

REMARKS ON DECAY ESTIMATES FOR ZERO ORDER HEAT EQUATIONS.

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ABSTRACT. In this paper we obtain bounds for the decay rate for solutions to the nonlocal problem $u_t(t, x) = \int_{\mathbb{R}^n} J(x, y)[u(t, y) - u(t, x)]dy$. Here we deal with bounded kernels J but with polynomial tails, that is, we assume a lower bound of the form $J(x, y) \geq c_1|x - y|^{n+2\sigma}$, for $|x - y| > c_2$. Our estimates takes the form $\|u(t)\|_{L^q(\mathbb{R}^n)} \leq Ct^{-\frac{n}{2\sigma}(1-\frac{1}{q})}$

1. INTRODUCTION.

Let $J(x, y)$ be a bounded, nonnegative, continuous, symmetric function defined for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Here we address nonlocal Cauchy problems of the form

$$(1.1) \quad u_t(t, x) = \int_{\mathbb{R}^n} J(x, y)[u(t, y) - u(t, x)]dy \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n$$

with the initial condition $u(0, x) = u_0(x)$ satisfying $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. On J we also assume the basic condition

$$(J1) \quad x \mapsto J(x, y) \in L^1(\mathbb{R}^n) \quad \text{for all } y \in \mathbb{R}^n$$

By the symmetry of J we have the same properties interchanging x and y .

Our main interest here is the asymptotic behavior of the solution of (1.1) as $t \rightarrow +\infty$. It is well known that the diffusive nature of the equation implies the solution goes to zero when $t \rightarrow +\infty$. For smooth kernels J with compact support, it is proven in [4] that the solution u of the equation (1.1) has the decay estimate

$$\|u(\cdot, t)\|_{L^q(\mathbb{R}^n)} \leq Ct^{-\frac{n}{2}(1-\frac{1}{q})}$$

for any $q \in [1, \infty)$. Note that this decay rate is the same as the one that holds for solutions of the classical Heat equation. In the case of an equation in convolution form, that is when $J(x, y) = K(x - y)$ with K a nonnegative radial function, not necessarily compactly supported, it is proven in [2] that the solutions of equations with the form (1.1) have the decay estimate

$$(1.2) \quad \|u(\cdot, t)\|_{L^q(\mathbb{R}^n)} \leq Ct^{-\frac{n}{2\sigma}(1-\frac{1}{q})},$$

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provided the function K has a Fourier transform satisfying the expansion $\widehat{K}(\xi) = 1 + A|\xi|^{2\sigma} + o(|\xi|^{2\sigma})$, where $A > 0$ is a constant. It is remarkable that in this case the decay estimate is analogous to the one for the σ -order fractional heat equation, $v_t = -(-\Delta)^\sigma v$, with $\sigma \in (0, 1)$. We also note that the convolution form of the equation allows the use of Fourier analysis to obtain this result.

In this work we consider kernels J not in convolution form and not compactly supported in order to investigate the diffusive role of the asymptotic tails of the kernels when doing decay estimates. We illustrate this feature asking for an asymptotic profile for J with the following lower bound

$$(J2) \quad J(x, y) \geq c_1|x - y|^{n+2\sigma}, \quad \text{for } |x - y| > c_2$$

for certain constants $c_1, c_2 > 0$ and $\sigma \in (0, 1)$.

Besides, we prevent the use of Fourier analysis whenever we do not ask our operators to be in convolution form. Despite of this difficulty, energy methods can be carried out in order to obtain the main result of this paper, which reads as follows:

Theorem 1.1. *Let $n \geq 2$ and J be a bounded, nonnegative, continuous, symmetric function defined in \mathbb{R}^n satisfying (J1) and (J2). Then, for each $q \in (1, +\infty)$, the solution of (1.1) associated to an initial condition $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ goes to zero in $L^q(\mathbb{R}^n)$ as $t \rightarrow +\infty$. Moreover, the solution has an asymptotic decay that can be bounded by*

$$(1.3) \quad \|u(\cdot, t)\|_{L^q(\mathbb{R}^n)} \leq C\|u_0\|_{L^1(\mathbb{R}^n)} t^{-\frac{n}{2\sigma}(1-\frac{1}{q})},$$

where the constant C depends on q, σ and n .

