

SYSTOLES OF 2-COMPLEXES, REEB GRAPH, AND GRUSHKO DECOMPOSITION

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ABSTRACT. Let X be a finite 2-complex with unfree fundamental group. We prove lower bounds for the area of a metric on X , in terms of the square of the least length of a noncontractible loop in X . We thus establish a uniform systolic inequality for all un-free 2-complexes. Our inequality improves the constant in M. Gromov's inequality in this dimension. The argument relies on the Reeb graph and the coarea formula, combined with an induction on the number of freely indecomposable factors in Grushko's decomposition of the fundamental group. More specifically, we construct a kind of a Reeb space "minimal model" for X , reminiscent of the "chopping off long fingers" construction used by Gromov in the context of surfaces. As a consequence, we prove the agreement of the Lusternik-Schnirelmann and systolic categories of a 2-complex.

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1. INEQUALITY FOR UNFREE COMPLEXES

Throughout the paper, complex means simplicial complex unless some other is said explicitly.

The homotopy 1-systole, denoted $\text{sys}\pi_1(X)$, of a compact metric space X is, by definition, the least length of a noncontractible loop in X . The first systematic study of this invariant and its generalisations, by M. Gromov [Gro83], built upon earlier pioneering work of C. Loewner, P. Pu [Pu52], R. Accola, C. Blatter, M. Berger [Berg65], J. Hebda [He82], and others. Such work was recently surveyed in the articles [CrK03, KL05].

The *optimal systolic ratio*, denoted $\text{SR}(X)$, of an n -dimensional complex X , is the supremum of the values of the scale-invariant ratio

$$\frac{\text{sys}\pi_1^n(\mathcal{G})}{\text{vol}_n(\mathcal{G})},$$

the supremum being over all piecewise flat metrics \mathcal{G} on X .

Let X be a connected finite complex. Let

$$f : X \longrightarrow K(\pi_1(X), 1)$$

be its classifying map, *i.e.* a map that induces an isomorphism of fundamental groups. Here $K(\pi, 1)$ denotes a connected CW -space with $\pi_1(K(\pi, 1)) = \pi$ and $\pi_n(K(\pi, 1)) = 0$ for all $n \geq 2$. If $f' : X \longrightarrow K'(\pi_1(X), 1)$ is another classifying map, then there exists a homotopy equivalence $h : K(\pi_1(X), 1) \rightarrow K'(\pi_1(X), 1)$ such that $hf \simeq f'$.

Definition 1.1. A complex X is said to be *n-essential* if there exists a classifying map $X \rightarrow K(\pi, 1)$ that cannot be homotoped into the $(n - 1)$ -skeleton of $K(\pi_1(X), 1)$, *cf.* [Gro99, p. 264], [KR04]. By the above, *every* classifying map has this property.

In the case $n = 2$, a 2-complex is 2-essential if and only if it is unfree, *cf.* Theorem 12.1.

M. Gromov [Gro83, Appendix 2] proved that every n -essential n -complex X satisfies a systolic inequality, *i.e.* there exists a (finite) constant $C_n > 0$ such that $\text{SR}(X) \leq C_n$. Note that a converse is true by [Ba93, Lemma 8.4].

When X is a surface, numerous systolic inequalities are now available. They include near-optimal asymptotic upper bounds for the optimal systolic ratio of surfaces of large genus [Gro83, KS05], as well as near-optimal asymptotic *lower* bounds for large genus [BS94, KSV05]. The genus 2 surface was recently shown to be Loewner [KS06a], and moreover admits an optimal systolic inequality in the CAT(0) class [KS06b]. A relative version of Pu's inequality [Pu52] was obtained in [BaCIK05]. However, many of the existing techniques, including the entropy technique of [KS05], are not applicable in the more general context of an arbitrary finite 2-complex.

Definition 1.2. A 2-complex is *unfree* if its fundamental group is not free.

Specifically in dimension 2, M. Gromov [Gro83, 6.7.A] (note a misprint in the exponent) showed that every unfree 2-dimensional complex X satisfies the inequality

$$\text{SR}(X) \leq 10^4. \quad (1.1)$$

Contrary to the case of surfaces, where a (better) systolic inequality can be derived by simple techniques, Gromov's proof of inequality (1.1) depends on the advanced filling techniques of [Gro83]. Note that the technique of cutting a surface open along a non-separating loop, and applying the coarea formula to the distance function from one of the boundary components, does not seem to generalize to arbitrary 2-complexes.

Recently, M. Gromov [Gro05] remarked that a systolic inequality for unfree 2-complexes should be a consequence merely of the coarea formula.

We have been able to obtain a *uniform* inequality for arbitrary unfree 2-complexes using merely the coarea formula, the properties of the Reeb graph of the distance function, as well as the classical Grushko decomposition of the fundamental group [St65, ScW79]. More specifically, we construct a kind of a Reeb space "minimal model" $M(X, r)$ for X , reminiscent of the "chopping off long fingers" construction used by M. Gromov [Gro83] in the context of surfaces. We thus improve the constant in (1.1) to the value 12. Namely, we show the following.

Theorem 1.3. *Every finite unfree piecewise flat 2-complex X satisfies the bound $\text{SR}(X) \leq 12$.*

Remark 1.4. Given an arbitrary 2-dimensional complex X , we consider a 2-dimensional complex Y whose fundamental group is the unfree factor of the Grushko decomposition of the fundamental group of X ,

cf. Section 2. Then there is a map $f : Y \rightarrow X$ inducing a monomorphism in π_1 , *cf.* [ScW79, Lemma 1.5]. Now in the absence of loose loops (see Section 8), for the source space Y , an inequality $\text{SR}(Y) \leq 4$ can be established relatively easily by means of the coarea formula. Furthermore, one can pushforward systolic inequalities by such a map f , by a technique pioneered by I. Babenko [Ba93], *cf.* [KR04, KR05]. This would prove a systolic inequality for the target space X , as well. However, the resulting inequality for X is not uniform, since pushforward affects the constant in the inequality. Namely, the constant is worsened by the number of faces of Y in the inverse image of a face of X . To overcome this difficulty, we will use an induction on the free index of Grushko, *cf.* (2.3).

Question 1.5. It is an open question whether all unfree 2-complexes satisfy Pu's inequality for $\mathbb{R}\mathbb{P}^2$, equivalently if the optimal constant in (1.1) is $\frac{\pi}{2}$.

This article is organized as follows. The corank and the free index of Grushko $\text{FIG}(X)$ of a 2-complex X are reviewed in Section 2. A useful application of the Seifert-van Kampen theorem appears in Section 3. The Reeb graph and its generalisation called Reeb space are described in Section 4. Section 5 introduces a minimal model for X , obtained by pruning suitable simply connected superlevel sets, as well as superfluous branches of the Reeb tree. Technical results on the level curves of the distance function are presented in Sections 6. We define loose loops and describe the intersection of pointed systoles with level curves in Section 7. The dichotomy loose loop/area lower bound is explained in Section 8. In Section 9 we prove the bound $\text{SR}(X) \leq 4$ when $\text{FIG}(X) = 0$, *cf.* Theorem 9.3. A suitable noncontractible level of the distance function is identified in Section 10. In Section 11, using such a noncontractible level and the coarea formula, we prove a corank-dependent systolic inequality for unfree 2-complexes. We compute the Lusternik-Schnirelmann category $\text{cat}_{\text{LS}}(X)$ of a 2-complex X in Section 12, and show that $\text{cat}_{\text{LS}}(X)$ and the systolic category of X agree. The proof of the uniform bound $\text{SR}(X) \leq 12$ occupies Sections 13 and 14.

All complexes are assumed to be finite, connected, and piecewise flat, unless explicitly mentioned otherwise.

2. CORANK AND FREE INDEX OF GRUSHKO

Recall that the *corank* of a group G is defined to be the maximal rank n of a free group F_n admitting an epimorphism $G \rightarrow F_n$ from G .

Clearly, every finitely generated group has finite corank, and therefore each finite CW -space has fundamental group of finite corank.

Grushko's theorem [St65, ScW79] asserts that every finitely generated group G has a decomposition as a free product of subgroups

$$G = F_p * H_1 * \cdots * H_q \quad (2.1)$$

such that F_p is free of rank p , while every H_i is nontrivial, non isomorphic to \mathbb{Z} and freely indecomposable. Furthermore, given another decomposition of this sort, say $G = F_r * H'_1 * \cdots * H'_s$, one necessarily has $r = p$, $s = q$ and, after reordering, H'_i is conjugate to H_i .

