On Courant's nodal domain property for linear combinations of eigenfunctions (after P. Bérard and B. Helffer).

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In Memoriam of A. El Soufi.

Figure: CIMPA Summer School in Damas (2004)

Abstract

We revisit Courant's nodal domain property for linear combinations of eigenfunctions, and propose new, simple and explicit counterexamples for domains in $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{T}^2$, or $\mathbb{R}^3$.

This work has been done in collaboration with P. Bérard and has benefitted from the precious help of V. Bonnaillie-Noël.

Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded open domain or, more generally, a compact Riemannian manifold with boundary.

Consider the eigenvalue problem

$$\begin{cases} -\Delta u &= \lambda \, u \quad \text{in } \Omega\,, \\ B(u) &= 0 \qquad \text{on } \partial\Omega\,, \end{cases} \tag{1}$$

where $B(u)$ is some boundary condition on $\partial\Omega$, so that we have a self-adjoint boundary value problem (including the empty condition if Ω is a closed manifold).
For example, $D(u) = u\big|_{\partial\Omega}$ for the Dirichlet boundary condition, or $N(u) = \frac{\partial u}{\partial \nu}|_{\partial\Omega}$ for the Neumann boundary condition.

Call $H(\Omega, B)$ the associated self-adjoint extension of $-\Delta$, and list its eigenvalues in nondecreasing order, counting multiplicities,

$$0 \le \lambda_1(\Omega, B) < \lambda_2(\Omega, B) \le \lambda_3(\Omega, B) \le \cdots \tag{2}$$

For any $n \ge 1$, define the number

$$\tau(\Omega, B, n) = \min\{k \mid \lambda_k(\Omega, B) = \lambda_n(\Omega, B)\}. \tag{3}$$

E_{λ_n} will denote the eigenspace associated with λ_n.

The Courant nodal theorem

For a real continuous function v on Ω, the *nodal set* is

$$\mathfrak{Z}(v) = \overline{\{x \in \Omega \mid v(x) = 0\}}\,, \tag{4}$$

and $\beta_0(v)$ denotes the number of connected components of $\Omega \setminus \mathfrak{Z}(v)$ i.e., the number of *nodal domains* of v.

Courant's nodal Theorem (1923)

For any nonzero $u \in E_{\lambda_n(\Omega,B)}$,

$$\beta_0(u) \leq \tau(\Omega, B, n) \leq n\,. \tag{5}$$

Courant's nodal domain theorem can be found in Courant-Hilbert [7].

The extended Courant nodal property

Given $r > 0$, denote by $\mathfrak{L}(\Omega, B, r)$ (or shortly $\mathfrak{L}_r$) the space

$$\mathfrak{L}(\Omega, B, r) = \left\{ \sum_{\lambda_j(\Omega,B) \leq r} c_j\, u_j \mid c_j \in \mathbb{R}, u_j \in E_{\lambda_j(\Omega,B)} \right\} . \qquad (6)$$

Extended Courant Property:= (ECP)

We say that $v \in \mathfrak{L}\,(\Omega, B, \lambda_n(\Omega, B))$ satisfies (ECP) if

$$\beta_0(v) \leq \tau(\Omega, B, n)\,. \qquad (7)$$

A footnote in Courant-Hilbert [7] indicates that (ECP) holds for any linear combination of the n first eigenfunctions, and refers to the PhD thesis of Horst Herrmann (Göttingen, 1932) [13].

Historical remarks : Sturm (1836), Pleijel (1956).

1. (ECP) is true for Sturm-Liouville equations. This was first announced by C. Sturm in 1833 [26] and the proof was published in 1836 in [27].
Other proofs were later on given by J. Liouville and Lord Rayleigh who both cite Sturm explicitly.

2. Å. Pleijel mentions (ECP) in his well-known paper [23] (1956) on the asymptotic behaviour of the number of nodal domains of a n-th Dirichlet eigenfunction with the following comment:
"In order to treat, for instance the case of the free three-dimensional membrane $[0,\pi]^3$*, it would be necessary to use, in a special case, the theorem quoted in [6].... However, as far as I have been able to find there is no proof of this assertion in the literature."*

Historical remarks: V. Arnold (1973)

3. As pointed out by V. Arnold [1], when $\Omega = \mathbb{S}^d$, (ECP) is related to Hilbert's $16-$th problem. Arnold [2] mentions that he actually discussed the footnote with R. Courant, that (ECP) cannot be true (contradiction with one of his theorems), and that O. Viro produced in 1979 counter-examples for the 3-sphere $\mathbb{S}^3$, and any degree (for the corresponding harmonic polynomials) larger than or equal to 6 [28].

