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Résumé of the five main publications

It is obviously uneasy to choose five publications among one hundred. Therefore, I have awarded their importance to the publications both because of their impact in the continuation of my scientific works and the importance of the questions they left open. Thus I have banished one article ([7] in the list of my works) which had a large numbers of citations because of the explicit estimates it contains. Although this publication has been put in reference by many authors, it had no theoritical influence in all my researches. The publications that I have kept are all dealing with the central theme of my works, which is the study of singularities and the precise description of blow-up problems in nonlinear partial differential equations

The five chosen publications are the following:

1- Singular solutions of some nonlinear elliptic equations, **Nonlinear Anal. T. M. & A 5**, 225-242 (1981).

2- Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations (*coll. M.F. Bidaut-Veron*), **Inventiones Math. 106**, 489-539 (1991).

3- Boundary singularities of solutions of nonlinear elliptic equations (*coll. A. Gmira*), **Duke J. Math. 64**, 271-324 (1991).

4- The boundary trace of positive solutions of semilinear elliptic equations: the subcritical case (*coll. M. Marcus*), **Arch. Rat. Mech. An. 144**, 201-231 (1998).

5- Capacitary estimates of positive solutions of semilinear elliptic equations with absorption (*coll. M. Marcus*), **J. Europ. Math. Soc. 6**, 483-527 (2004).

1 Singular solutions of some nonlinear elliptic equations [12]

This publication, announced in a Note de Comptes Rendus in 1979, deals with the study of isolated singularities of solutions, defined in an open subset of the space $\mathbb{R}^n$, of equation

$$-\Delta u + |u|^{q-1} u = 0, \tag{1.1}$$

in the superlinear case ($q > 1$). Because this type of equations modelizes important problems in theoritical physics (Thomas-Fermi equations in nuclear physics, with $q = 3/2$ or $q = 5/2$ and $n = 3$, the radial solutions have been studied since the thirties (works of Fowler, Sommerfeld, Thomas and Fermi). The study of the radial solutions was settled upon classical methods from the ordinary differential equations theory: asymptotic expansion, linearization. The methods introduced here for the treatment of the non-radial solutions were essentially new: for the positive solutions, the use of the a priori estimate of Keller and Osserman allowed me to give an upper bound of the coefficient which appears in the Harnack inequalities, which allowed me to point out the asymptotic isotropy of the singularities. For studying the signed solutions, the introduction of the logarithmic and spherical variables transformed the problem into an elliptic equation in the cylinder $]0,+\infty[\times S^{n-1}$

$$v_{tt} + av_t + \Delta_{S^{n-1}} v + \ell v - |v|^{q-1} v = 0, \tag{1.2}$$

where a and ℓ are constants and $\Delta_{S^{n-1}}$ the Laplace-Beltrami operator on the sphere S^{n-1}. The main result that I prove in this article is the following:

If u is a positive solution of (1.1) in $\Omega\backslash\{a\}$ with a singularity at 0 and if $1 < q < n/(n-2)$,
(i) either

$$u(x) \approx k \begin{cases} |x|^{2-n} & \textit{if } n \geq 3 \\ \ln(1/|x|) & \textit{if } n = 2 \end{cases} \quad \textit{when } x \to 0, \tag{1.3}$$

for a constant $k > 0$ which can take any value,
(ii) or

$$\lim_{x \to a} |x|^{2/(q-1)} u(x) = \ell(n,q). \tag{1.4}$$

When u is no longer a function with constant sign, such a dichotomy still holds (the constant becoming $\pm\ell(n,q)$ in (ii)), provided

$$(n+1)/(n-1) \leq q < n/(n-2).$$

Two critical cases appeared:

$$q = n/(n-2) \text{ and } q = (n+1)/(n-1).$$

The first case, already observed by Fowler in his study of radial problems, corresponds to the vanishing of the coefficient ℓ. In that case, H. Brezis and myself have shown in [9], that equation (1.1) admits no isolated singularity; due to the important phenomenological role of this equation, this result was considered as surprising by physicists. The second critical case is explained by a symmetry breaking. The coefficient ℓ achieves the value $n-1$ which is the first nonzero eigenvalue of $\Delta_{S^{n-1}}$, and a bifurcation occurs in the study of the stationary problem associated to (1.2),

$$\Delta_{S^{n-1}} w + \ell w - |w|^{q-1} w = 0. \tag{1.5}$$

The structure of the set of solutions of this equation becomes much more complicated, in particular the solutions are no longer isolated because of the action of the continuous subgroups of $O(n)$. The so-called Lyapounov-LaSalle method was unable to give the full characterization of the limit set of the $v(t,.)$ when $t \to +\infty$. It is only ten years after, in a common work with X.Y. Chen and H. Matano [43], and thanks to an intensive use of Sturm techniques (based upon the Jordan separation theorem), that I succeeded in solving completely the 2 dimensional case in the study of (1.2) by showing that the limit set is always a single solution of (1.5). This ended the study of the isolated singularities of (1.1) in the plane.