Definition 2.1. We will refer to the number p in decomposition (2.1) as the *free index of Grushko* of G , denoted $\text{FIG}(G)$.

Thus, every finitely generated group G with $\text{FIG}(G) = p$ can be decomposed as

$$G = F_p * H_G, \quad (2.2)$$

where F_p is free of rank p and $\text{FIG}(H_G) = 0$. The subgroup H_G is unique up to isomorphism. Its isomorphism class is called the *unfree factor* of (the isomorphism class of) G . If X is a finite complex, we set

$$\text{FIG}(X) = \text{FIG}(\pi_1(X)) \quad (2.3)$$

by definition. Note that every finitely generated group G satisfies $\text{FIG}(G) \leq \text{corank}(G)$.

Definition 2.2. Let G be a finitely generated group. We say that an element $g \in G$ *splits off* if G admits a decomposition as a free product with the infinite cyclic group generated by g .

3. AN APPLICATION OF THE SEIFERT-VAN KAMPEN THEOREM

We need the following well-known fact, *cf.* [Rud98, Prop. I.3.26].

Proposition 3.1. *Let (X, A) be a CW -pair. If A is a contractible space, then the quotient map $X \rightarrow X/A$ is a homotopy equivalence. \square*

Corollary 3.2. *Consider a CW -pair (X, A) and a CW -space $X \cup CA$ where CA is the cone over A . Then the quotient map*

$$X \cup CA \rightarrow (X \cup CA)/CA = X/A$$

is a homotopy equivalence. In particular, $X \cup CA$ and X/A are homotopy equivalent, $X \cup CA \simeq X/A$. \square

Lemma 3.3. *Let (X, A) be a CW-pair with X and A connected. Then the quotient map $q : X \rightarrow X/A$ induces an epimorphism of fundamental groups. Furthermore, if the inclusion $j : A \rightarrow X$ induces the zero homomorphism $j_* : \pi_1(A) \rightarrow \pi_1(X)$ of fundamental groups, then the quotient map $q : X \rightarrow X/A$ induces an isomorphism of fundamental groups.*

Proof. By the Seifert–van Kampen theorem, the inclusion

$$i : X \subset X \cup CA$$

induces an epimorphism $i_* : \pi_1(X) \rightarrow \pi_1(X \cup CA)$ of fundamental groups, and i_* is an isomorphism if j_* is the zero map. Finally, by Corollary 3.2, the quotient map $X \cup CA \rightarrow X/A$ is a homotopy equivalence, while the map q is the composition $X \subset X \cup CA \rightarrow X/A$, proving the lemma. \square

Lemma 3.4. *Let $A = \{a_0, a_1, \dots, a_k\}$ be a finite subset of a connected CW-space X . Then $X \cup CA$ is homotopy equivalent to the wedge of X and k circles, $X \cup CA \simeq X \vee S_1^1 \vee \dots \vee S_k^1$. In particular,*

$$\pi_1(X \cup CA) = \pi_1(X) * F_k$$

where F_k is the group of rank k . In other words, the free index of Grushko of $\pi_1(X \cup CA)$ is at least k .

Proof. Let $X^{(1)}$ be the 1-skeleton of X . Without loss of generality, we can assume (subdividing X) that $A \subset X^{(1)}$, $\alpha \subset X^{(1)}$ and that $X^{(1)}$ is connected. Since $X^{(1)}$ is connected, we can find a tree $T \subset X^{(1)}$ that contains α and A . Since T is contractible, we have

$$\begin{aligned} X \cup CA &\simeq (X \cup CA)/T \simeq (X/T) \vee (CA/A) \\ &\simeq X \vee (CA/A) \simeq X \vee S_1^1 \vee \dots \vee S_k^1 \end{aligned}$$

from Proposition 3.1. \square

Let X be a finite connected complex and let $f : X \rightarrow \mathbb{R}^+$ be a function on X . Let

$$[f \leq r] := \{x \in X \mid f(x) \leq r\} \text{ and } [f \geq r] := \{x \in X \mid f(x) \geq r\}$$

denote the sublevel and superlevel sets of f , respectively.

Definition 3.5. Given $r \in \mathbb{R}^+$, suppose that a single path-connected component of the superlevel set $[f \geq r]$ contains k path-connected components of the level set $f^{-1}(r)$. Then we will say that the k path-connected components *coalesce forward*.

For future needs, recall that the connected components of any complex are path-connected.

Lemma 3.6. *Given $r > 0$, assume that the pair $([f \geq r], f^{-1}(r))$ is homeomorphic to a CW-pair. Suppose that the set $[f \leq r]$ is connected and that $k + 1$ connected components of $f^{-1}(r)$ coalesce forward. Then the corank of $\pi_1(X)$ is at least k . Furthermore, if the inclusion*

$$[f \leq r] \subset X$$

of the sublevel set $[f \leq r]$ induces the zero homomorphism of fundamental groups, then $\text{FIG}(X) \geq k$.

Proof. Let C_0, C_1, \dots, C_k be the distinct components of $f^{-1}(r)$ that coalesce forward, and let Z be the component of $[f \geq r]$ that contains all C_i 's. Then $[f \geq r] = Z \sqcup W$ where W is the (finite) union of all components of $[f \geq r]$ other than Z . Let $Y = Z / \sim$ where $x \sim y$ if and only if x, y belong to the same component C_i of $f^{-1}(r)$. Let a_i be the image of C_i under the quotient map $Z \rightarrow Y$ and let

$$A = \{a_0, \dots, a_k\} \subset Y.$$

We set

$$W' = W / (W \cap f^{-1}(r)).$$

Then we have

$$X/[f \leq r] = W' \vee Z / (\cup C_i) = W' \vee Y/A \simeq W' \vee Y \vee S_1^1 \vee \dots \vee S_k^1,$$

where the last equivalence follows from Corollary 3.2 and Lemma 3.4. By the Seifert–van Kampen Theorem, we have $\text{FIG}(X/[f \leq r]) \geq k$. In particular, there exists an epimorphism

$$\pi_1(X/[f \leq r]) \rightarrow F_k.$$

By Lemma 3.3, the quotient map $X \rightarrow X/[f \leq r]$ induces an epimorphism of fundamental groups. Hence $\text{corank}(\pi_1(X)) \geq k$. Finally, if the inclusion $[f \leq r] \subset X$ induces the zero homomorphism of fundamental groups, then $\pi_1(X/[f \leq r])$ is isomorphic to $\pi_1(X)$ by Lemma 3.3, proving the lemma. \square

4. REEB GRAPH OF A PIECEWISE FLAT COMPLEX

Definition 4.1. Let X be a finite connected complex. Consider a function $f : X \rightarrow \mathbb{R}^+$. Let $r_0 > 0$ be a real number. The Reeb space of f in range r_0 , denoted $\text{Reeb}(f, r_0)$, is defined as the quotient

$$\text{Reeb}(f, r_0) = X / \sim$$

where we have equivalence $x \sim y$ if and only if $f(x) = f(y) \leq r_0$, and x and y lie in the same connected component of the level set $f^{-1}(f(x))$, cf. [Re46]. The Reeb space $\text{Reeb}(f, \infty)$ in “full range” will be denoted simply $\text{Reeb}(f)$.

The following fact is a consequence of standard results on the triangulation of semialgebraic functions [BCR98, §9], [Sh89].

Proposition 4.2. *Let X be a finite, 2-dimensional, piecewise flat complex. Then the Reeb space $\text{Reeb}(f)$ of the distance function $f = f_p$ from a point $p \in X$ is a finite graph.*

Furthermore, the finite graph $\text{Reeb}(f)$ can be subdivided so that the natural map $X \rightarrow \text{Reeb}(f)$ yields a trivial bundle over the interior of each edge of $\text{Reeb}(f)$.

The Reeb graph was used in [GL83, Ka88] to study 3-manifolds of positive scalar curvature. Other applications are discussed in [CoP04].

Remark 4.3. A more precise description (than that predicted by semireal algebraic geometry) can be given of the level curves themselves, cf. Section 6.

From now on, X and f will be as in Proposition 4.2. The Reeb space $\text{Reeb}(f, r)$ in range r can be thought of as a “hybrid” space, consisting of two pieces. One piece is the Reeb graph $\text{Reeb}(f|_B)$ of the ball $B = B(p, r)$ of radius r centered at p . The other piece $[f \geq r]$ is the complement, in X , of the open ball of radius r , attached to the graph by a map μ which collapses each connected component of the level curve $f^{-1}(r)$ to a point:

$$\text{Reeb}(f, r) = \text{Reeb}(f|_{B(p,r)}) \cup_{\mu} [f \geq r]. \quad (4.1)$$

The Reeb graph $\text{Reeb}(f)$ of f is endowed with the length structure induced from X . Let $\bar{p} \in \text{Reeb}(f)$ be the image of p . The ball

$$B(p, r) \subset X$$

of radius $r < \frac{1}{2} \text{sys}\pi_1(X)$ projects to the ball $B(\bar{p}, r) \subset \text{Reeb}(f)$.