Note that the kernel J defining the nonlocal operator in (1.1) shares, at one hand, the “fat” tails of the fractional Laplacian of order 2σ , and on the other, the integrability property of zero order operators. However, the decay rate of the solution of problem (1.1) is as in the case of the fractional Laplacian (see (1.2)) and not as in the classical zero order case defined by compactly supported kernels. This shows the diffusive effect of the fractional Laplacian is more related with the tails of the kernel, which allows to transport mass to infinity through long jumps, than the highly intense, but shorter jumps related with the singularity at the origin.

2. BASIC FACTS AND PRELIMINARIES.

Concerning the existence and uniqueness of problem (1.1), the symmetry, boundedness and integrability assumptions over J , allows us to perform a fixed point argument to obtain the following result whose proof is omitted.

Theorem 2.1 ([1], [4]). *Let $J : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a bounded and symmetric function satisfying assumption (J1) and (J2), and let $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then, there exists a unique solution $u \in C([0, +\infty), L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$ of equation (1.1). This solution satisfies $\|u(\cdot, t)\|_{L^1(\mathbb{R}^n)} \leq \|u_0\|_{L^1(\mathbb{R}^n)}$ and $\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq \|u_0\|_{L^\infty(\mathbb{R}^n)}$ for all $t \geq 0$.*

The symmetry assumption over J allows us to use an energy approach in order to get Theorem 1.1. Roughly speaking, this assumption allows us to “integrate by parts” equation (1.1). For $q > 1$ we multiply the equation by $q|u|^{q-2}u$ and integrate, obtaining the inequality

$$(2.1) \quad \begin{aligned} & \partial_t \int_{\mathbb{R}^n} |u(x)|^q dx \\ &= -\frac{q}{2} \int_{\mathbb{R}^{2n}} J(x, y)(u(y) - u(x))(|u(y)|^{q-2}u(y) - |u(x)|^{q-2}u(x)) dy dx, \end{aligned}$$

where we have avoided the dependence on t of the function u for simplicity.

Now we recall the following inequality: let $q \geq 2$ and $a, b \neq 0$. Then, there exists a constant C depending only on q , such that

$$(2.2) \quad (a - b)(|a|^{q-2}a - |b|^{q-2}b) \geq C|a - b|^q.$$

Hence, by the application this inequality into (2.1), we conclude

$$(2.3) \quad \partial_t \|u\|_{L^q(\mathbb{R}^n)}^q \leq -C_q \int_{\mathbb{R}^{2n}} J(x, y)|u(y) - u(x)|^q dy dx =: -C_q E_{J,q}(u).$$

Our strategy is to get an estimate for the energy $E_{J,q}(u)$ in the right-hand side of the above expression in terms of a higher L^p -norm of u . This can be accomplished due to the asymptotic behavior (J2) of the kernel. Being similar to the tails of the kernel of the fractional Laplacian on order 2σ , we compare $E_{J,q}$ in (2.3) with an ad-hoc fractional seminorm, for which Sobolev-type inequalities are available.

We recall that for $\sigma \in (0, 1)$ and $q \in [1, \infty)$, $W^{\sigma,q}(\mathbb{R}^n)$ is the fractional Sobolev space of all $L^q(\mathbb{R}^n)$ functions with bounded fractional seminorm $[v]_{\sigma,q}$, given by

$$(2.4) \quad [v]_{\sigma,q}^q = \iint_{\mathbb{R}^{2n}} \frac{|v(x+z) - v(x)|^q}{|z|^{n+q\sigma}} dx dz.$$

Under these definitions, the key step in our approach is to relate the energy in (2.3) with the corresponding fractional seminorm defined in (2.4). Once this is accomplished, we apply the following fractional Sobolev-type inequality, which asserts the existence of a constant $C > 0$ such that for each $v \in W^{\sigma,q}(\mathbb{R}^n)$ with $\sigma q < n$, it holds

$$(2.5) \quad \|v\|_{L^{q^*}(\mathbb{R}^n)}^q \leq C[v]_{\sigma,q}^q,$$

where $q^* = q^*(\sigma) = nq/(n - \sigma q)$ (see [3]). By the use of the above inequality we get the desired increment of the L^p -norm of the solutions, and after this the proof follows standard arguments.

