CONVERGENCE RESULTS FOR A CLASS OF NONLINEAR FRACTIONAL EQUATIONS.

PATRICIO FELMER AND ERWIN TOPP

1. Introduction.

Let $T > 0$. The type of equations we consider here has the form

$$(1.1)\begin{cases} u_t(t,x) - \mathcal{I}[u(t,\cdot),x] = g(t,x,u(t,x)) & (t,x) \in (0,T)\times\mathbb{R}^n \\ u(0,x) = u_0(x) & x \in \mathbb{R}^n \end{cases}$$

where $\mathcal{I}$ is a nonlocal nonlinear operator in the Isaac's form. The functions $u_0 \in C(\mathbb{R}^n)$, $g \in C([0,T)\times\mathbb{R}^n\times\mathbb{R})$ are given, with properties to be fixed later. This type of equations appears in the context of *two player stochastic games with jumps*, in wich the trajectory of the stochastic process governing the dynamics is modeled by a Lévy Process and is controled by two players, the first one controling the trayectory interested in maximize certain pay-off depending on it and the second one trying to minimize the same pay-off. In that context, the solution u of (1.1) represents the value function of the game. See for instance [7].

The operator $\mathcal{I}$ is a nonlocal, nonlinear operator defined as follows: Consider a function $w \in L^\infty(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$ and, for a given set $\mathcal{A}$, a family of functions $K_\alpha : \mathbb{R}^n\setminus\{0\} \to \mathbb{R}_+$, $\alpha \in \mathcal{A}$, satisfying

$$(1.2)\qquad \forall\ \alpha \in \mathcal{A},\ K_\alpha(z) \le C_r w(z), \quad if\ |z| \ge r,$$

where C_r depends on r, but not depends on α, and

$$(1.3)\qquad \sup_{\alpha\in\mathcal{A}} \int_B |z|^2 K_\alpha(z)dz \le A < +\infty.$$

where $A > 0$ is an universal constant and B is the unit ball centered at zero.

For each kernel K_α and a function $u : \mathbb{R}^n \to \mathbb{R}$ we define the linear non-local operator

$$L_K[u,x] = \int_{\mathbb{R}^n} (u(x+z) + u(x-z) - 2u(x))K(z)dz =: \int_{\mathbb{R}^n} \delta(u,x,z)K(z)dz$$

It is important to point out that we can write in an equivalent form this nonlocal term as

$$L_\alpha[u,x] = \int_{\mathbb{R}^n} (u(x+z) - u(x) - 1_B Du(x)\cdot z)K_\alpha(z)dz$$

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where the last kernel is twice the kernel of the former equivalent description. We will use both expressions whenever it is necessary.

Given a family $\{K_\alpha\}_{\alpha\in\mathcal{A}}$ as above, our operator $\mathcal{I}$ in (1.1) has the Isaacs structure which have the form

$$\mathcal{I}[u,x] \quad = \quad \inf_{i\in\mathcal{A}_1}\sup_{j\in\mathcal{A}_2} L_{(i,j)}[u,x]. \tag{1.4}$$

with $\mathcal{A}_1\times\mathcal{A}_2\subseteq\mathcal{A}$. This type of expression includes also the class of Bellman's nonlinear operators given by

$$\mathcal{I}[u,x]=\sup_{\alpha\in\mathcal{A}'} L_\alpha[u,x], \quad \text{for } \mathcal{A}'\subseteq\mathcal{A}.$$

We will call this type of operator *elliptic* in a sense to be fixed later, but which is the adaptation to the fractional framework to the usual concept.

In the Lévy Process context, the jumps of the trajectories of the process are uniquely defined by its *Lévy measure*, which in our case are cast by $K_\alpha(z)dz$. Usually, Lévy measures are singular and even nonintegrable at zero depicting in this case the high intensity of the small jumps. This type of condition is setted in (1.3). Condition (1.2) allows the definiteness of the operator $\mathcal{I}$ for a class of functions u with certain behavior at infinity depending on the integrability of the weight function w.

For the function g we ask to be $C-$Lipschitz for some positive constant C and decreasing in the sense that for all $t\in[0,T), x\in\mathbb{R}^n$

$$g(t,x,b)-g(t,x,a)\le b-a, \text{ if } a\le b. \tag{1.5}$$

This condition over the nonlinearity is also classic and it has to do with the ellipticity of the operator $\mathcal{I}$.

In this work we present a particular family of kernels satisfting the assumptions mentioned above. Fix $\sigma\in(0,1)$. We start considering a weight function w with the form

$$w_\sigma(z)=\frac{1}{1+|z|^{n+2\sigma}}, \quad z\in\mathbb{R}^n \tag{1.6}$$

emphasizing the dependence on σ of w to recall the growth condition to be considered over the functions u to be evaluated in the operator.

Denoted by $\{K_\alpha\}$ a family of kernels with the following form

$$\begin{cases} K_\alpha(z)=\frac{a_\alpha(\hat{z})}{|z|^{n+2\sigma}}, \\ a_\alpha\in L^\infty(\mathbb{S}^{n-1}), \ \forall\ \alpha, \\ a_\alpha(z)=a_\alpha(-z), \ \forall z\in\mathbb{S}^{n-1}, \\ a_\alpha(\hat{z})\in[\lambda,\Lambda], \ \forall\ \alpha \text{ and } z\in\mathbb{R}^n\setminus\{0\} \end{cases} \tag{1.7}$$

where $0<\lambda\le\Lambda<+\infty$ and $\hat{z}=z/|z|$ for $z\neq 0$. We also shall include in this family all the nonsingular kernels with the form

$$K_{\alpha,\epsilon}(z):=\epsilon^{-(n+2\sigma)}\frac{a_\alpha(\hat{z})}{1+|z/\epsilon|^{n+2\sigma}}, \tag{1.8}$$

for all α and $\epsilon\in(0,1]$. We adopt the convention $K_{\alpha,0}=K_\alpha$.

The singular kernel K_α is associated in the probabilistic literature with a process known as $2\sigma-$stable process.

In the case of a singular kernel, we need some minimal assumptions over the regularity at the point x of the function u in order to get the evaluation $L_\alpha[u,x]$ well defined. One of this assumptions are $u \in C^{1,1}(x)$ in the sense of Caffarelli and Silvestre ([5], [6]). See also [13] for a precise definition.

Given a set of indices $\mathcal{A}$ and corresponding kernels satisfying the hypotheses (1.7), we will denote

$$\mathcal{L}_\sigma(\mathcal{A}) = \{K_{\alpha,\epsilon} \ : \ (\alpha,\epsilon) \in \mathcal{A} \times [0,1]\}$$

or simply $\mathcal{L}_\sigma$ if there is no ambiguity in the set of indices. Note that this family satisfy the condition (1.2) and (1.3), and is formed by singular ($\epsilon = 0$) and nonsingular kernels ($\epsilon \in (0,1]$). In this case, we write the Isaacs operator as

$$\mathcal{I}_\epsilon[u,x] = \inf_i \sup_j L_{(i,j),\epsilon}[u,x] \tag{1.9}$$

for $\epsilon \in [0,1]$, $L_{(i,j),\epsilon}$ is the operator with kernel $K_{(i,j),\epsilon}$ and $(i,j) \in \mathcal{A}_1 \times \mathcal{A}_2 \subseteq \mathcal{A}$. In addition, we define the extremal operators of the family $\mathcal{M}_\sigma^-, \mathcal{M}_\sigma^+$, defined as

$$\mathcal{M}_\sigma^- u(x) := \inf_{\alpha\in\mathcal{A}, e\in[0,1]} L_{\alpha,e}[u,x]$$
$$\mathcal{M}_\sigma^+ u(x) := \sup_{\alpha\in\mathcal{A}, e\in[0,1]} L_{\alpha,e}[u,x]$$

A very important feature of this class of operators is the uniform ellipticity with respect to the extremal operators in the sense given in [5], [6], say, for each $x \in \mathbb{R}^n, u, v \in C^{1,1}(x)$ we have

$$\mathcal{M}_\sigma^-[(u-v),x] \leq \mathcal{I}_\epsilon[u,x] - \mathcal{I}_\epsilon[v,x] \leq \mathcal{M}_\sigma^+[(u-v),x]. \tag{1.10}$$

Using a family $\mathcal{L}_\sigma$, the main goal is to approximate solutions of a nonlinear nonlocal integro-differential equation by a sequence of solutions of nonlocal equations of zeroth order, namely, a sequence of solutions of an equation like (1.1), but with an operator $\mathcal{I}$ defined using bounded kernels. The rolle of this sequence will be cast by the family $(\mathcal{I}_\epsilon)_\epsilon$ defined as in (1.9).

Always considering the family $\mathcal{L}_\sigma$, we recall the limit equation as $\epsilon \to 0$

$$\left\{ \begin{array}{ll} u_t(t,x) - \mathcal{I}_0[u(t,\cdot),x] = g(t,x,u(t,x)) & (t,x) \in (0,T) \times \mathbb{R}^n \\ u(0,x) = u_0(x) & x \in \mathbb{R}^n \end{array} \right. \tag{1.11}$$

where $\mathcal{I}_0$ is the Isaacs operator in (1.9) but taking $\epsilon = 0$. Note that the family where the infimum and supremum is taken corresponds to unbounded, even nonintegrable kernels.

Instead the family of bounded solutions is a very confortable set of functions to be evaluated in the equations (1.1) and (1.11), we remark that the family of kernels in $\mathcal{L}_\sigma$ allow some growth at infinity of these functions. In our case, we can provide some interesting results considering unbounded functions u such that there exists $0 < \gamma < 2\sigma$ and $C > 0$ satisfying

$$\lim_{|x|\to 0} \sup_{t\in[0,T)} |u(t,x)||x|^{-\gamma} = 0 \tag{1.12}$$

The convergence result is the following

Theorem 1.1. *Let $\sigma \in (0,1)$ and $\gamma < \min\{2\sigma, 1\}$ fixed. Consider $u_0 : \mathbb{R}^n \to \mathbb{R}$ and $g : (0,T) \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ uniformly continuous functions. Assume also*

$$\lim_{|x|\to\infty} \sup_{t\in[0,T)} |g(t,x,0)||x|^{-\gamma} = 0 \tag{1.13}$$

and

$$\lim_{|x|\to\infty} |u_0(x)||x|^{-\gamma} = 0. \tag{1.14}$$

Let u_ϵ the unique viscosity solution of equation (1.1) and u the unique viscosity solution of the equation (1.11), both satisfying (1.12). Then, u_ϵ converges locally uniform in $[0,T) \times \mathbb{R}^n$ to u as $\epsilon \to 0$.

There is an interesting description of Theorem 1.1 in the context of stochastic games. In the case that the Lévy measure defining the trajectoy is integrable, we say that the process has *finite activity* and in characterized by the finiteness on the quantity of jumps of the trajectory in each finite interval of time. On the other hand, when the Lévy measure is not integrable, we say that the process has *infinte activity* and its jumps are infinite it each finite interval of time. Then, the result says that the value function of the game defined through controlled pure jump process of *infinite activity* can be approximated by a sequence of value functions of *finite activity* games.

The convergence is achieved through an equicontinuity argument over the sequences u_ϵ. However, we can provide a convergence rate when the data u_0, g are sufficiently regular. This regularity allows us to find more explicit bounds over the difference $u - u_\epsilon$ using comparison properties of the equation to be precised later. For instance, it is known (see [13]) that if we assume $f \in C^\gamma(\mathbb{R}^n)$ with $\gamma > 0$, then the solution u of a linear equation of the type $u + (-\Delta)^{2\sigma} u = f$ in $\mathbb{R}^n$ satisfy $u \in C^{2\sigma+\gamma}(\mathbb{R}^n)$. We do not obtain such improvement of regularity here, but assuming stronger regularity over the data f and u_0 we obtain the following results concerning the rate of convergence. Note that in this case we obtain a global uniform convergence.

Theorem 1.2. *Let u_ϵ, u be as in Theorem 1.1 and consider $\gamma > 0$. Assume $u_\epsilon \in C^{1,2\sigma+\gamma}(\mathbb{R}^n)$ with modulus of continuity that does not depend on ϵ. Then*

$$||u - u_\epsilon||_{L^\infty(\mathbb{R}^n)} = O(\epsilon^\gamma).$$

In the proofs of Theorem 1.1 and 1.2 the fact that $\sigma \in (0,1)$ is important in order to assure the finiteness of some integral quantities arising naturally in the computations related to the nonlocal operators. Nevertheless, it is well known that we can approximate local second order operators through nonlocal ones doing $\sigma \to 1$. For instance, in the linear case, $(1-\sigma)(-\Delta)^\sigma$ tends to a multiple of the local Laplace operator Δ when $\sigma \to 1$, see [5]. Note that the factor $(1-\sigma)$ is important in order to assure the finiteness of the limit. The idea is to approximate such type of local operators in the nonlinear setting by integro-differential operators (associated to singular kernels) and also by nonlocal operators of zeroth order (associated with nonsingular kernels).

Since now we need to move the exponent σ, we have to worry when this exponent is near 1. In this case, for a set of indices $\mathcal{A}$ and $s \in (s_0, 1)$ with $s_0 > 0$, we will consider a family of kernels $K_{s,\alpha}$ with the form

$$\begin{cases} K_{\alpha,s}(z) = (1-s)\frac{a_\alpha(\hat{z})}{|z|^{n+2s}}, \\ a_\alpha \in L^\infty(\mathbb{S}^{n-1}), \ \forall\, \alpha, \\ a_\alpha(z) = a_\alpha(-z), \ \forall z \in \mathbb{S}^{n-1}, \\ a_\alpha(\hat{z}) \in [\lambda, \Lambda], \ \forall\, \alpha \text{ and } z \in \mathbb{R}^n \setminus \{0\} \end{cases} \tag{1.15}$$

where $0 < \lambda \leq \Lambda < +\infty$ and $\hat{z} = z/|z|$ for $z \neq 0$. This family will satisfy condition (1.2) but relative to the weight function

$$\bar{w}(z) = (1 + |z|^n)^{-1}. \tag{1.16}$$

Fix $\mathcal{A}_1, \mathcal{A}_2$ two sets such that $\mathcal{A}_1 \times \mathcal{A}_2 \subseteq \mathcal{A}$. We define the operators $\mathcal{J}_s$ as

$$\mathcal{J}_s[u,x] = \inf_{i \in \mathcal{A}_1} \sup_{j \in \mathcal{A}_2} \int_{\mathbb{R}^n} \delta(u,x,z) K_{(i,j),s}(z) dz \tag{1.17}$$

for $u \in C^{1,1}(x)$. Clearly, this type of operator is fractional, in the sense that it is nonlocal, defined with kernels which are singular (not integrable) at zero. We will use this kind of operators to aproximate solutions of a local problem. We need to set the limit operator. Define, for each $(i,j) \in \mathcal{A}_1 \times \mathcal{A}_2$ and $k, l \in \{1, ..., n\}$

$$c_{k,l}^{i,j} = \int_{\mathbb{S}^{n-1}} w_k w_l a_{ij}(w) d\sigma(w).$$

The result when the exponent moves to 1 is the following

Theorem 1.3. *Assume the conditions over u_0 and g in the Theorem 1.1 holds. Let $0 < \gamma < 1$ and u_s be the unique viscosity solution of the problem*

$$\begin{cases} u_t(t,x) - \mathcal{J}_s[u(t,\cdot),x] = g(t,x,u(t,x)) & x \in \mathbb{R}^n, t \in (0,T) \\ u(0,x) = u_0(x) & x \in \mathbb{R}^n \end{cases} \tag{1.18}$$

satisfying (1.12) *with this γ.*

Then, u_s converges uniformly over compact sets of $[0,T) \times \mathbb{R}^n$ as $s \to 1^-$ to the unique viscosity solution satisfying (1.12) *(with respect to γ) of the* *problem*

(1.19)
$$\begin{cases} u_t(t,x) - F(D_x^2 u(t,x)) = g(t,x,u(t,x)) & x \in \mathbb{R}^n, t \in (0,T) \\ u(0,x) = u_0(x) & x \in \mathbb{R}^n \end{cases}$$

where F is the nonlinear operator

$$F(M) = \frac{w_n}{2} \inf_i \sup_j \sum_{k,l=1}^{n} c_{k,l}^{i,j} m_{kl}, \quad \text{for } M = (m_{kl})_{kl}$$

and $w_n = |B|$.

