 A priori estimates

Now we consider the case of a fully coupled Hamiltonian system and more generally of a multipower system. First consider the system

$$\begin{cases} -\Delta_p u = |x|^a\, u^S\, v^R, \\ -\Delta_m v = |x|^b u^Q\, v^T, \end{cases} \tag{5.1}$$

where $p, m > 1$, $Q, R > 0$, $S, T \geq 0$, and $a, b \in \mathbb{R}$. If $QR \neq (p-1-S)(m-1-T)$, it admits a particular solution (u^*, v^*), given by

$$u^*(x) = A^*|x|^{-\gamma}, \qquad v^*(x) = B^*|x|^{-\xi}, \tag{5.2}$$

where

$$\gamma = \frac{(a+p)(m-1-T)+(b+m)R}{QR-(p-1-S)(m-1-T)}, \qquad \xi = \frac{(b+m)(p-1)+(a+p)Q}{QR-(p-1-S)(m-1-T)}. \tag{5.3}$$

for some constants A^*, B^* depending on N, p, m, a, b, whenever $0 < \gamma < (N-p)/(p-1)$ and $0 < \xi < (N-m)/(m-1)$. The condition

$$QR > (p-1-S)(m-1-T) \qquad (\text{resp} <)$$

corresponds for the system to the condition

$$Q > p-1 \qquad (\text{resp} <)$$

for the scalar case of equation (3.3).

Theorem 5.1 *Let $N \geq p, m > 1$. Let $\mathcal{A}$ and $\mathcal{B} : \Omega \times \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^N$ be respectively* **W-p-C** *and* **W-m-C**. *Let $u, v \in C^0(\Omega)$ with $u \in W^{1,p}_{loc}(\Omega)$, $v \in W^{1,m}_{loc}(\Omega)$, be nonnegative solutions of*

$$\begin{cases} -div\,[\mathcal{A}(x,u,\nabla u)] \geq |x|^a u^S v^R, \\ -div\,[\mathcal{B}(x,v,\nabla v)] \geq |x|^b u^Q v^T, \end{cases} \tag{5.4}$$

in $\Omega = \Omega_i$ *(resp.* Ω_e*), with* $a, b \in \mathbb{R}$, $Q, R > 0$, $S, T \geq 0$. *Assume that*

$$p - 1 - S > 0, \qquad m - 1 - T > 0. \tag{5.5}$$

i) Case $QR > (p-1-S)(m-1-T)$. *Suppose for example* $Q > p-1-S$. *Then for small* $\rho > 0$ *(resp. for large* $\rho > 0$*),*

$$\left(\oint_{\mathcal{C}_{\rho/2,\rho}} v^R \right)^{1/R} \leq C\ \rho^{-\xi}, \tag{5.6}$$

and if $R > m-1-T$, *or if* $\mathcal{B}$ *is* **S-**m**-C,**

$$\left(\oint_{\mathcal{C}_{\rho/2,\rho}} u^Q \right)^{1/Q} \leq C\ \rho^{-\gamma}, \tag{5.7}$$

if $R \leq m-1-T$, *without this assumption on* $\mathcal{B}$,

$$\left(\oint_{\mathcal{C}_{\rho/2,\rho}} u^\ell \right)^{1/\ell} \leq C\ \rho^{-\gamma}, \qquad \forall \ell \in (0, QR/(m-1-T))\,. \tag{5.8}$$

ii) Case $QR < (p-1-S)(m-1-T)$. *Suppose that*$\mathcal{A}$ *is* **S-**p**-C** *and* $\mathcal{B}$ *is* **S-**m**-C**. *Then either* $u, v \equiv 0$, *or*

$$u(x) \geq C\ |x|^{-\gamma}\,, \qquad v(x) \geq C\ |x|^{-\xi}\,. \tag{5.9}$$

Proof i) Case $QR > (p-1-S)(m-1-T)$.

First case: $R > m-1-T$, $Q > p-1-S$. Let us apply Proposition 2.1 to the first line of system (5.4): for any $\alpha \in (1-p, 0]$ and any $\varepsilon > 0$,

$$\oint_{\Omega,\varphi_\rho} |x|^a u_\varepsilon^{S+\alpha} v^R \leq C\ \rho^{-p} \left(\oint_{\Omega,\varphi_\rho} u^\ell \right)^{(p-1+\alpha)/\ell},$$

for any $\ell > p-1+\alpha$. We choose $\alpha = -S$. Then

$$\oint_{\Omega,\varphi_\rho} v^R \leq C\ \rho^{-(a+p)} \left(\oint_{\Omega,\varphi_\rho} u^\ell \right)^{(p-1-S)/\ell}. \tag{5.10}$$

By hypothesis, $Q > p-1-S$, hence we can choose $\ell = Q$, so that

$$\oint_{\Omega,\varphi_\rho} v^R \leq C\ \rho^{-(a+p)} \left(\oint_{\Omega,\varphi_\rho} u^Q \right)^{(p-1-S)/Q}. \tag{5.11}$$

Similarly, since $R > m-1-T$, we obtain

$$\oint_{\Omega,\varphi_\rho} u^Q \leq C\ \rho^{-(b+m)} \left(\oint_{\Omega,\varphi_\rho} v^R \right)^{(m-1-T)/R}, \tag{5.12}$$

This in turn implies

$$\oint_{\Omega,\varphi_\rho} v^R \leq C\ \rho^{-[a+p+(b+m)(p-1-S)/Q]}\ \left(\oint_{\Omega,\varphi_\rho} v^R\right)^{(p-1-S)(m-1-T)/QR},$$

and (5.6), (5.7) follow from (5.11) and (5.12).