Once we have obtained the result for $q \geq 2$, the corresponding decay estimate for the remaining case $q \in (1, 2)$ can be obtained by interpolation. Hence, at the very end our arguments rely on estimates for the nonlinear operator in the right hand side of (2.3).

Notation. For $x \in \mathbb{R}^n$ and $r > 0$, we denote $B_r(x)$ the ball with center x and radius r . We write B_r when $x = 0$.

3. FRACTIONAL SEMINORM ESTIMATES.

First, we consider a positive smooth function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ with the following properties

$$(3.1) \quad \text{supp}(\psi) \subset B_1, \quad \text{and} \quad \int_{\mathbb{R}^n} \psi(x) dx = 1.$$

With the aid of this function, we split a function u into two parts. We will denote the “smooth” part of u as v and the remaining as w . We let

$$(3.2) \quad v(t, x) := \int_{\mathbb{R}^n} \psi(x - z)u(t, z) dz; \quad u(t, x) := v(t, x) + w(t, x).$$

As a first property of this decomposition we have that each L^q norm of the functions v and w is controlled by the corresponding norm of u .

Lemma 3.1. *Let v and w be given by definition (3.2). For each $q \in (0, +\infty)$ we have*

$$\|v\|_{L^q(\mathbb{R}^n)} \leq C\|u\|_{L^q(\mathbb{R}^n)}, \quad \text{and} \quad \|w\|_{L^q(\mathbb{R}^n)} \leq C\|u\|_{L^q(\mathbb{R}^n)}.$$

Proof. We start with v . Using its definition, we have

$$\begin{aligned} \int_{\mathbb{R}^n} |v(x)|^q dx &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \psi(x - y)u(y) dy \right|^q dx \\ &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \psi(x - y)^{1/q'} \psi(x - y)^{1/q} u(y) dy \right|^q dx \\ &\leq \int_{\mathbb{R}^n} \left[\left(\int_{\mathbb{R}^n} \psi(x - y) dy \right)^{1/q'} \left(\int_{\mathbb{R}^n} \psi(x - y) |u(y)|^q dy \right)^{1/q} \right]^q dx \\ &\leq C(q, \psi) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi(x - y) |u(y)|^q dy dx \\ &= C(q, \psi) \int_{\mathbb{R}^n} |u(y)|^q \int_{\mathbb{R}^n} \psi(x - y) dx dy \\ &\leq C(q, \psi) \int_{\mathbb{R}^n} |u(y)|^q dy. \end{aligned}$$

The inequality corresponding to w easily follows from the triangular inequality in L^q . \square

Now we state a key results to get the desired estimate on the decay rate.

Proposition 3.2. *Let $J : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ a bounded and symmetric function satisfying hypotheses (J1) and (J2). Let ψ satisfying (3.1) and $q \in (2\sigma, +\infty)$. Then, there exists a constant $C > 0$ such that for all $u \in L^q(\mathbb{R}^n)$ and v, w defined in (3.2), we have*

$$(3.3) \quad [v]_{2\sigma q^{-1}, q}^q + \|w\|_{L^q(\mathbb{R}^n)}^q \leq C \iint_{\mathbb{R}^{2n}} J(x, y) |u(x) - u(y)|^q dx dy.$$

The constant C depends on ψ, σ, q and n .