Note that in this case, we are approximating solutions of a local equation by solutions of a sequence of fractional equations. As in the Theorem 1.1, there is a interpretation in the context of stochastic games for the Theorem 1.3. When the trajectory of the game which is controlled by tha players is defined by a diffusion process, then the infinitesimal generator of this process is a linear second order differential operator. It is known also that the trajectory of a diffusion process is continuous. Then, Theorem 1.3 says that it is possible to approximate the value function of a game defined by a controlled diffusion process through a sequence of value functions defined by controlled jump processes. In other words, since diffusion process is also called $2-$stable process, the result says that a value function associated to such as process is approximated by the value function associated to $2s-$stable process as $s \to 1^-$.

It is possible to obtain an identical result as Theorem 1.3, but taking a sequence of solutions of nonlocal nonlinear equations associated to nonsingular kernels. In that case, we consider

$$K^b_{(i,j),s}(z) := (1-s)^{-(n+2s)+1} a_{ij}(\hat{z}) \Big(1 + |z/(1-s)|^{n+2s}\Big)^{-1}$$

and the associated operator

$$\mathcal{J}_s^b := \inf_i \sup_j \int_{\mathbb{R}^n} \delta(u,x,z) K^b_{(i,j),s}(z) dz$$

We can prove also a convergence rate associated to the Theorem 1.3 when we assume certain regularity.

Theorem 1.4. *Let u_s, u be as in Theorem 1.3 and consider $\gamma > 0$. Assume $u_s \in C^{1,2+\gamma}(\mathbb{R}^n)$ with modulus of continuity that does not depend on s. Then*

$$||u - u_s||_{(0,T)\times L^\infty(\mathbb{R}^n)} \leq C(\lambda, \Lambda, n, T, ||u||_{C^2}, [u]_{2+\gamma}) \frac{1-s}{\gamma}.$$

Note that related with the evaluation of the operator $\mathcal{I}$ in each equation we are allowed to consider functions with a certain growth at infinity, this is (1.12). Conditions (1.13) and (1.14) have to do with the existence of the solutions of the equations considered and we don't know if they are optimal in some sense. The uniqueness of the solutions has to do with suitable Comparison Principles. In this sense, as it is stated in the Theorems above we use the notion of viscosity solution to obtain, which we recall next

Definition 1.5. *An upper semicontinuous function* $u : [0,T) \times \mathbb{R}^n \to \mathbb{R}^n$ *is a viscosity subsolution to the problem* (1.1) *if*

- $u(0,x) \leq u_0(x)$ *for all* $x \in \mathbb{R}^n$.
- $\forall$ $(t_0,x_0) \in (0,T) \times \mathbb{R}^n$, $\varphi \in C^2((0,T) \times \mathbb{R}^n)$ *such that* $u - \varphi$ *has a maximum at* (t_0,x_0) *in the set* $B_\delta(t_0,x_0)$ *with* $\delta > 0$, *we have*

$$\varphi_t(t_0,x_0) - \inf_\alpha \sup_\beta \left(L^{1,\delta}_{\alpha,\beta}[x_0, \varphi(t_0,\cdot)] + L^{2,\delta}_{\alpha,\beta}[x_0, D_x\varphi(t_0,x_0), u(t_0,\cdot)] \right) \leq f(t_0,x_0) \tag{1.20}$$

where, for a kernel K*;* $x,p \in \mathbb{R}^n$, $\varphi \in C^2(\mathbb{R}^n)$, $v \in L^\infty(\mathbb{R}^n)$

$$L^{1,\delta}_K[x,\varphi] := \int_{B_\delta} (\varphi(x+y) - \varphi(x) - 1_B D\varphi(x) \cdot y) K(y) dy$$

$$L^{2,\delta}_K[x,p,v] := \int_{B_\delta^c} (v(x+y) - v(x) - 1_B p \cdot y) K(y) dy.$$

In a similar way, we define viscosity supersolution. A solution shall be a function which is sub and supersolution simultaneously. Note that a solution must be a continuous function. This is one of the several equivalent settings of the definition of viscosity solution. See for instance [4], [1], [2]. We shall use the notation

$$\mathcal{I}_\delta[u, \varphi, (t_0,x_0)]$$

to refer the evaluation in the nonlocal term made in (1.20).

Instead we are able to conclude the result for solutions satisfying (1.12), the framework of sub and supersolutions we will use is defined through the following sets: Subsolutions will be thought to be in the set

$$\mathcal{L}^1_* := \left\{ u \in USC((0,T)\times\mathbb{R}^n) \cap L^\infty_{loc}((0,T)\times\mathbb{R}^n) \ : \ \sup_{t\in[0,T)} \int_{\mathbb{R}^n} |u(t,x)| w_\sigma(x) dx < \infty \right\}$$

while supersolutions will be in the set

$$\mathcal{L}^{1*} := \left\{ u \in LSC((0,T)\times\mathbb{R}^n) \cap L^\infty_{loc}((0,T)\times\mathbb{R}^n) \ : \ \sup_{t\in[0,T)} \int_{\mathbb{R}^n} |u(t,x)| w_\sigma(x) dx < \infty \right\}$$

Note this familiy allow some growth at infinity for its elements, depending on the power σ of the maximal kernel w_σ.

2. Stability Lemma.

Note that in the context of Theorem 1.1 we will need a notion of convergence of operators. The next Lemma will provide us the desired convergence under some assuptioms over the components of the equation. One of this assumptions will be a kind of operator's convergence defined next, see also [6].

Definition 2.1. (Weak Convergence of Operators) *We say that a sequence of operators I_j converges weakly to I with respect to the function w if for each $\rho > 0$, $\bar{x} \in \mathbb{R}^n$ and functions v with the form*

$$v(x) = \begin{cases} p(x), & x \in B_\rho(\bar{x}), \\ u(x), & x \in B_\rho(\bar{x})^c. \end{cases}$$

where p is a polynomial of degree 2 and $u \in L^1(w(x)dx)$, then $I_j[u, x] \to I[u, x]$ uniformly in $B_{\rho/2}(\bar{x})$.

Lemma 2.2. (Stability Lemma) *Let $w \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Let $\mathcal{L}$ a family of kernels satisfying (1.2) and (1.3) and consider a sequence of operators I_j of the form (1.4) uniformly elliptic in j relative to the family $\mathcal{L}$. Let u, u_j functions in $[0, T) \times \mathbb{R}^n$ and an operator I such that*

- (s1) $\partial_t u_j(t, x) - I_j[u_j(t, \cdot), x] \geq g(t, x, u_j)$ *in $(0, T) \times \mathbb{R}^n$ in the viscosity sense.*
- (s2) $u_j \to u$ *locally uniform in* $(0, T) \times \mathbb{R}^n$.
- (s3) *u is continuous in $(0, T) \times \mathbb{R}^n$.*
- (s4) $I_j \to I$ *weakly in $\mathbb{R}^n$ (w.r.t. w).*
- (s5) $|u_j(t, \cdot)|_{L^1(w)}, |u(t, \cdot)|_{L^1(w)} \leq C$, *for all j and $t \in (0, T)$.*
- (s6) *g is continuous.*

Then, $\partial_t u(t, x) - I[u(t, \cdot), x] \geq g(t, x, u)$ in $(0, T) \times \mathbb{R}^n$ in the viscosity sense.

Proof. Let $p(t, x)$ be a polynomial of degree two touching u from below at a point $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$ in a neighborhood $N_r := (t_0 - r, t_0 + r) \times B_r(x_0)$. Define also

$$v(t, x) = \begin{cases} p(t, x), & (t, x) \in N_r \\ u(t, x), & (t, x) \in N_r^c \end{cases}$$

Since $u_j \to u$ locally uniform in $(0, T) \times \mathbb{R}^n$, it is possible to find sequences $t_j \to t_0$, $x_j \to x_0$, $d_j \to 0$ such that $p + d_j$ touches u_j from below at x_j. Define

$$v_j(t, x) = \begin{cases} p(t, x) + d_j, & (t, x) \in N_r \\ u_j(t, x), & (t, x) \in N_r^c \end{cases}$$

Since p is smooth, $v_j(t_j, x_j) \to v(t_0, x_0)$.

Let $y \in B_{r/4}(x_0)$ and j large so that $t_j \in (t_0 - r/2, t_0 + r/2)$. Then

(2.1)

$$\begin{aligned} |I_j[v_j(t_j, \cdot), y] - I[v(t_0, \cdot), y]| \leq\ & |I_j[v_j(t_j, \cdot), y] - I_j[v(t_j, \cdot), y]| \\ & + |I_j[v(t_j, \cdot), y] - I_j[v(t_0, \cdot), y]| \\ & + |I_j[v(t_0, \cdot), y] - I[v(t_0, \cdot), y]| \end{aligned}$$

We will estimate each term on the right hand side of (2.1). By (s5), the third term goes to zero uniformly in j for all $y \in B_{r/4}(x_0)$.

Define $U = B_r(x_0 - y) \cap B_r(x_0 + y) \neq \emptyset$ and note that $U^c \subset B^c_{r/2}(0)$. Since $v_j(t_j, \cdot)$ and $v(t_j, \cdot)$ differ only by a constant in U, we have $\delta(v(t_j, \cdot) - v_j(t_j, \cdot), y, z) = 0$ if $z \in U$. Using this and the ellipticity we have for the first term at the right hand side of (2.1)

$$
\begin{aligned}
|I_j[v_j(t_j,\cdot),y] - I_j[v(t_j,\cdot),y]| &\leq \max\left\{\left|\mathcal{M}^+[v_j(t_j,\cdot) - v(t_j,\cdot),y]\right|; \left|\mathcal{M}^+[v(t_j,\cdot) - v_j(t_j,\cdot),y]\right|\right\} \\
&\leq C_r \int_{B^c_{r/2}(0)} |\delta(v(t_j,\cdot) - v_j(t_j,\cdot),y,z)| w(z)dz
\end{aligned}
$$

where we have used the condition (1.2). But the last term can be estimated as

$$
\begin{aligned}
\int_{B^c_{r/2}(0)} |\delta((v_j - v)(t_j,\cdot),y,z)| w(z)dz &\leq \int_{B^c_{r/2}(0)} |v_j(t_j,z) - v(t_j,z)| w(z+y)dz \\
&\quad + \int_{B^c_{r/2}(0)} |v_j(t_j,z) - v(t_j,z)| w(z-y)dz \\
&\quad + 2|v_j(t_j,y) - v(t_j,y)| \int_{B^c_{r/2}(0)} w(z)dz. \\
&\leq ||v_j(t_j,\cdot) - v(t_j,\cdot)||_{L^1(\mathbb{R}^n,w)} + C|d_j|.
\end{aligned}
$$

Combining (s2) and (s5) we conclude the above L^1 norm goes to zero with j and so the first term in the right hand side of (2.1). For the second term in the right hand side of (2.1), using again the ellipticity, we have

$$
\begin{aligned}
|I_j[v(t_j,\cdot),y] - I_j[v(t_0,\cdot),y]| &\leq \sup_{K \in \mathcal{L}} \int_U |\delta(p(t_j,\cdot) - p(t_0,\cdot),y,z)| K(z)dz \\
&\quad + C_r \int_{B^c_{r/2}(0)} |u(t_j,z) - u(t_0,z)| w(z)dz \\
&\quad + C_r |u(t_j,y) - u(t_0,y)| \int_{B^c_{r/2}(0)} w(z)dz.
\end{aligned}
$$

By the smoothness of p in (t, x) and the condition (1.3) the first term goes to zero with j. Using the continuity of the limit function u and the integrability of w we conclude that the second and third term in the right hand side of the last inequality tend to zero with j. This concludes that the left hand side of (2.1) tends to zero when $j \to \infty$ uniformly in $y \in B_{r/4}(x_0)$.

By (s1) it is possible to write

$$\begin{aligned}\partial_t p(t_0,x_0) - I[v(t_0,\cdot),x_0] &\geq g(t_j,x_j,v_j(t_j,x_j)) \\ &+|I_j[v_j(t_j,\cdot),x_j] - I[v(t_0,\cdot),x_j]| \\ &+|\partial_t p(t_0,x_0) - \partial_t p(t_j,x_j)|\end{aligned}$$

The smoothness of p, the uniform convergence on $B_{r/4}(x_0)$ of (2.1) proved above and (s6) concludes the result. □

Lemma 2.3. *Let $J_{\alpha,\epsilon}$ the family of kernels given by* (1.8) *and $\mathcal{I}_\epsilon$ its asocited nonlineal nonlocal operator. Then $\mathcal{I}_\epsilon \to \mathcal{I}_0$ weakly as $\epsilon \to 0$.*

Proof: Let $x_0 \in \mathbb{R}^n$ and v be a function with the form

$$v(x) = \begin{cases} p(x), & x \in B_r(x_0) \\ u(x), & x \in B_r^c(x_0) \end{cases}$$

with $r > 0$, p a polinomial of degree 2 and $u \in L^1(\mathbb{R}^n, w)$. First, we will see the result for a Bellman-type operator. Fix $\epsilon > 0$, $x \in B_{r/4}(x_0)$. We have for all $\eta > 0$

$$\begin{aligned}\mathcal{I}_\epsilon[v,x] - \mathcal{I}_0[v,x] &= \sup_{\alpha\in\mathcal{A}} L_{\alpha,\epsilon}[u,x] - \sup_{\alpha\in\mathcal{A}} L_\alpha[u,x] \\ &\leq \eta + L_{\alpha(\eta),\epsilon}[u,x] - \sup_{\alpha\in\mathcal{A}} L_\alpha[u,x] \\ &\leq \eta + L_{\alpha(\eta),\epsilon}[u,x] - L_{\alpha(\eta)}[u,x] \\ &\leq \eta + \epsilon^{n+2\sigma}\int_{\mathbb{R}^n} \delta(u,x,z)\frac{a_{\alpha(\eta)}(\hat{y})}{(\epsilon^{n+2\sigma}+|z|^{n+2\sigma})|z|^{n+2\sigma}}dz \\ &\leq \eta + \epsilon^{n+2\sigma} C_n\Lambda||D^2p||_{L^\infty(B_r(x_0))}\int_0^{r/2}\frac{r^{1-2\sigma}dr}{(\epsilon^{n+2\sigma}+r^{n+2\sigma})} \\ &\quad + \epsilon^{n+2\sigma}\int_{B^c_{r/2}(0)} \delta(u,x,z)\frac{a_{\alpha(\eta)}(\hat{y})}{(\epsilon^{n+2\sigma}+|z|^{n+2\sigma})|z|^{n+2\sigma}}dz \\ &\leq \eta + \epsilon^{2(1-\sigma)}C(n,\sigma)\Lambda||D^2p||_{L^\infty(B_r(x_0))} + \epsilon^{n+2\sigma}C(n,r,\Lambda,\sigma,||u||_{L^1(\mathbb{R}^n,w)})\end{aligned}$$

Since this happens for any $\eta > 0$, we conclude

$$\mathcal{I}_\epsilon[v,x] - \mathcal{I}_0[v,x] \leq C(n,\sigma,p,u,\Lambda)\epsilon^{2(1-\sigma)}.$$

Note that the constant appearing in the previous inequality does not depend on x. Analogously, for all $\eta > 0$

$$\begin{aligned}\mathcal{I}_\epsilon[v,x] - \mathcal{I}_0[v,x] &= \sup_{\alpha\in\mathcal{A}} L_{\alpha,\epsilon}[u,x] - \sup_{\alpha\in\mathcal{A}} L_\alpha[u,x] \\ &\geq -\eta + L_{\alpha(\eta),\epsilon}[u,x] - L_{\alpha(\eta)}[u,x] \\ &\geq -\eta + C(n,\sigma,p,u,\lambda)\epsilon^{2(1-\sigma)}\end{aligned}$$

Then, $|\mathcal{I}_\epsilon[v,x]-\mathcal{I}_0[v,x]|\le C(n,\sigma,p,u,\lambda,\Lambda)\epsilon^{2(1-\sigma)}$ for all $x\in B_{r/4}(x_0)$ and the result follows taking $\epsilon\to 0$.