More precisely V. Arnold wrote:

"Having read all this, I wrote a letter to Courant: "Where can I find this proof now, 40 years after Courant announced the theorem?". Courant answered that one can never trust one's students: to any question they answer either that the problem is too easy to waste time on, or that it is beyond their weak powers."

Note here that R. Courant was eighty years old (He died in 1972).

And V. Arnold continues:

The point is that for the sphere $\mathbb{S}^2$ *(with the standard Riemannian metric) the eigenfunctions (spherical functions) are polynomials. Therefore, their linear combinations are also polynomials, and the zeros of these polynomials are algebraic curves (whose degree is bounded by the number n of the eigenvalue). Therefore, from the generalized Courant theorem one can, in particular, derive estimates for topological invariants of the complements of projective real algebraic curves (in terms of the degrees of these curves).*

Knowing this, I immediately deduced from the generalized Courant theorem new results in Hilbert's famous 16th-problem: "Study the topological properties of the arrangement of real algebraic curves of degree n on the real projective plane."

This Hilbert problem (for $n > 7$) is still unsolved, although many interesting estimates for different invariants have been obtained by Petrovskii, Oleinik, Gudkov, and others. About 1970, I associated this theory with the topology of four-dimensional manifolds, and my successors (Rokhlin, Viro, Kharlamov, Givental, Gromov, Witten, Floer, McDuff, and others) included all this into symplectic topology and quantum field theory. And then it turned out that the results of the topology of algebraic curves that I had derived from the generalized Courant theorem contradict the results of quantum field theory. Nevertheless, I knew that both my results and the results of quantum field theory were true. Hence, the statement of the generalized Courant theorem is not true (explicit counterexamples were soon produced by Viro).

Historical remarks: Gladwell-Zhu (2003)

4. In [9], Gladwell and Zhu refer to (ECP) as the *Courant-Herrmann conjecture*.

They claim that this extension of Courant's theorem is not stated, a fortiori proved, in Herrmann's thesis or subsequent publications.

They consider the case in which Ω is a rectangle in $\mathbb{R}^2$, stating that they were not able to find a counter-example to (ECP) in this case.

They also provide numerical evidence that there are counter-examples for more complicated (non convex) domains.

They suggest that may be the conjecture could be true in the convex case.

Historical remarks: looking for the PHD thesis of H. Herrmann

5. After a personal investigation, what we can add, after getting from the BNF, the manuscript of the PHD-thesis is that Herrmann's thesis has three parts.
Only the second part was accepted by the evaluating committee for publication. This part does not contain any mention of (ECP).
The first part, was published later in [14] in Math. Z. in 1936 in a different form.
The third part was never published. The title of this chapter indicates that this part was devoted to the analysis of the Fourier-Robin problem and to analyze how the eigenvalues tend to the eigenvalues of the Dirichlet problem as the Robin parameter tends to $+\infty$. Nothing to do with (ECP).

The purpose in this talk is to provide simple counter-examples to the *Extended Courant Property* for domains in $\mathbb{R}^2$, $\mathbb{S}^2$ or $\mathbb{R}^3$, including convex domains.

Let us observe that:

- Numerics is involved for some examples.
- For other we give purely mathematical proofs.
- No algebraic topology will be involved.

Rectangle with a crack

Let $\mathfrak{R}_0$ be the rectangle $]0,4\pi[\times]0,2\pi[$. For $0 < a \leq 1$, let $C_a :=]0,a] \times \{\pi\}$ and $\mathfrak{R}_a := \mathfrak{R}_0 \setminus C_a$ and consider the Neumann condition. The setting is described in Dauge-Helffer [8].