Proof. For the estimate concerning w , we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |w(x)|^q dx &= \int_{\mathbb{R}^n} |u(x) - v(x)|^q dx \\
&= \int_{\mathbb{R}^n} \left| u(x) - \int_{\mathbb{R}^n} \psi(x-z)u(z)dz \right|^q dx \\
&= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \psi(x-z)(u(x) - u(z))dz \right|^q dx \\
&= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \psi(x-z)^{1/q'} \psi(x-z)^{1/q} (u(x) - u(z))dz \right|^q dx.
\end{aligned}$$

Applying Holder's inequality, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} |w(x)|^q dx \\
&\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \psi(x-z)dz \right)^{q/q'} \left(\int_{\mathbb{R}^n} \psi(x-z)|u(x) - u(z)|^q dz \right) dx \\
&\leq C(q, q', \psi) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi(x-z)|u(x) - u(z)|^q dz dx.
\end{aligned}$$

Since ψ is supported in B_1 , clearly we have that for all $|x-z| \geq 1$, $\psi(x-z) \leq J(x, z)$. Meanwhile, when $|x-z| < 1$ there exists a constant C depending only on $|\psi|_\infty$ such that $\psi(x-z) \leq CJ(x, z)$. Then

$$\|w\|_{L^q(\mathbb{R}^n)}^q \leq C \iint_{\mathbb{R}^{2n}} J(x, y)|u(x) - u(y)|^q dx dy.$$

Now we deal with the term concerning v . We split the fractional seminorm as

$$\begin{aligned}
[v]_{2\sigma q^{-1}, q}^q &= \iint_{|x-y|>1} \frac{|v(x) - v(y)|^q}{|x-y|^{n+2\sigma}} dx dy + \iint_{|x-y|\leq 1} \frac{|v(x) - v(y)|^q}{|x-y|^{n+2\sigma}} dx dy \\
&=: I_{ext} + I_{int}.
\end{aligned}$$

We look at these integrals separately. For I_{ext} , using the definition of v we have

$$I_{ext} = \iint_{|x-y|>1} \left| \int_{\mathbb{R}^n} (u(x-z) - u(y-z))\psi(z)dz \right|^q |x-y|^{-(n+2\sigma)} dx dy$$

Now, we can look the measure $\mu(dz) = \psi(z)dz$ as a probability measure because of (3.1). Clearly, the function $t \mapsto |t|^q$ is convex in \mathbb{R} . Then, we apply Jensen's inequality on the dz -integral in right-hand side of the last expression, concluding

$$I_{ext} \leq \iint_{|x-y|>1} \int_{\mathbb{R}^n} |u(x-z) - u(y-z)|^q \psi(z)dz |x-y|^{-(n+2\sigma)} dx dy,$$

which, after an application of Fubini's Theorem, gives

$$I_{ext} \leq \int_{\mathbb{R}^n} \psi(z) \left(\iint_{|x-y|>1} |u(x-z) - u(y-z)|^q |x-y|^{-(n+2\sigma)} dx dy \right) dz,$$

Then, applying the change $\tilde{x} = x - z, \tilde{y} = y - z$ in the $dx dy$ integral and using (3.1), we conclude

$$\begin{aligned} I_{ext} &\leq \int_{\mathbb{R}^n} \psi(z) \left(\iint_{|\tilde{x}-\tilde{y}|>1} |u(\tilde{x}) - u(\tilde{y})|^q |\tilde{x} - \tilde{y}|^{-(n+2\sigma)} d\tilde{x} d\tilde{y} \right) dz \\ &= \iint_{|\tilde{x}-\tilde{y}|>1} |u(\tilde{x}) - u(\tilde{y})|^q |\tilde{x} - \tilde{y}|^{-(n+2\sigma)} d\tilde{x} d\tilde{y} \\ &\leq C \iint_{|\tilde{x}-\tilde{y}|>1} \frac{|u(\tilde{x}) - u(\tilde{y})|^q}{(1 + |\tilde{x} - \tilde{y}|)^{-(n+2\sigma)}} d\tilde{x} d\tilde{y}. \end{aligned}$$

Using this last expression, we obtain from the assumption (J2) that

$$(3.4) \quad I_{ext} \leq C \iint_{\mathbb{R}^{2n}} J(x, y) |u(x) - u(y)|^q dx dy.$$