Second case: $R < m-1-T$, $Q > p-1-S$. For any $\alpha \in (1-m, 0)$, from (2.16),

$$\oint_{\Omega,\varphi_\rho} v_\varepsilon^{\alpha+T}\ u^Q \leq C\ \rho^{-(b+m)}\ \left(\oint_{\Omega,\varphi_\rho} v^\lambda\right)^{(m-1+\alpha)/\lambda},$$

for any $\lambda > m-1+\alpha$. Let us set $\delta = \alpha + T$. Then if

$$\delta \in (1-m+T, R-m+1-T) \tag{5.13}$$

we can take $\lambda = R$, and get again (5.6). Now for any $\ell \in (p-1-S, Q)$,

$$\begin{aligned}\oint_{\Omega,\varphi_\rho} u^\ell &= \oint_{\Omega,\varphi_\rho} v_\varepsilon^{\delta\ell/Q}\ u^\ell\ v_\varepsilon^{-\delta\ell/Q} \\ &\leq \left(\oint_{\Omega,\varphi_\rho} v_\varepsilon^{\delta}\ u^Q\right)^{\ell/Q} \left(\oint_{\Omega,\varphi_\rho} v_\varepsilon^{-\delta\ell/(Q-\ell)}\right)^{1-\ell/Q}.\end{aligned}$$

Since $QR > (p-1-S)(m-1-T)$, we can choose ℓ such that

$$\max(p-1-S, \frac{QR}{R+m-1-T}) < \ell < \frac{QR}{m-1-T} \tag{5.14}$$

Then we determine δ by $\delta\ell/(Q-\ell) = -R$. By construction it satisfies (5.13). Thus

$$\oint_{\Omega,\varphi_\rho} u^\ell \leq C\ \rho^{-(b+m)\ell/Q}\ \left(\oint_{\Omega,\varphi_\rho} v^R\right)^{(m-1+\delta+T)\ell/QR+1-\ell/Q},$$

which means after computation

$$\oint_{\Omega,\varphi_\rho} u^\ell \leq C\ \rho^{-(b+m)\ell/Q}\ \left(\oint_{\Omega,\varphi_\rho} v^R\right)^{(m-1-T)\ell/QR}. \tag{5.15}$$

Reporting (5.15) into (5.10), we deduce

$$\oint_{\Omega,\varphi_\rho} v^R \leq C\ \rho^{-[a+p+(b+m)(p-1-S)/Q]}\ \left(\oint_{\Omega,\varphi_\rho} v^R\right)^{(p-1-S)(m-1-T)/QR},$$

as $\varepsilon \to 0$, and (5.6) follows. We deduce (5.8) from (5.15), for any ℓ verifying (5.14). At last consider any

$$\ell \leq \max(p-1, QR/(R+m-1)),$$

and choose $\lambda > 1$ such that $\lambda\ell$ satisfies (5.14). Then

$$\left(\oint_{\Omega,\varphi_\rho} u^\ell\right)^{1/\ell} \leq \left(\oint_{\Omega,\varphi_\rho} u^{\lambda\ell}\right)^{1/\lambda\ell} \leq C\ \rho^{-\gamma},$$

from the Jensen inequality, and (5.8) follows.

Now assume that $\mathcal{B}$ is **S-**m-**C**. We have (5.6), and want to obtain the corresponding estimate for u. We start from

$$\oint_{\Omega,\varphi_\rho} |x|^b u^Q \leq C\ \rho^{-m}\ \left(\oint_{\Omega,\varphi_\rho} v^\lambda\right)^{(m-1-T)/\lambda}$$

for any $\lambda > m-1-T$. If $v \equiv 0$, then also $u \equiv 0$, since $\mathcal{A}(x,0,0) = 0$, and reciprocally. Now assume that $u \neq 0, v \neq 0$. Then $v > 0$, from the strong maximum principle. Moreover from the weak Harnack inequality, choosing

$$\lambda \in (m-1, N(m-1)/(N-m))\,, \tag{5.16}$$

and changing slightly the function φ_ρ, we get

$$\int_{\mathcal{C}_{3\rho/4,5\rho/4}} u^Q \leq C\ \rho^{N-(m+b)}\ (\min_{|x|=\rho} v)^{m-1-T}. \tag{5.17}$$

Now we consider $\varepsilon > 0$ small enough, we take the power $R/(m-1-T)$ and integrate between $\rho(1-\varepsilon)$ and $\rho(1+\varepsilon)$: denoting $k = (N-m-b)R/(m-1-T)$,

$$\begin{aligned}\int_{\rho(1-\varepsilon)}^{\rho(1+\varepsilon)} \left(\int_{\mathcal{C}_{3r/4,5r/4}} u^Q\right)^{R/(m-1-T)} &\leq C \int_{\rho(1-\varepsilon)}^{\rho(1+\varepsilon)} r^k\ (\min_{|x|=r} v)^R \\ &\leq C_\varepsilon\ \rho^{k-N+1} \int_{\mathcal{C}_{\rho(1-\varepsilon),\rho(1+\varepsilon)}} v^R, \end{aligned} \tag{5.18}$$