Lemma 4.4. *Let $r < \frac{1}{2} \text{sys}\pi_1(X)$. The subgraph $B(\bar{p}, r) \subset \text{Reeb}(f)$ is a tree, denoted T_r . Thus, the decomposition (4.1) becomes*

$$\text{Reeb}(f, r) = T_r \cup_{\mu} [f \geq r]. \quad (4.2)$$

Proof. Every edge of $\text{Reeb}(f)$ lifts to a minimizing path in X , given by a segment of the minimizing path from p . Given an embedded loop $\gamma \subset \text{Reeb}(f)$, we can lift each of its edges to X . The endpoints of adjacent edges lift to a pair of points lying in a common connected component of a level curve of f (by definition of the Reeb graph), and can therefore be connected in X by a path which projects to a constant path in $\text{Reeb}(f)$. We thus obtain a loop in X whose image in $\text{Reeb}(f)$ is homotopic to γ . The lemma now follows from the fact that the inclusion $B(p, r) \subset X$ induces the trivial homomorphism of fundamental groups, cf. Remark 7.2. \square

Lemma 4.5. *Let $r < \frac{1}{2} \text{sys}\pi_1(X)$. Then the natural projection map $h : X \rightarrow \text{Reeb}(f, r)$ induces an isomorphism of fundamental groups,*

$$h_* : \pi_1(X) \rightarrow \pi_1(\text{Reeb}(f, r)). \quad (4.3)$$

Proof. The obvious map $g : X/B(p, r) \rightarrow \text{Reeb}(f, r)/T_r$ (induced by h) is a homeomorphism. Consider the commutative diagram

$$\begin{array}{ccc} \pi_1(X) & \xrightarrow{h_*} & \pi_1(\text{Reeb}(f, r)) \\ \downarrow & & \downarrow \\ \pi_1(X/B(p, r)) & \xrightarrow{g_*} & \pi_1(\text{Reeb}(f, r)/T_r) \end{array}$$

The vertical maps are isomorphisms by Lemma 3.3 and Remark 7.2, while g_* is induced by a homeomorphism. \square

5. A MINIMAL MODEL AND TREE ENERGY

We describe a pruning of the Reeb space which results in a kind of a “minimal model”, *cf.* (5.1), for our 2-complex X .

Let $r < \frac{1}{2} \text{sys}\pi_1(X)$. Consider the function $\bar{f} : \text{Reeb}(f, r) \rightarrow \mathbb{R}$ naturally defined by f . Denote by $\mathcal{C} \subset \text{Reeb}(f, r)$ the union of the connected components C of the subset $[\bar{f} \geq r]$ with the following two properties:

- (1) C is simply connected;
- (2) C is attached to the tree $T_r \subset \text{Reeb}(f, r)$ at a single leaf.

The pullback of the standard 1-form dr on \mathbb{R} by the map

$$\text{Reeb}(f) \rightarrow \mathbb{R}$$

back to an edge of $\text{Reeb}(f)$ defines a positive direction on each edge. We prune the tree T_r by removing edges of the following two types:

- (1) edges that do not reach the set $[\bar{f} \geq r]$ when we follow the positive direction;
- (2) edges that lead only to components in $\mathcal{C} \subset [\bar{f} \geq r]$.

Denote by $T'_r \subset T_r$ the resulting pruned tree. In other words, T'_r is the union of embedded paths connecting p to components of the complement $[\bar{f} \geq r] \setminus \mathcal{C}$.

A vertex of T'_r is by definition a point whose open neighborhood in T'_r is not homeomorphic to an open interval. An edge is a connected component of the complement of the set of vertices.

Define the *energy* $E(\Gamma)$ of a graph Γ to be the sum of the squares of the lengths of its edges. The height of a tree with a distinguished root leaf p is the *least* distance from p to any other leaf of the tree.

Proposition 5.1. *The energy of a tree Γ of height h satisfies the bound $E(\Gamma) \geq \frac{1}{2}h^2$.*

Proof. For a simplest tree shaped as the letter Y, the least energy is given by the metric for which the bottom interval is twice as long as each of the top two intervals. Arguing inductively, we see that the lower bound is attained by the infinite tree where the length of an edge is halved after each branching. \square

Definition 5.2. We introduce a kind of a minimal model $M(X, r) \subset \text{Reeb}(f, r)$ for X by setting

$$M(X, r) = T'_r \cup_\mu ([f \geq r] \setminus \mathcal{C}). \quad (5.1)$$

Proposition 5.3. *The map $X \rightarrow M(X, r)$ collapsing all superfluous material to the vertex of T_r where it is attached to the pruned tree $T'_r \subset T_r$, induces an isomorphism $\pi_1(X) \simeq \pi_1(M(X, r))$.*

Definition 5.4. If the pruned tree T'_r has exactly one edge attached to its root leaf, we call this edge the *root edge* of T'_r . If T'_r has more than one edge attached to its root leaf, the root edge is defined as the trivial edge reduced to the root leaf of T'_r .

Proposition 5.5. *Let X be an unfree 2-complex. Let $e \subset T'_r$ be an open non-root edge. Consider the space*

$$Y = M(X, r) \setminus e$$

obtained from the minimal model by removing e . Then there exists an unfree connected component $Y_ \subset Y$ satisfying*

$$\text{FIG}(Y_*) \leq \text{FIG}(X) - 1. \quad (5.2)$$

Proof. If removing e does not disconnect the space $\text{Reeb}(f, r)$, then, by Lemma 3.4, we have

$$\pi_1(M(X, r)) \simeq \pi_1(\text{Reeb}(f, r)) \simeq \pi_1(Y) * \mathbb{Z},$$

proving equality in (5.2). If removing e disconnects $\text{Reeb}(f, r)$, then Y decomposes as $Y_- \cup Y_+$, where each of Y_\pm is connected. Neither component is simply connected by definition of the pruned tree. We have $\pi_1(X) \simeq \pi_1(Y_-) * \pi_1(Y_+)$ by the Seifert-van Kampen theorem. Hence $\text{FIG}(X) = \text{FIG}(Y_-) + \text{FIG}(Y_+)$. If both Y_+ and Y_- are unfree, we choose the one with the least FIG. If one of them is free, then the other necessarily satisfies (5.2). \square

6. LEVEL CURVES

In this section, we present technical results concerning the level curves of the distance function on a piecewise flat 2-complex. The 2-simplices of such a complex will be referred to as *faces*.

Theorem 6.1. *A level curve of the distance function from a point in a finite piecewise flat 2-dimensional complex X is a finite union of circular arcs and isolated points.*

The claim is obvious for level curves at sufficiently small distance from a vertex $p \in X$. Such a level curve is known as the *link* of X at p , cf. [Mu84, §2].

The structure of the proof is 3-fold:

- (1) a finiteness result on the number of geodesic arcs of bounded length joining a pair of points of X ;
- (2) the construction of a finite graph Λ_p , cf. (6.1), containing a level curve S ;
- (3) an argument showing that S is a subgraph of Λ_p .

Lemma 6.2. *Let $C > 0$. Then there are finitely many geodesic arcs of length at most C between any pair of points of a finite piecewise flat 2-complex X .*

Proof. Let X' be the complement of the set of vertices in X . Let $p, q \in X'$. By the flatness of the complex, there is at most one geodesic path from p to q in a given homotopy class of such paths in X' . Note that X' is homotopy equivalent to a compact 2-complex, by removing an open disk around each vertex. The lemma now follows from the finiteness of the number of homotopy classes of bounded length in a compact complex. \square

Remark 6.3. Consider a geodesic $\gamma \subset \tilde{X}'$ in the universal cover \tilde{X}' of X' . Then the union of all faces meeting γ , includes a flat strip containing γ .

Continuing with the proof of Theorem 6.1, note that the link at q is a graph which can be identified with the set of unit tangent vectors at q . To describe the graph Λ of step 2, we refine the link at q as follows. Consider the tangent vectors of the geodesics of length smaller than the diameter of X , connecting q to all vertices of X . We subdivide the link recursively. At each step, we add to the link the vertices corresponding to such tangent vectors, so that the edge containing such a vertex is split into two. By Lemma 6.2, the resulting graph V_q is finite.

