Closed curves with prescribed curvature

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Seillac, September 7, 2012
Classical results

- Theorem (Lyusternik and Fet, 1951)
  Any compact Riemannian manifold \((M, g)\) contains a closed geodesic.

- Theorem (Lyusternik and Schnirelmann, 1929)
  Any \((S^2, g)\) contains at least three embedded closed geodesics.

What happens, if you consider closed curves with constant or prescribed curvature instead of closed geodesics?
The problem

Given

- an oriented (compact) surface \((M, g)\) with a Riemannian metric \(g\),
- a smooth (positive) function \(k : M \rightarrow \mathbb{R}\).

**Problem:** Existence of a closed \(k\)-curve, i.e. a closed immersed curve \(\gamma : S^1 \rightarrow M\) with geodesic curvature \(k_g(\gamma, t) = k(\gamma(t))\).
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**Geodesic curvature:**
\[
k_g(\gamma, t) := |\dot{\gamma}(t)|^{-3} \langle D_t g \dot{\gamma}(t), J_g(\gamma(t)) \dot{\gamma}(t) \rangle_g,
\]

\(J_g(p)\): rotation by \(+\pi/2\) in \(T_p M\) w.r.t. the given orientation and metric.
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- **Geodesic curvature:**
  \[ k_g(\gamma, t) := |\dot{\gamma}(t)|_g^{-3}\langle D_{t,g} \dot{\gamma}(t), J_g(\gamma(t))\dot{\gamma}(t)\rangle_g, \]

  \(J_g(p)\): rotation by \(+\pi/2\) in \(T_pM\) w.r.t. the given orientation and metric.

- These curves are called **magnetic geodesics**. They correspond to trajectories of a charged particle on \(M\) in a magnetic field with magnetic form \(kd\mu_g\) and solve

  \[ D_{t,g} \dot{\gamma} = |\dot{\gamma}|_g k(\gamma) J_g(\gamma) \dot{\gamma}. \]
Methods and approaches

- **Theory of dynamical systems:** Arnold ’86, Ginzburg ’96
  Magnetic geodesics are periodic orbits of a *twisted* Hamiltonian flow.
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Magnetic geodesics are periodic orbits of a *twisted* Hamiltonian flow.

- **Morse-Novikov theory:** Novikov ’84, Taimanov ’92
Minimize $E(\gamma) := \int_{S^1} |\dot{\gamma}| + \int_B k \mu_g$ where $\partial B = \gamma$.
If $d\theta = k \mu_g$, then $\int_B k \mu_g = \int_\gamma \theta$.
In general, the functional $E$ is multi-valued.
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  In general, the functional $E$ is multi-valued.

- **Aubry-Mather-theory:** Contreras, Macarini, Paternain ’04
  Existence of closed $k$-curves on compact surfaces, if $k\mu_g$ is exact.
Existence results for large curvature

Let $(M, g)$ be a compact oriented Riemannian surface and $k : M \rightarrow \mathbb{R}$ positive.

- (Ye '91, Pacard-Xu '09)
  - If $k \equiv \text{const}$ is large, there are at least two embedded closed $k$-curves for $M = S^2$ and at least three otherwise.
  - These curves are perturbations of geodesic circles.
  - Any nondegenerate critical point of the Gauss curvature $K_g$ of $(M, g)$ leads to a local foliation with $k$-curves.
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  ▶ If \(k\) is large, there are at least two embedded closed \(k\)-curves for \(M = S^2\) and at least three otherwise.
  ▶ If, moreover, \(k\) is a Morse function, these closed \(k\)-curves are near critical points of \(k\).
Conjectures

- **Conjecture (**1**: (Arnold ’81)
  If \((M, g)\) is compact oriented surface and \(k\) is positive, then there is a closed \(k\)-curve.
  More precisely, there are at least two for \(M = S^2\) and at least three in all other cases.