Hence in particular

$$\left(\int_{\mathcal{C}_{3\rho(1+\varepsilon)/4,5\rho(1-\varepsilon)/4}} u^Q\right)^{R/(m-1-T)} \leq C_\varepsilon\ \rho^{k-N} \int_{\mathcal{C}_{\rho(1-\varepsilon),\rho(1+\varepsilon)}} v^R,$$

that means

$$\oint_{\mathcal{C}_{3\rho(1+\varepsilon)/4,5\rho(1-\varepsilon)/4}} u^Q \leq C_\varepsilon\ \rho^{-(b+m)} \left(\oint_{\mathcal{C}_{\rho(1-\varepsilon),\rho(1+\varepsilon)}} v^R\right)^{(m-1-T)/R}.$$

But from (5.10), after another change of φ_ρ, we find

$$\oint_{\mathcal{C}_{\rho(1-\varepsilon),\rho(1+\varepsilon)}} v^R \leq C_\varepsilon\ \rho^{-(a+p)}\ \left(\oint_{\mathcal{C}_{\rho(1-\varepsilon),\rho(1+\varepsilon)}} u^Q\right)^{(p-1-S)/Q}.$$

Taking $\varepsilon = 1/10$, we have $\mathcal{C}_{\rho(1-\varepsilon),\rho(1+\varepsilon)} \subset \mathcal{C}_{3\rho(1+\varepsilon)/4,5\rho(1-\varepsilon)/4}$, hence

$$\oint_{\mathcal{C}_{33\rho/40,45\rho/40}} u^Q \leq C\rho^{-[b+m+(a+p)(m-1-T)/R]} \left(\oint_{\mathcal{C}_{33\rho/40,45\rho/40}} u^Q \right)^{(p-1-S)(m-1-T)/QR}, \tag{5.19}$$

hence (5.8) follows by a simple covering.

ii) Case $QR < (p-1-S)(m-1-T)$. Here $\mathcal{A}$ and $\mathcal{B}$ are strongly coercive. Then $u, v > 0$, and in the same way, for any $\varepsilon' > 0$ small enough,

$$\left(\int_{\mathcal{C}_{3\rho(1+\varepsilon')/4,5\rho(1-\varepsilon')/4}} v^R \right)^{Q/(p-1-S)} \leq C_{\varepsilon'}\ \rho^{(N-p-a)Q/(p-1-S)-N} \int_{\mathcal{C}_{\rho(1-\varepsilon'),\rho(1+\varepsilon')}} u^Q.$$

Then taking for example $\varepsilon = \varepsilon' = 1/10$, we have $\mathcal{C}_{\rho(1-\varepsilon),\rho(1+\varepsilon)} \subset \mathcal{C}_{3\rho(1+\varepsilon')/4,5\rho(1-\varepsilon')/4}$, and $\mathcal{C}_{\rho(1-\varepsilon'),\rho(1+\varepsilon')} \subset \mathcal{C}_{3\rho(1+\varepsilon)/4,5\rho(1-\varepsilon)/4}$, hence we get again (5.19). It implies

$$\left(\oint_{\mathcal{C}_{33\rho/40,45\rho/40}} v^R \right)^{1/R} \geq C\ \rho^{-\xi},$$

$$\left(\oint_{\mathcal{C}_{33\rho/40,45\rho/40}} u^Q \right)^{1/Q} \geq C\ \rho^{-\gamma}.$$

We can assume that $R < m-1-T$. Then

$$C\ \rho^{-\xi} \leq \left(\oint_{\mathcal{C}_{3\rho/4,5\rho/4}} v^R \right)^{1/R} \leq \left(\oint_{\mathcal{C}_{3\rho/4,5\rho/4}} v^\lambda \right)^{1/\lambda}$$

for any $\lambda \geq R$. Choosing $\lambda \in (m-1, N(m-1)/(N-m))$, we deduce that

$$v(x) \geq C\ |x|^{-\xi}$$

from the Harnack inequality. Then

$$u(x) \geq C\ |x|^{-\gamma}$$

from (5.17).■

Remark 5.1 In the case of the system with two degenerated Laplacian operators

$$\begin{cases} -\Delta_p u \geq |x|^a\ u^S\ v^R, \\ -\Delta_m v \geq |x|^b\ u^Q\ v^T, \end{cases} \tag{5.20}$$

the proof of Theorem 5.1 can be shortened by a reduction to the case $S = T = 0$, which means to the case of a Hamiltonian system. Indeed, from the strict maximum principle, we can assume that u, v are positive. Let $\theta, \tau \in (0,1)$. We set

$$w = u^\theta, \quad z = v^\tau.$$

Then w is super-p-harmonic, z is super-m-harmonic, and

$$\begin{cases} -\Delta_p w \geq C\ |x|^a\ w^{[S-(1-\theta)(p-1)]/\theta}\ z^{R/\tau}, \\ -\Delta_m z \geq C\ |x|^b\ w^{Q/\theta}\ z^{[T-(1-\tau)(m-1)]/\tau}, \end{cases}$$

for some $C > 0$. As $S < p-1, T < m-1$, we can choose $\theta = 1 - S/(p-1)$, $\tau = 1 - T/(m-1)$, and we find

$$\begin{cases} -\Delta_p w \geq C\ |x|^a\ z^{R_0}, \\ -\Delta_m z \geq C\ |x|^b\ w^{Q_0}\ , \end{cases} \tag{5.21}$$

with $R_0 = R/\tau$ and $Q_0 = Q/\theta$. And the condition $Q_0 > p-1$ (resp $R_0 > m-1$) is equivalent to $Q > p-1-S$ (resp.$R > m-1-T$).