The extension of the results of classification of isolated sigularities of quasilinear equations of the type

$$-div\left(|Du|^{p-2}\,Du\right)=0, \tag{1.6}$$

for $p>1$, and

$$-div\left(|Du|^{p-2}\,Du\right)+u^q=0, \tag{1.7}$$

for $0<p-1<q$ has been achieved in the articles [31] and [33] written with S. Kichenassamy and A. Friedman respectively.

2 Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations [49]

In 1981, B. Gidas et J. Spruck published an important article dealing with the local and global properties of the positive solutions to

$$\Delta u+u^q=0, \tag{2.1}$$

for exponent which where not covered by P.L. Lions article, namely

$$n/(n-2)\leq q<(n+2)/(n-2).$$

The three most spectacular results in their work are :
1- Non-existence of positive and regular solutions to (2.1) in $\mathbb{R}^n$, property which is not valid if $q\geq(n+2)/(n-2)$ because of Pohozaev's identity.
2- The a priori upper bound satisfied by any positive solution of (2.1) in $B_2\setminus\{0\}$,

$$u(x)\leq C(n,q)|x|^{-2/(q-1)},\quad \forall x\in B_1\setminus\{0\}. \tag{2.2}$$

3- The uniqueness of the positive solution to the following equation on the unit sphere

$$-\Delta_{S^{n-1}}\omega+\ell\omega-\omega^q=0, \tag{2.3}$$

when $1<q<(n+1)/(n-3)$ and $(q-1)\ell\leq n-1$.
The proofs of these two last results are surprising by their level of technical difficulties, both settled upon the "miraculous" introduction of some vector fields associated to the problems. In the case of the a priori estimate, they derived some integral identities from which they were able to compute the Harnack coefficient in equation (2.1) thanks to Serrin's celebrated result.

By using the two last results, they proved that, if u satisfies (2.1) in a punctured open set $\Omega\setminus\{a\}$, either u is regular, or

$$\lim_{x\to a}|x-a|^{2/(q-1)}u(x)=\ell(q,n). \tag{2.4}$$

In our article, which is one of my first excursion in the world of Riemannian geometry, and a part of the results of which has been announced in 1989 in [47], we return to questions

1 et 2 treated by Gidas and Spruck. We also consider the isothermal gas sphere equation (already studied in the radial case by Chandrashekar fifty years before)

$$-\Delta u = \lambda e^u \tag{2.5}$$

in a 3 dimensional open subset Ω, for $\lambda > 0$. First we notice the existence of a parametrisation of the singular solutions of (2.5) by the conformal group of the sphere S^2: introducing the spherical variables (r, σ) in $\mathbb{R}^3$, and setting

$$u(r, \sigma) = \ln(1/r^2) + \ln(2/\lambda) + 2\omega,$$

the function ω satisfies on S^2,

$$-\Delta_{S^2}\omega + 1 - e^{2\omega} = 0. \tag{2.6}$$

Since the solutions of this equation can be expressed by

$$\omega = \frac{1}{2}\ln(\det|d\phi|),$$

where ϕ is a conformal transformation of S^2, the set of the solutions of this equation can be endowed with a structure of a noncompact 3-dim manifold (even a structure of Lie group can be put on this set). By understanding how to utilize L. Simon's results on the asymptotics of geometric functional with analytic coefficients, extremely hard results indeed which are the infinite dimensional extension of Lojasiewicz classical results, we prove in particular. *Let u be a solution of (2.5) in $B_1 \setminus \{0\}$, such that $|x|^2 e^u \in L^1_{loc}(B_1)$. Then*
(i) either there exists $\gamma \leq 0$ such that

$$-\Delta u = \lambda e^u + 4\pi\gamma\delta_0, \tag{2.7}$$

from which follows

$$u(x) = \gamma |x|^{-1} + O(1), \tag{2.8}$$

near 0 *(in particular u is regular if $\gamma = 0$),*
(ii) or there exist a solution ω of (2.6) such that