Arnold ’84: (**1** is true for a flat torus \((T^2, g_0)\).

Hedlund ’36: Nonexistence of \(k\)-curves (“Horocycle flow”) (**1** is wrong for \((H/\Gamma, g_0)\) with \(K g_0 \equiv -1\) and \(k \equiv 1\).

- **Conjecture (**2**: (Novikov ’82, Rosenberg and Smith ’11)
  On \((S^2, g)\) with \(k\) positive, there is an embedded closed \(k\)-curve.
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On \((S^2, g)\) with \(k\) positive, there is an embedded closed \(k\)-curve.
Nonexistence of $k$-curves

*Hedlund '36:* There is no closed $k_0$-curve in $(\mathbb{H}/\Gamma, g_0)$, where $k_0 \equiv 1$ and $\Gamma$ is a subgroup of orientation preserving isometries of $(\mathbb{H}, g_0)$, such that $(\mathbb{H}/\Gamma, g_0)$ is a compact oriented surface with Gauss curvature $K_{g_0} \equiv -1$.

\[ \mathbb{H} = B_1(0) \subset \mathbb{R}^2 \]
\[ g_0 = 4(1 - |x|^2)^{-2} g_{\mathbb{R}^2} \]
Results

- (*1) is true for \((S^2, g)\), if \(K_g \geq 0\) and \(k > 0\), i.e. there are two closed \(k\)-curves in this case. (S. ’12)

- There is a closed \(k\)-curve on \((H/\Gamma, g)\), if \(K_g \geq -1\) and \(k > 1\). (S. ’12)

- (*2) holds for \((S^2, g)\), i.e. there are two closed embedded \(k\)-curves, under each of the following assumptions:
  1. \(k > 0\) and \(g\) is 14-pinched (\(\sup K_g < 4 \inf K_g\)). (S. ’11)
  2. \(K_g > 0\) and \(k\) is small enough. (Rosenberg & S. ’11)
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- (*2) holds for \((S^2, g)\), i.e. there are two closed embedded \(k\)-curves, under each of the following assumptions:

  1. \(k > 0\) and \(g\) is \(\frac{1}{4}\)-pinched (\(\sup K_g < 4 \inf K_g\)). (S. ’11)
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2. \(K_g > 0\) and \(k\) is small enough. (Rosenberg & S. ’11)
An outline of the proof

Closed $k$-curves correspond to zeros of the vector field $X_{g,k}$ on $H^{2,2}(S^1, M)$ given by

$$X_{k,g}(\gamma) := (-D^2_{t,g} + 1)^{-1}(-D_{t,g}\dot{\gamma} + |\dot{\gamma}|_g k(\gamma)J_g(\gamma)\dot{\gamma}).$$

Note that

$$T_{\gamma}H^{2,2}(S^1, M) = \{ \text{Periodic } H^{2,2} - \text{vector fields along } \gamma \}.$$

For $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$ consider the action on $H^{2,2}(S^1, M)$ defined by

$$(\theta \ast \gamma)(t) := \gamma(t + \theta).$$

The vector field $X_{k,g}$ is invariant under the $S^1$-action and any zero comes along with a $S^1$ orbit of zeros.
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- **Step 1:** Count zero orbits, i.e. define a $S^1$-degree.
- **Step 2:** Compute the $S^1$-degree in an unperturbed situation. e.g. constant curvature and $k$.
- **Step 3:** Prove compactness of the set of zero orbits and use a homotopy argument.
Step1: The $S^1$-degree

We follow the degree theory of Tromba ’78 for Fredholm vector fields on Banach manifolds and give a $S^1$-equivariant version.

Degree theories in the context of geometric problems have also been developed by White ’87 and Rosenberg and Smith ’11.

There are two equivalent ways to define the $S^1$-degree in our case:

- **functional analytic setting**
  - Leray-Schauder degree
  - count negative eigenvalues of Jacobi operators

- **geometric setting**
  - fixed point index of the Poincaré map
The $S^1$-degree (functional analytic setting)

Fix a zero $\gamma$ of $X_{k,g}$.