Remark 5.2 In the case of the system

$$\begin{cases} -\Delta u \geq |x|^a\ u^S\ v^R, \\ -\Delta v \geq |x|^b u^Q\ v^T, \end{cases}$$

theorem 5.1 can be obtained in another way: we observe that the mean values of u and v satisfy

$$\begin{cases} -\Delta \overline{u} \geq r^a\ \overline{u^S\ v^R}, \\ -\Delta \overline{v} \geq r^b \overline{u^Q\ v^T}. \end{cases}$$

Now u, v are superharmonic, hence there exists a constant $C = C(N)$ such that

$$u(x) \geq C\ \overline{u}(|x|), \qquad v(x) \geq C\ \overline{v}(|x|),$$

from [9], Lemma 2.2. Then

$$\begin{cases} -\Delta \overline{u} \geq r^a\ \overline{u}^S \overline{v}^R, \\ -\Delta \overline{v} \geq r^b \overline{u}^Q \overline{v}^T, \end{cases}$$

so that the study reduces to the radial case.

5.2 Case of a system of equations

Here we give an extension of Corollary 3.2 to the case of an Hamiltonian system of equations. Here we suppose that $m = p$.

Theorem 5.2 *Assume that $N > p = m > 1$, and*

$$p-1 < Q < Q_0, \qquad p-1 < R < Q_0,$$

and $\mathcal{A}, \mathcal{B}$ are **S**-p-**C**. *Let $u, v \in C^0(\Omega_i) \cap W^{1,p}_{loc}(\Omega_i)$, be nonnegative solutions of*

$$\begin{cases} -div\,[\mathcal{A}(x,u,\nabla u)] = |x|^a\ v^R, \\ -div\,[\mathcal{B}(x,v,\nabla v)] = |x|^b\ u^Q, \end{cases} \tag{5.22}$$

in Ω_i. Then there exists $C > 0$ such that near 0,

$$u(x) \le C\ |x|^{-\gamma}\,, \qquad v(x) \le C\ |x|^{-\xi}\,. \tag{5.23}$$

If moreover $R < Q_a$ and $Q < Q_b$, then

$$u(x) + v(x) \le C\ |x|^{(p-N)/(p-1)} \tag{5.24}$$

Proof Since u is a solution of an equation, it is also a subsolution. By using Moser's iterative methods as in [32], or [36], for any ball $B(x, 4r) \subset \Omega_i$, and any $\ell > 1$, and $s > N/p$,

$$\begin{aligned}\sup_{B(x,r)} u \quad &\le \quad C \left(\oint_{B(x_0,2r)} u^\ell \right)^{1/\ell} \\ &\qquad + C \left(r^{p-N/s} \left\| |x|^a\, v^R \right\|_{L^s(B(x_0,4r))} \right)^{1/(p-1)},\end{aligned}$$

where $C = C(N, p, K)$, see also [**?**]. In the same way, for any $\varepsilon \in (0, 1/2)$ and any ball $B(x, 2r) \subset \Omega_i$,

$$\begin{aligned}\sup_{B(x,r)} u \quad &\le \quad C\, \varepsilon^{-N/p} \left(\oint_{B(x,r(1+\varepsilon))} u^\ell \right)^{1/\ell} \\ &\qquad + C\, \varepsilon^{-N/p} \left(r^{p-N/s} \left\| |x|^a\, v^R \right\|_{L^s(B(x,r(1+4\varepsilon)))} \right)^{1/(p-1)}.\end{aligned}$$

In particular for any $x \in (1/2)\Omega_i$,

$$\begin{aligned}\sup_{B(x,|x|/2)} u \quad &\le \quad C\, \varepsilon^{-N/p} \left(\oint_{B(x,(1+\varepsilon)\frac{|x|}{2})} u^\ell \right)^{1/\ell} \\ &\qquad + C\, \varepsilon^{-N/p} \left(|x|^{a+p-N/s} \left\| v^R \right\|_{L^s(B(x,(1+4\varepsilon)\frac{|x|}{2}))} \right)^{1/(p-1)}.\end{aligned}$$

Now $R < N(p-1)/(N-p)$, hence we can find $\beta \in (0, 1)$ such that

$$s = R/(R - (p-1)\beta) > N/p.$$

And $v^R = v^{(p-1)\beta} v^{R/s}$. Then

$$\left\| v^R \right\|^{1/(p-1)}_{L^s(B(x,(1+4\varepsilon)\frac{|x|}{2}))} \le \Big(\sup_{B(x,(1+4\varepsilon)\frac{|x|}{2})} v \Big)^\beta \times \left(\int_{B(x,(1+4\varepsilon)\frac{|x|}{2})} v^R \right)^{1/(p-1)s}.$$