For an Isaacs operator, define

$$I_{\alpha,\epsilon}[u,x]=\sup_\beta L_{(\alpha,\beta),\epsilon}[u,x],$$

so

$$\mathcal{I}_\epsilon[u,x]=\inf_\alpha I_{\alpha,\epsilon}[u,x]$$

and we adopt the analogous notation to the operator associated to $\epsilon=0$. Then, taking ϵ, x and η as before

$$\begin{aligned}\mathcal{I}_\epsilon[u,x]-\mathcal{I}_0[u,x] &= \inf_\alpha I_{\alpha,\epsilon}[u,x]-\inf_\alpha I_{\alpha,*}[u,x]\\ &\le \eta+I_{\alpha(\eta),\epsilon}[u,x]-I_{\alpha(\eta),*}[u,x]\\ &= \eta+\sup_\beta L_{(\alpha(\eta),\beta),\epsilon}[u,x]-\sup_\beta L_{\alpha(\eta),\beta}[u,x]\end{aligned}$$

and the result follows in the same way as in the Bellman case. □

For the local limit case considered in Theorem 1.3, we have

Lemma 2.4. *Let $\mathcal{J}_s$ the sequence of nonlocal operators given by* (1.17) *and $F(D^2\cdot)$ defined in* (1.20)*. Then $\mathcal{J}_s\to F(D^2\cdot)$ weakly (with respect to w_1) as $s\to 1^-$.*

Proof: Let $u\in L^1(\mathbb{R}^n,w_1(x)dx)$. Let $x_0\in\mathbb{R}^n$ and $\rho>0$. Consider p a polynomial of degree 2 and define

$$v(x)=\begin{cases} p(x), & x\in B_\rho(x_0)\\ u(x), & x\in B^c_\rho(x_0)\end{cases}$$

Let $x\in B_{\rho/2}(x_0)$ and $\epsilon>0$ arbitrary. We have

$$\begin{aligned}&\mathcal{J}_s[v,x]-F(D^2v(x))\\ =\;&\mathcal{J}_s[v,x]-F(D^2p(x))\\ =\;&\inf_\alpha\sup_\beta(1-s)\int_{\mathbb{R}^n}\delta(v,x,z)|DetA_{\alpha,\beta}|^{-1}|A^{-1}_{\alpha,\beta}z|^{n+2s}dz-w_n/2\inf_\alpha\sup_\beta a^{\alpha,\beta}_{ij}\partial_{ij}p(x)\\ =\;&\inf_\alpha\sup_\beta(1-s)\int_{\mathbb{R}^n}\delta(v_{\alpha,\beta},x_{\alpha,\beta},z)|z|^{n+2s}dz-w_n/2\inf_\alpha\sup_\beta a^{\alpha,\beta}_{ij}\partial_{ij}p(x)\end{aligned}$$

with $v_{\alpha,\beta}(y)=v(A_{\alpha,\beta}y)$ and $x_{\alpha,\beta}=A^{-1}_{\alpha,\beta}x$. Following the steps of Lemma 2.3, we conclude the existence of $\alpha(\epsilon),\beta(\epsilon)$ such that

$$\mathcal{J}_s[v,x]-F(D^2v(x))\le\epsilon+(1-s)\int_{\mathbb{R}^n}\delta(v_{\alpha,\beta},x_{\alpha,\beta},z)|z|^{-(n+2s)}dz-w_n/2a^{\alpha,\beta}_{ij}\partial_{ij}p(x)$$

But for the first term in the right hand side of the last equation we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \delta(v_{\alpha,\beta}, x_{\alpha,\beta}, z)|z|^{-(n+2s)}dz \\
= &\int_{B_\rho(0)} < D^2 v_{\alpha,\beta}(x) > |z|^{-(n+2s)+2}dz \\
&+ \int_{B^c_{\rho/2}(0)} v_{\alpha,\beta}(x_{\alpha,\beta}+z)w_1(z)dz + \int_{B^c_{\rho/2}(0)} v_{\alpha,\beta}(x_{\alpha,\beta}-z)w_1(z)dz \\
&-2p(x) \int_{B^c_{\rho/2}(0)} w_1(z)dz \\
= & w_n a_{ij}^{\alpha,\beta}\partial_{ij}p(x)\int_0^\rho t^{1-2s}dt - 2C_\rho p(x) \\
&+ \int_{B^c_{\rho/2}(0)} v(x+A_{\alpha,\beta}z)w_1(z)dz + \int_{B^c_{\rho/2}(0)} v(x-A_{\alpha,\beta}z)w_1(z)dz \\
\leq & w_n a_{ij}^{\alpha,\beta}\partial_{ij}p(x)\int_0^\rho t^{1-2s}dt - 2C_\rho p(x) + C(\Lambda, n) \int_{B_{\lambda\rho/4}(0)} |v(z)|w_1(z)dz \\
\leq & \frac{\rho^{2(1-s)}}{2(1-s)} w_n a_{ij}^{\alpha,\beta}\partial_{ij}p(x) - 2C_\rho p(x) + C(\Lambda, \lambda, n, \rho)||v||_{L^1(\mathbb{R}^n, w_1)}.
\end{aligned}
$$

This allow us to write

$$
\begin{aligned}
\mathcal{J}_s[v,x] - F(D^2v(x)) \;\; \leq \;\; & \epsilon + \frac{w_n}{2} a_{ij}^{\alpha,\beta}\partial_{ij}p(x)(\rho^{2(1-s)}-1) \\
& +(1-s)C(\lambda, \Lambda, n, \rho, |p|_{L^\infty(B_\rho(x_0))}, ||v||_{L^1(\mathbb{R}^n,w_1)}) \\
\leq \;\; & \epsilon + (\rho^{2(1-s)}-1)C + (1-s)C
\end{aligned}
$$

where the constant C depends on $\lambda, \Lambda, n, \rho, |p|_{C^2(B_\rho(x_0))}, |u|_{L^1(\mathbb{R}^n,w_1)}$ but not on ϵ and x. A similar lower bound can be found. Taking $s \to 1^-$ and recalling that ϵ is arbitrary, we conclude the result. □

3. Comparison Principles.

Comparison Principles are the heart of the arguments to come. We will show that equations like (1.1) satisfies Comparison Principle when the sub and supersolutions are in the sets $\mathcal{L}^1_*$, $\mathcal{L}^{1*}$ and are bounded above, below, respectively.

Proposition 3.1. (Comparison Principle) *Let $g \in C((0,T)\times\mathbb{R}^n\times\mathbb{R})$ satisfying* (1.5)*, $u_0 \in C(\mathbb{R}^n)$. Let $w \in L^\infty(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$ and $\mathcal{L}$ a family of kernels $\{K_{ij}\}$ satisfying* (1.2) *(relative to w) and* (1.3)*. Consider also $\mathcal{I}$ a* *nonlocal operator with the form* (1.4) *relative to* $\mathcal{L}$. *Let* $u \in \mathcal{L}^1_*$ *and* $v \in \mathcal{L}^{1*}$ *sub and supersolution of the problem* (1.1) *associated to this operator* $\mathcal{I}$, *respectively, with* u *bounded above and* v *bounded below. Then,* $u \leq v$ *in* $[0,T) \times \mathbb{R}^n$.

Proof: First, we need to define a special function before start the proof. Consider $\psi(x)$ a smooth function with $D\psi, D^2\psi$ uniformly bounded in $\mathbb{R}^n$ and such that $\psi(x) = 0$ if $|x| \leq 1$ and $\psi(x) > |\sup u| + |\inf v|$ if $|x| > 2$. For $\beta > 0$, we define $\psi_\beta(x) = \psi(\beta x)$. In relation to ψ_β, we can say:

(1) $D\psi_\beta, D^2\psi_\beta \to 0$ as $\beta \to 0$ and the convergence is uniform in $\mathbb{R}^n$.

(2) For each $K \in \mathcal{L}$, $L_K[\psi_\beta, x] \to 0$ as $\beta \to 0$ uniformly in $\mathbb{R}^n$. In fact, for all $x \in \mathbb{R}^n$ we have

$$\begin{aligned}
L_K[\psi_\beta, x] &= \int_{\mathbb{R}^n} (\psi_\beta(x+z) + \psi_\beta(x-z) - 2\psi_\beta(x))K(z)dz \\
&= \int_{B} (\psi(\beta x+\beta z) + \psi(\beta x-\beta z) - 2\psi(\beta x))K(z)dz \\
&\quad + \int_{B^c} (\psi(\beta x+\beta z) + \psi(\beta x-\beta z) - 2\psi(\beta x))K(z)dz \\
&\leq |D^2\psi|_\infty \int_B |\beta z|^2 K(z)dz + \int_{B^c} |\delta(\psi, \beta x, \beta z)|w(z)dz \\
&\leq \beta^2 |D^2\psi|_\infty A + C \int_{B_{1/\sqrt{\beta}} \setminus B} |\delta(\psi, \beta x, \beta z)|w(z)dz + C \int_{B^c_{1/\sqrt{\beta}}} |\delta(\psi, \beta x, \beta z)|w(z)dz \\
&\leq \beta^2 |D^2\psi|_\infty A + C|D^2\psi|_\infty \int_{B_{1/\sqrt{\beta}} \setminus B} |\beta z|^2 w(z)dz + 4C|\psi|_\infty \int_{B^c_{1/\sqrt{\beta}}} w(z)dz \\
&\leq \beta^2 |D^2\psi|_\infty A + C|D^2\psi|_\infty \beta \int_{B_{1/\sqrt{\beta}} \setminus B} w(z)dz + 4C|\psi|_\infty \int_{B^c_{1/\sqrt{\beta}}} w(z)dz \\
&\leq \beta^2 |D^2\psi|_\infty A + C|D^2\psi|_\infty \beta |w|_{L^1} + 4C|\psi|_\infty o_\beta(1).
\end{aligned}$$

where we have used the conditions (1.2) (with $r = 1$) and (1.3). Then

$$L_K[\psi_\beta, x] \leq C(\psi, A, w) o_\beta(1) \tag{3.1}$$

It is important to remark that the constant appearing in (3.1) is uniform in x and K. It is possible to find a similar lower bound for $L_K[\psi_\beta, x]$.

Now we start the proof. We will argue by contradiction, assuming that

$$M := \sup_{\substack{t \,\in\, [0,T) \\ x \,\in\, \mathbb{R}}} (u(t,x) - v(t,x)) > 0.$$

With the function ψ_β defined above, dedoubling variables we consider the penalization

(3.2)
$$M_{\epsilon,\beta,\eta} := \sup_{\substack{s,\,t\,\in\,[0,T)\\ x,\,y\,\in\,\mathbb{R}}} \Big(u(t,x) - v(t,y) - \frac{|x-y|^2}{2\epsilon} - \frac{(s-t)^2}{2\epsilon} - \psi_\beta(x) - \frac{\eta}{T-t} \Big),$$

for ϵ, β, η which we will think small.

By the contradiction assumption, there exists a $(t_0, x_0) \in (0,T) \times \mathbb{R}^n$ such that
$$u(t_0, x_0) - v(t_0, x_0) \geq M/2,$$
hence, if we fix $\beta_0 < \frac{1}{2(1+|x_0|)}$ and $\eta < \frac{(T-t_0)M}{16}$, then we have

$$M_{\epsilon,\beta,\eta} > M/4 > 0, \tag{3.3}$$

for all ϵ, $\beta < \beta_0$. Hereafter, the parameter η shall be fixed as above.

Consider a sequence of points (s,t,x,y) such that its evaluation over the penalization converges to the supremum. By the construction of ψ_β, the x component of this sequence belongs to the ball $B_{1/\beta}(0)$. Using (3.3), we conclude for ϵ suitably small

$$|x-y|^2 \leq 2(|\sup u| + |\inf v|)\epsilon \leq C\epsilon \tag{3.4}$$

for all η and β. Then, we have the y component of the sequence belongs to a compact set also. Using (3.3) we have the t component of the sequence satisfies

$$T - t > \eta(|\sup u| + |\inf v|)^{-1} > 0, \quad \forall \epsilon, \beta, \tag{3.5}$$

and following (3.4), it is possible to prove that

$$|s-t|^2 \leq 2(|\sup u| + |\inf v|)\epsilon \leq C\epsilon, \tag{3.6}$$

which means that for ϵ small, s is away T also.

Then, we can assure that supremum $M_{\epsilon,\beta,\eta}$ is attained at a possibly non unique point $(\bar{s}_{\epsilon,\beta,\eta}, \bar{t}_{\epsilon,\beta,\eta}, \bar{x}_{\epsilon,\beta,\eta}, \bar{y}_{\epsilon,\beta,\eta}) \in [0,T)^2 \times \mathbb{R}^n \times \mathbb{R}^n$.