Let $\rho > 0$. Given an edge e of the refined link V_q at q , consider the corresponding pencil of geodesics, of length ρ , issuing from q . By

Remark 6.3, their endpoints trace out a single compact (possibly self-intersecting) circular arc, denoted

$$CA(q, e, \rho) \subset X,$$

of “radius” ρ and geodesic curvature $\frac{1}{\rho}$, at least if X is a surface. In general, the structure of $CA(q, e, \rho)$ may be more complicated due to branching at a point where a geodesic issuing from q encounters a 1-cell of X , but in any case the number of circular arcs forming the graph CA can be controlled in terms of the total number of faces.

Now let $p \in X$, and let $r > 0$. Let (q_i) be an enumeration of the vertices of X , and (e_{ij}) an enumeration of the edges of the refined link V_{q_i} .

Lemma 6.4. *Let $S = S(r) \subset X$ be the level r curve of the distance function from $p \in X$, namely, $S = \{x \in X \mid \text{dist}(p, x) = r\}$. Consider the finite union*

$$\Lambda_p = \Lambda_p(r) = \bigcup_{i,j} CA(q_i, e_{ij}, r - \text{dist}(p, q_i)). \quad (6.1)$$

Then Λ_p is a finite graph, containing S , and at distance at most r from p .

Proof. To describe the graph structure of Λ_p , note that a pair of circular arcs in a given simplex of X meet in at most a pair of points. This elementary algebraic-geometric observation is thus the basis of the proof of Theorem 6.1; cf. Remark 6.9.

Whether the arcs are transverse or tangent, we include the common points as vertices of Λ_p . Also, if a circular arc of Λ_p meets an edge of X , then the common point is declared to be a vertex of Λ_p . \square

To complete the proof of Theorem 6.1, we need a notion of a cut locus. The notion of a cut locus for a complex is ill-defined. However, we define an analogue CL_p of the cut locus from a point $p \in X$ inside an open face $\Delta \subset X$, as the set of points $q \in \Delta$ with at least a pair of distinct unit tangent vectors at q to minimizing geodesics from p .

Remark 6.5. The vertices of CL_p seem to be related to the critical points of the distance function from p in the sense of Grove-Shiohama-Gromov-Cheeger [Che91]. We will not pursue this direction.

Lemma 6.6. *The set $CL_p \subset X$ is relatively closed inside each open face of X .*

Proof. Consider a sequence of points $\{x_n\}$ in CL_p converging to $x \in \Delta$. Let u_n be a unit vector at x_n , and tangent to a minimizing geodesic

from p . By choosing a subsequence, we can assume that the minimizing geodesics corresponding to the vectors u_n lead to the same vertex $A \in X$, and, furthermore, are homotopic as relative paths from A to Δ for the pair $(\{A\} \cup X', \Delta)$, where X' is the complement of the set of vertices in X . Now, we use a kind of a developing map and consider the face Δ as lying in the Euclidean plane \mathbb{R}^2 . Then there is a point $\tilde{A} \in \mathbb{R}^2$ such that the circular arcs of Λ_p corresponding to the vectors u_n at $x_n \in \Delta \subset \mathbb{R}^2$ are arcs of concentric circles with common center \tilde{A} .

If $\{v_n\}$ is another sequence of tangent vectors at $\{x_n\}$, we similarly obtain points $B \in X$ and $\tilde{B} \in \mathbb{R}^2$ for the sequence v_n . Now suppose the two sequences have a common limit vector $\lim_n u_n = \lim_n v_n$. Then the points $\tilde{A}, \tilde{B}, x \in \mathbb{R}^2$ are collinear, while $x \notin [\tilde{A}, \tilde{B}]$. Hence

$$\begin{aligned} d(\tilde{A}, \tilde{B}) &= |d(\tilde{A}, x) - d(\tilde{B}, x)| \\ &= |d(A, p) - d(B, p)|. \end{aligned}$$

Since the geodesics leading to x_n are assumed minimizing, we have

$$\begin{aligned} |d(\tilde{A}, x_n) - d(\tilde{B}, x_n)| &= |d(A, p) - d(B, p)| \\ &= d(\tilde{A}, \tilde{B}). \end{aligned}$$

Therefore, the points $\tilde{A}, \tilde{B}, x_n \in \mathbb{R}^2$ are collinear, as well, and $x_n \notin [\tilde{A}, \tilde{B}]$. Thus $u_n = v_n$, proving the lemma. \square

The proof of Theorem 6.1 is now completed by means of the following lemma.

Lemma 6.7. *The level curve S is a subgraph of the finite graph Λ_p .*

Proof. At least two circular arcs corresponding to edges of Λ_p pass through each point of $CL_p \cap \Lambda_p$. Therefore, the points of $CL_p \cap \Lambda_p$ are vertices of the graph Λ_p . In particular, every open edge $e \subset \Lambda_p$ is disjoint from CL_p .

Every point $x \in e \cap S$ is away from the relatively closed set CL_p , cf. Lemma 6.6. Hence the level curves (of the distance function from p) in the neighborhood of x form a family of concentric circular arcs. The circular arc of this family passing through x is clearly contained in S , but also in e . Otherwise, it would intersect transversely the circular arc e and the point x would lie in CL_p . We conclude that $e \cap S$ is open in e .

On the other hand, the level curve S of a distance function is a closed set, hence the intersection $e \cap S$ is closed in e . Therefore, this intersection coincides with e or is empty. It follows that $S \subset \Lambda_p$ is a subgraph. \square

Given a point $p \in X$, set $f(x) = \text{dist}(p, x)$.

Corollary 6.8. *The triangulation of X can be refined in such a way that the sets $[f \leq r]$, $f^{-1}(r)$, and $[f \geq r]$ become CW-subspaces of X .*

Proof. We add the graph $S = f^{-1}(r)$ to the 1-skeleton of X . Some of the resulting faces may not be triangles, therefore a further (obvious) refinement may be necessary. The sets $[f \leq r]$ and $[f \geq r]$ are connected components of the complement $X \setminus S$, and hence CW-subspaces. \square

Remark 6.9. Alternatively, Corollary 6.8 (but *not* Theorem 6.1) can be deduced from standard results in real semialgebraic geometry, as follows, *cf.* [BCR98]. First, note that X can be embedded into some \mathbb{R}^N as a semialgebraic set and that the distance function f is a semialgebraic function on X . Thus, the level curve $f^{-1}(r)$ is a semialgebraic subset of X and, therefore, a finite graph, *cf.* proof of Lemma 6.4.

7. LOOSE LOOPS AND POINTED SYSTOLES

Definition 7.1. Let $p \in X$. A shortest noncontractible loop of X based at p is called a *pointed systolic loop* at p . Its length, denoted by $\text{sys}\pi_1(X, p)$, is called the *pointed systole* at p .

Remark 7.2. Alternatively, $\text{sys}\pi_1(X, p)$ could be defined as twice the upper bound of the reals $r > 0$ such that induced map $\pi_1(B(p, r)) \rightarrow \pi_1(X)$ is zero. In other words, every loop in $B(p, r)$ is contractible in X .

The following lemma describes the structure of a pointed systolic loop.

Lemma 7.3. *Let γ be a pointed systolic loop at $p \in X$, and let $L = \text{length}(\gamma) = \text{sys}\pi_1(X, p)$.*

- (i) *The loop γ is formed of two distance-minimizing arcs, starting at p and ending at a common endpoint, of length $L/2$.*
- (ii) *Any point of self-intersection of the loop γ is no further than $\frac{1}{2}(L - \text{sys}\pi_1(X))$ from p .*

Proof. Consider an arclength parametrisation $\gamma(s)$ with $\gamma(0) = \gamma(L) = p$. Let $q = \gamma(\frac{L}{2}) \in X$ be the ‘‘midpoint’’ of γ . Then q splits γ into a pair of paths of the same length $\frac{L}{2}$, joining p to q . By Remark 7.2, if q were contained in the open ball $B(p, \frac{L}{2})$, the loop γ would be contractible. This proves item (i).