Now from the upper estimate (5.6) and (5.8), taking $\ell = Q > p - 1$,

$$\begin{aligned}\sup_{B(x,\frac{|x|}{2})} u \quad &\le \quad C\, \varepsilon^{-N/p} \left(|x|^{-\gamma} + \left(|x|^{a+p-\xi R/s} \right)^{1/(p-1)} \Big(\sup_{B(x,(1+4\varepsilon)\frac{|x|}{2})} v \Big)^\beta \right) \\ &\le \quad C\, \varepsilon^{-N/p} \left(|x|^{-\gamma} + |x|^{(a+p-\xi R/s)/(p-1)} \Big(\sup_{B(x,(1+4\varepsilon)\frac{|x|}{2})} v \Big)^\beta \right).\end{aligned}$$

And $\xi R = a + p + (p-1)\gamma$, hence $(a + p - \xi R/s)/(p-1) = \beta\xi - \gamma$, so that

$$\sup_{B(x,\frac{|x|}{2})} |x|^\gamma\, u \leq C\, \varepsilon^{-N/p} \left(1 + (\sup_{B(x,(1+4\varepsilon)|x|/2)} |x|^\xi\, v)^\beta \right).$$

This implies with a new constant C

$$1 + \sup_{B(x,\frac{|x|}{2})} |x|^\gamma\, u \leq C\, \varepsilon^{-N/p} \left(1 + \sup_{B(x,(1+4\varepsilon)\frac{|x|}{2})} |x|^\xi\, v \right)^\beta.$$

In the same way, if $\varepsilon < 1/8$, choosing such that $Q/(R-(p-1)\beta') > N/p$,

$$1 + \sup_{B(x,(1+4\varepsilon)\frac{|x|}{2})} |x|^\xi\, v \leq C\, \varepsilon^{-N/p} \left(1 + \sup_{B(x,(1+16\varepsilon)\frac{|x|}{2})} |x|^\gamma\, u \right)^{\beta'},$$

since $p-1 < Q < N(p-1)/(N-p)$. Then

$$1 + \sup_{B(x,|x|/2)} |x|^\gamma\, u \leq C\, \varepsilon^{-N(1+\beta)/p} \left(1 + \sup_{B(x,(1+16\varepsilon)|x|/2)} |x|^\gamma\, u \right)^{\beta\beta'}.$$

Using the bootstrap technique of [8], Lemma 2.2, we deduce that

$$1 + \sup_{B(x,\frac{|x|}{2})} |x|^\gamma\, u \leq C \left(1 + \sup_{B(x,\frac{|x|}{2})} |x|^\gamma\, u \right)^{\beta\beta'}$$

for another constant C, since $\beta\beta' < 1$. It follows that u, and similarly v satisfy the punctual estimate (5.23). Moreover, from the weak Harnack inequality,

$$\begin{aligned} \sup_{B(x,\frac{|x|}{2})} u \;\leq\; & C\, \varepsilon^{-N/p} \inf_{B(x,(1+\varepsilon)\frac{|x|}{2})} u \\ & + C\, \varepsilon^{-N/p}\, |x|^{(a+p)/(p-1)} (\sup_{B(x,(1+4\varepsilon)\frac{|x|}{2})} v)^\beta \times (\inf_{B(x,(1+4\varepsilon)\frac{|x|}{2})} v)^{R/(p-1)-\beta} \end{aligned}$$

But from the estimate (2.27) of Proposition 2.3, if $N > p$,

$$\begin{aligned} \sup_{B(x,\frac{|x|}{2})} u \;\leq\; & C\, \varepsilon^{-N/p}\, |x|^{-(N-p)/(p-1)} \\ & + C\, \varepsilon^{-N/p}\, |x|^{(a+p)/(p-1)-((N-p)/(p-1))(R/(p-1)-\beta)} (\sup_{B(x,(1+4\varepsilon)\frac{|x|}{2})} v)^\beta, \end{aligned}$$

$$\begin{aligned} \sup_{B(x,\frac{|x|}{2})} |x|^{(N-p)/(p-1)}\, u \;\leq\; & C\, \varepsilon^{-N/p} \\ & + C\, \varepsilon^{-N/p} \times |x|^m (\sup_{B(x,(1+4\varepsilon)\frac{|x|}{2})} |x|^{(N-p)/(p-1)}\, v)^\beta, \end{aligned}$$

whith

$$m = [(N+a)(p-1) - ((N-p)R]\,/(p-1).$$

If $R < Q_a$, then $m > 0$, hence

$$\sup_{B(x,|x|/2)} |x|^{(N-p)/(p-1)}\, u \le C\, \varepsilon^{-N/p} \left(1 + \sup_{B(x,(1+4\varepsilon)|x|/2)} |x|^{(N-p)/(p-1)}\, v\right)^{\beta}.$$

If $Q < Q_b$, we get in the same way

$$\sup_{B(x,(1+4\varepsilon)\frac{|x|}{2})} |x|^{(N-p)/(p-1)}\, v \le C\, \varepsilon^{-N/p} \left(1 + \sup_{B(x,(1+16\varepsilon)\frac{|x|}{2})} |x|^{(N-p)/(p-1)}\, u\right)^{\beta'}$$

and we deduce (5.24) by applying the bootstrap technique.■

Remark 5.3 This result is new for quasilinear operators $\mathcal{A}, \mathcal{B}$, even if it is not optimal. Suppose for simplicity that $a = b = 0$. We get estimates of the type 5.24 in the square $p-1 < Q, R < Q_0$. We presume that it remains true in the region

$$\max(\gamma, \xi) > N(p-1)/(N-p),$$

(even for a multipower system), when $\mathcal{A} = \mathcal{B}$ satisfy $(\mathbf{H}_p)$. Indeed this has been proved in [7] in case $p = 2$, $\mathcal{A} = \mathcal{B} = -\Delta$. The proof lies on a comparison property of the solutions, which cannot exist in the general case, where $\mathcal{A} = \mathcal{B}$ depend on u, v. In the general case the result could be obtained by precising the weak Harnack inequality as in [1].