We call

$$\begin{cases} 0 < \delta_1(0) < \delta_2(0) \leq \delta_3(0) \leq \cdots \\ \text{resp.} \\ 0 = \nu_1(0) < \nu_2(0) \leq \nu_3(0) \leq \cdots \end{cases} \tag{8}$$

the Neumann eigenvalues of $-\Delta$ in $\mathfrak{R}_0$.
They are given by the $\frac{m^2}{16} + \frac{n^2}{4}$ for pairs (m,n) of non-negative integers.
Corresponding eigenfunctions are products of cosines.
Similarly, the Neumann eigenvalues of $-\Delta$ in $\mathfrak{R}_a$ are denoted by

$$\begin{cases} 0 < \delta_1(a) < \delta_2(a) \leq \delta_3(a) \leq \cdots \\ \text{resp.} \\ 0 = \nu_1(a) < \nu_2(a) \leq \nu_3(a) \leq \cdots \end{cases} . \tag{9}$$

The first three Neumann eigenvalues for the rectangle $\mathfrak{R}_0$ are as follows.

$\nu_1(0)$	0	$(0,0)$	$\psi_1(x,y)=1$
$\nu_2(0)$	$\frac{1}{16}$	$(1,0)$	$\psi_2(x,y)=\cos(\frac{x}{4})$
$\nu_3(0)$		$(0,1)$	$\psi_3(x,y)=\cos(\frac{y}{2})$
$\nu_4(0)$	$\frac{1}{4}$	$(2,0)$	$\psi_4(x,y)=\cos(\frac{x}{2})$

(10)

Dauge-Helffer (1993) prove:

Theorem

For $i \geq 1$,

1. $[0,1] \ni a \mapsto \nu_i(a)$ is non-increasing.
2. $]0,1[\ni a \mapsto \nu_i(a)$, is continuous.
3. $\lim_{a\to 0+} \nu_i(a) = \nu_i(0)$.

It follows that for $0 < a$, small enough, we have

$$0 = \nu_1(a) = \nu_1(0) < \nu_2(a) \leq \nu_2(0) < \nu_3(a) \leq \nu_4(a) \leq \nu_3(0)\,. \tag{11}$$

Observe that for $i = 1$ and 2, $\frac{\partial \psi_i}{\partial y}(x, y) = 0$. Hence for a small enough, ψ_1 and ψ_2 are the first two eigenfunctions for $\mathfrak{R}_a$ with the Neumann condition with associated eigenvalues 0 and $\frac{1}{4}$. We have

$$\alpha \psi_1(x, y) + \beta \psi_2(x, y) = \alpha + \beta \cos(\frac{x}{4}).$$

We can choose the coefficients α, β in such a way that these linear combinations of the first two eigenfunctions have two or three nodal domains in $\mathfrak{R}_a$.

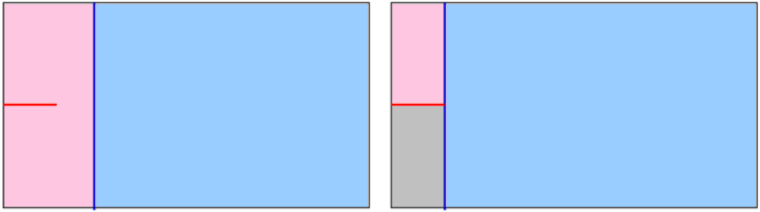


Figure: Rectangle with a crack (Neumann condition)

This proves that (ECP) is false in $\mathfrak{R}_a$ with Neumann condition.

Notice that we can introduce several cracks

$$\{(x, b_j) \mid 0 < x < a_j\}_{j=1}^{k}$$

so that for any $d \in \{2, 3, \dots k + 2\}$ there exists a linear combination of 1 and $\cos(\frac{x}{4})$ with d nodal domains.

Sphere $\mathbb{S}^2$ with cracks

On the round sphere $\mathbb{S}^2$, we consider the geodesic lines $(x, y, z) \mapsto (\sqrt{1-z^2}\cos\theta_i, \sqrt{1-z^2}\sin\theta_i, z)$ through the north pole $(0,0,1)$, with distinct $\theta_i \in [0, \pi[$.
Removing the geodesic segments $\theta_0 = 0$ and $\theta_2 = \frac{\pi}{2}$ with $1 - z \leq a \leq 1$, we obtain a sphere $\mathbb{S}^2_a$ with a crack in the form of a cross.
We consider the Neumann condition on the crack.
We then easily produce a function in the space generated by the two first eigenspaces of the sphere with a crack having five nodal domains.