It only remains to prove that the maximum is away 0. In fact, if we consider β fixed and assume that $\bar{t} \to 0$ as $\epsilon \to 0$ (which implies that $\bar{s} \to 0$ by (3.6)), taking limsup in (3.3) we conclude
$$0 < M/4 < u(0, x^*) - v(0, x^*) \leq 0$$
for some $|x^*| \leq \frac{1}{2\beta}$ by the initial condition, which is a contradiction. Then we conclude that for $0 < \beta < \beta_0$, $M_{\epsilon,\beta,\eta}$ is in fact a maximum point attained at a point
$$(\bar{s}_{\epsilon,\beta,\eta}, \bar{t}_{\epsilon,\beta,\eta}, \bar{x}_{\epsilon,\beta,\eta}, \bar{y}_{\epsilon,\beta,\eta}) \in (0,T)^2 \times \mathbb{R}^n \times \mathbb{R}^n$$
uniformly in ϵ small enough. Ommiting for the momment all the subscipts of the maximum point, we can write

(3.7)
$$0 < M/4 < u(\bar{t}, \bar{x}) - v(\bar{s}, \bar{y}), \quad \text{for all } \epsilon, \text{ and all } \beta \text{ small enough.}$$

The idea is to use all the above penalizations introduced as test functions. Hence, the function

$$(t,x) \mapsto u(t,x) - (v(\bar{s},\bar{y}) + \frac{|x-\bar{y}|^2}{2\epsilon} + \frac{(t-\bar{s})^2}{2\epsilon} + \psi_\beta(x) + \frac{\eta}{T-t})$$

has a maximum point at $(\bar{t},\bar{x})$. In the same way, the function

$$(s,y) \mapsto v(s,y) - (u(\bar{t},\bar{x}) - \frac{|\bar{x}-y|^2}{2\epsilon} - \frac{(\bar{t}-s)^2}{2\epsilon} - \psi_\beta(\bar{x}) - \frac{\eta}{T-\bar{t}})$$

Denote $\varphi(s,t,x,y) := \frac{|x-y|^2}{2\epsilon} + \frac{(t-s)^2}{2\epsilon} + \psi_\beta(x) + \frac{\eta}{T-t}$ for simplicity. Using that u is a subsolution and v is a supersolution for (1.1), then we can add both testing suitably to conclude

$$\begin{aligned}\frac{\eta}{(T-\bar{t})^2} \leq\;& g(\bar{t},\bar{x},u(\bar{t},\bar{x})) - g(\bar{s},\bar{y},v(\bar{s},\bar{y}))\\ &+ \inf_i \sup_j L^{1,\delta}_{ij}[\bar{x},\varphi(\bar{s},\bar{t},\cdot,\bar{y})] + L^{2,\delta}_{ij}[\bar{x},D_x\varphi(\bar{s},\bar{t},\bar{x},\bar{y}),u(\bar{t},\cdot)]\\ &- \inf_i \sup_j L^{1,\delta}_{ij}[\bar{y},-\varphi(\bar{s},\bar{t},\bar{x},\cdot)] + L^{2,\delta}_{ij}[\bar{y},-D_y\varphi(\bar{s},\bar{t},\bar{x},\bar{y}),v(\bar{s},\cdot)]\end{aligned}$$

for all $\delta > 0$.

It is possible to proceed as in the proof of Lemma 2.3, looking for the Bellman case and then passing to the Isaacs case to find $K \in \mathcal{L}$ (depending on T,η,ϵ,β) such that

(3.8)

$$\begin{aligned}\frac{\eta}{(T-\bar{t})^2} \leq\;& \frac{\eta}{2T^2} + g(\bar{t},\bar{x},u(\bar{t},\bar{x})) - g(\bar{s},\bar{y},v(\bar{s},\bar{y}))\\ &+ L^{1,\delta}_K[\bar{x},\varphi(\bar{s},\bar{t},\cdot,\bar{y})] + L^{2,\delta}_K[\bar{x},D_x\varphi(\bar{s},\bar{t},\bar{x},\bar{y}),u(\bar{t},\cdot)]\\ &- L^{1,\delta}_K[\bar{y},-\varphi(\bar{s},\bar{t},\bar{x},\cdot)] - L^{2,\delta}_K[\bar{y},-D_y\varphi(\bar{s},\bar{t},\bar{x},\bar{y}),v(\bar{s},\cdot)]\end{aligned}$$

We need to estimate the right hand side of (3.8). Recalling $(\bar{s},\bar{t},\bar{x},\bar{y})$ is a global maximum point attaining $M_{\epsilon,\beta,\eta}$, it is possible to write down the following inequalities for each $z \in \mathbb{R}^n$

(3.9)

$$\begin{aligned}u(\bar{t},\bar{x}+z) - u(\bar{t},\bar{x}) - \Big(\frac{\bar{x}-\bar{y}}{\epsilon} + D\psi_\beta(\bar{x})\Big)\cdot z \;\leq\;& v(\bar{s},\bar{y}+z) - v(\bar{s},\bar{y})\\ &+\psi_\beta(\bar{x}+z) - \psi_\beta(\bar{x}) - D\psi_\beta(\bar{x})\cdot z\\ &-\frac{\bar{x}-\bar{y}}{\epsilon}\cdot z\end{aligned}$$

and

(3.10)

$$u(\bar{t},\bar{x}+z) - u(\bar{t},\bar{x}) \leq v(\bar{s},\bar{y}+z) - v(\bar{s},\bar{y}) + \psi_\beta(\bar{x}+z) - \psi_\beta(\bar{x}).$$

For any $K \in \mathcal{L}$ and its corresponding linear operator L_K (which we denote simply as L), we have

(3.11)

$$L^{1,\delta}[\bar{x}, \varphi(\bar{s}, \bar{t}, \cdot, \bar{y})] = L^{1,\delta}[\bar{y}, -\varphi(\bar{s}, \bar{t}, \bar{x}, \cdot)] + \frac{1}{\epsilon}\int_{B_\delta} |z|^2 K(z)dz + L^{1,\delta}[\bar{x}, \psi_\beta].$$

Using (3.9) we obtain

(3.12)

$$\begin{aligned} L_1^{2,\delta}[\bar{x}, D_x\varphi(\bar{s}, \bar{t}, \bar{x}, \bar{y}), u(\bar{t}, \cdot)] \quad &\leq \quad L_1^{2,\delta}[\bar{y}, -D_y\varphi(\bar{s}, \bar{t}, \bar{x}, \bar{y}), v(\bar{s}, \cdot)] \\ &\quad\; + L_1^{2,\delta}[\bar{x}, D\psi_\beta(\bar{x}), \psi_\beta] \end{aligned}$$

Similarly, now using (3.10) we obtain

(3.13)

$$L_2^{2,\delta}[\bar{x}, u(\bar{t}, \cdot)] \leq L_2^{2,\delta}[\bar{y}, v(\bar{s}, \cdot)] + L_2^{2,\delta}[\bar{x}, \psi_\beta].$$

Plugging (3.11), (3.12) and (3.13) in (3.8) and recalling the assumption (1.3) we conclude

$$\begin{aligned} \frac{\eta}{(T-\bar{t})^2} \quad &\leq \quad \frac{\eta}{2T^2} + g(\bar{t}, \bar{x}, u(\bar{t}, \bar{x})) - g(\bar{s}, \bar{y}, v(\bar{s}, \bar{y})) \\ &\quad\; + \frac{1}{\epsilon}o_\delta + L^{1,\delta}[\bar{x}, \psi_\beta] + L_1^{2,\delta}[\bar{x}, D\psi_\beta(\bar{x}), \psi_\beta] + L_2^{2,\delta}[\bar{x}, \psi_\beta]. \end{aligned}$$

Using (3.1) to control the integral terms applied to ψ_β and then taking $\delta \to 0$ we have

$$\frac{\eta}{(T-\bar{t})^2} \quad \leq \quad \frac{\eta}{2T^2} + g(\bar{t}, \bar{x}, u(\bar{t}, \bar{x})) - g(\bar{s}, \bar{y}, v(\bar{s}, \bar{y})) + C(A, \psi, w)o_\beta(1)$$

At this point, we can use (1.5) with (3.7) in the folowing way

$$\begin{aligned} g(\bar{t}, \bar{x}, u(\bar{t}, \bar{x})) - g(\bar{s}, \bar{y}, v(\bar{s}, \bar{y})) \quad &= \quad g(\bar{t}, \bar{x}, u(\bar{t}, \bar{x})) - g(\bar{s}, \bar{y}, u(\bar{t}, \bar{x})) \\ &\quad\; + g(\bar{s}, \bar{y}, u(\bar{t}, \bar{x})) - g(\bar{s}, \bar{y}, v(\bar{s}, \bar{y})) \\ &\leq \quad g(\bar{t}, \bar{x}, u(\bar{t}, \bar{x})) - g(\bar{s}, \bar{y}, u(\bar{t}, \bar{x})) \end{aligned}$$

to conclude

$$\frac{\eta}{(T-\bar{t})^2} \leq \frac{\eta}{2T^2} + g(\bar{t}, \bar{x}, u(\bar{t}, \bar{x})) - g(\bar{s}, \bar{y}, u(\bar{t}, \bar{x})) + C(A, \psi, w)o_\beta(1).$$

For β fixed, we know $\bar{t} > 0$ uniformly in ϵ small enough. Then, we can take $\epsilon \to 0$ and the continuity of g together with the bounds (3.4) and (3.6) to obtain

(3.14)
$$\frac{\eta}{T^2} \leq \frac{\eta}{2T^2} + C(A, \psi, w)o_\beta(1).$$

Then sending $\beta \to 0$ in (3.14) to conclude

$$0 < \frac{\eta}{T^2} \leq \frac{\eta}{2T^2}$$

This is a contradiction. □

When u_0, g are bounded, it is possible to use Perron's method applying the previous comparison principle to conclude the existence of a unique bounded solution of the equation of the type (1.1).

We are able to drop the boundeness condition in the last Proposition and allow unboundeness on the sub and supersolutions, but under slightly stronger condition over the integrability of the sub and supersolutions than the one's appeared in the definition of $\mathcal{L}^1_*, \mathcal{L}^{1*}$. This unboundeness depends on the behavior of the family of kernels defining the nonlocal nonlinear operator $\mathcal{I}$.

Proposition 3.2. (Comparison for Unbounded Solutions in $\mathbb{R}^n$) *Let $g \in C((0,T)\times\mathbb{R}^n\times\mathbb{R})$ satisfying* (1.5)*, $u_0 \in C(\mathbb{R}^n)$. Consider the equation*

(3.15)
$$\begin{cases} u_t(t,x) - \mathcal{I}[u(t,\cdot),x] = g(t,x,u(t,x)) & (t,x)\in(0,T)\times\mathbb{R}^n \\ u(0,x) = u_0(x) & x\in\mathbb{R}^n \end{cases}$$

where $\mathcal{I}$ is an Isaacs operator associated to a family of kernels of the form (1.7)*. Let $u \in \mathcal{L}^1_*$ and $v \in \mathcal{L}^{1*}$ sub and supersolutions of the equation* (3.15) *respectively, and such that, for some $0<\gamma<2\sigma$,*

$$\limsup_{|x|\to\infty}\sup_{(0,T)} |x|^{-\gamma}u(t,x) \le 0 \tag{3.16}$$

$$\liminf_{|x|\to\infty}\inf_{(0,T)} |x|^{-\gamma}v(t,x) \ge 0 \tag{3.17}$$

Then, $u \le v$ in $[0,T)\times\mathbb{R}^n$.

Proof: Let $k>0$ to be fixed. Define
$$\tilde{w}(t,x) = e^{kt}(1+|x|^2)^{\gamma/2}$$

For each $\epsilon>0$, it is clear by the condition (3.16) $u-\epsilon\tilde{w}$ is locally bounded and bounded from above.

On the other hand, for each kernel K_α considered in (1.7) we have

(3.18)
$$\begin{aligned}
&\int_{\mathbb{R}^n}\Big((1+|x+z|^2)^{\gamma/2} - (1+|x|^2)^{\gamma/2} - 1_B D(1+|x|^2)^{\gamma/2}\cdot z\Big)K(z)dz \\
&\le C(\gamma,A)(1+|x|)^{\gamma-1} + \int_{\mathbb{R}^n}\Big((1+|x+z|^2)^{\gamma/2} - (1+|x|^2)^{\gamma/2}\Big)K(z)dz \\
&= C(\gamma,A)(1+|x|)^{\gamma-1} + (1+|x|^2)^{\gamma/2}\int_{\mathbb{R}^n}\Big[\Big(\frac{1+|x+z|^2}{1+|x|^2}\Big)^{\gamma/2} - 1\Big]K(z)dz \\
&\le C(\gamma,A)(1+|x|^2)^{\gamma/2}.
\end{aligned}$$

Now we prove that $u-\epsilon\tilde{w}$ is subsolution of the equation (3.15) if we take k large enough. Let $(t_0,x_0)\in(0,T)\times\mathbb{R}^n$ and ϕ smooth such that $u-\epsilon\tilde{w}-\phi$

has a global maximum in (t_0, x_0). However, we can see that $u - (\epsilon\tilde{w} + \phi)$ has a maximum point at (t_0, x_0). Since u is subsolution of (3.15), we have

$$\begin{aligned} 0 \geq\ & \partial_t\phi(t_0,x_0) + \epsilon k\tilde{w}(t_0,x_0) \\ & - \inf_i \sup_j L^{1,\delta}_{ij}[\phi,(t_0,x_0)] + L^{2,\delta}_{ij}[u, \epsilon D\tilde{w} + D\phi, (t_0,x_0)] \\ & -g(t_0,x_0,u(t_0,x_0)). \end{aligned} \tag{3.19}$$

But, for each i, j

$$L^{1,\delta}_{ij}[\phi,(t_0,x_0)] = L^{1,\delta}_{ij}[\epsilon\tilde{w} + \phi,(t_0,x_0)] - \epsilon L^{1,\delta}_{ij}[\tilde{w},(t_0,x_0)]. \tag{3.20}$$

and

(3.21)

$$L^{2,\delta}_{ij}[u - \epsilon\tilde{w}, D\phi, (t_0,x_0)] = L^{2,\delta}_{ij}[u, \epsilon D\tilde{w} + D\phi, (t_0,x_0)] - \epsilon L^{2,\delta}_{ij}[\tilde{w}, D\tilde{w}, (t_0,x_0)].$$

Using (3.20), (3.21),(3.18) and then applying the inequality (3.19), we have

$$\begin{aligned} & \partial_t\phi(t_0,x_0) + \epsilon k\tilde{w}(t_0,x_0) \\ & - \inf_i \sup_j L^{1,\delta}_{ij}[\phi,(t_0,x_0)] + L^{2,\delta}_{ij}[u - \epsilon\tilde{w}, D\phi, (t_0,x_0)] \\ & -g(t_0,x_0,u(t_0,x_0) - \epsilon\tilde{w}(t_0,x_0)) \\ \leq\ & \partial_t\phi(t_0,x_0) + \epsilon k\tilde{w}(t_0,x_0) \\ & - \inf_i \sup_j L^{1,\delta}_{ij}[\phi,(t_0,x_0)] + L^{2,\delta}_{ij}[u, \epsilon D\tilde{w} + D\phi, (t_0,x_0)] + \epsilon C(\gamma, A)(1+|x|^2)^{\gamma/2} \\ & -g(t_0,x_0,u(t_0,x_0) - \epsilon\tilde{w}(t_0,x_0)) + g(t_0,x_0,u(t_0,x_0)) - g(t_0,x_0,u(t_0,x_0)) \\ \leq\ & -\epsilon k\tilde{w}(t_0,x_0) + \epsilon C(\gamma, A)(1+|x_0|^2)^{\gamma/2} + \\ & -g(t_0,x_0,u(t_0,x_0) - \epsilon\tilde{w}(t_0,x_0)) + g(t_0,x_0,u(t_0,x_0)) \end{aligned}$$

Taking $k > C(\gamma, A)$ and using the monotonicity of g, we conclude that $u - \epsilon\tilde{w}$ is a bounded above, locally bounded subsolution of (3.15) for each $\epsilon > 0$. In the same way, it is possible to prove that $v + \epsilon\tilde{w}$ is a bounded below, locally bounded supersolution of (3.15). Then, for all $\epsilon > 0$, using Proposition (3.1), we obtain $u - \epsilon\tilde{w} \leq v + \epsilon\tilde{w}$ in $[0,T) \times \mathbb{R}^n$. Taking $\epsilon \to 0$ we conclude the proof □

4. Existence of Unbounded Solutions.

Using the last comparison principle, it is possible to obtain existence of a solution u of the equation (3.15) satisfying

$$\lim_{|x|\to\infty} \sup_{t\in[0,T)} |u(t,x)||x|^{-\gamma} = 0.$$

but as in [3], we need to ask for the data (1.13) and (1.14).