5.3 Nonexistence results

Now let us give nonexistence results for system (5.4) in $\mathbb{R}^N$, $\mathbb{R}^N \setminus \{0\}$, Ω_e, Ω_i.

Theorem 5.3 *Let us make the assumptions of Theorem 5.1, with $N \ge p, m > 1$, and $QR > (p-1-S)(m-1-T)$.*

i) Assume that

$$\max\left((p-1)\gamma - (N-p), (m-1)\xi - (N-m)\right) \ge 0, \tag{5.25}$$

and $Q > p-1-S$ and ($R > m-1-T$ or $\mathcal{B}$ is **S**-m-**C***). If (u,v) is a solution of 5.4 in $\mathbb{R}^N$, then $u \equiv 0$ or $v \equiv 0$.*

ii) Assume that the inequality is strict in (5.25), $N > p, m$ and $\mathcal{A}$ is **S**-p-**C***, and $\mathcal{B}$ is* **S**-m-**C***, then the same result holds in $\mathbb{R}^N \setminus \{0\}$.*

iii) Assume that $\mathcal{A}, \mathcal{B}$ satisfy $(\mathbf{H}_p)$, $(\mathbf{H}_m)$. If (5.25) holds and (u,v) is a solution of (5.4) in Ω_e , then $u \equiv 0$ or $v \equiv 0$. If

$$\max(\gamma, \xi) > 0 \tag{5.26}$$

or

$$\max(\gamma,\xi)\geq 0 \text{ and } \mathcal{A} \text{ is } \mathbf{S}\text{-}p\text{-}\mathbf{C}, \text{ and } \mathcal{B} \text{ is } \mathbf{S}\text{-}m\text{-}\mathbf{C},$$

and (u,v) *is a solution of (5.4) in* Ω_i *, then* $u\equiv 0$ *or* $v\equiv 0$.

Proof i) We can assume that $Q>p-1-S$. Then from (2.16) with $\alpha=0$, and (5.8),

$$\int_{B_\rho}|x|^a v^R \leq C\ \rho^{N-p-N(p-1-S)/Q}\left(\int_{\mathcal{C}_{\rho,2\rho}} u^Q\right)^{(p-1-S)/Q} \leq C\ \rho^{\theta_1},$$

with

$$\theta_1 = N-p-(p-1)\gamma.$$

First assume $R>m-1-T$. Then, from (5.6),

$$\int_{B_\rho}|x|^b u^Q \leq C\ \rho^{N-m-N(m-1-T)/R}\left(\int_{\mathcal{C}_{\rho,2\rho}} v^R\right)^{(m-1-T)/R} \leq C\ \rho^{\theta_2}, \tag{5.27}$$

with

$$\theta_2 = N-m-(m-1)\xi.$$

And (5.25) means that $\theta_1\leq 0$ or $\theta_2\leq 0$. If $\theta_2<0$, then as $\rho\to+\infty$ we get $\int_{\mathbb{R}^N}|x|^b u^Q=0$, hence $u\equiv 0$. In the same way, if $\theta_1<0$, then $v\equiv 0$. If $\theta_2=0\leq\theta_1$, then $|x|^\sigma u^Q\in L^1(\mathbb{R}^N)$, hence

$$\lim\int_{\mathcal{C}_{2^n,2^{n+1}}}|x|^b u^Q = 0.$$

Now we also have

$$\int_{B_\rho}|x|^a v^R \leq C\ \rho^{N-p-N(p-1-S)/Q}\left(\int_{\mathcal{C}_{\rho,2\rho}} u^Q\right)^{(p-1-S)/Q}$$

from (5.10), since $Q>p-1-S$. Then we find

$$\int_{B_\rho}|x|^b u^Q \leq C\ \left(\int_{\mathcal{C}_{\rho,2\rho}}|x|^b u^Q\right)^{(p-1-S)(m-1-T)/QR},$$

hence again $u\equiv 0$.