Now we take care about I_{int} . In this case, using the definition of ψ , we can write

$$(3.5) \quad I_{int} = \iint_{|x-y|<1} \left| \int_{\mathbb{R}^n} u(z) (\psi(x-z) - \psi(y-z)) dz \right|^q |x-y|^{-(n+2\sigma)} dx dy.$$

Note that by using (3.1), we have for all $x, y \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} u(x) (\psi(x-z) - \psi(y-z)) dz = u(x) \left(\int_{\mathbb{R}^n} \psi(x-z) dz - \int_{\mathbb{R}^n} \psi(y-z) dz \right) = 0,$$

and then

$$\int_{\mathbb{R}^n} u(z) (\psi(x-z) - \psi(y-z)) dz = \int_{\mathbb{R}^n} (u(z) - u(x)) (\psi(x-z) - \psi(y-z)) dz.$$

Thus, using this equality into (3.5), we get

$$I_{int} = \iint_{|x-y|<1} \left| \int_{\mathbb{R}^n} (u(z) - u(x)) (\psi(x-z) - \psi(y-z)) dz \right|^q |x-y|^{-(n+2\sigma)} dx dy.$$

However, note that if $|x - z| \geq 2$ in the dz integral, since $|x - y| < 1$ necessarily $|y - z| > 1$. Then, due to the fact that ψ is supported in the unit ball, the contribution of the integrand when $|x - z| \geq 2$ is null in the dz integral. Taking this into account, applying Hölder's inequality into the

dz -integral, we have

$$\begin{aligned}
I_{int} &= \iint_{|x-y|<1} \left| \int_{|x-z|<2} (u(z) - u(x))(\psi(x-z) - \psi(y-z)) dz \right|^q \\
&\quad \times |x-y|^{-(n+2\sigma)} dx dy \\
&\leq \iint_{|x-y|<1} \left(\int_{|x-z|<2} |u(z) - u(x)|^q dz \right) \\
&\quad \times \left(\int_{|x-\tilde{z}|<2} |\psi(x-\tilde{z}) - \psi(y-\tilde{z})|^{q'} d\tilde{z} \right)^{q/q'} |x-y|^{-(n+2\sigma)} dx dy,
\end{aligned}$$

where $q' = q/(q-1)$ is the Hölder conjugate to q . By Fubini's Theorem we can write

$$I_{int} = \int_{x \in \mathbb{R}^n} \left(\int_{|x-z|<2} (u(z) - u(x))^2 dz \right) \Psi(x) dx,$$

where

$$\Psi(x) = \int_{|x-y|<1} \left(\int_{|x-\tilde{z}|<2} |\psi(x-\tilde{z}) - \psi(y-\tilde{z})|^{q'} d\tilde{z} \right)^{q/q'} |x-y|^{-(n+2\sigma)} dy.$$

Using the regularity of ψ , we have

$$\begin{aligned}
\Psi(x) &\leq \int_{|x-y|<1} \left(\int_{|x-\tilde{z}|<2} \|D\psi\|_{\infty}^{q'} |x-y|^{q'} d\tilde{z} \right)^{q/q'} |x-y|^{-(n+2\sigma)} dy \\
&\leq \|D\psi\|_{\infty}^q |B_2|^{q/q'} \int_{|x-y|<1} |x-y|^q |x-y|^{-(n+2\sigma)} dy,
\end{aligned}$$

and since $q > 2\sigma$, we conclude the last integral is convergent, obtaining

$$\Psi(x) \leq C_{n,\sigma,q} \|D\psi\|_{\infty}^q |B_2|^{q/q'},$$

which leads us to the following estimate for I_{int}

$$I_{int} \leq C \int_{x \in \mathbb{R}^n} \int_{|x-z|<2} |u(z) - u(x)|^q dz dx.$$