- Due to the $S^1$-action, $\dot{\gamma}$ is in the kernel of $D_{g \gamma}X_{k,g}$.
The $S^1$-degree (functional analytic setting)

Fix a zero $\gamma$ of $X_{k,g}$.

- Due to the $S^1$-action, $\dot{\gamma}$ is in the kernel of $D_g X_{k,g}|_{\gamma}$.
- Define a vector field $W_g$ on $H^{2,2}(S^1, M)$ by

$$W_g(\gamma) = \left( -(D_t, g)^2 + 1 \right)^{-1} \dot{\gamma}.$$  

The vector field $X_{k,g}$ is orthogonal to $W_g$. Consequently,

$$D_g X_{k,g}|_{\gamma} : T_{\gamma}H^{2,2}(S^1, M) \to \langle W_g(\gamma) \rangle^\perp.$$
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$$D_{g}X_{k,g}\big|_{\gamma} : T_{\gamma}H^{2,2}(S^1, M) \rightarrow \langle W_{g}(\gamma) \rangle^\perp.$$ 

- $\dot{\gamma} \notin \langle W_{g}(\gamma) \rangle^\perp$. 
The $S^1$-degree (functional analytic setting)

**Definition**
The zero orbit $S^1 \ast \gamma$ is called *nondegenerate*, if

$$D_{gX_k,g|\gamma} : \langle W_g(\gamma) \rangle^\perp \rightarrow \langle W_g(\gamma) \rangle^\perp$$

is an isomorphism. In this case we define the local $S^1$-degree by

$$\text{deg}_{\text{loc},S^1}(X_{k,g}, S^1 \ast \gamma) := \text{sgn}D_{gX_k,g|\gamma},$$

where $\text{sgn}D_{gX_k,g|\gamma}$ denotes the usual Leray-Schauder degree.
The $S^1$-degree (functional analytic setting)

Let $\mathcal{M}$ be an open $S^1$-invariant subset of curves in $H^{2,2}(S^1, \mathcal{M})$. We assume that $X_{k,g}$ is proper in $\mathcal{M}$, i.e.

$$\{ \gamma \in \mathcal{M} : X_{k,g}(\gamma) = 0 \}$$

is compact.

Using an equivariant version of the Sard-Smale lemma, the $S^1$-degree $\chi_{S^1}(X_{k,g}, \mathcal{M}) \in \mathbb{Z}$ is defined by

$$\chi_{S^1}(X_{k,g}, \mathcal{M}) := \sum_{\{ S^1 \ast \gamma \subset \mathcal{M} \ where Y_{k,g}(S^1 \ast \gamma) = 0 \} \ deg_{loc,S^1}(Y_{k,g}, S^1 \ast \gamma),}$$

where $Y_{k,g}$ is a small perturbation of $X_{k,g}$ with only finitely many critical orbits in $\mathcal{M}$, that are all nondegenerate.
The Poincaré map

Let $S^1 \ast \gamma$ be an isolated zero orbit of $X_{k,g}$ and consider the corresponding periodic orbit in the unit tangent bundle

$$\Sigma_1 M := \{ (x, V) \in TM : |V|_g = 1 \}.$$  

We fix a transversal section $E$ in $\Sigma_1 M$ at the point $\theta := (\gamma(0), |\dot{\gamma}(0)|^{-1} \dot{\gamma}(0))$ and denote by $P : B_1 \cap E \to B_2 \cap E$ the Poincaré map, where $B_1$, $B_2$ are open neighborhoods of $\theta$. 
The Poincaré map

If $S^1 \ast \gamma$ is an isolated zero orbit of $X_{k,g}$ then

- $\theta$ is an isolated fixed point of $P$
The Poincaré map

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- $\theta$ is an isolated fixed point of $P$
- $\text{deg}_{loc,S^1}(X_{k,g}, S^1 \ast \gamma) = -i(P, \theta)$, where $i(P, \theta)$ denotes the index of the isolated fixed point $\theta$. 