We consider $\tilde{w}(t,x) = e^{kt}(1+|x|^2)^{\gamma/2}$ as before and define, for each $n \in \mathbb{N}$

$$\tilde{w}_n(t,x) = C_n(t+1) + \frac{1}{n}\tilde{w}_n(t,x).$$

with C_n to be fixed later. We have

$$\begin{aligned}
& \partial_t \tilde{w}_n(t,x) - \mathcal{I}[\tilde{w}_n,(t,x)] - g(t,x,\tilde{w}_n) \\
\geq \quad & C_n + \frac{k}{n}\tilde{w} - C(A,\gamma)(1+|x|^2)^{\gamma/2} - g(t,x,\tilde{w}_n) \\
\geq \quad & C_n + \frac{k}{n}\tilde{w} - \frac{1}{n}C(A,\gamma)(1+|x|^2)^{\gamma/2} - g(t,x,0)
\end{aligned}$$

since g is decreasing in its third variable.

By the assumption over g, it is possible to get C_n so that

$$g(t,x,0) \leq C_n + \frac{1}{n}\tilde{w} \tag{4.1}$$

and then, taking $k = 1 + C(A,\gamma)$ we conclude that

$$\partial_t \tilde{w}_n(t,x) - \mathcal{I}[\tilde{w}_n,(t,x)] - g(t,x,\tilde{w}_n) \geq 0.$$

for each $n \in N$.

In a smiliar way, due to the assumption over u_0 we can get C_n so large such that

$$u_0(x) \leq C_n + \frac{1}{n}\tilde{w}(0,x).$$

This implies that $\bar{w} = \min \tilde{w}_n$ is a supersolution of the equation.

Also, it is possible to take C_n large such that $\tilde{w}_{n+1} \geq \tilde{w}_n + 1$ in the set $[0,T) \times B_n(0)$?????.

• para que es esto?

The supersolution $\bar{w}$ satisfies the condition (3.16). In fact, if we assume that there is a sequence $x_k \to \infty$ such that

$$\sup_{[0,T)} (\min \tilde{w}_n)|x|^{-\gamma} = l > 0.$$

This implies that there exists $k_0 \in \mathbb{N}$ and $\tilde{t} \in (0,T)$ such that for all $j > k_0$ and $n \in \mathbb{N}$

$$C_n(\tilde{t}+1)|x_j|^{-\gamma} + \frac{1}{n}e^{kj}(1+|x_j|^2)^{\gamma/2}|x_j|^{-\gamma} > l/2.$$

Taking limits as $j \to \infty$ and then taking limits as $n \to \infty$, we conclude

$$0 > l/2 > 0$$

which is a contradiction.

Similarly, we can construct a subsolution $\underline{\mathrm{w}}$ satisfying the inequality (3.17). Then, for each subsolution u of (3.15) with $u \leq \bar{w}$, we have comparison because in that case u will satisfy (3.16). We have analogous consequence in the case of a supersolution v with $v \geq \underline{\mathrm{w}}$. Therefore, we apply Perron's method to conclude the existence of a solution of (3.15) with

$$\lim_{|x|\to\infty} \sup_{[0,T)} u(t,x)|x|^{-\gamma} = 0.$$

By the comparison principle, this solution is unique.

5. Comparison Principle in Bounded Domains.

Here we will provide a Comparison Principle for problems defined in a bounded domain. This result will allow us to conclude a desired modulus of continuity for the sequence of approximating equations. Throughout all this section, we will consider $\Omega \subset \mathbb{R}^n$ a bounded domain. We will consider an equation in $(0,T)\times\Omega$ of the type

$$u_t(t,x) - \mathcal{I}[u(t,\cdot),x] = g(t,x,u(t,x)) \qquad (t,x)\in(0,T)\times\Omega \tag{5.1}$$

and its corresponding *Cauchy Problem with Exterior Condition*

$$\begin{cases} u_t(t,x) - \mathcal{I}[u(t,\cdot),x] = g(t,x,u(t,x)) & (t,x)\in(0,T)\times\Omega \\ u(t,x) = f(t,x) & (t,x)\in[0,T)\times\Omega^c \\ u(0,x) = u_0(x) & x\in\bar{\Omega}. \end{cases} \tag{5.2}$$

A very natural condition over f is the following

$$\sup_{t\in[0,T)} ||f(t,\cdot)||_{L^1(w_\sigma)} < +\infty \tag{5.3}$$

Since we do not adress here the existence of the solution of the problem (5.2), we shall avoid some technical assumptions over the data g and u_0 concerning its growth at infinity, used in the existence result of unbounded solutions for the problem in the entire space.

We will prove Comparison Principle for the problem (5.2) under suitable hypotheses over the sub and supersolutions and the data f and u_0. The first result has to do with sub and supersolutions bounded above and below respectively.

Theorem 5.1. *Consider f, g, u_0 (5.3), g satisfying (1.5). Let $u\in\mathcal{L}^1_*, v\in\mathcal{L}^{1*}$ sub and supersolution to the problem (5.2), u bounded above, v bounded below in $(0,T)\times\mathbb{R}^n$. Then, $u\le v$ in $(0,T)\times\Omega$.*

Using this result, we extend the result for unbounded sub and supersolutions satisfying the same condition asked in the Proposition 3.2.

Theorem 5.2. *Consider f, g, u_0 (5.3), g satisfying (1.5). Let $u\in\mathcal{L}^1_*, v\in\mathcal{L}^{1*}$ sub and supersolution to the problem (5.2), u satisfying (3.16), v satinfying (3.17). Then, $u\le v$ in $(0,T)\times\Omega$.*

For this, we will need some technical results over the equation (5.1) in order to obtain a auxiliary equation for the difference of its super and subsolutions. An important tool for this purposes are the following definitions of inf and sup convolutions. First, we will consider functions u, v upper and lower semicontinuous in $[0,T)\times\mathbb{R}^n$ respectively, and extend its definition to all $[0,T]\times\mathbb{R}^n$ taking the corresponding semicontonuous extension. Namely,

$$u(T,x) = \limsup_{t\to T^-} u(t,x), \quad v(T,x) = \liminf_{t\to T^-} v(t,x).$$

With this setting, for $\alpha > 0$ we define for $(t,x) \in [0,T] \times \Omega$

$$u^\alpha(t,x) := \sup_{\substack{s \in [0,T] \\ y \in \bar{\Omega}}} u(s,y) - \frac{1}{\alpha}(s-t)^2 - \frac{1}{\alpha}|y-x|^2.$$

$$u_\alpha(t,x) := \inf_{\substack{s \in [0,T] \\ y \in \bar{\Omega}}} v(s,y) + \frac{1}{\alpha}(s-t)^2 + \frac{1}{\alpha}|y-x|^2.$$

It is known that for all $\alpha > 0$, u^α, v_α are continuous functions in $[0,T] \times \bar{\Omega}$. We have as a first result

Lemma 5.3. *Let $u \in \mathcal{L}^1_*, v \in \mathcal{L}^{1*}$, u bounded above in $(0,T) \times \mathbb{R}^n$ and v bounded below in $(0,T) \times \mathbb{R}^n$ viscosity sub and supersolution to equation* (5.1)*, respectively, with g satisfying* (1.5)*. Then,*

(i) *u^α (resp. v_α) is a viscosity solution for the problem*

$$u_t(t,x) - \mathcal{I}[u(t,\cdot),x] - g(t,x,u(t,x)) \leq \ (resp. \ \geq) \quad d_{\alpha,1} \ (resp. \ -d_{\alpha,2}), \quad (t,x) \in (0,T) \times \Omega,$$

(ii) *$u^\alpha - v_\alpha$ is a viscosity solution for the problem*

(5.4)
$$u_t(t,x) - \mathcal{M}^+[u(t,\cdot),x] + Cu(t,x)^- \leq \tilde{d}_\alpha, \quad (t,x) \in (0,T) \times \Omega,$$

where $d_{\alpha,1}, d_{\alpha,2}, \tilde{d}_\alpha \to 0^+$ locally uniform in $(0,T) \times \Omega$ as $\alpha \to 0^+$.

Proof: *Proof of Assertion (i).* We will give the proof for the subsolution. The proof for the supersolution is similar. Let $(t_0,x_0) \in (0,T) \times \Omega$. Define

$$d(t_0,x_0) := \max\{t_0, T-t_0, dist(x_0,\partial\Omega)\} > 0.$$

Note that for each y such that $|y-x_0|^2 \geq 4\alpha(|\sup_{(0,T)\times\mathbb{R}^n} u| + 1)$, then for each $s \in (0,T)$

$$u(s,y) - \frac{1}{\alpha}(s-t_0)^2 - \frac{1}{\alpha}|y-x_0|^2 \leq u(s,y) - 4(|\sup_{(0,T)\times\mathbb{R}^n} u| + 1) \leq -2|\sup_{(0,T)\times\mathbb{R}^n} u|.$$

In the same way, now for s such that $(s-t_0)^2 \geq 4\alpha(|\sup\limits_{(0,T)\times\mathbb{R}^n} u| + 1)$, then for each $y \in \mathbb{R}^n$ we have

$$u(t_0+s, x_0+y) \leq -2|\sup_{(0,T)\times\mathbb{R}^n} u|.$$

So, if we consider α satisfying

$$4\alpha(|\sup_{(0,T)\times\mathbb{R}^n} u| + 1) < \frac{1}{4} d(t_0,x_0), \tag{5.5}$$

then, by definition of u^α, it is clear that the supremum is achieved at some point $(s_1,y_1) \in (0,T) \times \Omega$, namely

$$u^\alpha(t_0,x_0) = u(s_1,y_1) - \frac{1}{\alpha}(s_1-t_0)^2 - \frac{1}{\alpha}|x_0-y_1|^2$$

with

$$(s_1 - t_0)^2, |x_0 - y_1|^2 \leq 4\alpha(|\sup_{(0,T)\times\mathbb{R}^n} u| + 1).$$

Consider a function ϕ smooth such that $u^\alpha - \phi$ has a strict global maximum point at (t_0, x_0). For all (t, x) such that

$$(5.6)\ (t, x) \in (t_0 - \frac{1}{4}d(t_0, x_0), t_0 + \frac{1}{4}d(t_0, x_0)) \times B(x_0, d(t_0, x_0)/4),$$

if we denote by $s_0 = s_1 - t_0 \in (0, T)$ and $y_0 = y_1 - x_0 \in \Omega$ we have, by definition of the sup convolution

$$\begin{aligned} & u(t_0 + s_0, x_0 + y_0) - \frac{1}{\alpha}|y_0|^2 - \frac{1}{\alpha}s_0^2 - \phi(t_0, x_0) \\ & \geq u^\alpha(t, x) - \phi(t, x) \\ & \geq u(t + s_0, x + y_0) - \frac{1}{\alpha}|y_0|^2 - \frac{1}{\alpha}s_0^2 - \phi(t, x) \end{aligned}$$

By (5.5), it is clear that $x + y_0 \in \Omega$ and $t + s_0 \in (0, T)$ taking into account (t, x) as in (5.6). Denote

$$z_0 = x_0 + y_0;\ z = x + y_0 \quad \text{and} \quad \tau_0 = t_0 + s_0;\ \tau = t + s_0,$$

then, for any (τ, z) satisfying

$$(\tau, z) \in (t_0 - d(t_0, x_0)/2, t_0 + d(t_0, x_0)/2) \times B(x_0, d(t_0, x_0)/2)$$

the testing inequality becomes

$$u(\tau_0, z_0) - \phi(\tau_0 - s_0, z_0 - y_0) \geq u(\tau, z) - \phi(\tau - s_0, z - y_0)$$

Using that u is subsolution to (5.1), we conclude for all $\delta < d(t_0, x_0)/4$

$$
\begin{aligned}
0 \;\geq\; & \partial_t \phi(\tau_0 - s_0, z_0 - y_0) - g(\tau_0, z_0, u(\tau_0, z_0)) \\
& - \inf_i \sup_j L^{1,\delta}_{ij}[\phi(\tau_0 - s_0, \cdot - y_0), z_0] + L^{2,\delta}[u(\tau_0, \cdot), D_x\phi(\tau_0 - s_0, z_0 - y_0), z_0] \\
= \;& \partial_t \phi(t_0, x_0) - g(\tau_0, z_0, u(\tau_0, z_0)) \\
& - \inf_i \sup_j \int_{B_\delta} \Big(\phi(t_0, x_0 + z) - \phi(t_0, x_0) - D_x\phi(t_0, x_0)\cdot z\Big) K_{ij}(z)dz \\
& + \int_{B_\delta^c} \Big(u(\tau_0, z_0 + z) - u(\tau_0, z_0) - D_x\phi(t_0, x_0)\cdot z\Big) K_{ij}(zdz) \\
\geq \;& \partial_t \phi(t_0, x_0) - g(\tau_0, z_0, u(\tau_0, z_0)) \\
& - \inf_i \sup_j \int_{B_\delta} \Big(\phi(t_0, x_0 + z) - \phi(t_0, x_0) - D_x\phi(t_0, x_0)\cdot z\Big) K_{ij}(z)dz \\
& + \int_{B_\delta^c} \Big(u(t_0 + s_0, x_0 + y_0 + z) - u(t_0 + s_0, x_0 + y_0) - D_x\phi(t_0, x_0)\cdot z\Big) K_{ij}(z)dz \\
\geq \;& \partial_t \phi(t_0, x_0) - g(\tau_0, z_0, u(\tau_0, z_0)) \\
& - \inf_i \sup_j \int_{B_\delta} \Big(\phi(t_0, x_0 + z) - \phi(t_0, x_0) - D_x\phi(t_0, x_0)\cdot z\Big) K_{ij}(z)dz \\
& + \int_{B_\delta^c} \Big(u^\alpha(t_0, x_0 + z) - u^\alpha(t_0, x_0) - D_x\phi(t_0, x_0)\cdot z\Big) K_{ij}(z)dz \\
\geq \;& \partial_t \phi(t_0, x_0) - g(\tau_0, z_0, u(\tau_0, z_0)) \\
& - \inf_i \sup_j L^{1,\delta}_{ij}[\phi(t_0, \cdot), x_0] + L^{2,\delta}[u^\alpha(t_0, \cdot), D_x\phi(t_0, x_0), x_0]
\end{aligned}
$$

Then we conclude that

$$
\begin{aligned}
& \partial_t \phi(t_0, x_0) - g(t_0, x_0, u^\alpha(t_0, x_0)) \\
& - \inf_i \sup_j L^{1,\delta}_{ij}[\phi(t_0, \cdot), x_0] + L^{2,\delta}[u^\alpha(t_0, \cdot), D_x\phi(t_0, x_0), x_0] \\
\leq \;& g(\tau_0, z_0, u(\tau_0, z_0)) - g(t_0, x_0, u(\tau_0, z_0)) + g(t_0, x_0, u(\tau_0, z_0)) - g(t_0, x_0, u^\alpha(t_0, x_0)) \\
\leq \;& g(\tau_0, z_0, u(\tau_0, z_0)) - g(t_0, x_0, u(\tau_0, z_0)) \\
\leq \;& d_{\alpha,1},
\end{aligned}
$$

where $d_{\alpha,1}$ can be defined as

$$
d_{\alpha,1}(t,x) = \sup_{|y|,|s| \leq C(t,x,\alpha); |r| \leq C_u(t,x)} |g(t+s, x+y, r) - g(t,x,r)|
$$

with

$$
C(t,x,\alpha) := \frac{1}{4} \min\{t, T-t, dist(x, \partial\Omega), 4\alpha(|\sup_{(0,T)\times\mathbb{R}^n} u| + 1)\}.
$$

and

$$C_u(t,x) = \sup_{C(s,y,\alpha)<1/2C(t,x,\alpha)} |u(s,y)|$$

Clearly, $d_{\alpha,1} \to 0$ locally uniform in $(0,T)\times\Omega$ as $\alpha \to 0^+$ by the continuity of g.