From this, it is easy to get

$$I_{int} \leq C \int_{|x-z|\leq 2} \frac{|u(z) - u(x)|^q}{(1 + |x-z|)^{n+2\sigma}} dz dx,$$

which by the use of (J2), let us conclude that

$$I_{int} \leq C \iint_{\mathbb{R}^{2n}} J(x, y) |u(x) - u(y)|^q dx dy.$$

This last estimate together with (3.4) concludes the proof. \square

4. PROOF OF THEOREM 1.1

For simplicity, we use the following notation: for $q > 1$, we denote $\tilde{\sigma} = \tilde{\sigma}_q = 2\sigma q^{-1}$. Hence, the fractional critical exponent of q relative to $\tilde{\sigma}$ is denoted by \tilde{q}^* and is given by

$$\tilde{q}^* = \frac{nq}{n - q\tilde{\sigma}} = \frac{nq}{n - 2\sigma}.$$

We need an intermediate result.

Lemma 4.1. *Let us assume the hypotheses of Theorem 2.1 and Proposition 3.2. Then, there exists two constants $C_1, C_2 > 0$ depending on n, q and σ such that*

$$(4.1) \quad \|u(\cdot, t)\|_{L^q(\mathbb{R}^n)}^q \leq C_1 \|u_0\|_{L^1(\mathbb{R}^n)}^{q(1-\theta)} E_{J,q}(u)^\theta + C_2 E_{J,q}(u),$$

where $\theta \in (0, 1)$ satisfies the equality

$$(4.2) \quad \frac{1}{q} = \frac{\theta}{\tilde{q}^*} + (1 - \theta), \quad \text{that is} \quad \theta = 1 - \frac{2\sigma}{n(q-1) + 2\sigma}.$$

Proof: Using the definition of v and w in (3.2), we have

$$(4.3) \quad \|u\|_{L^q(\mathbb{R}^n)}^q \leq 2^{q-1} (\|v\|_{L^q(\mathbb{R}^n)}^q + \|w\|_{L^q(\mathbb{R}^n)}^q).$$

By the definition of v , it belongs to L^p for all p . Hence, we can interpolate, obtaining

$$\|v\|_{L^q(\mathbb{R}^n)} \leq \|v\|_{L^{\tilde{q}^*}(\mathbb{R}^n)}^\theta \|v\|_{L^1(\mathbb{R}^n)}^{1-\theta},$$

where θ is given by (4.2). Recalling the Sobolev-type inequality (2.5), we have

$$\|v\|_{L^q(\mathbb{R}^n)} \leq C [v]_{\tilde{\sigma}, q}^\theta \|v\|_{L^1(\mathbb{R}^n)}^{1-\theta},$$

and then, using this last inequality, (4.3), Proposition 3.2 and the property $\|u(\cdot, t)\|_{L^1(\mathbb{R}^n)} \leq \|u_0\|_{L^1(\mathbb{R}^n)}$ given in Theorem 2.1, we get (4.1). \square

Proof of Theorem 1.1. First we look for the case $q \geq 2$. Consider the constants C_1, C_2 of the previous lemma and denote

$$K_1 = C_1 \|u_0\|_{L^1(\mathbb{R}^n)}^{q(1-\theta)}, \quad \text{and} \quad H(x) = K_1 x^\theta + C_2 x, \quad \text{for } x \geq 0.$$

With these definitions, (4.1) can be written as

$$\|u\|_{L^q(\mathbb{R}^n)}^q \leq H(E_{J,q}(u)),$$

and this leads us to the inequality

$$H^{-1}(\|u\|_{L^q(\mathbb{R}^n)}^q) \leq E_{J,q}(u).$$

We need to look more carefully the behavior of H^{-1} . Note that if $x \geq (K_1^{-1}C_2)^{\frac{1}{\theta-1}}$, we have

$$H(x) \leq 2C_2 x,$$

meanwhile, if $x < (K_1^{-1}C_2)^{\frac{1}{\theta-1}}$ we have

$$H(x) \leq 2K_1 x^\theta.$$

The above estimates allow us to write

$$H^{-1}(x) \geq \begin{cases} (x/(2K_1))^{1/\theta} & x \leq C^*, \\ x/(2C_2) & x > C^*, \end{cases}$$