If p' is a point of self-intersection of γ , the loop γ decomposes into two loops γ_1 and γ_2 based at p' , with $p \in \gamma_1$. Since the loop γ_1 is

shorter than the pointed systolic loop γ at p , it must be contractible. Hence γ_2 is noncontractible, so that

$$\text{length}(\gamma_2) \geq \text{sys}\pi_1(X).$$

Therefore,

$$\text{length}(\gamma_1) = L - \text{length}(\gamma_2) \leq L - \text{sys}\pi_1(X),$$

proving item (ii). \square

Definition 7.4. Let γ be a pointed systolic loop at $p \in X$, and let $L = \text{length}(\gamma) = \text{sys}\pi_1(X, p)$. Let r be a real number satisfying

$$L - \text{sys}\pi_1(X) < 2r < L.$$

If for at least one such r , the loop γ meets two distinct connected components of the level curve $S(r) = \{x \in X \mid \text{dist}(x, p) = r\} \subset X$, then we say that γ is a *loose loop*.

Proposition 7.5. *The loop homotopy class $[\gamma]$ of a loose loop γ splits off in $\pi_1(X, p)$, cf. Definition 2.2. In particular, $\text{FIG}(X) > 0$.*

In other words, there is a 2-complex Y and a map $X \rightarrow Y \vee S^1$ inducing an isomorphism $\varphi : \pi_1(X, p) \rightarrow \pi_1(Y, y_0) * \mathbb{Z}$ where the image of $[\gamma]$ is a generator of the free factor: $\varphi([\gamma]) = 1 \in \pi_1(S^1) = \mathbb{Z}$.

Proof. Consider the natural projection $h : X \rightarrow \text{Reeb}(f, r)$, where r is as in Definition 7.4 so that γ meets two different components of $S(r)$. By Lemma 7.3, the image $h(\gamma)$ of γ is homotopic to the simple loop $c = \alpha \cup \beta$ where α agrees with the unique embedded arc of T_r with the same endpoints as $\beta = \gamma \cap [f > r]$. Therefore, the loop homotopy class of $h(\gamma)$ splits off in $\pi_1(\text{Reeb}(f, r))$. This yields the desired result since the natural projection h induces an isomorphism of fundamental groups by Lemma 4.5. \square

8. LOOSE LOOPS vs AREA LOWER BOUNDS

The following proposition provides a lower bound for the length of level curves in a 2-complex X .

Proposition 8.1. *Let $L = \text{sys}\pi_1(X, p)$ be the pointed systole of a finite 2-complex. Let r be a real number satisfying*

$$L - \text{sys}\pi_1(X) < 2r < L.$$

Consider the level curve $S \subset X$ at distance r from $p \in X$. Let γ be a pointed systolic loop at p . If γ is not loose, then

$$\text{length } S \geq 2r - L + \text{sys}\pi_1(X). \quad (8.1)$$

Proof. By Lemma 7.3, the loop γ is formed of two distance-minimizing arcs which do not meet at distance r from p . Thus, the loop γ intersects S at exactly two points. Let $\gamma' = \gamma \cap B$ be the subarc of γ lying in $B = B(p, r)$.

Since γ meets exactly one connected component of S , there exists an embedded arc $\alpha \subset S$ connecting the endpoints of γ' . By Remark 7.2, every loop based at p and lying in $B(p, r)$ is contractible in X . Hence γ' and α are homotopic (keeping endpoints fixed), and the loop $\alpha \cup (\gamma \setminus \gamma')$ is homotopic to γ . Hence,

$$\text{length}(\alpha) + \text{length}(\gamma) - \text{length}(\gamma') \geq \text{sys}\pi_1(X). \quad (8.2)$$

Meanwhile, $\text{length}(\gamma) = L$ and $\text{length}(\gamma') = 2r$, proving the lower bound (8.1), since $\text{length}(S) \geq \text{length}(\alpha)$. \square

9. THE BOUND $\text{SR}(X) \leq 4$ FOR GRUSHKO UNFREE COMPLEXES

The length estimate of the previous section easily yields a uniform bound for the systolic ratio of any Grushko unfree 2-complex, *i.e.* a complex X with $\text{FIG}(X) = 0$.

Theorem 9.1. *Let X be a finite piecewise flat 2-complex. Let $\gamma \subset X$ be a pointed systolic loop at $x \in X$, and assume γ is not loose. For every real number r such that*

$$\text{sys}\pi_1(X, x) - \text{sys}\pi_1(X) \leq 2r \leq \text{sys}\pi_1(X, x), \quad (9.1)$$

the area of the ball $B(x, r)$ of radius r centered at x satisfies

$$\text{area } B(x, r) \geq \left(r - \frac{1}{2}(\text{sys}\pi_1(X, x) - \text{sys}\pi_1(X))\right)^2. \quad (9.2)$$

Proof of Theorem 9.1. Let $L = \text{sys}\pi_1(X, x)$. Denote by $S = S(r)$ and $B = B(r)$, respectively, the level curve and the ball of radius r satisfying $L - \text{sys}\pi_1(X) < 2r < L$, centered at $x \in X$.

The non-loose loop γ meets a single connected component of S . Let $\varepsilon = L - \text{sys}\pi_1(X)$. Now, Proposition 8.1 implies that

$$\text{length } S(r) \geq 2r - L + \text{sys}\pi_1(X) = 2r - \varepsilon. \quad (9.3)$$

Following [Gro83, 5.1.B] and [He82], we use the coarea formula, [Fe69, 3.2.11], [Cha93, p. 267] to obtain

$$\begin{aligned} \text{area } B(x, r) &\geq \int_{\frac{\varepsilon}{2}}^r \text{length } S(\rho) d\rho \\ &\geq \int_{\frac{\varepsilon}{2}}^r (2\rho - \varepsilon) d\rho \\ &\geq \left(r - \frac{\varepsilon}{2}\right)^2 \end{aligned}$$

for every r satisfying (9.1). \square

Corollary 9.2. *If X admits a systolic loop which is not loose, then $\text{SR}(X) \leq 4$.*

Proof. If γ is a systolic loop of X , then $\varepsilon = 0$. By hypothesis, γ is not loose, and we let r tend to $\frac{1}{2} \text{sys}\pi_1(X)$. \square

Corollary 9.3. *Let X be a Grushko unfree 2-complex. Then the area of every ball $B(x, r) \subset X$ of radius $r < \frac{1}{2} \text{sys}\pi_1(X)$ centered at a point x lying on a systolic loop of X , satisfies the lower bound*

$$\text{area } B(x, r) \geq r^2. \quad (9.4)$$

In particular, we have the bound $\text{SR}(X) \leq 4$.

Example 9.4. The Moore spaces M_n with $\pi_1(M_n) = \mathbb{Z}_n$ satisfy the bound $\text{SR}(X) \leq 4$ for all n .

10. NONCONTRACTIBLE FIBERS

Corollary 9.3 is limited by its hypothesis on the free index of Grushko. As a warm-up to the more exotic uses of the coarea formula in later sections, we first present a simple argument in Section 11, to obtain an explicit upper bound for $\text{SR}(X)$ dependent on the corank of the fundamental group. The topological result of this section will be used in Section 11.

Lemma 10.1. *Let $h : Y \rightarrow [0, 1]$ be a map that yields a trivial bundle over the half-interval $(0, 1]$. Let $F = h^{-1}(0)$ and assume that F is compact. Assume also that the pair (Y, F) is (homeomorphic to) a CW-pair. If every fiber of h is contractible in Y and path connected then Y is simply connected.*

Proof. Since (Y, F) is a CW-pair, there exists a neighborhood U of F such that F is a deformation retract of U . Since F is compact, there exists $\delta > 0$ such that $h^{-1}[0, \delta) \subset U$. Let $A = h^{-1}[0, \delta)$, $B = h^{-1}(0, 1]$ and $C = A \cap B$. The homomorphism $\pi_1(C) \rightarrow \pi_1(B)$ (induced by the inclusion) is an isomorphism, while the homomorphism $\pi_1(C) \rightarrow \pi_1(A)$ is trivial since every loop in C is homotopic in A to a loop in U , and therefore to a loop in F . So, by the Seifert – van Kampen Theorem, Y is simply connected. \square

Corollary 10.2. *Let T be a tree and $h : Y \rightarrow T$ be a map that yields a trivial bundle over the interior of each of edges of T . Assume also that for every $t \in T$ the pair $(Y, h^{-1}(t))$ is a CW-pair. Finally, assume that $h^{-1}(T')$ is compact for every finite subtree T' of T . If every fiber of h is contractible in Y and path connected then Y is simply connected.*

Proof. We first consider the case of $T = [0, 1]$. Then $Y = A \cup B$ with $A = h^{-1}[0, 1)$ and $B = h^{-1}(0, 1]$. Notice that all the fibers of h are compact. So, A and B are simply connected by Lemma 10.1. Now the result follows from the Seifert-van Kampen Theorem.