Now assume that $R\leq m-1-T$ and $\mathcal{B}$ is $\mathbf{S}$-m-$\mathbf{C}$, and choose λ as in 5.16). We get

$$\int_{B_\rho}|x|^b u^Q \leq C\ \rho^{N-m-N(m-1-T)/\lambda}\left(\int_{\mathcal{C}_{\rho,2\rho}} v^\lambda\right)^{(m-1-T)/\lambda} \leq C\ \rho^{N-m}\ (\min_{|x|=\rho} v)^{m-1-T}.$$

Hence

$$\begin{aligned}\left(\int_{B_\rho}|x|^b u^Q\right)^{R/(m-1-T)} &\leq C\rho^{-1}\int_\rho^{2\rho}\left(\int_{B_r}|x|^b u^Q\right)^{R/(m-1-T)}\\ &\leq C\rho^{-1}\int_\rho^{2\rho} r^{(N-m)R/(m-1-T)}\,(\min_{|x|=r} v)^R.\end{aligned}$$

That means

$$\int_{B_\rho}|x|^b u^Q \leq C\ \rho^{N-m-N(m-1-T)/R}\left(\int_{\mathcal{C}_{\rho,2\rho}} v^R\right)^{(m-1-T)/R}$$

so that (5.27) is still valid, from (5.6). We get the conclusions as above.

ii) The conclusion follows as in Theorem 3.3.

iii) Here we observe that from (5.6), (5.7), (5.8) and the lower estimate (2.34)

$$C_1\rho^{(m-N)/(m-1)} \leq \left(\oint_{\mathcal{C}_{\rho/2,\rho}} v^R\right)^{1/R} \leq C\ \rho^{-\xi}, \tag{5.28}$$

$$C_2\rho^{(p-N)/(p-1)} \leq \left(\oint_{\mathcal{C}_{\rho/2,\rho}} u^\ell\right)^{1/\ell} \leq C\ \rho^{-\gamma}, \tag{5.29}$$

with

$$\ell = Q \quad \text{if } R > m-1-T, \qquad \ell \in (0, QR/(m-1-T)) \quad \text{if } R \leq m-1-T.$$

This is impossible if the inequality is strict in (5.25). Now suppose for example that

$$\theta_2 = N-m-(m-1)\xi = 0\ \leq\ \theta_1 = N-p-(p-1)\gamma.$$

Then we observe that

$$L_{\mathcal{A}}u \geq |x|^a\, u^S v^R \geq C\ \ |x|^{a-R\xi}\, u^S$$

If $S=0$, then, from (2.37),

$$u \geq C\ \ |x|^{(a-R\xi+p)/(p-1)} = C\ \ |x|^{-\gamma}\,.$$

If $S \neq 0$, then, from Theorem 3.1,

$$u \geq C\ \ |x|^{-(a-R\xi+p)/(S-p+1)} = C\ \ |x|^{-\gamma}\,.$$

In turn we get

$$L_{\mathcal{A}}v \geq |x|^b\, u^Q v^T \geq C\ \ |x|^{b-\gamma Q}\, v^T.$$

If $T=0$, then $b-\gamma Q=-N$, hence

$$v \geq C\ |x|^{-\xi}\,(\ln|x|)^{1/(m-1)}\,. \tag{5.30}$$

from (2.38), which contradicts (5.28). If $T\neq 0$, then $b-\gamma Q=-N+T\xi$,

$$L_{\mathcal{A}}v \geq C\ |x|^{-N+T\xi}\,v^T \geq C\ |x|^{-N}\,,$$

hence again (5.30) holds, which is impossible. The proof is similar if $\theta_1=0$.

In Ω_i we obtain in the same way

$$C_1 \leq \left(\oint_{\mathcal{C}_{\rho/2,\rho}} v^R\right)^{1/R} \leq C\,\rho^{-\xi}, \tag{5.31}$$

$$C_2 \leq \left(\oint_{\mathcal{C}_{\rho/2,\rho}} u^\ell\right)^{1/\ell} \leq C\,\rho^{-\gamma}, \tag{5.32}$$

from (2.33). This is impossible if $\max(\gamma,\xi)>0$. Now suppose for example $\xi=0\geq\gamma$, and $\mathcal{A}$ is **S-**p-**C**, and $\mathcal{B}$ is **S-**m-**C**. Then

$$L_{\mathcal{A}}u \geq |x|^a\,u^S,$$

hence if $S=0$,

$$u \geq C\ |x|^{(a+p)/(p-1)}\,,$$

from (2.37). If $S\neq 0$, then from Theorem 3.1,

$$u \geq C\ |x|^{(a+p)/(p-1-S)}\,,$$

hence in any case

$$L_{\mathcal{A}}v \geq |x|^b\,u^Q v^T \geq C\ |x|^{b+(a+p)Q/(p-1-S)}\,v^T.$$

If $T=0$, then $b+(a+p)Q/(p-1)=-m$ since $\xi=0$, hence

$$L_{\mathcal{A}}v \geq |x|^{-m}\,,$$

from (2.37), hence

$$v \geq C\ |\ln|x||\,,$$

which contradicts (5.31). If $T\neq 0$, then $b+(a+p)Q/(p-1-S)=-m$, hence the same result holds. ■

Theorem 5.4 *We make the assumptions of Theorem 5.1, with $N>p,m$, and suppose that$\mathcal{A}$ is* **S-**p-**C**, *and $\mathcal{B}$ is* **S-**m-**C**, *and $QR<(p-1-S)(m-1-T)$.*

i) If (u,v) is a solution of (5.4) in Ω_e and

$$\min(\gamma,\xi)<0 \tag{5.33}$$

then $u \equiv 0$ *or* $v \equiv 0$.

ii) If (u, v) *is a solution of (5.4) in* Ω_i *and*

$$\max\left((p-1)\gamma - (N-p), (m-1)\xi - (N-p)\right) > 0, \tag{5.34}$$

then $u \equiv 0$ *or* $v \equiv 0$.

iii) If moreover $\mathcal{A}, \mathcal{B}$ *satisfy* $(\mathbf{H}_p)$,$(\mathbf{H}_m)$, *the same results hold in case of equality.*

Proof i) Suppose that the problem has a nontrivial solution, then $u > 0$. And for any $\ell \in (0, N(p-1)/(N-p))\,, \lambda \in (0, N(m-1)/(N-m))$ and large ρ,

$$C_1 \rho^{-\gamma} \leq \left(\oint_{\mathcal{C}_{\rho/2,\rho}} u^\ell \, dx \right)^{1/\ell} \leq C_2,$$

$$C_1 \rho^{-\xi} \leq \left(\oint_{\mathcal{C}_{\rho/2,\rho}} v^\lambda \, dx \right)^{1/\lambda} \leq C_2,$$

from (2.30) and (3.8). This is impossible if (5.33) holds. Now suppose for example $\xi = 0 \leq \gamma$ and $\mathcal{A}, \mathcal{B}$ satisfy $(\mathbf{H}_p)$,$(\mathbf{H}_m)$. Then

$$L_{\mathcal{A}} v \geq |x|^b \, u^Q \geq C \ |x|^{b - Q\gamma}$$

and $b - Q\gamma = -m$. This contradicts the Proposition 2.7.

ii) In the same way, for small ρ,

$$C_1 \rho^{-\gamma} \leq \left(\oint_{\mathcal{C}_{\rho/2,\rho}} u^\ell \, dx \right)^{1/\ell} \leq C_2 \ \rho^{-(N-p)/(p-1)},$$

$$C_1 \rho^{-\xi} \leq \left(\oint_{\mathcal{C}_{\rho/2,\rho}} v^\lambda \, dx \right)^{1/\lambda} \leq C_2 \ \rho^{-(N-m)/(m-1)},$$

from (2.27) and (3.8), which is impossible if (5.34) holds. If $\mathcal{A}, \mathcal{B}$ satisfy $(\mathbf{H}_p)$, $(\mathbf{H}_m)$, and for example $N - m - (m-1)\xi = 0$, then

$$L_{\mathcal{A}} v \geq C \ |x|^{b - Q\gamma} \, v^T \geq C \ |x|^{b - Q\gamma - T\xi}$$

but $b - Q\gamma - T\xi = -N$, hence again a contradiction. ■

Remark 5.4 In the case of the system (5.20) we can also reduce a part of the results to the radial case: as in Propositions 2.6 and 2.7, for any positive solution (u, v) in Ω_e (resp. Ω_i), we can construct a radial positive solution of the system in $2\Omega_e$ (resp. $(1/2)\Omega_i$)

$$\begin{cases} -\Delta_p U = r^a \, U^S V^R, \\ -\Delta_m V = r^b \, U^Q V^T, \end{cases}$$

such that $u \geq U$ and $u \geq V$. A radial analysis of this system allows to find again the results of Theorem 5.3, see for example [16]. But it does not give the upper estimates of Theorem 5.1.

Remark 5.5 The conditions given in Theorems 5.3, 5.4 are not the unique conditions of nonexistence. Suppose for simplicity that $\mathcal{A}, \mathcal{B}$ satisfy $(\mathbf{H}_p)$, $(\mathbf{H}_m)$ and are **S-**p**-C**, **S-**m**-C.**

i) Let (u, v) be a solution of (5.4) in Ω_e. If

$$R \leq (p+a)(m-1)/(N-p), \text{ or } Q \leq (m+b)(p-1)/(N-m),$$

then $u \equiv 0$ or $v \equiv 0$. Indeed

$$-div\,[\mathcal{A}(\nabla u)] \geq |x|^{a-R(N-m)/(m-1)}\, u^s,$$

from Proposition 2.6 and the conclusion comes from Theorem 3.4 if $s \in (0,1)$, or from Proposition 2.7 if $s = 0$.

ii) Let (u, v) be a solution of (5.4) in Ω_i. If

$$s \geq (a+N)(p-1)/(N-p) \text{ or } t \geq (b+N)(m-1)/(N-m)$$

then $u \equiv 0$ or $v \equiv 0$. Indeed

$$-div\,[\mathcal{A}(\nabla u)] \geq |x|^{a}\, u^s,$$

and we conclude as above. This was noticed in the radial case in [16].The same phenomenon appears for systems with the other sign, or for systems of mixed type, see [9].

In the case of half spaces, we can extend the results of Sections (4.1), (4.2). We get for example the following. The proof is left to the reader.

Theorem 5.5 *Assume that* $N \geq p, m > 1$, *and* $Q > p-1$.

i) If $\gamma > \beta_{p,N}$, *or* $\xi > \beta_{m,N}$,*and* $(u,v) \in (C^1(\overline{\Omega_e^+}))^2$ *is a solution of 5.1 in* Ω_e^+, *then* $u \equiv 0$ *or* $v \equiv 0$.

ii) If $\min(\gamma, \xi) < -1$, *and* $(u,v) \in (C^1(\overline{\Omega_i^+}))^2$ *is a solution of 5.1 in* Ω_i^+, *then* $u \equiv 0$ *or* $v \equiv 0$.

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