The function z is also an eigenfunction of $\mathbb{S}^2_a$ with eigenvalue 2.
For a small enough, $\lambda_4(a) = 2$, with eigenfunction z.
For $0 < b < a$, the linear combination $z - b$ has five nodal domains in $\mathbb{S}^2_a$, see Figure below in spherical coordinates.

It follows that (ECP) does not hold on the sphere with cracks.

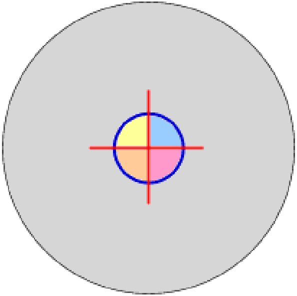


Figure: Sphere with crack, five nodal domains

Remark. Removing more geodesic segments around the north pole, we can obtain a linear combination $z - b$ with as many nodal domains as we want.

The cube with Dirichlet boundary condition

We can adapt a method of Gladwell-Zhu, which has no success for the square in $2D$, to the $3D$-case.

Consider the cube $\mathfrak{C}_\pi =]0,\pi[^3$. The eigenvalues are the numbers

$$q_3(k,m,n) = k^2 + m^2 + n^2\,, \quad k,m,n \in \mathbb{N}.$$

A corresponding complete set of eigenfunctions is given by

$$\phi_{k,m,n}(x,y,z) = \sin(kx)\,\sin(my)\,\sin(nz)\,, \quad k,m,n \in \mathbb{N}.$$

The first Dirichlet eigenvalues of the cube are given by

$$\delta_1 \; [3] < \delta_2 = \delta_3 = \delta_4 \; [6] < \delta_5 = \delta_6 = \delta_7 \; [9] < \\ < \delta_8 = \delta_9 = \delta_{10} \; [11] < \delta_{11} \cdots .$$

Using Chebyshev polynomials, for $k, m, n \in \mathbb{N}$ we have

$$\phi_{k,m,n}(x,y,z) = \phi_{1,1,1}(x,y,z)\, U_{k-1}(\cos x)\, U_{m-1}(\cos y)\, U_{n-1}(\cos z)\,.$$

The factor $\phi_{1,1,1}$ does not vanish in the cube $\mathfrak{C}_\pi$. The map

$$\mathfrak{C}_\pi \ni (x,y,z) \mapsto (X,Y,Z) := (\cos(x), \cos(y), \cos(z)) \in]-1,1[^3$$

is a diffeomorphism from $\mathfrak{C}_\pi$ to the cube $]-1,1[^3$.

The counting of the nodal domains of a linear combination $\Phi \in \mathfrak{L}_r$,

$$\Phi = \sum_{q_3(k,m,n) \leq r} c_{k,m,n}\, \phi_{k,m,n}$$

in the cube $\mathfrak{C}_\pi$, is the same as the counting for,

$$\Psi = \sum_{q_3(k,m,n) \leq r} c_{k,m,n}\, U_{k-1}(X) U_{m-1}(Y) U_{n-1}(Z)$$

in the cube $]-1,1[^3$.

Using the formulas for the Chebyshev polynomials, one gets that the linear combinations Ψ for $k^2 + m^2 + n^2 \leq 11 = \delta_{10}$ correspond to the polynomials of degree ≤ 2 in the variables X, Y and Z.

In particular, $f_a(X, Y, Z) := X^2 + Y^2 + Z^2 - a$ is a linear combination Ψ with $k^2 + m^2 + n^2 \leq 11$.

Since $11 = \delta_8 = \delta_9 = \delta_{10}$, Courant's upper bound is

$$8 = \tau(\mathfrak{C}_\pi, D, 10)\,.$$

It follows that when $\sqrt{2} < a < \sqrt{3}$, the zero set of the function ϕ_a determines 9 nodal domains and this provides a counter-example to (ECP) for the $3D$-cube with Dirichlet boundary condition.