Now consider the case of a finite tree T . The result follows from the Seifert-van Kampen Theorem by an obvious induction.

In the general case, consider a loop $\varphi : S^1 \rightarrow Y$. The image of $h\varphi$, which is compact, is contained in some finite subtree T' of T . Now let $Y' = h^{-1}(T')$. By Step 2, the loop φ is contractible in Y' . Hence the result. \square

Proposition 10.3. *Let X be a connected 2-dimensional complex, and let $f : X \rightarrow \mathbb{R}$ be a semialgebraic function. If the group $\pi_1(X)$ is not free, then, for a suitable $\rho \in \mathbb{R}$, the preimage $f^{-1}(\rho)$ contains a noncontractible loop of X .*

Proof. Consider the Reeb graph $\text{Reeb}(f)$ of f and let $\widehat{f} : X \rightarrow \text{Reeb}(f)$ be the lifted function. Let $p : T \rightarrow \text{Reeb}(f)$ be the universal cover of $\text{Reeb}(f)$. Consider the pull-back diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ h \downarrow & & \downarrow \widehat{f} \\ T & \xrightarrow{p} & \text{Reeb}(f) \end{array}$$

Since $\pi_1(X)$ is not free, the group $\pi_1(Y)$ is non-trivial. Hence, by Corollary 10.2 and in view of Proposition 4.2, there exist $t \in T$ and a loop in $h^{-1}(t)$ that is homotopy non-trivial in Y . Therefore, there exists a loop in $\widehat{f}^{-1}(p(t))$ which is non-contractible in X . This loop is contained in $f^{-1}(\rho)$ with $\rho = f(\widehat{f}^{-1}(p(t)))$. \square

11. CORANK-DEPENDENT INEQUALITIES

In this section, we apply the coarea formula to obtain an explicit bound for $\text{SR}(X)$ dependent on the corank of the fundamental group. More refined applications of the coarea formula will be presented in later sections, to obtain a uniform bound.

We will say that a space X is *unfree* if its fundamental group is not free, and that X is of zero corank if $\pi_1(X)$ is. The first theorem of this section is a special case of Corollary 9.3 (but see Remark 11.3). We include its proof as preparation for the more general result of Theorem 11.5.

Theorem 11.1. *Every 2-dimensional, piecewise flat complex X of zero corank satisfies the bound $\text{SR}(X) \leq 4$.*

Proof of Theorem 11.1. Choose a point $p \in X$ and consider the function $f(x) = \text{dist}(p, x)$. The idea is to apply the coarea formula “inward”, *i.e.* toward the basepoint $p \in X$, so as to obtain the desired lower bound for the area, along the lines of [Gro83, p. 129, line 9].

In fact, we will use the coarea formula to show that the area of the 2-complex is bounded below by the area of a right triangle with base $\frac{1}{2} \text{sys}\pi_1(X)$ and altitude $\text{sys}\pi_1(X)$.

Consider the Reeb graph $\text{Reeb}(f)$ of f and note that the preimages of f are finite graphs and CW -subspaces of X , *cf.* Theorem 6.1 and Corollary 6.8. By Proposition 10.3, there exists a connected component

$$C_\rho \subset f^{-1}(\rho) \subset X,$$

containing a loop which is noncontractible in X .

Since X is a compact path metric space, it satisfies the Hopf–Rinow theorem, [Gro99, p. 9]. For each $x \in C_\rho$, consider a minimizing geodesic path joining p to x . Given $r < \rho$, let x^r be a point where this geodesic meets the level set $f^{-1}(r)$. Then

$$\text{dist}(x^r, x) = f(x) - f(x^r) = \rho - r.$$

The points x^r lie in the boundary of the connected component of the superlevel set $[f \geq r]$ containing C_ρ . Thus, the connected components C_1^r, \dots, C_k^r of this boundary coalesce forward. By Lemma 3.6, there is only one such component, denoted C^r (that is, $k = 1$ and $C^r = C_1^r$).

Lemma 11.2. *We have*

$$\rho \geq \frac{1}{2} \text{sys}\pi_1(X) \tag{11.1}$$

$$\text{length}(C^r) \geq \text{sys}\pi_1(X) - 2(\rho - r) \tag{11.2}$$

Proof. The first inequality follows from Remark 7.2. For the second inequality, we can assume that C^r does not contain a loop noncontractible in X , otherwise the result is obvious. Then, there is a pair of the points $x^r, y^r \in C^r$ with $x = y \in C_\rho \subset X$, and a path $[x^r, y^r] \subset C^r$ between them, such that the loop $[x^r, y^r] \cup [y^r, y] \cup [y, x^r]$ is noncontractible in X . Indeed, if all such loops were contractible, then the noncontractible loop contained in C_ρ would admit a continuous retraction to C^r , which is impossible. Therefore, $\text{length}([x^r, y^r]) \geq \text{sys}\pi_1(X) - 2(\rho - r)$. \square

Returning to the proof of Theorem 11.1, we use the coarea formula, [Fe69, 3.2.11], [Cha93, p. 267], and exploit (11.2) to write

$$\begin{aligned} \text{area}(X) &= \int_0^\infty \text{length}(f^{-1}(r)) \, dr \\ &\geq \int_0^\rho \text{length}(C^r) \, dr \\ &\geq \int_{\rho - \frac{1}{2} \text{sys}\pi_1(X)}^\rho \text{sys}\pi_1(X) - 2(\rho - r) \, dr \\ &\geq \frac{1}{4} \text{sys}\pi_1(X)^2, \end{aligned}$$

cf. [Gro83, p. 129, line 9], proving the theorem. \square

Remark 11.3. The estimate $\text{SR}(X) \leq 4$ can be improved in terms of the difference $\rho - \frac{1}{2} \text{sys}\pi_1(X)$, where ρ is the noncontractible level. Note that the method of proof of Corollary 9.3 does not allow for such an improvement.

Lemma 11.4. *Let $C_\rho \subset X$ be a level curve containing a loop which is noncontractible in X . Normalize the systole to the value 2. Then for every $n \in \mathbb{N}$, the level C_ρ admits a $\frac{1}{n}$ -separated set containing at least $n + 1$ elements.*

Proof. Choose an essential loop $\ell \subset C_\rho$. Let $A, B \in \ell$ be a pair of points realizing the diameter of ℓ . Thus $\text{dist}(A, B) \geq 1$. Let α be a connected component of the complement $\ell \setminus \{A, B\}$. Choose a maximal finite sequence of points $x_i \in \alpha$, $i = 0, 1, 2, \dots$ satisfying $\text{dist}(x_i, A) = \frac{i}{n} \leq 1$. By the triangle inequality, for all $i \neq j$, we have $\text{dist}(x_i, x_j) \geq \frac{1}{n}$. \square

Theorem 11.5. *Every unfree, 2-dimensional, piecewise flat complex X satisfies the bound $\text{SR}(X) \leq 16(\text{corank}(\pi) + 1)^2$.*

Proof. Let $n - 1$ be the corank of $\pi_1(X)$. Normalize the systole to the value 2. By Lemma 11.4, there exists an n -separated set $\{x_i\} \subset C_\rho$, such that $\text{dist}(x_i, x_j) \geq \frac{1}{n}$ when $i \neq j$. For each x_i , consider a minimizing geodesic path joining p to x_i . Given $r < \rho$, let x_i^r be a point where this geodesic meets the level set $f^{-1}(r)$. Then for each $i = 0, 1, \dots, n$, we have

$$\text{dist}(x_i^r, x_i) = f(x_i) - f(x_i^r) = \rho - r.$$

Note that the connected set C_ρ is contained in the superlevel set $[f \geq \rho]$. Thus the set $\{x_i\} \subset C_\rho$ coalesces forward. Hence by Lemma 3.6, the number of components of $f^{-1}(r)$ is at most n . Hence there is a pair of

points x_k^r, x_ℓ^r in a common connected component of $f^{-1}(r)$. Let C_r be such a component. By the triangle inequality,

$$\begin{aligned} \text{dist}(x_k^r, x_\ell^r) &\geq -\text{dist}(x_k^r, x_k) + \text{dist}(x_k, x_\ell) - \text{dist}(x_\ell, x_\ell^r) \\ &\geq \frac{1}{n} - 2(\rho - r). \end{aligned} \quad (11.3)$$

Therefore

$$\text{length}(C_r) \geq \text{diam } C_r \geq \text{dist}(x_k^r, x_\ell^r) \geq \frac{1}{n} - 2(\rho - r). \quad (11.4)$$