(Nikishin '74, Simon '74): $-i(P, \theta) \geq -1$, $i(P, \theta) = 1 + \frac{1}{2}(E - H)$. 

$P$ is area preserving.
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Step 2: Computation of the $S^1$ degree

We consider the set of curves

$$\mathcal{M}_A := \{ \gamma \in H^{2,2}(S^1, M) : \dot{\gamma} \neq 0, \gamma \text{ is Alexandrov embedded} \}$$

and additionally for $M = S^2$

$$\mathcal{M}_E := \{ \gamma \in H^{2,2}(S^1, S^2) : \dot{\gamma} \neq 0, \gamma \text{ is embedded} \}$$
Step 2: Computation of the $S^1$ degree

Fix $(M, g_0)$ with a constant curvature metric $g_0$ and

$$\begin{cases} k_0 > 0, & \text{if } M = S^2 \\ k_0 >> 1, & \text{if } M \neq S^2, \end{cases}$$

The zero orbits in $\mathcal{M}_A$ as well as in $\mathcal{M}_E$ are parametrized by $M$.

The round sphere $(S^2, g_0)$ with $k \equiv k_0$.

The hyperbolic plane $(\mathbb{H}, g_\mathbb{H})$ with $K_{g_\mathbb{H}} \equiv -1$ and $k \equiv k_0 > 1$. 
Step 2: Computation of the $S^1$ degree

The zero orbits in $\mathcal{M}_A$ as well as in $\mathcal{M}_E$ are parametrized by $M$. To compute the $S^1$-degree, choose a Morse function $k_1$ on $M$ and replace $k_0$ by $k_0 + \varepsilon k_1$. As $\varepsilon \to 0^+$ we find:

- To every critical point $w \in M$ of $k_1$ corresponds exactly one zero orbit $S^1 \ast \gamma_w$ of $X_{k_0 + \varepsilon k_1, g_0}$. 

These are all zero orbits in $\mathcal{M}_A$ or $\mathcal{M}_E$. Hence

$$\chi_{S^1}(X_{k_0, g_0}, \mathcal{M}_E) = \chi_{S^1}(X_{k_0 + \varepsilon k_1, g_0}, \mathcal{M}_A) = -\chi(M).$$
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Hence

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\chi_{S^1}(X_{k_0, g_0}, \mathcal{M}_E) = \chi_{S^1}(X_{k_0, g_0}, \mathcal{M}_A) \\
= \chi_{S^1}(X_{k_0 + \varepsilon k_1, g_0}, \mathcal{M}_A) = -\chi(M).
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Step 3: Compactness

**A priori estimates:** Fix \((M, g)\) and \(\gamma \in \mathcal{M}_A\), i.e. there is an immersion \(F : B \to M\) such that \(\gamma = F(\partial B)\). Gauss-Bonnet applied to \((B, F^*g)\) yields

\[
2\pi = \int_B K_{F^*g} dV_{F^*g} + \int_{\partial B} k_{F^*g} dS_{F^*g}
\geq \min(K_g) \text{vol}(B) + \int_{\gamma} k_g dS_g
\geq \min(K_g) \text{vol}(B) + L(\gamma) \min(k)
\]

If \(K_g \geq 0\), then \(L(\gamma) \leq 2\pi (\min(k) - 1)\).

If \(K_g \equiv -1\), then the (hyperbolic) isoperimetric inequality gives \(L(\gamma) \geq \text{vol}(B)\). Hence \(L(\gamma) \leq 2\pi (\min(k) - 1) - 1\).

For the general case \(K_g \geq -1\) we use a conformal change of the metric in \(B\).
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For the general case \(K_g \geq -1\) we use a conformal change of the metric in \(B\).
Step 3: Compactness

This yields compactness in $\mathcal{M}_A$, because the limit of Alexandrov embedded locally convex curves remains Alexandrov embedded. Using the homotopy invariance of the $S^1$-degree, we find

$$-\chi(M) = \chi_{S^1}(X_{k_0, g_0}, \mathcal{M}_A) = \chi_{S^1}(X_k, g, \mathcal{M}_A).$$

In particular, if $M = S^2$, the $S^1$-degree is $-2$. Since the local $S^1$-degree of an isolated zero orbit is greater than or equal to $-1$, there are at least two zero orbits.
Step 3: Compactness

To prove compactness of embedded curves in \((S^2, g)\) of small positive geodesic curvature \(k\) with positive Gauss curvature \(K_g > 0\), we use Reilly’s formula (see also Choi and Wang ’83):
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To prove compactness of embedded curves in \((S^2, g)\) of small positive geodesic curvature \(k\) with positive Gauss curvature \(K_g > 0\), we use Reilly’s formula (see also Choi and Wang ’83):

- Let \((M, g)\) be a compact Riemannian manifold with boundary \(\partial M\), \(f \in C^\infty(M)\), \(z = f|_{\partial M}\) and \(u = \frac{\partial f}{\partial n}\) on \(\partial M\), where \(n\) denotes the outer normal. Then

\[
\int_M (\bar{\Delta} f)^2 - |\bar{\nabla}^2 f|^2 = \int_M \text{Ric}(\bar{\nabla} f, \bar{\nabla} f)
\]

\[
+ \int_{\partial M} (\Delta z + Hu) u - \langle \nabla z, \nabla u \rangle + \Pi(\nabla z, \nabla z),
\]

where we denote by \(\bar{\Delta}, \Delta\) and \(\bar{\nabla}, \nabla\) the Laplacians and covariant derivatives on \(M\) and \(\partial M\) respectively; \(H\) is the mean curvature and \(\Pi\) is the second fundamental form of \(\partial M\).
Step 3: Compactness

If $\gamma$ is embedded, we are in the above situation with $\partial M = \gamma$.

- **Reilly’s formula**

\[
\int_M (\bar{\Delta} f)^2 - |\bar{\nabla}^2 f|^2 = \int_M \text{Ric}(\bar{\nabla}f, \bar{\nabla}f)
\]
\[
+ \int_{\partial M} (\Delta z + Hu) u - \langle \nabla z, \nabla u \rangle + \Pi(\nabla z, \nabla z).
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Step 3: Compactness

If $\gamma$ is embedded, we are in the above situation with $\partial M = \gamma$.

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- We let $z$ be an eigenfunction to the first nontrivial eigenvalue $\lambda_1$ of $\Delta z + \lambda_1 z = 0$ on $\partial M$, and $f$ its harmonic extension to $M$. In dimension two, this leads to

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0 \geq \left( \inf_M K_g \right) \int_M |\nabla f|^2 - 2\lambda_1 \int_{\partial M} zu.
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- Integration by parts yields

\[
\int_{\partial M} zu = \int_M |\bar{\nabla} f|^2 + f \bar{\Delta} f = \int_M |\bar{\nabla} f|^2 > 0.
\]
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Consequently, if $\gamma$ is an embedded closed curve with nonnegative geodesic curvature on $(S^2, g)$ with positive Gauss curvature $K_g$, then

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  \[ \lambda_1 = \frac{4\pi^2}{L(\gamma)^2}. \]
- Hence, we get a uniform bound
  \[ L(\gamma) \leq 2\pi \sqrt{2} \left( \inf_{S^2} K_g \right)^{-\frac{1}{2}}. \]
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- Using the uniform bound from Reilly’s formula this continues to hold for curves with small positive geodesic curvature.
- This allows to carry out the degree argument within the class of embedded curves.
Open problems: Flow approach

Consider \((S^2, g)\), \(k : S^2 \to \mathbb{R}\) positive and

\[
\frac{\partial \gamma}{\partial t} = (k_g - k) N
\]

\[
\gamma|_{t=0} = \gamma_0
\]

- Embedded curves need not to remain embedded.
- Alexandrov embedded curves remain Alexandrov embedded.

**Conjecture:** There are at least two Alexandrov embedded closed \(k\)-curves on \((S^2, g)\) for any metric \(g\) and any \(k > 0\).
Open problems: Higher dimensional versions

Theorem (Jost, 1989)
Every \((S^3, g)\) contains four embedded minimal spheres.

Theorem (White, 1989)
Every \((S^3, g)\) with positive Ricci curvature contains an embedded minimal torus.

What happens, if you consider surfaces with prescribed mean curvature instead of minimal surfaces?

**Conjecture:** (Rosenberg and Smith ’11)
Given \((S^3, g)\) with positive (Ricci) curvature and \(H : S^3 \to \mathbb{R}\) positive, then there is an (embedded) sphere with prescribed mean curvature \(H\).
Hopf tori in $S^3$

Following *Pinkall* '85 and *Barros et al.* '99 we consider

$$S^3 = \{(z_1, z_2) : z_1, z_2 \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 4\}$$

and the Hopf map $\mathcal{H} : S^3 \to S^2$ defined by

$$\mathcal{H}(z_1, z_2) := \frac{1}{4}(|z_1|^2 - |z_2|^2, 2z_1 \overline{z_2}) \in \partial B_1(0) \subset \mathbb{R}^3.$$ 

For any metric metric $g$ on $S^2$, we define a metric $\tilde{g}$ on $S^3$ by

$$\tilde{g}(V, W) := \mathcal{H}^* g(V, W) + \theta(V)\theta(W),$$

where $\theta|_x(V) := \frac{1}{2}\langle ix, V \rangle_{\mathbb{R}^4}$. 

If $\gamma$ is a closed (embedded) curve in $S^2$, then $\mathcal{H}^{-1}(\gamma)$ is a flat (embedded) torus in $S^3$ with mean curvature

$$H_{\tilde{g}}\mathcal{H}^{-1}(\gamma(t)) = \frac{1}{2} k_g(\gamma, t).$$
Hopf tori in $S^3$

Consequently we find (Wojtowytsch ’11)

- Given $k : S^2 \to \mathbb{R}$ positive, then there are two embedded tori in the round $S^3$ with prescribed mean curvature $k \circ \mathcal{H}$.

- Given a $\frac{1}{4}$-pinched metric $g$ on $S^2$ and $c > 0$, then there are two embedded tori in $(S^3, \tilde{g})$ with constant mean curvature $c$. 
Closed $k$-curves correspond to periodic trajectories of a vector field $\tilde{\Phi}$ on the unit tangent bundle of $(S^2, g)$

$$\Sigma_1 S^2 := \{(x, V) \in TS^2 : |V|^g = 1\} \cong \mathbb{RP}^3 \xleftarrow{1:2} S^3$$

Lift $\tilde{\Phi}$ to a vector field $\Phi$ on $S^3$. 

The Legendrian Seifert conjecture
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- **Hofer ’93**: Every Reeb vector field $V$ on $(S^3, \lambda)$ has a periodic orbit.
  
  contact structure: $\lambda \wedge d\lambda \neq 0$, Reeb vector field: $d\lambda(V, \cdot) = 0$ and $\lambda(V) = 1$. (*Weinstein conjecture, Taubes ’07.*)
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- **Φ** is Legendrian for the (standard) contact structure $\lambda$ lifted from $\Sigma_1 S^2$, i.e. $\lambda(\Phi) = 0$.

- **Arnold ’96:** Open problem: Existence of periodic orbits of Legendrian vector fields on $S^3$ (with standard contact structure)?