The result for bounded below supersolutions is analogous, taking into account in the proof the definition of inf-convolution.

Proof of Assertion (ii). Define $w_\alpha := u^\alpha - v_\alpha$ and $(t_0, x_0) \in (0,T)\times\Omega$, ϕ smooth function such that $w_\alpha - \phi$ has a strict global maximum at (t_0, x_0). Doubling variables and penalising, for each $\epsilon > 0$ we consider the function

(5.7)

$$\Psi_{\epsilon,\eta}(s,t,x,y) := u^\alpha(t,x) - v_\alpha(s,y) - \frac{|x-y|^2}{2\epsilon} - \frac{(s-t)^2}{2\epsilon} - \phi(t,x);$$

with $x, y \in \Omega,\ s,t \in (0,T)$. Consider $R > 0$ such that $\Omega \subset\subset B_{R-1}$ and

$$C_R := \sup_{[0,T]\times\bar{B}_R} \{|u^\alpha| + |v_\alpha| + |\phi|\} < +\infty.$$

Denote

$$M_\epsilon := \sup_{[0,T]^2\times\bar{\Omega}^2} \Psi_\epsilon \geq w_\alpha(t_0,x_0) - \phi(t_0,x_0).$$

which is a supremum attained in some point $(\bar{s}_\epsilon, \bar{t}_\epsilon, \bar{x}_\epsilon, \bar{y}_\epsilon) \in [0,T]^2 \times \bar{\Omega}^2$.

If $|x-y|^2 > 4C_R\epsilon$ or $(s-t)^2 > 4C_R\epsilon$, then $\Psi_\epsilon(s,t,x,y) < -2C_R$, meanwhile $|\Psi_\epsilon(t_0,t_0,x_0,x_0)| \leq C_R$. This means that

$$|\bar{x}-\bar{y}|^2 \leq 4C_R\epsilon, \quad (\bar{s}-\bar{t})^2 \leq 4C_R\epsilon$$

and this implies that as $\epsilon \to 0$, each cluster point of the sequence has the form $(\hat{t},\hat{t},\hat{x},\hat{x})$. Hence

$$w_\alpha(t_0,x_0)-\phi(t_0,x_0) \leq \liminf_{\epsilon\to 0} M_\epsilon \leq \limsup_{\epsilon\to 0} M_\epsilon \leq w_\alpha(\hat{t},\hat{x})-\phi(\hat{t},\hat{x}) \leq w_\alpha(t_0,x_0)-\phi(t_0,x_0)$$

which implies

$$\lim_{\epsilon\to 0} M_\epsilon = w_\alpha(t_0,x_0) - \phi(t_0,x_0),$$

and then

$$\lim_{\epsilon\to 0} w_\alpha(\bar{t}_\epsilon, \bar{x}_\epsilon) = w_\alpha(t_0,x_0).$$

Since (t_0,x_0) is a strict global maximum for $w_\alpha - \phi$,

$$(\bar{s}_\epsilon, \bar{t}_\epsilon, \bar{x}_\epsilon, \bar{y}_\epsilon) \to (t_0,t_0,x_0,x_0) \quad as\ \epsilon \to 0.$$

We can take ϵ small to get $(\bar{s}_\epsilon, \bar{t}_\epsilon, \bar{x}_\epsilon, \bar{y}_\epsilon)$ uniformly away from the boundary of $(0,T)^2\times\Omega^2$. As before, it is possible to freeze the variables s, y and look the penalization as a testing for u^α over the variables t, x at the maximum point $(\bar t_\epsilon, \bar x_\epsilon)$. Using Lemma 5.3 with u^α, we get for $\delta < d(t_0, x_0)/2$

$$\begin{aligned} d_{\alpha,1}(\bar t_\epsilon, \bar x_\epsilon) \;\geq\;& \frac{\bar t_\epsilon - \bar s_\epsilon}{\epsilon} + \partial_t \phi(\bar t_\epsilon, \bar x_\epsilon) \\ & - \inf_i \sup_j L^{1,\delta}_{ij}[\phi(\bar t_\epsilon, \cdot) + \frac{|\cdot - \bar y_\epsilon|^2}{2\epsilon}, \bar x_\epsilon] + L^{2,\delta}_{ij}[u^\alpha(\bar t_\epsilon, \cdot), D_x\phi(\bar t_\epsilon, \bar x_\epsilon) + \frac{\bar x_\epsilon - \bar y_\epsilon}{\epsilon}, \bar x_\epsilon)] \\ & - g(\bar t_\epsilon, \bar x_\epsilon, u^\alpha(\bar t_\epsilon, \bar x_\epsilon)). \end{aligned}$$

where $d_{\alpha,1} \to 0^+$ locally uniform in $(0,T)\times\Omega$, by the assertion (i). Similarly, using Lemma 5.3 with v_α at the minimum point $(\bar s_\epsilon, \bar y_\epsilon)$ we obtain

$$\begin{aligned} -d_{\alpha,2}(\bar s_\epsilon, \bar x_\epsilon) \;\leq\;& \frac{\bar t_\epsilon - \bar s_\epsilon}{\epsilon} \\ & - \inf_i \sup_j L^{1,\delta}_{ij}[\frac{|\bar x_\epsilon - \cdot|^2}{2\epsilon}, \bar y_\epsilon] + L^{2,\delta}_{ij}[v_\alpha(\bar s_\epsilon, \cdot), \frac{\bar x_\epsilon - \bar y_\epsilon}{\epsilon}, \bar y_\epsilon)] \\ & - g(\bar s_\epsilon, \bar y_\epsilon, v_\alpha(\bar s_\epsilon, \bar y_\epsilon)). \end{aligned}$$

where, as before, $d_{\alpha,2} \to 0^+$ locally uniform in $(0,T)\times\Omega$ as $\alpha \to 0^+$.

But using the same arguments as in (3.11), (3.12) and (3.13) we can get for all i, j

(5.8)

$$L^{1,\delta}_{ij}[\phi(\bar t_\epsilon, \cdot) + \frac{|\cdot - \bar y_\epsilon|^2}{2\epsilon}, \bar x_\epsilon] = L^{1,\delta}_{ij}[\frac{|\bar x_\epsilon - \cdot|^2}{2\epsilon}, \bar y_\epsilon] + L^{1,\delta}_{ij}[\phi(\bar t_\epsilon, \cdot), \bar x_\epsilon];$$

and

(5.9)

$$\begin{aligned} & L^{2,\delta}_{ij}[u^\alpha(\bar t_\epsilon, \cdot), \frac{\bar x_\epsilon - \bar y_\epsilon}{\epsilon} + D_x\phi(\bar t_\epsilon, \bar x_\epsilon), \bar x_\epsilon)] \\ & = L^{2,\delta}_{ij}[v_\alpha(\bar s_\epsilon, \cdot), \frac{\bar x_\epsilon - \bar y_\epsilon}{\epsilon}, \bar y_\epsilon)] \\ & \quad + \int\limits_{B^c_\delta} \Big(u^\alpha(\bar t_\epsilon, \bar x_\epsilon + z) - v_\alpha(\bar s_\epsilon, \bar y_\epsilon + z) - (u^\alpha(\bar t_\epsilon, \bar x_\epsilon) - v_\alpha(\bar s_\epsilon, \bar y_\epsilon)) - 1_B D_x\phi(\bar t_\epsilon, \bar x_\epsilon \cdot z) \Big) K_{ij}(z) dz \end{aligned}$$

Using (5.8) and (5.9) on the respective testings of u^α and v_α, we obtain for each $a > 0$ and $\delta > 0$ small, but independent of ϵ

$$\begin{aligned} - \;& (d_{\alpha,1}(\bar t_\epsilon, \bar x_\epsilon) + d_{\alpha,2}(\bar s_\epsilon, \bar y_\epsilon)) \\ \leq \;& a - \partial_t\phi(\bar t_\epsilon, \bar x_\epsilon) + g(\bar t_\epsilon, \bar x_\epsilon, u^\alpha(\bar t_\epsilon, \bar x_\epsilon)) - g(\bar s_\epsilon, \bar y_\epsilon, v_\alpha(\bar s_\epsilon, \bar y_\epsilon)) \\ & + \int\limits_{B_\delta} \Big(\phi(\bar t_\epsilon, \bar x_\epsilon + z) - \phi(\bar t_\epsilon, \bar x_\epsilon) - D_x\phi(\bar t_\epsilon, \bar x_\epsilon) \cdot z \Big) K_a(z) dz \\ & + \int\limits_{B^c_\delta} \Big(u^\alpha(\bar t_\epsilon, \bar x_\epsilon + z) - v_\alpha(\bar s_\epsilon, \bar y_\epsilon + z) - (u^\alpha(\bar t_\epsilon, \bar x_\epsilon) - v_\alpha(\bar s_\epsilon, \bar y_\epsilon)) - 1_B D_x\phi(\bar t_\epsilon, \bar x_\epsilon) \cdot z \Big) K_a(z) dz \end{aligned}$$

where K_a is some kernel depending on a and ϵ. Then

$$\begin{aligned}
- & \quad (d_{\alpha,1}(\bar{t}_\epsilon,\bar{x}_\epsilon)+d_{\alpha,2}(\bar{s}_\epsilon,\bar{y}_\epsilon)) \\
\leq & \quad a-\partial_t\phi(\bar{t}_\epsilon,\bar{x}_\epsilon)+g(\bar{t}_\epsilon,\bar{x}_\epsilon,u^\alpha(\bar{t}_\epsilon,\bar{x}_\epsilon))-g(\bar{s}_\epsilon,\bar{y}_\epsilon,v_\alpha(\bar{s}_\epsilon,\bar{y}_\epsilon)) \\
& \quad +\sup_K\Big\{L_K^{1,\delta}[\phi(\bar{t}_\epsilon,\cdot),\bar{x}_\epsilon] \\
& \quad +\int_{B_\delta^c}\Big(u^\alpha(\bar{t}_\epsilon,\bar{x}_\epsilon+z)-v_\alpha(\bar{s}_\epsilon,\bar{y}_\epsilon+z)-(u^\alpha(\bar{t}_\epsilon,\bar{x}_\epsilon)-v_\alpha(\bar{s}_\epsilon,\bar{y}_\epsilon))-1_BD_x\phi(\bar{t}_\epsilon,\bar{x}_\epsilon)\cdot z\Big)K(z)dz\Big\} \\
\leq & \quad 2a-\partial_t\phi(\bar{t}_\epsilon,\bar{x}_\epsilon)+g(\bar{t}_\epsilon,\bar{x}_\epsilon,u^\alpha(\bar{t}_\epsilon,\bar{x}_\epsilon))-g(\bar{s}_\epsilon,\bar{y}_\epsilon,v_\alpha(\bar{s}_\epsilon,\bar{y}_\epsilon)) \\
& \quad +\sup_K\Big\{L_K^{1,\delta}[\phi(t_0,\cdot),x_0]+L_K^{2,\delta}[w_\alpha(t_0,\cdot),D_x\phi(t_0,x_0),x_0]\Big\} \\
& \quad +\theta(\epsilon,\delta) \qquad\qquad (5.10)
\end{aligned}$$

where $\theta(\epsilon,\delta)\to 0$ as $\epsilon\to 0$ when δ is fixed. This conclusion is postponed to the Appendix A. Using the continuity of u^α, v_α, ϕ and g it is possible to take limits as $\epsilon\to 0$ to conclude, for each $a>0$ and $\delta>0$ small

$$\begin{aligned}
-\tilde{d}_\alpha(t_0,x_0) \quad \leq & \quad 2a-\partial_t\phi(t_0,x_0)+g(t_0,x_0,u^\alpha(t_0,x_0))-g(t_0,x_0,v_\alpha(t_0,x_0)) \\
& \quad +\sup_K\Big\{L_K^{1,\delta}[\phi(t_0,\cdot),x_0]+L_K^{2,\delta}[w_\alpha(t_0,\cdot),D_x\phi(t_0,x_0),x_0]\Big\} \\
\leq & \quad 2a-\partial_t\phi(t_0,x_0)-Cw_\alpha(t_0,x_0)^- \\
& \quad +\sup_K\Big\{L_K^{1,\delta}[\phi(t_0,\cdot),x_0]+L_K^{2,\delta}[w_\alpha(t_0,\cdot),D_x\phi(t_0,x_0),x_0]\Big\}
\end{aligned}$$

where

$$\tilde{d}_\alpha(t,x)=\sup_{(s,y),(t,x)\in N(t_0,x_0)}(d_{\alpha,1}(t,x)+d_{\alpha,2}(s,y))$$

with

$$N(t_0,x_0)=\{(t,x)\in(0,T)\times\Omega\ :\ d(t,x)<d(t_0,x_0)/2\}.$$

Since this inequality holds for any $a>0$, we conclude that w_α satisfies in the viscosity sense the equation (5.4). □

The following result has to do with the stability of the equation satisfied by $u^\alpha-v_\alpha$.