As before, we apply the coarea formula to obtain

$$\begin{aligned} \text{area}(X) &= \int_0^\infty \text{length}(f^{-1}(r)) \, dr \\ &\geq \int_0^\infty \text{length}(C_r) \, dr \\ &\geq \int_{\rho-(2n)^{-1}}^\rho \left(\frac{1}{n} - 2(\rho - r) \right) \, dr \\ &= \frac{1}{4}n^{-2}, \end{aligned}$$

cf. [Gro83, p. 129, line 9]. We conclude that $4n^2 \text{area}(X) \geq 1$, proving the theorem. \square

12. COMPARISON WITH LUSTERNIK-SCHNIRELMANN CATEGORY

Originally we were led to consider the systoles of 2-complexes in the context of the comparison with the Lusternik-Schnirelmann category cat_{LS} , *cf.* [KR04]. Since the latter equals 2 unless the group is free, *cf.* Theorem 12.1, the question arose whether an unfree 2-complex always satisfies a systolic inequality. Eventually we found an affirmative answer in [Gro83], *cf.* inequality (1.1). Thus, the systolic category cat_{sys} of X , defined in [KR04], coincides with $\text{cat}_{\text{LS}}(X)$ for every 2-complex X , *cf.* Theorem 12.6:

$$\text{cat}_{\text{sys}} = \text{cat}_{\text{LS}}. \quad (12.1)$$

Note that the two categories coincide, as well, for arbitrary closed 3-manifolds [KR05]. An open question is whether all Poincaré 3-complexes also satisfy equality (12.1). Recent examples due to J. Hillman [Hi04] show that the answer may not be easy to obtain.

The Lusternik-Schnirelmann category $\text{cat}_{\text{LS}}(X)$ of a 2-complex X can similarly be characterized in terms of its fundamental group. Thus, we have $\text{cat}_{\text{LS}} X = 1$ if $\pi_1(X)$ is free, and $\text{cat}_{\text{LS}} X = 2$ otherwise (see Theorem 12.1 for a detailed statement).

Theorem 12.1. *Let X be a 2-dimensional connected finite CW-space, and let $\pi \neq 0$ denote the fundamental group of X . The following conditions are equivalent:*

- (1) *the group π is free;*
- (2) *the space X is homotopy equivalent to a wedge of a finite number of circles and 2-spheres;*
- (3) *one has $\text{cat}_{\text{LS}} X = 1$;*
- (4) *for every group τ , every map $f : X \rightarrow K(\tau, 1)$ can be deformed into the 1-skeleton of $K(\tau, 1)$.*

Remark 12.2. Thus, a finite 2-dimensional complex satisfies a systolic inequality if and only if none of the four equivalent conditions of Theorem 12.1 holds.

Proof. We prove the following implications: (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1) \Rightarrow (4) \Rightarrow (1).

The implication (1) \Rightarrow (2) is proved by C. Wall [Wa65, Proposition 3.3]. We recall his argument for completeness. Let $\mathbb{Z}[\pi]$ denote the group ring of the group π that we assumed to be free. First, Wall proved that $\pi_2(X)$ is a finitely generated projective $\mathbb{Z}[\pi]$ -module. Now, by a theorem of H. Bass [Bass64], every finitely generated projective module over $\mathbb{Z}[\pi]$ is free. Consider a wedge Y of k circles and l spheres, where k is the number of free generators of π and l is a number of free generators of the $\mathbb{Z}[\pi]$ -module $\pi_2(X)$. Now we map Y to X by mapping circles to free generators of $\pi_1(X)$ and spheres to $\mathbb{Z}[\pi]$ -free generators of $\pi_2(X)$, and this map is a homotopy equivalence.

The implication (2) \Rightarrow (3) is obvious. The implication (3) \Rightarrow (1) is well known, cf. [CLOT03, Exercise 1.21]. The implication (1) \Rightarrow (4) holds for $\tau = \pi$ since $K(\pi, 1)$ is a wedge of circles if π is free. For general τ , we notice that every map $f : X \rightarrow K(\tau, 1)$ factors through $K(\pi, 1)$. To prove that (4) \Rightarrow (1), it suffices to prove the implication for a map f that induces an isomorphism of fundamental groups. Let K denote the 1-skeleton of $K(\pi, 1)$. Then the map $X \rightarrow K \subset K(\pi, 1)$ induces an isomorphism of fundamental groups. Thus, π is a subgroup of a free group $\pi_1(K)$. Hence π is free. \square

Proposition 12.3. *Let X be a connected, finite, n -dimensional CW-complex, where $n > 2$. Let $\pi_1(X) = \pi$ and assume that $H^k(\pi; G) = 0$ for all $k \geq n$ and all $\mathbb{Z}[\pi]$ -modules G . Then $\text{cat}_{\text{LS}} X < n$.*

Proof. By theorem of I. Berstein [Ber76], cf. [CLOT03, Theorem 2.51], we have $\text{cat}_{\text{LS}} X = n$ only if

$$u^n \neq 0 \text{ for some } u \in H^1(X; I(\pi)^{\otimes n}), \quad (12.2)$$

where $I(\pi)$ is the augmentation ideal of $\mathbb{Z}[\pi]$. We can obtain the classifying space $K(\pi, 1)$ by attaching k -cells with $k \geq 3$ to X . Now, the inclusion $X \rightarrow K(\pi, 1)$ induces an isomorphism $H^1(\pi; G) \rightarrow H^1(X; G)$ for any $\mathbb{Z}[\pi]$ -module G . Therefore $u^n = 0$. \square

Corollary 12.4. *Every free, connected, finite CW-complex X of dimension at least 2 satisfies $\text{cat}_{\text{LS}}(X) \leq n - 1$.*

Proof. Notice that $H^i(F; G) = 0$ for every free group F and $i > 1$. Now, for $n > 2$ the claim follows from Proposition 12.3, while for $\dim X = 2$ the claim follows from Theorem 12.1, item (3). \square

An invariant called *systolic category*, $\text{cat}_{\text{sys}}(X)$, of X was defined in [KR04]. It is a homotopy invariant, which, furthermore, coincides with the Lusternik-Schnirelmann category $\text{cat}_{\text{LS}}(X)$ for all 3-manifolds [KR05]. We now calculate it for 2-dimensional complexes. For technical reasons, we need to describe the 1-dimensional case first.

Proposition 12.5. *Every graph X satisfies $\text{cat}_{\text{sys}} X = \text{cat}_{\text{LS}} X$. The common value is 0 if X is contractible, and 1 otherwise.*

Proof. A graph Γ which contains nontrivial cycles, satisfies the obvious systolic inequality $\text{sys}\pi_1(\Gamma) \leq \text{length}(\Gamma)$. The 1-systole of a tree is infinite, being an infimum over an empty set. \square

Theorem 12.6. *Let X be a 2-dimensional complex that is not homotopy equivalent to a wedge of circles. Then we have $\text{cat}_{\text{sys}}(X) = \text{cat}_{\text{LS}}(X)$.*

Proof. Every free X is homotopy equivalent to a wedge of circles and 2-spheres, cf. [Wa65] and Theorem 12.1. We replace X by such a wedge W . For any $K > 0$, we can find a metric with $\text{sys}\pi_1(W)^2 \geq K \text{area}(W)$. Clearly, the 2-systole of W satisfies $\text{sys}_2(W) \leq \text{area}(W)$. Therefore $\text{cat}_{\text{sys}}(W) = \text{cat}_{\text{LS}}(W) = 1$ in this case.

The theorem now follows from (1.1) (or Theorem 12.1), combined with the homotopy invariance of both categories [KR05]. \square

13. A USEFUL AUXILIARY SPACE

The main goal of this section is the construction of a space Z obtained by cutting a loose loop of X along a graph, and folding the graph to a tree.

We assume that X is connected. Let $p \in X$ be a point on a systolic loop of X . Let f be the distance function from p . Let

$$T'_r \subset \text{Reeb}(f, r)$$

be the pruned Reeb tree of the ball of radius $r < \frac{1}{2} \text{sys}\pi_1(X)$, as in Section 5. Let $e \subset T'_r$ be an open non-root edge of length t_0 , isometrically identified with $(0, t_0)$.

Let $t \in (0, t_0) = e$. Note that the connected component $C_t \subset X$ of the level set of f corresponding to t is contractible in X by Remark 7.2. Let $\lambda = f(C_t) = \bar{f}(t) \in \mathbb{R}$ be the value taken by f on C_t .