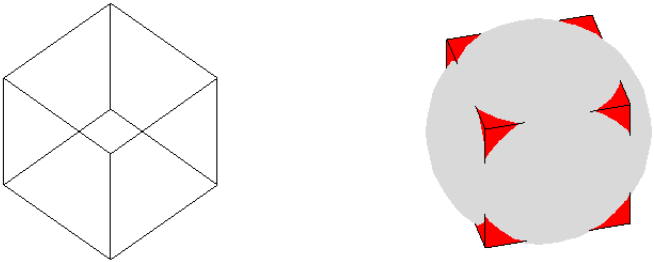


Figure: Cube with Dirichlet boundary condition

The equilateral triangle (Dirichlet or Neumann)

Let $\mathcal{T}_e$ denote the equilateral triangle with sides 1, see Figure 5.
The eigenvalues and eigenfunctions of $\mathcal{T}_e$, with either Dirichlet or Neumann condition on the boundary $\partial\mathcal{T}_e$, can be completely described.
We show that the equilateral triangle provides a counterexample to the *Extended Courant Property* for both the Dirichlet and the Neumann boundary condition.

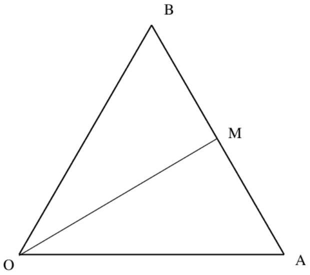


Figure: Equilateral triangle $\mathcal{T}_e = [OAB]$

Neumann boundary condition

The sequence of Neumann eigenvalues of the equilateral triangle $\mathcal{T}_e$ begins as follows,

$$ytretl0 = \lambda_1(\mathcal{T}_e, N) < \frac{16\pi^2}{9} = \lambda_2(\mathcal{T}_e, N) = \lambda_3(\mathcal{T}_e, N) < \lambda_4(\mathcal{T}_e, N)\,. \tag{12}$$

The second eigenspace has dimension 2, and contains one invariant eigenfunction φ_2^N under the mirror symmetry w.r.t OM, and another antiinvariant eigenfunction φ_3^N.
φ_2^N is given by

$$\varphi_2^N(x,y) = 2\cos\left(\frac{2\pi x}{3}\right)\left(\cos\left(\frac{2\pi x}{3}\right) + \cos\left(\frac{2\pi y}{\sqrt{3}}\right)\right) - 1\,. \tag{13}$$

The set $\{\varphi_2^N + 1 = 0\}$ consists of the two line segments $\{x = \frac{3}{4}\} \cap \mathcal{T}_e$ and $\{x + \sqrt{3}y = \frac{3}{2}\} \cap \mathcal{T}_e$, which meet at the point $(\frac{3}{4}, \frac{\sqrt{3}}{4})$ on $\partial\mathcal{T}_e$. The sets $\{\varphi_2 + a = 0\}$, with $a \in \{0, 1-\varepsilon, 1, 1+\varepsilon\}$, and small positive ε, are shown in Figure 7. When a varies from $1-\varepsilon$ to $1+\varepsilon$, the number of nodal domains of $\varphi_2 + a$ in $\mathcal{T}_e$ jumps from 2 to 3, with the jump occurring for $a = 1$.

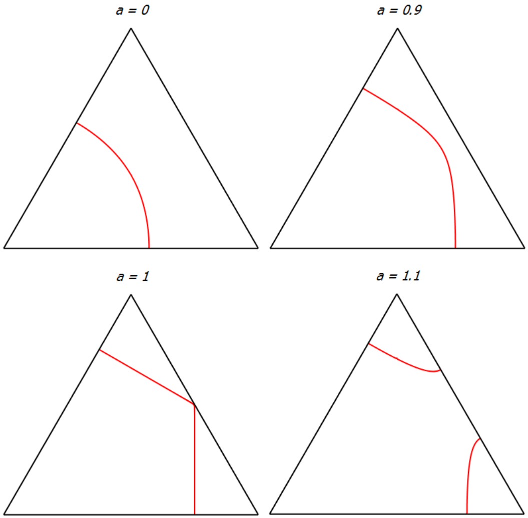


Figure: Levels sets $\{\varphi_2^N + a = 0\}$ for $a \in \{0\,;\, 0.9\,;\, 1\,;\, 1.1\}$

If we take the coordinates $X = \cos\frac{2\pi}{3}x$ and $Y = \cos\frac{2\pi}{\sqrt{3}}y$ we are reduced to the level sets of $(X, Y) \mapsto X(X + Y)$:

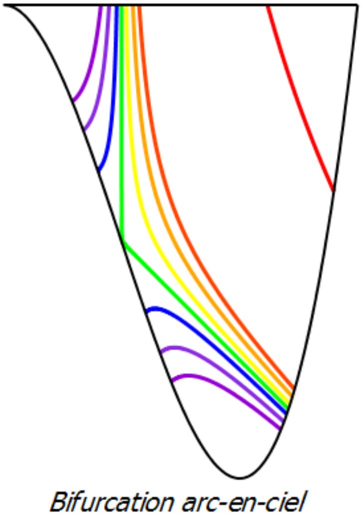


Figure: Levels sets in the X, Y variables

It follows that $\varphi_2^N + a = 0$, for $1 \leq a \leq 1.2$, provides a counterexample to the Extended Courant Property for the equilateral triangle with Neumann boundary condition.

Dirichlet boundary condition

The sequence of Dirichlet eigenvalues of the equilateral triangle $\mathcal{T}_e$ begins as follows,

$$\delta_1(\mathcal{T}_e) = \frac{16\pi^2}{3} < \delta_2(\mathcal{T}_e) = \delta_3(\mathcal{T}_e) = \frac{112\pi^2}{9} < \delta_4(\mathcal{T}_e). \tag{14}$$

Up to scaling, the first eigenfunction φ_1^D is given by

$$\varphi_1^D(x,y) = -8\,\sin\frac{2\pi y}{\sqrt{3}}\,\sin\frac{\pi(\sqrt{3}x+y)}{\sqrt{3}}\,\sin\frac{\pi(\sqrt{3}x-y)}{\sqrt{3}}\,, \tag{15}$$

which shows that it does not vanish inside $\mathcal{T}_e$.

A surprising (new ?) formula.

The second eigenvalue has multiplicity 2, with one eigenfunction φ_2^D symmetric with respect to the median OM, and the other φ_3^D anti-symmetric. Up to scaling, φ_2^D is given by

$$
\begin{aligned}
\varphi_2^D(x,y) = \;& \sin\left(\tfrac{2\pi}{3}(5x+\sqrt{3}y)\right) - \sin\left(\tfrac{2\pi}{3}(5x-\sqrt{3}y)\right) \\
& + \sin\left(\tfrac{2\pi}{3}(x-3\sqrt{3}y)\right) - \sin\left(\tfrac{2\pi}{3}(x+3\sqrt{3}y)\right) \\
& + \sin\left(\tfrac{4\pi}{3}(2x+\sqrt{3}y)\right) - \sin\left(\tfrac{4\pi}{3}(2x-\sqrt{3}y)\right) .
\end{aligned} \tag{16}
$$

First astonished by some numerics, we arrive to the conclusion that the following surprising result could be true:

Lemma

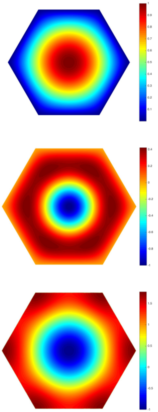
$$\varphi_2^D = -\varphi_1^D \varphi_2^N \,.$$

Proof Express everything in terms of $X = \cos \frac{2\pi}{3} x$ and $Y = \cos \frac{2\pi}{\sqrt{3}} y$. We have then to verify an equality between two polynomials of the variables X and Y.

We deduce from the lemma that the counterexample for Neumann is identical to the counterexample for Dirichlet ! The level sets of φ_2^N and $\varphi_2^D / \varphi_1^D$ are the same.

Numerical simulations for Regular polygons (Virginie Bonnaillie-Noël).

In (2D) Gladwell-Zhu were not successful for the square. One can be successfull for the hexagone for Neumann and for Dirichlet (Numerics).



Heptagon

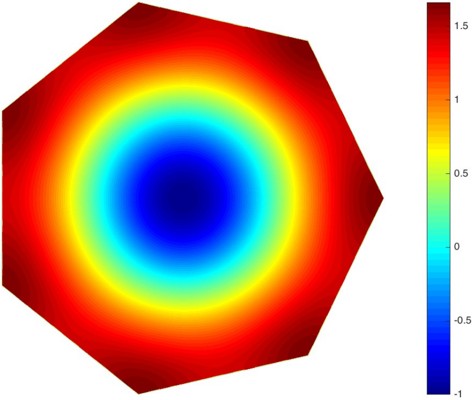


Figure: Level lines of $\frac{w_{6,D}}{w_{1,D}}$ for the Dirichlet problem in the regular heptagon

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