Proposition 5.4. *Let $\mathcal{I}$, g as in the Theorem 1.1, g $C-$Lipschitz satisfying (1.5). Let $u\in\mathcal{L}^1_*, v\in\mathcal{L}^{1*}$ viscosity sub and supersolution to the equation (5.1), respectively. Assume also u is bounded above and v is bounded below in $(0,T)\times\mathbb{R}^n$. Then, the function $u-v$ is a viscosity solution to the problem*

$$w_t-\mathcal{M}^+w+Cw^-\leq 0,\quad (0,T)\times\Omega. \tag{5.11}$$

Proof: We will use the equation (5.4) and use some known convergence properties of the inf and sup concolutions. Let $(t_0,x_0)\in(0,T)\times\Omega$ and a smooth function ϕ such that $u-v-\phi$ has strict global maximum at (t_0,x_0). It is known that $u^\alpha\to u$ and $v_\alpha\to v$ as $\alpha\to 0$ in the $\Gamma-$sense. This implies that there exists a sequence $(t_\alpha,x_\alpha)\to(t_0,x_0)$ such that $u^\alpha-v_\alpha-\phi$ has a maximum point at (t_α,x_α) for all α small enough, maximum point in the interior of a neighborhood $N \subset (0,T)\times\Omega$ of (t_0,x_0). We can take this neighborhood independent of α. It is possible to prove that for each $(t,x)\in(0,T)\times\Omega$,

(5.12)
$$\limsup_{\substack{\alpha\to 0^+\\ y_\alpha\to x\\ s_\alpha\to t}} u^\alpha(s_\alpha,y_\alpha)-v_\alpha(s_\alpha,y_\alpha)-\phi(s_\alpha,y_\alpha)\le u(t,x)-v(t,x)-\phi(t,x).$$

This inequality will provide us, in our current setting, the following important fact

(5.13)
$$\lim_{\alpha\to 0} u^\alpha(t_\alpha,x_\alpha)-v_\alpha(t_\alpha,x_\alpha)-\phi(t_\alpha,x_\alpha)=u(t_0,x_0)-v(t_0,x_0)-\phi(t_0,x_0).$$

The assertions (5.12) and (5.13) will be proved in the Appendix B. Note that in particular, (5.13) implies

$$\lim_{\alpha\to 0} u^\alpha(t_\alpha,x_\alpha)-v_\alpha(t_\alpha,x_\alpha)=u(t_0,x_0)-v(t_0,x_0) \tag{5.14}$$

by the continuity of ϕ. Then

$$\begin{aligned}
&\partial_t\phi(t_0,x_0)-\mathcal{M}^+[(u-v)(t_0,\cdot),x_0]+C(u(t_0,x_0)-v(t_0,x_0))^+\\
\le\ &\tilde{d}_\alpha(t_\alpha,x_\alpha)+\partial_t\phi(t_0,x_0)-\partial_t\phi(t_\alpha,x_\alpha)\\
&+C(u(t_0,x_0)-v(t_0,x_0))^- - C(u^\alpha(t_\alpha,x_\alpha)-v_\alpha(t_\alpha,x_\alpha))^-\\
&+\mathcal{M}^+[(u^\alpha-v_\alpha)(t_\alpha,\cdot),x_\alpha]-\mathcal{M}^+[(u-v)(t_0,\cdot),x_0]
\end{aligned}$$

Using the smoothness of ϕ, the fact that $\tilde{d}_\alpha$ vanishes locally uniform in $(0,T)\times\Omega$ and (5.14) we conclude

$$\begin{aligned}
&\partial_t\phi(t_0,x_0)-\mathcal{M}^+[(u-v)(t_0,\cdot),x_0]+C(u(t_0,x_0)-v(t_0,x_0))^+\\
\le\ &o_\alpha(1)+\mathcal{M}^+[(u^\alpha-v_\alpha)(t_\alpha,\cdot),x_\alpha]-\mathcal{M}^+[(u-v)(t_0,\cdot),x_0]
\end{aligned} \tag{5.15}$$

So, it remains to estimate only the maximal term. But it is possible to prove the following estimate, using (5.14). Its proof is postponed to the Appendix B.

$$\mathcal{M}^+[(u^\alpha-v_\alpha)(t_\alpha,\cdot),x_\alpha]-\mathcal{M}^+[(u-v)(t_0,\cdot),x_0]\le o_\alpha(1) \tag{5.16}$$

Taking limsup as $\alpha\to 0^+$ in (5.15) we conclude the desired inequality

$$\partial_t\phi(t_0,x_0)-\mathcal{M}^+[(u-v)(t_0,\cdot),x_0]+C(u(t_0,x_0)-v(t_0,x_0))^+\le 0$$

which implies that $u-v$ is subsolution to (5.11).

□

In order to obtain the desired comparison principle, it is enough to show a maximum principle for the problem (5.11). We will state the result in the following way

Proposition 5.5. *Let $u \in \mathcal{L}^1_*$ bounded above viscosity solution to the problem* (5.11)*. Denote by $\Gamma = \{0\} \times \Omega \cup [0,T) \times \mathbb{R}^n \setminus \Omega$. Then*

$$\sup_{(0,T)\times\Omega} u \leq \sup_{\Gamma} u^+.$$

Proof: Let $\epsilon > 0$ and $M \in \mathbb{R}$. Consider the function $\varphi_M = \frac{\epsilon}{T-t} + M$. For $M \geq -\epsilon/T$, we have

(5.17)
$$\partial_t \varphi_M - \mathcal{M}^-[\varphi_M] + C\varphi_M^- = \frac{\epsilon}{(T-t)^2} + C\Big(\frac{\epsilon}{T-t} + M\Big)^- > 0.$$

Since u is bounded above, we define $M_0 \geq -\epsilon/T$ the infimum of the real numbers M satisfying

$$\varphi_M \geq u^+, \ \forall\ (t,x) \in [0,T) \times \mathbb{R}^n.$$

Assume

$$M_0 > \sup_{\Gamma} u^+.$$

This implies in particular that $M_0 > 0$. Under this assumption, we have the existence of a point $(t_0, x_0) \in (0,T) \times \Omega$ where $M_0 + \epsilon/(T - t_0) = u^+(t_0, x_0) = u(t_0, x_0)$, using the semicontinuity of u. We have that φ_{M_0} is a test function for the subsolution u of the equation (5.11) at (t_0, x_0). Then, we have for each $\epsilon > 0$

$$\partial_t \varphi_{M_0}(t_0, x_0) - \mathcal{M}^-[x_0, \varphi_{M_0}(t_0, \cdot)] + C(\varphi_{M_0})^- \leq 0,$$

a contradiction with (5.17). Then, necessarily we have

$$M_0 \leq \sup_{\Gamma} u^+,$$

but in this case, we conclude that for each $(t,x) \in [0,T) \times \Omega$

$$u(t,x) \leq \varphi_{M_0} \leq M_0 + \frac{\epsilon}{T-t} \leq \sup_{\Gamma} u^+ + \frac{\epsilon}{T-t}.$$

Taking $\epsilon \to 0$, we conclude the result.

□

Proof of Theorem 5.1: By Proposition 5.4 we have that $u - v$ satisfies equation (5.11) and since u is subsolution and v is supersolution to the problem (5.2), we have $u - v \leq 0$ in Γ. Applying Proposition 5.5, we conclude the result. □

Proof of Theorem 5.2: Since u satisfies (3.16), if we define

$$\tilde{u}_\epsilon(t,x) := u(t,x) - \epsilon e^{kt}(1 + |x|^2)^{\gamma/2}$$

we have that this function is bounded above for each $\epsilon > 0$. As in the proof of the Proposition 3.2 it is possible to prove that taking k large enough, $\tilde{u}_\epsilon$ is a viscosity subsolution to the equation (5.1). Obviously, $\tilde{u}_\epsilon \leq u$ and satisfies the conditions for f and u_0 asked in the statement of the Theorem 5.1. On the other hand,

$$\tilde{v}_\epsilon := v(t,x) + \epsilon e^{kt}(1 + |x|^2)^{\gamma/2}$$

is bounded below viscosity supersolution for (5.1) and the exterior and initial condition satisfy (5.3) also. We apply Theorem 5.1 to conclude

$$\tilde{u}_\epsilon \leq \tilde{v}_\epsilon, \quad in \ (0,T)\times\Omega$$

which concludes the result. □

6. Modulus of Continuity for Unbounded Solutions.

We are assuming that g, u_0 are uniformly continuous. This assumption is restrictive, in the sense that we cannot allow a superlinear growth for the data. Therefore, we have to think γ to be less than 1 (this implies assumptions over σ too, say $\sigma < 1/2$). However, we can provide the desired modulus of continuity for the family in the case of unbounded solutions in this restrictive case.

we start with the spatial modulus of continuity. Denote for $y \in \mathbb{R}^n$

$$C_{g,y} = \sup_{(t,x,a)\in(0,T)\times\mathbb{R}^n\times\mathbb{R}} |g(t,x+y,a) - g(t,x,a)| \tag{6.1}$$

$$C_{0,y} = \sup_{x\in\mathbb{R}^n} |u_0(x+y) - u_0(x)|. \tag{6.2}$$

Define $w_\epsilon(t,x) := u_\epsilon(t,x+y) + tC_{g,y} + C_{0,y}$. Clearly, $w_\epsilon(0,x) = u_0(x+y) + C_{0,y} \geq u_0(x)$. We can prove that w_ϵ is a supersolution for the eqaution satisfied by u_ϵ. Let $(t_0,x_0) \in (0,T)\times\mathbb{R}^n$ and ϕ smooth function such that $w_\epsilon - \phi$ has a maximum at the point (t_0,x_0). Then, there exists $r > 0$ such that for all $(t,x) \in B_r(t_0,x_0)$

$$u_\epsilon(t_0,x_0+y) + t_0C_{y,g} + C_{0,y} - \phi(t_0,x_0) \geq u_\epsilon(t,x+y) + tC_{y,g} + C_{0,y} - \phi(t,x)$$

Putting $z_0 = x_0 + y$ and $z = x + y$ in the last inequality, we get that for all $z \in B_r(t_0,z_0)$

$$u_\epsilon(t_0,z_0) + t_0C_{y,g} + C_{0,y} - \phi(t_0,z_0-y) \geq u_\epsilon(t,z) + tC_{y,g} + C_{0,y} - \phi(t,z-y)$$

Denoting $\tilde\phi(t,z)=\phi(t,z-y)$ we can use that u_ϵ is solution of the equation to get for all $\delta>0$

$$\begin{aligned} g(t_0,z_0,u_\epsilon(t_0,z_0)) &= \partial_t\tilde\phi(t_0,z_0)-C_{y,g} \\ &\quad -\inf_i\sup_j\Big[\int\limits_{B_\delta}(\tilde\phi(t_0,z_0+e)-\tilde\phi(t_0,z_0)-1_B D\tilde\phi(t_0,z_0)\cdot e)de \\ &\quad +\int\limits_{B_\delta^c}(u_\epsilon(t_0,z_0+e)-u_\epsilon(t_0,z_0)-1_B D\phi(t_0,z_0)\cdot)de\Big] \\ &= \partial_t\phi(t_0,x_0)-C_{y,g} \\ &\quad -\inf_i\sup_j\Big[\int\limits_{B_\delta}(\phi(t_0,x_0+e)-\phi(t_0,x_0)-1_B D\tilde\phi(t_0,x_0)\cdot e)de \\ &\quad +\int\limits_{B_\delta^c}(w_\epsilon(t_0,x_0+e)-w_\epsilon(t_0,x_0)-1_B D\phi(t_0,x_0)\cdot e)de \end{aligned}$$

But by definition of $C_{y,g}$ and the monotonicity of g

$$\begin{aligned} g(t_0,x_0+y,u_\epsilon(t_0,z_0))+C_{y,g} &\geq g(t_0,x_0+y,u_\epsilon(t_0,z_0))-g(t_0,x_0+y,u_\epsilon(x_0+y)+t_0C_{y,g}+C_{0,y}) \\ &\quad +g(t_0,x_0,w_\epsilon(t_0,x_0)) \\ &\geq +g(t_0,x_0,w_\epsilon(t_0,x_0)) \end{aligned}$$

This concludes that w_ϵ is a supersolution. In the same way, it is possible to find an analogous subsolution to u_ϵ. Then

$$|u_\epsilon(t,x+y)-u_\epsilon(t,x)|\leq TC_{y,g}+C_{0,y}=:m(|y|)$$

is the desired spatial modulus of continuiy for u_ϵ, which is independent of ϵ and t.

Since m is a modulus of continuity, we can assume it sublinear. Then, for each $t\in(0,T)$ and $\rho>0$ we have

(6.3)
$$u_\epsilon(t,x)\leq u_\epsilon(t,x_0)+m(\rho)+\frac{m(\rho)}{\rho}|x-x_0|,\ \forall x,x_0\in\mathbb{R}^n$$

Without lost of generality, it is possible to assume $m(\rho)/\rho>1$ (if it is not, we add 1 to $m(\rho)/\rho$). Fix $(t_0,x_0)\in(0,T)\times\mathbb{R}^n$ Since we know that

$$\lim\sup_{[0,T)}|u_\epsilon(t,x)||x|^{-\gamma}=0$$

we can take $R_0>1$ so large $\sup\limits_{[0,T)}|u_\epsilon(t,x)|\leq|x|^\gamma$ for all $|x|>R_0$. If we consider

$$R=\max\{2|x_0|,2R_0\},\tag{6.4}$$

then we have

$$\sup_{[0,T)} |u_\epsilon(t,x)| \leq 2^\gamma |x-x_0|^\gamma, \ \forall |x| \geq R. \tag{6.5}$$

From the sub and supersolution built in the existence of unbounded solution (Chapter 4), it is possible to get an upper and lower bound $\tilde{C} = C(g, u_0, \gamma, A, R)$ for the function u_ϵ in $(0,T) \times B_R$ depending on R, the data g, u_0, the growth parameter γ, and the uniform integrabilty of the kernels, but not on ϵ.

Denote

$$C_R := 2^\gamma + \tilde{C} + (2R)^{1-\gamma}$$

(we stress the dependence on R, but by the definition it is clear the dependence on the other parameters) and using the notation $< x >_a = (a^2 + |x|^2)^{1/2}$, we define

$$v_\rho(t,x) = u_\epsilon(t_0,x_0) + m(\rho) + C_R \frac{m(\rho)}{\rho} < x - x_0 >^\gamma_{\rho^{1/\gamma}} + (e^{\theta(t-t_0)} - 1) < x - x_0 >^\gamma_1$$

where $\theta > 0$ is a constant to be fixed later.

For a point $(t,x) \in [t_0, T) \times B_R^c$, we have

$$\begin{aligned} v_\rho(t,x) &\geq u_\epsilon(t_0,x_0) + C_R \frac{m(\rho)}{\rho} < x - x_0 >^\gamma_{\rho^{1/\gamma}} \\ &\geq -\tilde{C} + C_R |x-x_0|^\gamma \\ &= \tilde{C}(|x-x_0|^\gamma - 1) + 2^\gamma |x-x_0|^\gamma \\ &\geq 2^\gamma |x-x_0| \\ &\geq u_\epsilon(t,x) \end{aligned}$$

where we have used that $|x - x_0| > 1$ when $x \in B_R^c$ by the definition of R in (6.4), and (6.5).