Next, we define new spaces W and Z as follows. Consider the complement $X \setminus C_t$. Glue back two copies of $C = C_t$ to $X \setminus C$ in order to compactify the two open sets

$$U_- = \pi^{-1}(e) \cap [f < \lambda] \quad \text{and} \quad U_+ = \pi^{-1}(e) \cap [f > \lambda]$$

in the neighborhood of C , where $\pi : X \rightarrow \text{Reeb}(f, r)$ is the quotient map. Denote by W the resulting complex. If C is a tree, no further modifications need to be made.

Otherwise we introduce further identifications on W as follows. If the graph C contains a nontrivial embedded loop, we map this loop isometrically to a concentric circle in the complex plane. We then use complex conjugation, folding the circle to an interval, to introduce the same identification on both copies of C in W .

Inductively, we can eliminate all cycles of both copies of C in W while preserving the structure of a piecewise flat complex (up to subdivision). Denote by C_\pm the pair of trees thus obtained after elimination of all cycles. Here C_\pm is attached to U_\pm as before. By construction, the new space

$$Z = (X \setminus C) \cup C_- \cup C_+$$

has the same fundamental group as $\text{Reeb}(f, r) \setminus e$, cf. Proposition 5.3. The space Z is equipped with a piecewise flat structure induced from X , and $\text{area}(Z) = \text{area}(X)$.

Let $M(X, r)$ be the minimal model, cf. Section 5.

Lemma 13.1. *Let $e \subset M(X, r)$ be an open non-root edge. Then the natural map*

$$i : Z \rightarrow Y = M(X, r) \setminus e \tag{13.1}$$

sending $U_- \cup C_-$ and $U_+ \cup C_+$ to the endpoints of e induces an isomorphism of fundamental groups (on each connected component).

Proof. The proof is immediate from the construction. \square

Lemma 13.2. *There exists an unfree connected component Z_* of Z with $\text{FIG}(Z_*) \leq \text{FIG}(X) - 1$.*

Proof. The lemma follows from Lemma 13.1 and Proposition 5.5. \square

Remark 13.3. By Proposition 4.2, the fundamental group of the connected components C_{\pm} of C does not change as t runs over e . Thus, we can choose Z_* so that

- (1) either C_- is contained in Z_* for every $t \in e$;
- (2) or C_+ is contained in Z_* for every $t \in e$.

14. THE UNIFORM BOUND

Theorem 14.1. *Every finite unfree piecewise flat 2-complex X satisfies the bound*

$$\text{SR}(X) \leq 12.$$

Proof. By Corollary 9.3, the bound holds if $\text{FIG}(X) = 0$. Assume that the bound holds if $\text{FIG}(X) < n$. We will prove the bound for the case $\text{FIG}(X) = n$. We use the notation of Section 13. Let $e \subset T'_r$ be an open non-root edge of the pruned tree $T'_r \subset M(X, r)$ where $r < \frac{1}{2} \text{sys}\pi_1(X)$. Consider the space $Z_* = Z_*(e)$. By Lemma 13.2, $\text{FIG}(Z_*) < n$, while $\text{area}(Z_*) = \text{area}(X)$. Suppose there is a connected component $C = C_t \subset X$ of a level curve of the distance function, with $t \in e$, such that $\text{sys}\pi_1(X) \leq \text{sys}\pi_1(Z_*)$. Then,

$$\text{SR}(X) = \frac{\text{sys}\pi_1(X)^2}{\text{area}(X)} \leq \frac{\text{sys}\pi_1(Z_*)^2}{\text{area}(Z_*)} = \text{SR}(Z_*) \leq 12,$$

by the inductive hypothesis. Now let us assume that $\text{sys}\pi_1(X) \geq \text{sys}\pi_1(Z_*(e))$, for every non-root edge $e \subset T'_r$ and all $t \in e$. Then every systolic loop of Z_* must meet either C_- or C_+ . Indeed, if a systolic loop $\gamma \subset Z_*$ lies in

$$Z \setminus (C_- \cup C_+) = X \setminus C \subset X,$$

then

$$\text{sys}\pi_1(X) \leq \text{length}(\gamma) = \text{sys}\pi_1(Z_*).$$

Denote by X_τ the connected component of the level set of f corresponding to the point $\tau \in e = (0, t_0)$. We have the following lemma.

Lemma 14.2. *Let γ be a systolic loop of Z_* .*

- (1) *If γ meets C_- , then, for every $\tau \in (0, t)$,*

$$\text{length}(X_\tau) \geq 2t - 2\tau;$$

- (2) *If γ meets C_+ , then, for every $\tau \in (t, t_0)$,*

$$\text{length}(X_\tau) \geq 2\tau - 2t.$$

Proof. Suppose that γ meets C_- . By Lemma 13.1, the image $i(\gamma)$ must leave the corresponding endpoint of the edge $e \subset M(X, r)$. In particular, γ must meet the level X_τ . Moreover, since $i(\gamma)$ covers $(0, t)$ at least twice, γ meets X_τ at least twice.

Choose a subarc $\alpha \subset U_- \cup C_-$ of γ which meets C_- , and with its endpoints in X_τ . Note that $\text{length}(\alpha) \geq 2(t - \tau)$. Let $\beta \subset X_\tau$ be an embedded path joining the endpoints of α . By Lemma 13.1, the loop $(\gamma \setminus \alpha) \cup \beta$ is homotopic to the systolic loop γ in Z_* . Since it cannot be shorter than γ , we have

$$\text{length}(\beta) \geq \text{length}(\alpha).$$

Hence,

$$\text{length}(X_\tau) \geq 2(t - \tau),$$

which proves item (1). Item (2) is proved using similar arguments. \square

Lemma 14.3. *Assume no connected component (of a level curve of the distance function) $C \subset X$ can be found such that $\text{sys}\pi_1(X) \leq \text{sys}\pi_1(Z_*)$. Then we have*

$$\text{area}(\pi^{-1}(e)) \geq \frac{1}{2} \text{length}(e)^2.$$

Proof. Let A_- and A_+ denote the set of values t for which there exists a systolic loop of Z_* that meets C_- and C_+ , respectively. Then A_- and A_+ are relatively closed subsets of $(0, t_0)$. Hence, if $A_+ \cap A_- = \emptyset$ then at least one of the sets A_- , A_+ is the full interval $(0, t_0)$.

Suppose $A_+ \cap A_-$ is nonempty. Then for some $t \in [0, t_0]$, there exist two systolic loops γ_- and γ_+ of Z_* such that γ_\pm meets C_\pm . Now items (1) and (2) of Lemma 14.2 provide a lower bound for the length of X_τ for every $\tau \in (0, t_0)$. Integrating this lower bound from 0 to t_0 leads, through the coarea formula, to the lower bound

$$\text{area}(\pi^{-1}(e)) \geq \frac{1}{2} t_0^2.$$

Suppose $A_- = (0, t_0)$. Then for all $t \in e$, there exists a systolic loop of Z_* that meets C_- . Integrating the lower bound provided by item (1) of Lemma 14.2 over $[0, t_0]$, we obtain, through the coarea formula, the bound

$$\text{area}(\pi^{-1}(e)) \geq t_0^2.$$

The same bound holds if $A_+ = (0, t_0)$. \square

The pruned tree T'_r decomposes into the root edge e_p of length $\ell \geq 0$ (possibly zero) based at $\pi(p)$, and two trees Γ and Γ' , of height $r - \ell$, attached to e_p at the other endpoint:

$$T'_r = e_p \cup \Gamma \cup \Gamma'.$$

The bound of Lemma 14.3 can be improved for the root edge e_p to

$$\text{area}(\pi^{-1}(e_p)) \geq \ell^2.$$

Indeed, for every $\tau \in [0, \ell]$, the level curve $S_\tau = f^{-1}(\tau)$ is connected. Furthermore, every systolic loop through p meets S_τ , cf. Lemma 7.3. The lower bound of Proposition 8.1 on the length of S_τ leads, via the coarea formula, to the desired bound.

Thus, each edge of Γ and Γ' makes a contribution of one half of its length squared to the total area of X , while the root edge e_p makes a contribution equal to the square of its length. Hence,

$$\begin{aligned} \text{area}(X) &\geq \ell^2 + \frac{1}{2}E(\Gamma) + \frac{1}{2}E(\Gamma') \\ &\geq \ell^2 + \frac{1}{2}(r - \ell)^2 \\ &\geq \frac{1}{3}r^2, \end{aligned}$$

where the second inequality comes from Proposition 5.1. The proof of Theorem 14.1 is then completed by letting r tend to $\frac{1}{2} \text{sys}\pi_1(X)$. \square

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