where $C^* = C_0^* \|u_0\|_{L^1(\mathbb{R}^n)}^q$ with

$$(4.4) \quad C_0^* = 2C_1^{1/(1-\theta)} C_2^{\theta/(\theta-1)}.$$

Hence, denoting $\phi(t) = \phi[u](t) := \|u(\cdot, t)\|_{L^q(\mathbb{R}^n)}^q$ and recalling the above definition of K_1 , we use (2.3) together with the above estimates to conclude ϕ satisfies the differential inequality

$$(4.5) \quad \partial_t \phi(t) + C_3 \|u_0\|_{L^1(\mathbb{R}^n)}^{q(1-\theta^{-1})} \phi(t)^{1/\theta} \mathbf{1}_{\{\phi(t) \leq C^*\}} + C_4 \phi(t) \mathbf{1}_{\{\phi(t) > C^*\}} \leq 0,$$

where $C_3 = (2C_1)^{-1/\theta}$, $C_4 = (2C_2)^{-1}$.

At this point, we split the analysis in two cases. First, we consider the case $\phi(0) \leq C^*$, that is when $\|u_0\|_{L^q(\mathbb{R}^n)} \leq (C_0^*)^{1/q} \|u_0\|_{L^1(\mathbb{R}^n)}$. In this case, by (4.5) and since ϕ is nonincreasing, we get

$$\partial_t \phi(t) + C_3 \|u_0\|_{L^1(\mathbb{R}^n)}^{q(1-\theta^{-1})} \phi(t)^{1/\theta} \leq 0$$

for all $t > 0$. Integrating from 0 to t , we conclude that

$$\phi(t) \leq \left(\phi(0)^{\theta^{-1}(\theta-1)} + \theta^{-1}(1-\theta) C_3 \|u_0\|_{L^1(\mathbb{R}^n)}^{q(1-\theta^{-1})} t \right)^{1/(1-\theta^{-1})}.$$

From this, using the expression (4.2) for θ we have

$$\frac{\theta}{\theta-1} = -\frac{n(q-1)}{2\sigma},$$

and recalling $\phi(t) = \|u(t)\|_{L^q(\mathbb{R}^n)}^q$, we conclude

$$\|u(\cdot, t)\|_{L^q(\mathbb{R}^n)} \leq C_\theta \|u_0\|_{L^1(\mathbb{R}^n)} t^{-\frac{n}{2\sigma}(1-q^{-1})}.$$

In the case $q \in (1, 2)$, we can use interpolation. We have

$$\|u\|_{L^q(\mathbb{R}^n)} \leq \|u\|_{L^1(\mathbb{R}^n)}^{1-\alpha} \|u\|_{L^2(\mathbb{R}^n)}^\alpha,$$

with $\alpha = 2(q-1)q^{-1}$. Then, we use that the L^1 -norm does not increase with time and with the decay estimates for $q = 2$, to conclude

$$\|u(\cdot, t)\|_{L^q(\mathbb{R}^n)} \leq C \|u_0\|_{L^1(\mathbb{R}^n)} t^{-\frac{n}{2\sigma}(1-\frac{1}{q})},$$

which is the desired inequality. \square

5. APPENDIX

Proof of Inequality (2.2): Clearly the inequality holds if $a = b$, then, without loss of generality, we assume $|a| > |b|$. If this is the case, (2.2) is equivalent to

$$(1 - x)(1 - |x|^{q-2}x) \geq C|1 - x|^q$$

for all $|x| < 1$. Now, in the case $x < 0$ we have the left-hand side is bounded from above by 1, meanwhile the right-hand side is bounded from above by 2^q . Hence, it is sufficient to consider $C = 2^{-q}$. For the case $x \in (0, 1)$, the desired inequality is equivalent to

$$1 \geq x^{q-1} + C(1 - x)^{q-1}.$$

When $q \geq 2$, this inequality holds with any $C \leq 1$ because

$$1 \geq x + (1 - x) \geq x^{q-1} + (1 - x)^{q-1}.$$

Hence, we have the result with $C = 2^{-q}$. \square

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