$$u(x) = \ln(1/|x|^2) + \ln(2/\lambda) + 2\omega(x/|x|) + \circ(1), \tag{2.9}$$

when $x \to 0$. We unify the techniques of the proof of Gidas and Spruck results I and II by remplacing their vector fields by a systematic use of the Bochner-Lichnerowicz-Weitzenböck's formula, which, on a Riemannian manifold (M^d, g), is as follows

$$\frac{1}{2}\Delta_g|\nabla f|^2 = |Hess\, f|^2 + \langle\nabla_g\Delta_g f, \nabla_g f\rangle + Ricc_g(\nabla_g f, \nabla_g f). \tag{2.10}$$

We prove that *if (M^d, g) is a compact and complete manifold and $q > 1$, any positive solution of*

$$\Delta_g\omega - \lambda\omega + u^q = 0,$$

on M, is constant if

$$Ricc_g \geq \frac{d-1}{d}(q-1)\lambda, \tag{2.11}$$

$$q \leq (d+2)/(d-2), \tag{2.12}$$

with at least one strict inequality if (M^d, g) *is conformaly diffeomorphic to* (S^d, g_0). Thanks to this result we give new estimates of the Sobolev quotient

$$S_{\lambda,q} = \inf\{Q_{\lambda,q}(v) : v \in W^{1,2}(M) \setminus \{0\}\}, \tag{2.13}$$

where

$$Q_{\lambda,q}(v) = \frac{\displaystyle\int_M \left(|\nabla v|^2 + \lambda v^2\right) dv_g}{\left(\displaystyle\int_M |v|^{q+1} dv_g\right)^{2/(q+1)}}, \tag{2.14}$$

for $1 < q \leq (n+2)/(n-2)$, with respect to the Ricci curvature of the manifold. Later on with my PhD student J.R. Licois, I have extended this result in [65], [75], by showing that it is possible to replace (2.11) by

$$(q-1)\lambda \leq \lambda_1(M) + \frac{qd(d-1)}{q+d(d+2)}\left(Ricc_g - \frac{d-1}{d}\lambda_1(M)\right). \tag{2.15}$$

Notice that the second term on the right is always non-positive by Lichnerowicz-Obata's theorem. In particular this allows to consider flat manifolds such as flat tori (think for example of doubly periodic functions in the plane). By a systematic use of formula (2.10), we simplify the demonstration of the a priori estimate (2.4) and extend them to equations with a singular potential such as

$$\Delta u + u^q - k\,|x|^{-2}\,u = 0. \tag{2.16}$$

The systematic use of the a priori estimates and Simon's method allows use to characterize the isolated singularities at 0 of the positive solutions of these equations in $B_2 \setminus \{0\}$. In the conformaly invariant case $q = (n+2)/(n-2)$ we show the existence of *elliptic waves* which are solutions of (2.16) under the form

$$u(x) = u(r,\sigma) = r^{(2-n)/2}\omega(exp[\ln r A](\sigma)) \tag{2.17}$$

for some non-zero skew symetric matrix A. The role of these elliptic waves in describing the set of positive solutions of (2.16) is still a mystery.

3 Boundary singularities of solutions of nonlinear elliptic equations [48]

This work, essentially written in 1989, is probably the one which has the deepest impact on my present researches in the sense that it opened the systematic studies of the boundary trace of solutions of nonlinear elliptic equations. In his recent book *Super Diffusion and*

Positive Solutions of Nonlinear Partial Differential Equations, **Univ. Series Lectures 34**, B. Dynkin quotes (p. 6) the pioneering role of this article. This work is threefolds

I- The solvability, in a n-dimensionnal open subset, of nonlinear equations with Radon measures as boundary value

$$\begin{cases} -\Delta u + g(u) = 0 \text{ in } \Omega, \\ \qquad\qquad u = \mu \text{ on } \partial\Omega, \end{cases} \tag{3.1}$$

where g is a continuous function, most often non-decreasing, and μ is a Radon measure on $\partial\Omega$. The solution is expressed in a weak sense: $u \in L^1(\Omega)$ and $g(u)\rho(x) \in L^1(\Omega)$ where $\rho(x) = dist(x, \partial\Omega)$, sufficient actually to prove uniqueness thanks to the use of $W_0^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega)$ as space of test functions, space which was already introduced by H. Brezis in the formulation of a similar problem. Existence is ensured under the assumption

$$\int_1^\infty g(r^{1-n})r^n dr < \infty \tag{3.2}$$

which expresses that, for any $a \in \partial\Omega$, the Poisson kernel P_a, an harmonic function in Ω having a Dirac mass at a, satisfies

$$\int_\Omega g(P_a(x))\rho(x)dx < \infty.$$

The construction of solutions is performed by approximation, using as a fundamental tool, some estimates of equi-integrability given by the introduction of Marcinkiewicz spaces of the type $M^p(\Omega, \rho^\alpha dx)$. If $g(r) = |r|^{q-1} r$ the condition (3.2) is fulfilled if and only if $0 < q < (n+1)/(n-1)$.

II- The removability of boundary isolated singularities, which are problems under the form: $u \in C^2(\Omega) \cap C(\bar{\Omega} \setminus F)$ satisfies

$$\begin{cases} -\Delta u + g(u) = 0 \text{ in } \Omega, \\ \qquad\qquad u = \phi \text{ on } \partial\Omega \setminus F, \end{cases} \tag{3.3}$$

where $F = \{a_1, \ldots, a_p\}$ is a discrete subset of the boundary and $\phi \in C(\partial\Omega)$. The result that we obtain here is the analogous for the boundary value problems of a similar result proved by Brezis and myself concerning internal isolated singularities. It reads as follows: *if g satisfies*

$$\liminf_{r\to+\infty} r^{-(n+1/(n-1)} g(r) > 0 \textit{ and } \limsup_{r\to-\infty} |r|^{-(n+1/(n-1)} g(r) < 0, \tag{3.4}$$

the function u can be extended as a continuous function defined in whole $\bar{\Omega}$, moreover it satisfies the regular Dirichlet problem with boundary data ϕ. Roughly speaking, if the growth of the nonlinearity, which is here an absorbing term, is too strong, the isolated singularities cannot exist.

III- In the model case of equation

$$-\Delta u + |u|^{q-1} u = 0, \tag{3.5}$$

the preceding results read as follows:

If $0 < q < (n+1)/(n-1)$, the boundary value problem with a Radon measure as boundary data can be solved in a unique way, for *any* Radon measure on $\partial\Omega$.

If $q \geq (n+1)/(n-1)$ the boundary isolated singularities are removable. As a consequence the boundary value problem with a Dirac mass δ_a as data, where $a \in \partial\Omega$, cannot be solved.

In the third part of this article, I give a precise description of the asymptotic profile of boundary isolated singularities of solutions of (3.5) (thus in the case $1 < q < (n+1)/(n-1)$), showing that, in the positive case those singularities are of two types: *let the singularity be fixed at $a \in \partial\Omega$*

(i) either

$$u(x) \approx kP_a(x) \text{ when } x \to a \in \partial\Omega, \tag{3.6}$$

for some constant $k > 0$ which can take any value,

(ii) or

$$\lim_{x\to a} |x-a|^{2/(q-1)} u(x) = \omega((x-a)/|x-a|) \tag{3.7}$$

where, up to a spatial rotation, ω is the unique positive solution of equation (1.5) on the hemisphère S_+^{n-1}, with zero Dirichlet data on the equator.

In the case where the function u has not a constant sign, a similar result of dichotomy still holds (with two possible values $\pm\omega$ in (ii)), provided

$$(n+2)/n \leq q < (n+1)/(n-1).$$

The value $q = (n+2)/n$ corresponds to a new symmetry breaking, more complicated than the one previously seen in the study of the solutions of (1.5). The methods that we use are based upon the asymptotic study of the solutions, in the infinite cylinder $]0, +\infty[\times S_+^{n-1}$, of an equation having the following form

$$\begin{aligned}(1 + \epsilon_1 v_{tt} + (a+\epsilon_2)v_t + (1+\epsilon_3)\Delta_{S^{n-1}} v + \epsilon_4 \nabla_{S^{n-1}} v_t + \epsilon_5 \nabla^2_{S^{n-1}} v \\ +(\ell + \epsilon_6)v - (1+\epsilon_7)|v|^{q-1} v = 0,\end{aligned} \tag{3.8}$$

where the functions ϵ_j, which go to zero at infinity, come from the parametric representation of $\partial\Omega$ in a neighborhood of a. The results are obtained by combining energy methods with the sharp asymptotic techniques introduced in the work with X.Y. Chen et H. Matano in [43].

In collaboration with R. Borghol, results on the characterization of boundary isolated singularities for equations of the type

$$-div\left(|Du|^{N-2} Du\right) + u^q = 0 \tag{3.9}$$

are obtained in [113] in the case $q > N-1$. The main tools are the conformal invariance of the N-harmonique equation in $\mathbb{R}^N$, the construction of fundamental solutions having an isolated singularity on the boundary, the boundary Harnack [105] inequalities, and the solution of the spherical p-harmonic spectral problem, which is the research of solutions under the form $u(r,\sigma) = r^{-\beta}\omega(\sigma)$ in the half space H which vanish on ∂H (or $\partial H \setminus \{0\}$). Thanks to this, we derived a description of isolated singularities, or removability results similar to those published in [48].

4 The boundary trace of positive solutions of semilinear elliptic equations: the subcritical case [76]

The study of the boundary trace of solutions of equations is very natural, and it can be assimilated to the physical notion of observable. The mathematical problem has two aspects:

First, being given a function u satisfying some partial differential equation, in a n-dimensional domain Ω, is it possible to define a notion of "boundary value" for this solution ? This genralized boundary value is called a **boundary trace**.

Then, the set of all possible traces having been identified, does a element of this set define in a unique way a solution of the equation, the boundary trace of which it is ?

The case of positive harmonic function, in a bounded regular domain (for simplicity) is known since the works of Riesz and Herglotz. Any positive harmonic function u admits a boundary trace which is a Radon measure μ on $\partial\Omega$. Furthermore this function is uniquely determined by its boundary trace thanks to Poisson's formula

$$u(x) = \int_{\partial\Omega} P_\Omega(x,y)d\mu(y), \quad \forall x \in \Omega, \tag{4.1}$$

where $P_\Omega(x,y)$ is the Poisson kernel in Ω. At end, Fatou's theorem allows to give pointwise behaviour almost everywhere on the boundary. These results have been extended to positive superharmonic functions by Doob. The case of subharmonic functions remains open. The work which is presented here is at the origin of the researches on the trace theory that I am performing with M. Marcus. *First we prove that if u is a positive solution of*

$$-\Delta u + u^q = 0 \quad \text{in } \Omega, \tag{4.2}$$

(we suppose $q > 1$ so that the problem is superlinear), for any point $a \in \partial\Omega$ the following dichotomy occurs:
(i) either for any open neighborhood V_a of a

$$\lim_{\beta\to 0} \int_{\Sigma_\beta \cap V_a} u(x)dS = \infty, \tag{4.3}$$

where, for $\beta > 0$,

$$\Sigma_\beta = \{x \in \Omega : \rho(x) = dist(x, \partial\Omega) = \beta\},$$

(ii) or there exists an open neighborhood V of a and a positive linear functional ℓ on $C^\infty(V \cap \partial\Omega)$ such that

$$\lim_{\beta \to 0} \int_{\Sigma_\beta \cap V} u(x)\zeta dS = \ell(\zeta), \tag{4.4}$$

for any function ζ of class C^∞, with compact support in $V \cap \partial\Omega$. The regular points are those for which (i) occurs. Their set, denoted by $\mathcal{S}(u)$, is a closed subset of the boundary. On the complementary set $\mathcal{R}(u) = \partial\Omega \backslash \mathcal{S}(u)$ there exists a unique positive Radon measure μ such that

$$\lim_{\beta \to 0} \int_{\Sigma_\beta \cap V} u(x)\zeta dS = \int_{\mathcal{R}(u)} \zeta d\mu, (\zeta), \tag{4.5}$$

for any continuous function ζ with compact support in $\mathcal{R}(u)$. To the couple $(\mathcal{S}(u), \mu)$ is associated a unique generalised Borel measure ν called the boundary trace of u and denoted by

$$Tr_{\partial\Omega}(u) = \nu \approx (\mathcal{S}(u), \mu). \tag{4.6}$$

Conversely we solve the problem

$$\left\{ \begin{array}{l} -\Delta u + u^q = 0, \quad \text{in } \Omega, \\ \quad Tr_{\partial\Omega}(u) = \nu \end{array} \right. \tag{4.7}$$

where ν is a generalized Borel measure on $\partial\Omega$, and we prove: *Let$1 < q < (n+1)/(n-1)$, then for any generalized Borel measure ν on $\partial\Omega$ the problem (4.7) admits one and only one solution.* The proof uses in a fundamental manner the results on isolated strong singularities brought into light in the preceding article. In particular, we prove that if $a \in \mathcal{S}(u)$, $u(x)$ is bounded from below by the solution of (4.2) described by the behaviour (3.7). If $\nu \approx (\mathcal{S}, \mu)$ where $\mathcal{S}$ is a closed subset of $\partial\Omega$ and μ a positive Radon measure on $\mathcal{R} = \partial\Omega \setminus \mathcal{S}$, we construct a maximal solution $\bar{u}_{\mathcal{S},\mu}$ and a minimal solution $\underline{u}_{\mathcal{S},\mu}$ both with trace ν. Succesively we show

$$\bar{u}_{\mathcal{S},\mu} - \underline{u}_{\mathcal{S},\mu} \leq \bar{u}_{\mathcal{S},0} - \underline{u}_{\mathcal{S},0}, \tag{4.8}$$

and

$$\bar{u}_{\mathcal{S},0} \leq K \underline{u}_{\mathcal{S},0}, \tag{4.9}$$

for some $K > 0$ depending on q and the dimension. Starting from this last estimate, and assuming, by contradiction,

$$\underline{u}_{\mathcal{S},0} \neq \bar{u}_{\mathcal{S},0}$$

we are able to construct a solution of (4.2) with trace $(\mathcal{S}, 0)$ smaller than the minimal solution $\underline{u}_{\mathcal{S},0}$. The results that we obtain extend by a purely analytic method, a remarquable previous work due to J.F. Le Gall dealing with the positive solutions of

$$-\Delta u + u^2 = 0 \quad \text{in } D_1 = \{x \in \mathbb{R}^2 : |x| < 2\}. \tag{4.10}$$

Le Gall's approach was essentially probabilistic and utilized, in a fundamental manner the theory of super-processes, already studied by E.B. Dynkin. Almost always in collaboration with M. Marcus I have developed the theory of the initial trace, the lateral boundary trace for semilinear parabolic equation the model of which is

$$u_t - \Delta u + u^q = 0 \quad \text{in } \Omega\times]0,+\infty[, \tag{4.11}$$

(cf. [74], [79], [88], [91] [93], [95]) or for geometric equations [81], or nonlinear equations which degenerate on the boundary [96], for example

$$-\Delta u + \rho^\alpha(x)u^q = 0 \quad \text{in } \Omega, \tag{4.12}$$

or even supersolutions [101] such as

$$-\Delta u + g(x,0) \geq 0 \quad \text{dans } \Omega. \tag{4.13}$$

Each time results of the existence of a trace in the set of generalized Borel measure are obtained. Moreover the inverse problem of characterizing a solution from its trace is treated up to a critical value of the exponent. The techniques become more and more delicate. Finaly, we started the study of the trace problem for solutions of (4.2) in open domains which are no longer regular, but piecewise C^1. The singularities of the boundary interfere in a very direct way in the description of the trace.

5 Capacitary estimates of positive solutions of semilinear elliptic equations with absorption [102]

Before presenting this work, I shall recall the context in which it has to be understood. The above mentioned work has pointed out the role of the notion of critical exponent in the search of the full description of positive solutions of semilinear equations. In the case of equation (4.2), the critical exponent is $q_c = (n+1)/(n-1)$. The works of Le Gall ($q = 2$), Dynkin and Kuznetsov ($q_c \leq q \leq 2$), then Marcus et myself [77] ($\max\{2, q_c\} \leq q$), and finally [90] (the general case $q_c \leq q$) led to the characterization (in terms of $C_{2/q,q'}$ Bessel capacities) the boundary sets which are removable for equation (4.2), which means, by definition,

$$\big(u \in C(\bar\Omega \setminus K) \cap C^2(\Omega)\,\text{solution de (4.2) et } u = 0\,\text{sur } \partial\Omega \setminus K\big) \Longrightarrow (u \equiv 0)\,.$$

The Radon measures μ for which the problem

$$\left\{\begin{array}{l} -\Delta u + u^q = 0, \quad \text{in } \Omega, \\ \quad u = \mu \quad \text{on } \partial\Omega \end{array}\right. \tag{5.1}$$

admits a weak solution $u = u_\mu$ (always unique) are also characterized: they are the Radon mesures which are absolutely continuous with respect to the $C_{2/q,q'}$-capacity. Les methods of Le Gall and Dynkin and Kuznetsov where essentially settled upon the intensive of various notions of probability theory (respectively the Brownian snake some super processes), while our methods are entirely analytic. Thanks to this result, we give, in [77], necessary and sufficient conditions on a Borel measure ν on the boundary for solving problem (4.7), by constructing a maximal solution. However a striking phenomenon in this case is the absence of uniqueness (Le Gall when $q = 2$ and Marcus and myself in the general case). Then Dynkin the notions of σ-moderate solution of (4.2), a solution which is the limit of an increasing sequence of weak solutions $u = u_{\mu_n}$ de (5.1), and of " fine trace ", described by using a boundary topology finer than the usual one. In a common work with Kuznetsov, he proved that, when $q_c \leq q \leq 2$, any solution σ-moderate of (4.2) is uniquely defined by its fine trace. *All the problem is now to prove that any solution solution is σ-moderate.* By a remarkable construction, B. Mselati, a PhD student of Le Gall, has succeeded in proving that it is truly the case when $q = 2$. Mselati's construction uses in a crucial way the properties of Le Gall's Brownian snake. The key-stone of Mselati's construction relies on the fact that the maximal solution $\bar{u}_K$ of (4.2), the trace of which (in the sense of [76]) is the Borel measure associated to the indicatrix of a boundary compact subset K, is actually σ-moderate. The whole thesis has been published in the **Memoirs of the Amer. math. Soc. 168** in 2004. In [102] we extend this result to all the cases of supercritical exponent q, using only analytic methods although extremely technical. We use in particular the new lifting that we have already introduced in [90] (*the optimal lifting*) together with a series of new sharp estimates for the Poisson kernel in piecewise regular domains. A key step in our work is a pointwise bilateral estimate of $\bar{u}_K$ thanks to a Wiener-type integral. If we define the largest of all the σ-moderate solution which blow-up on K and vanishes on $\partial\Omega \setminus K$ by

$$\underline{u}_K = \sup\{u_\mu : \mu \in W_+^{-2/q,q}(\partial\Omega),\ \mu(K^c) = 0\},$$

we show that, if $x \in \Omega$, there exists a constant $C = C_{\Omega,q,\alpha}$ such that

$$\begin{aligned}\frac{\lambda(x)}{C}\int_{\lambda(x)}^{\Lambda(x)} s^{-1-2/(q-1)} C_{2/q,q'}\left(\frac{1}{s}(B_s(x)\cap K)\right)\frac{ds}{s} \leq \underline{u}_K \\ \leq \bar{u}_K(x) \leq C\lambda(x)\int_{\lambda(x)}^{\Lambda(x)} s^{-1-2/(q-1)} C_{2/q,q'}\left(\frac{1}{s}(B_s(x)\cap K)\right)\frac{ds}{s},\end{aligned} \tag{5.2}$$

in any interior cone $\lambda(x) \leq \alpha \mathrm{dist}(x, \partial\Omega) = \alpha\rho(x)$ where $\alpha > 0$, $\lambda(x) = \mathrm{dist}(x, K)$ and $\Lambda(x) = \max\{|x - z| : z \in K\}$. In order to emphasize the difference between the supercritical and the subcritical cases, notice that in this last case we have used in [76] the bilateral estimate

$$\frac{1}{C}\left(\lambda(x)\right)^{-2/(q-1)} \leq \underline{u}_K(x) \leq \bar{u}_K(x) \leq C\left(\lambda(x)\right)^{-2/(q-1)} \tag{5.3}$$

which can be directly obtained by comparing $\underline{u}_K$ and $\bar{u}_K$ with explicit singular solution of (4.2), from which estimate (4.9) follows. These specific estimates do not exist in the super critical case, this is why a completely different approach is needed. This estimation allows us to characterise the strong blow-up points of $\bar{u}_K$. The derivation of (5.2) is inspired by the celebrated Wiener construction. By a judicious choice of test functions associated to our optimal lifting, we start by showing that if $K \subset B_\gamma(a) \cap \partial\Omega$, the following integral estimate holds

$$\int_{\Omega\setminus B_{2\gamma}(a)} \left(\bar{u}_K^q \rho + \bar{u}_K\right) dx \leq C\gamma^{n-2/(q-1)} C_{2/q,q'}(K/\gamma). \tag{5.4}$$

By estimating the value of the Poisson kernel in $\Omega \setminus B_{2\gamma}(a)$, we derive the next pointwise estimate

$$\bar{u}_K(x) \leq C\rho(x)\lambda^{1-2/(q-1)}(x)C_{2/q,q'}(K/\lambda(x)). \tag{5.5}$$

The Wiener slicing method, which consists in dominating $\bar{u}_K$ by $\sum_j \bar{u}_{K_j}$ where $K_j = K \cap \left(B_{2^{j+1}\lambda(x)}(x) \setminus B_{2^j\lambda(x)}(x)\right)$, gives rise to an estimate from above which takes into account (5.4) the fine structure of K. The estimate from above of $\underline{u}_K$ is obtained by noticing that any solution u_μ is smaller than the Poisson potential $\mathbb{P}_\Omega[\mu]$ of μ, thus it is larger than

$$\mathbb{P}_\Omega[\mu] - \mathbb{G}_\Omega[\mathbb{P}^q_\Omega[\mu]], \tag{5.6}$$

(where $\mathbb{G}_\Omega[.]$ is the Green potential in Ω), of a mesure $\mu \in W^{-2/q,q}_+(\partial\Omega)$ such that $\mu(K^c) = 0$. Using again Wiener's slicing technique, we construct a capacitary measure endowed with the following properties:
(i) the nonlinear term in (5.6) is dominated by lthe linear one,
(ii) the estimate of the linear term is similar to the one in (5.2).
As soon as (5.2) is derived, equality of $\underline{u}_K$ and $\bar{u}_K$ is infered by the same method as the one in [76]. However several consequences follow from these estimates. Two of the most strinking results in this direction are the following:
1- *For any thick point $\sigma \in K$, in the sense of the fine topology associated to the $C_{2/q,q'}$-capacity on the boundary, the following path integral is infinite*

$$\int_0^1 \bar{u}_K^{q-1}(\gamma(t))tdt = \infty$$

for any C^1 curve γ such that $\gamma(]0,1]) \subset \Omega$, $\gamma(0) \in \partial\Omega$, $\gamma'(0)$ transversal to $\partial\Omega$.
2-*For any $a > 0$ and any $\sigma \in \partial\Omega$,*

$$\limsup_{\substack{x \to \sigma \\ |x-\sigma| \leq a\rho(x)}} |x-\sigma|^{2/(q-1)}\,\bar{u}_K(x) \approx \limsup_{s \to 0} C_{2/q,q'}\left(s^{-1}(K \cap B_s(\sigma))\right).$$

The results presented in this work may look very technical, but they are the first pointwise quasi representation of singular solutions of super-critical problems. They are the major step for obtaining a general representation theorem of every positive solution of (4.2) thanks to the new notion non-probabilistic of *precise trace*, valid in any range of exponent. The construction of the recise trace and the uniqueness in the class of σ-moderate solutions is given in [110]. The proof that any solution is σ-moderate is on the way. Notice that, contrary to the notion of *fine trace* developed by Dynkin and Kuznetsov, the precise trace is point-wise and not defined up to a set of zero $C_{2/q,q'}$-capacity.

Finally, we have very recently obtained similar forms for other similinear problems, in particular for semilinear parabolic equations . In [109], [116] we have succeded to prove, up to extreme technical difficulties (much arder than in the elliptic case), bilateral estimates over the solution u_F of (4.11) in $\mathbb{R}^N \times]0, +\infty[$ with initial trace is the indicatrix function of a closed subset F of $\mathbb{R}^N$. This formula is valid for any $q > 1$ and can be understood as a quasi-capacity representation of u_F: there exist two positive constants C_1 and C_2 depending only on N and q such that

$$C_1 W_F(x,t) \leq u_F(x,t) \leq C_2 W_F(x,t) \quad \forall (x,t) \in \mathbb{R}^N \times]0,+\infty[\tag{5.7}$$

where W_F is the $C_{2/q,q'}$-capacitary potential of F, defined in $\mathbb{R}^N \times]0,+\infty[$ by

$$W_F(x,t) = t^{-1/(q-1)} \sum_{n=0}^{\infty} (n+1)^{N/2-1/(q-1)} e^{-n/4} C_{2/q,q'} \left(\frac{F_n}{\sqrt{(n+1)t}} \right), \tag{5.8}$$

where $F_n = F \cap \{y : \sqrt{nt} < |x-y| \leq \sqrt{(n+1)t}\}$. A first consequence of this résult is that the function u_F is σ-moderate. Following the method developed in the elliptic case [110], this result will surely lead us to the elaboration of a theory of the *precise initial trace.*