On the other hand, for a point $x \in B_R$

$$\begin{aligned} v_\rho(t_0,x) &\geq u_\epsilon(t_0,x_0) + m(\rho) + C_R \frac{m(\rho)}{\rho} < x - x_0 >^\gamma_{\rho^{1/\gamma}} \\ &\geq u_\epsilon(t_0,x_0) + m(\rho) + m(\rho)/\rho (2R)^{1-\gamma} |x-x_0|^\gamma \\ &\geq u_\epsilon(t_0,x_0) + m(\rho) + m(\rho)/\rho |x-x_0| \\ &\geq u_\epsilon(t_0,x) \end{aligned}$$

where in the last inequality we have used (6.3). Finally, for each $\rho > 0$ and for any point $(t,x) \in (0,T) \times B_R$

$$\begin{aligned} \partial_t v_\rho - \mathcal{I}[v_\rho] - g(t,x,v_\rho) &= \theta e^{\theta(t-t_0)} < x - x_0 >^\gamma_1 - C_R m(\rho)/\rho \mathcal{I}[< \cdot - x_0 >^\gamma_{\rho^{1/\gamma}}] \\ &\quad - (e^{\theta(t-t_0)} - 1)\mathcal{I}[< x - x_0 >^\gamma_1] - g(t,x,v_\rho) \end{aligned}$$

But for each K, we have for $a > 0$ and $x \in \mathbb{R}^n$:

$$|L_K[x, < \cdot >^\gamma_a]| \leq C(\gamma, n, \sigma)\Big((a^2 + |x|^2)^{\gamma/2} + (2\sigma - \gamma)^{-1}\Big) + C(n,\gamma) a^{\gamma - 2}$$

Then, using the previous bound and the monotonicity of g

$$\begin{aligned}\partial_t v_\rho - \mathcal{I}[v_\rho] - g(t,x,v_\rho) \quad \geq \quad & \theta e^{\theta(t-t_0)} < x - x_0 >_1^\gamma \\ & -C_R m(\rho)/\rho C(n,\gamma,\sigma)\Big(< x - x_0 >_{\rho^{1/\gamma}}^\gamma + (2\sigma-\gamma)^{-1} + \rho^{2(\gamma-2)/\gamma}\Big) \\ & -e^{\theta(t-t_0)} C(n,\gamma,\sigma)\Big(< x - x_0 >_1^\gamma + (2\sigma-\gamma)^{-1} + 1\Big) \\ & -g(t,x,u_\epsilon(t_0,x_0)).\end{aligned} \tag{6.6}$$

We can take

$$\theta = \theta_R \geq C(n,\gamma,\sigma)((2\sigma-\gamma)^{-1}+2)\Big(1+C_R m(\rho)/\rho\Big) + \sup_{(t,x)\in(0,T)\times B_R} g(t,x,-\tilde{C}),$$

a constant which is independent of ϵ, x, t so that the expression (6.6) becomes possitive. Hence, we have proven that v_ρ is a supersolution to the equation satisfied by u_ϵ in $(t_0,T)\times B_R$. Then, for all $\rho > 0$, applying Theorem 5.2:

$$u_\epsilon(t,x_0) \leq u_\epsilon(t_0,x_0) + m(\rho) + C_R \frac{m(\rho)}{\rho} < x - x_0 >_{\rho^{1/\gamma}}^\gamma + (e^{\theta_R(t-t_0)} - 1) < x - x_0 >_1^\gamma,$$

for all $(t,x) \in [t_0,T)\times B_{\bar{R}}$. Taking $x = x_0$ in the last expression, we have

$$u_\epsilon(t,x_0) \leq u_\epsilon(t_0,x_0) + m(\rho)(C_R+1) + (e^{\theta_R(t-t_0)} - 1) \tag{6.7}$$

Observing carefully the derivation of this quantity, we note that for all $\rho > 0$ and $R > 0$, this inequality holds for any $x_0 \in B_R$.

In the same way it is possible to contruct an analogous subsolution to u_ϵ in $[t_0,T)\times B_R$. Hence, for all $\rho > 0$ and $R > 1$,

$$|u_\epsilon(t,x) - u_\epsilon(t_0,x)| \leq m(\rho)(C_R+1) + (e^{\theta_R(t-t_0)} - 1)$$

for all $t \in [t_0,T)$, $x \in B_R$ and $\epsilon > 0$. Then, taking infimum in $\rho > 0$ we assure that

$$|u_\epsilon(t,x) + u_\epsilon(s,x)| \leq \inf_{\rho>0} \Big(m(\rho)(C_R+1) + (e^{\theta_R|t-s|} - 1)\Big) =: m_R(|t-s|)$$

is the desired modulus of continuity in t, local for $|x| < R$. □

7. Proof of Theorem 1.1.

Using that the sequence of solutions u_ϵ of the equation (1.1) is locally uniformly continuous, we have that $u_\epsilon \to u$ locally uniform in $(0,T)\times\mathbb{R}^n$. Using Lemma 2.3, we have all the conditions to apply Lemma 2.2 to conclude the result.

□

Appendix A. Proof of the Estimate (5.10) and (5.16)

We have, for each $a, \delta > 0$ fixed

$$
\begin{aligned}
& \sup_K \Big\{ L_K^{1,\delta}[\phi(\bar{t}_\epsilon, \cdot), \bar{x}_\epsilon] \\
& + \int_{B_\delta^c} \Big(u^\alpha(\bar{t}_\epsilon, \bar{x}_\epsilon + z) - v_\alpha(\bar{s}_\epsilon, \bar{y}_\epsilon + z) - (u^\alpha(\bar{t}_\epsilon, \bar{x}_\epsilon) - v_\alpha(\bar{s}_\epsilon, \bar{y}_\epsilon)) - 1_B D_x\phi(\bar{t}_\epsilon, \bar{x}_\epsilon) \cdot z \Big) K(z) dz \Big\} \\
\leq \; & a + \sup_K \Big\{ L_K^{1,\delta}[\phi(t_0, \cdot), x_0] + L_K^{2,\delta}[w_\alpha(t_0, \cdot), D_x\phi(t_0, x_0), x_0] \Big\} + I_1 + I_2 + I_3 + I_4
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &:= \int_{B_\delta} \Big\{ \phi(\bar{t}_\epsilon, \bar{x}_\epsilon + z) - \phi(t_0, x_0 + z) - \phi(\bar{t}_\epsilon, \bar{x}_\epsilon) + \phi(t_0, x_0) \\
&\qquad -(D_x\phi(\bar{t}_\epsilon, \bar{x}_\epsilon) - D_x\phi(t_0, x_0)) \cdot z \Big\} K_a(z) dz \\
I_2 &:= \int_{B_\delta^c} \Big(u^\alpha(\bar{t}_\epsilon, \bar{x}_\epsilon + z) - v_\alpha(\bar{s}_\epsilon, \bar{y}_\epsilon + z) - u^\alpha(t_0, x_0 + z) + v_\alpha(t_0, x_0 + z) \Big) K_a(z) dz \\
I_3 &:= -\Big(u^\alpha(\bar{t}_\epsilon, \bar{x}_\epsilon) - v_\alpha(\bar{s}_\epsilon, \bar{y}_\epsilon) - u^\alpha(t_0, x_0) + v_\alpha(t_0, x_0) \Big) \int_{B_\delta^c} K_a(z) dz \\
I_4 &:= \int_{B \setminus B_\delta} \Big(D_x\phi(t_0, x_0) - D_x\phi(\bar{t}_\epsilon, \bar{x}_\epsilon) \Big) \cdot z K_a(z) dz
\end{aligned}
$$

Using the continuity of u^α, v_α and the smoothness of ϕ together with the uniform integrability of the kernels away 0 described in (1.2), we obtain that $I_3, I_4 \to 0$ as $\epsilon \to 0$. By (1.3), we can apply the Dominated Convergence Theorem to conclude that $I_1 \to 0$ as $\epsilon \to 0$ also. It remains to control the more difficult term I_2. In this case, we will use the fact that for each $(t, x) \in (0, T) \times \mathbb{R}^n$, $u^\alpha(t, x) \geq u(t, x)$ and $v_\alpha(t, x) \leq v(t, x)$. In addition, since u is bounded above and v is bounded below, we can obtain the following inequalities for each $(t, x) \in (0, T) \times \mathbb{R}^n$

$$
u^\alpha(t, x) \leq \sup_{(0,T) \times \mathbb{R}^n} u < +\infty, \quad v_\alpha(t, x) \geq \inf_{(0,T) \times \mathbb{R}^n} v > -\infty.
$$

Since $u \in \mathcal{L}_*^1$ and $v \in \mathcal{L}^{1*}$, then it is easy to see that the integrand of I_2 is bounded by a function which is integrable with respect to w_σ. Using (1.2) and noting that the integrand of I_2 vanishes by the continuity of u^α and v_α, we apply Dominated Convergence Theorem as $\epsilon \to 0$ to conclude the definition of $\theta(\epsilon, \delta)$ in (5.10).

Now we will prove the inequality (5.16). For this, we take $a > 0$ arbitrary to write, for each $\delta > 0$

$$
\begin{aligned}
& \mathcal{M}^+[(u^\alpha - v_\alpha)(t_\alpha, \cdot), x_\alpha] - \mathcal{M}^+[(u-v)(t_0,\cdot), x_0] \\
\leq \; & a + L^{1,\delta}_{K_a}[\phi(t_\alpha,\cdot), x_\alpha] + L^{2,\delta}_{K_a}[(u^\alpha - v_\alpha)(t_\alpha,\cdot), D_x\phi(t_\alpha, x_\alpha), x_\alpha] \\
& - L^{1,\delta}_{K_a}[\phi(t_0,\cdot), x_0] - L^{2,\delta}_{K_a}[(u-v)(t_0,\cdot), D_x\phi(t_0,x_0), x_0] \\
= \; & a + I_1 + I_2 + I_3 + I_4
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 \; &:= \int_{B_\delta} \Big\{ \phi(t_\alpha, x_\alpha + z) - \phi(t_0, x_0 + z) - \phi(t_\alpha, x_\alpha) + \phi(t_0, x_0) \\
& \qquad - (D_x\phi(t_\alpha, x_\alpha) - D_x\phi(t_0,x_0)) \cdot z \Big\} K_a(z) dz \\
I_2 \; &:= \int_{B_\delta^c} \Big(u^\alpha(t_\alpha, x_\alpha + z) - v_\alpha(t_\alpha, x_\alpha + z) - v_\alpha(t_\alpha, x_\alpha + z) - (u(t_0,x_0) - v(t_0,x_0)) \Big) K_a(z) dz \\
I_3 \; &:= -\Big(u^\alpha(t_\alpha, x_\alpha) - v_\alpha(t_\alpha, x_\alpha) - (u(t_0,x_0) - v_(t_0,x_0)) \Big) \int_{B_\delta^c} K_a(z) dz \\
I_4 \; &:= -\int_{B \setminus B_\delta} \Big(D_x\phi(t_\alpha, x_\alpha) - D_x\phi(t_0,x_0) \Big) \cdot z K_a(z) dz
\end{aligned}
$$

The terms I_1, I_3 and I_4 clearly goes to zero as $\alpha \to 0^+$ because of the smoothness of ϕ and the equality (5.14). For the term I_2 we apply Fatou Lemma with the inequality (5.12) to conclude the result. □

Appendix B. Proof of (5.12) and (5.13)

We start with (5.12). Let $\alpha_n \to 0^+$ a sequence such that $y_n := y_{\alpha_n}$, $s_n := s_{\alpha_n}$ satisfy

$$
\lim_{n\to\infty} u^{\alpha_n}(s_n, y_n) - v_{\alpha_n}(s_n, y_n) - \phi(s_n, y_n) = \limsup_{\substack{\alpha \to 0^+ \\ y_\alpha \to x \\ s_\alpha \to t}} u^\alpha(s_\alpha, y_\alpha) - v_\alpha(s_\alpha, y_\alpha) - \phi(s_\alpha, y_\alpha)
$$

For n sufficienlty large the sequence (s_n, y_n) is uniformly away the boundary of $(0,T)\times\Omega$. Moreover, it is possible to obtain $z_n^1, z_n^2 \in \Omega$, $\tau_n^1, \tau_n^2 \in (0,T)$ such that

$$
\begin{aligned}
u^{\alpha_n}(s_n, y_n) &= u(\tau_n^1, z_n^1) - \frac{1}{\alpha_n}|y_n - z_n^1|^2 - \frac{1}{\alpha_n}|s_n - \tau_n^1|^2. \\
v_{\alpha_n}(s_n, y_n) &= v(\tau_n^2, z_n^2) + \frac{1}{\alpha_n}|y_n - z_n^2|^2 + \frac{1}{\alpha_n}|s_n - \tau_n^2|^2.
\end{aligned}
$$

Taking n larger if it is necessary, we can assume the sequences $(z_n^1), (z_n^2)$ are uniformly away $\partial\Omega$ and $(\tau_n^1), (\tau_n^2)$ are uniformly away $\{0, T\}$. Extracting a subsequence of each one if it is necessary, we can assume that there exist $(\hat{\tau}^1, \hat{\tau}^2, \hat{z}^1, \hat{z}^2) \in (0,T)^2 \times \Omega^2$ such that

$$\tau_n^1 \to \hat{\tau}^1, \quad \tau_n^2 \to \hat{\tau}^2, \quad z_n^1 \to \hat{z}^1, \quad z_n^2 \to \hat{z}^2$$

as $n \to \infty$. But by the local boundeness of u, v and ϕ, we have for all n

$$\begin{aligned} -\infty &< u^{\alpha_n}(s_n, y_n) - v_{\alpha_n}(s_n, y_n) - \phi(s_n, y_n) \\ &= u(\tau_n^1, z_n^1) - \frac{1}{\alpha_n}|y_n - z_n^1|^2 - \frac{1}{\alpha_n}|s_n - \tau_n^1|^2 \\ &\quad -v(\tau_n^2, z_n^2) - \frac{1}{\alpha_n}|y_n - z_n^2|^2 - \frac{1}{\alpha_n}|s_n - \tau_n^2|^2 \\ &\quad -\phi(s_n, y_n) \end{aligned}$$

which implies that in fact, $\hat{\tau}^1, \hat{\tau}^2 = t$ and $\hat{z}^1, \hat{z}^2 = x$.

Then

$$\begin{aligned} &\lim_{n\to\infty} u^{\alpha_n}(s_n, y_n) - v_{\alpha_n}(s_n, y_n) - \phi(s_n, y_n) \\ = &\limsup_{n\to\infty} \Big\{ u(\tau_n^1, z_n^1) - \frac{1}{\alpha_n}|y_n - z_n^1|^2 - \frac{1}{\alpha_n}|s_n - \tau_n^1|^2 \\ &-v(\tau_n^2, z_n^2) - \frac{1}{\alpha_n}|y_n - z_n^2|^2 - \frac{1}{\alpha_n}|s_n - \tau_n^2|^2 \\ &-\phi(s_n, y_n) \Big\} \\ \le &\limsup_{n\to\infty} u(\tau_n^1, z_n^1) - v(\tau_n^2, z_n^2) - \phi(s_n, y_n) \\ \le &\; u(t,x) - v(t,x) - \phi(t,x) \end{aligned}$$

by the semicontinuity of u and v and the continuity of ϕ. This concludes (5.12).

Now we will prove (5.13). Assume (t_0, x_0) is the maximum point of $u - v - \phi$. Since $u \le u^\alpha$ and $v \ge v_\alpha$ in $(0,T) \times \mathbb{R}^n$ and if we denote (t_α, x_α) the maximum point of $u^\alpha - v_\alpha - \phi$ in $(0,T) \times \Omega$, then clearly, for each $\alpha > 0$

$$\begin{aligned} u(t_0,x_0) - v(t_0,x_0) - \phi(t_0,x_0) &\le u^\alpha(t_0,x_0) - v_\alpha(t_0,x_0) - \phi(t_0,x_0) \\ &\le u^\alpha(t_\alpha,x_\alpha) - v_\alpha(t_\alpha,x_\alpha) - \phi(t_\alpha,x_\alpha) \end{aligned}$$

Hence

$$\begin{aligned} u(t_0,x_0) - v(t_0,x_0) - \phi(t_0,x_0) &\le \liminf_{\alpha\to 0^+} u^\alpha(t_\alpha,x_\alpha) - v_\alpha(t_\alpha,x_\alpha) - \phi(t_\alpha,x_\alpha) \\ &\le \limsup_{\alpha\to 0^+} u^\alpha(t_\alpha,x_\alpha) - v_\alpha(t_\alpha,x_\alpha) - \phi(t_\alpha,x_\alpha) \\ &\le u(t_0,x_0) - v(t_0,x_0) - \phi(t_0,x_0) \end{aligned}$$

where the last inequality is justified by (5.12). This concludes the proof.

□

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Patricio Felmer - Departamento de Ingeniería Matemática and CMM (UMI 2807 CNRS), Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile.

Erwin Topp - Departamento de Ingeniería Matemática, Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile.