HYPERELLIPTIC SURFACES ARE LOEWNER

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Abstract. We prove that C. Loewner's inequality for the torus is satisfied by all hyperelliptic surfaces X, as well. We first construct the Loewner loops on the (mildly singular) companion tori, locally isometric to X away from Weierstrass points. The loops are then transplanted to X, and surgered to obtain a Loewner loop on X.

Contents

1. Introduction

The systole, $\mathrm{sys}\pi_1(g)$, of a compact non simply connected Riemannian manifold (X, g) is the least length of a noncontractible loop $\gamma \subset X$:

$$\mathrm{sys}\pi_1(g) = \min_{[\gamma]\neq 0 \in \pi_1(X)} \mathrm{length}(\gamma). \tag{1.1}$$

This notion of systole is apparently unrelated to the systolic arrays of [Ku78]. We will be concerned with comparing this Riemannian invariant to the total area of the metric, as in Loewner's inequality (2.2). Higher dimensional optimal generalisations of Loewner's inequality are studied in [BK03, BK04, IK04, BCIK2]. The defining text for this material is [Gr99], with more details in [Gr83, Gr96]. See also the recent survey [CK03], as well as [KR04].

We will review the relevant literature in Section 2, state the main theorem in Section 3, and prove it in Sections 3 and 4.

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2. Inequalities of Loewner and Pu

The Hermite constant, denoted γ_n, can be defined as the optimal constant in the inequality

$$\mathrm{sys}\pi_1(\mathbb{T}^n)^2 \leq \gamma_n \,\mathrm{vol}(\mathbb{T}^n)^{2/n}, \tag{2.1}$$

over the class of all *flat* tori $\mathbb{T}^n$. Here γ_n is asymptotically linear in n, *cf.* [LLS90, pp. 334, 337]. The precise value is known for small n, *e.g.* $\gamma_2 = \frac{2}{\sqrt{3}}$, $\gamma_3 = 2^{\frac{1}{3}}, \ldots$. An inequality of type (2.1) remains valid in the class of *all* metrics, but with a nonsharp constant on the order of n^{2n} [Gr83].

Around 1949, Charles Loewner proved the first systolic inequality, *cf.* [Pu52]. He showed that every Riemannian metric g on the torus $\mathbb{T}^2$ satisfies the inequality

$$\mathrm{sys}\pi_1(g)^2 \leq \gamma_2 \,\mathrm{area}(g), \tag{2.2}$$

while a metric satisfying the boundary case of equality in (2.2) is necessarily flat, and is homothetic to the quotient of $\mathbb{C}$ by the lattice spanned by the cube roots of unity.

M. Gromov [Gr83, p. 50] (*cf.* [Ko87, Theorem 4, part (1)]) proved a general estimate which implies that if Σ_s is a compact orientable surface of genus s with a Riemannian metric, then

$$\frac{\mathrm{sys}\pi_1(\Sigma_s)^2}{\mathrm{area}(\Sigma_s)} < \frac{64}{4\sqrt{s}+27}. \tag{2.3}$$

Thus the optimal systolic ratio tends to 0 as the genus increases without bound.

Remark 2.1. It was shown in [Gr83] (see also [KS04]) that asymptotically the optimal systolic ratio behaves as $C\frac{(\log s)^2}{s}$.

Another helpful estimate is found in [Gr83, Corollary 5.2.B]. Namely, every aspherical compact surface (Σ, g) admits a metric ball $B = B_p\left(\frac{1}{2}\,\mathrm{sys}\pi_1(g)\right) \subset \Sigma$ of radius $\frac{1}{2}\,\mathrm{sys}\pi_1(g)$, which satisfies

$$\mathrm{sys}\pi_1(g)^2 \leq \frac{4}{3}\,\mathrm{area}(B). \tag{2.4}$$

Furthermore, whenever a point $x \in \Sigma$ lies on a two-sided loop which is minimizing in its free homotopy class, the metric ball $B_x(r) \subset \Sigma$ of radius $r \leq \frac{1}{2}\,\mathrm{sys}\pi_1(g)$ satisfies the estimate

$$2r^2 < \mathrm{area}\left(B_x(r)\right). \tag{2.5}$$

Question 2.2. It follows from Gromov's estimate (2.3) that orientable surfaces Σ_s satisfy Loewner's inequality (2.2) if $s > 50$. This is improved in [KS04] to $s \geq 20$. Can the genus assumption be removed altogether?

A similar question for Pu's inequality [Pu52] has an affirmative answer. The generalisation is immediate from Gromov's inequality (2.4). Namely, every surface (X, g) which is not a 2-sphere satisfies

$$\mathrm{sys}\pi_1(g)^2 \leq \frac{\pi}{2}\,\mathrm{area}(g), \tag{2.6}$$

where the boundary case of equality in (2.6) is attained precisely when, on the one hand, the surface X is a real projective plane, and on the other, the metric g is of constant Gaussian curvature.

3. Hyperelliptic surfaces and Loewner surfaces

Recall that a Riemann surface X is called hyperelliptic if it admits a degree 2 meromorphic function, *cf.* [Mi95, p. 60-61] as well as [Mi95, Proposition 4.11, p. 92]. The associated ramified double cover

$$Q : X \to S^2$$

over the sphere S^2 is conformal away from the $2s+2$ ramification points, where s is the genus. Its deck transformation $J : X \to X$ is called the *hyperellitic involution*. Such a holomorphic involution, if it exists, is uniquely characterized by the property of having precisely $2s + 2$ fixed points. The fixed points of J are called Weierstrass points. Their images under Q will be referred to as *ramification points*.

We provide the following partial answer in the direction of Question 2.2. We will say that a surface is *Loewner* if it satisfies inequality (2.2). We prove that every hyperelliptic surface is Loewner. More precisely, we prove the following.

Theorem 3.1. *Let (X, g) be an orientable surface, where the metric g belongs to a hyperelliptic conformal class. Then (X, g) is Loewner.*

Since every genus 2 surface is hyperelliptic [FK92, Proposition III.7.2, page 100], we obtain the following corollary.

Corollary 3.2. *Every metric on the genus 2 surface is Loewner.*

Note that this is the first improvement, known to the authors, on Gromov's 3/4 bound (2.4) in over 20 years, for surfaces of genus below 50, *cf.* Question 2.2. No extremal metric has as yet been conjectured in this genus, but it cannot be flat with conical singularities [Sa04]. The best available lower bound for the optimal systolic ratio in genus 2 can be found in [CK03, section 2.2]. For genus $s \geq 3$, the theorem follows from the following proposition, *cf.* Remark 2.1 and [Kon03].

Proposition 3.3. *Every hyperelliptic surface Σ_s of genus s satisfies the estimate*

$$\frac{\mathrm{sys}\pi_1(\Sigma_s)^2}{\mathrm{area}(\Sigma_s)} \leq \frac{4}{s+1}.$$

Proof. Averaging the metric by the hyperelliptic involution $J : X \to X$ improves the systolic ratio, *cf.* [BCIK1]. Thus we may assume that the metric g is invariant under J. The distance between any pair of Weierstrass points is then at least $\frac{1}{2}\,\mathrm{sys}\pi_1(\Sigma_s)$. Thus, the disks of radius $R = \frac{1}{4}\,\mathrm{sys}\pi_1(\Sigma_s)$ centered at the Weierstrass points are disjoint. M. Gromov (and J. Hebda before him) proved that if the metric is extremal for the systolic inequality, the area of such a disk is at least

$$2R^2 = \frac{1}{8}\,\mathrm{sys}\pi_1(\Sigma_s)^2,$$

cf. (2.5). The existence of an extremal metric was proved in [Gr83]. The latter result is still true in the class of hyperelliptic surfaces, proving the proposition. □

4. Proof of Theorem 3.1 in genus 2

Let X be a genus 2 surface. Recall that X has a hyperelliptic involution J with 6 Weierstrass points.

The idea of the proof of Theorem 3.1 in genus 2 is to apply Loewner's inequality to certain *companion tori* of X, and to surger the resulting loops so as to obtain a Loewner loop on X. We may need the following lemma.

Lemma 4.1. *Let $\mathbb{T}^2$ be a torus endowed with a metric invariant under its hyperelliptic involution J_{T^2}, with conical singularities with total angle less than 2π around each. Then the image of a systolic loop of $\mathbb{T}^2$ in S^2 under the hyperelliptic projection is a simple loop.*

Proof. Let $\gamma \subset \mathbb{T}^2$ be a systolic loop. Since J_{T^2} induces minus the identity homomorphism on $\pi_1(\mathbb{T}^2)$, the loops γ and $-J_{T^2}(\gamma)$ are homotopic. In the hypotheses of our lemma, two homotopic systolic loops are necessarily disjoint. Hence the image of γ on S^2 is simple. □

Definition 4.2. A *companion torus* $\mathbb{T}(a,b,c,d)$ of X is a torus whose ramification locus $\{a,b,c,d\} \subset S^2$ is a subset of the ramification locus of X.

As in the proof of Proposition 3.3, we can assume that the metric on X is invariant under J (see [BCIK1]). Therefore g descends to a metric g_0, of half the area, on S^2. Let's choose four of the 6 ramification points, say $a, b, c, d \in S^2$. Choose a double cover with ramification locus $\{a, b, c, d\}$, denoted

$$\mathbb{T}^2(a, b, c, d) \to S^2.$$

Pulling back the metric g_0 to the torus $\mathbb{T}^2(a, b, c, d)$, we obtain a metric of the same area as the surface X itself. This metric on the torus is smooth away from the two remaining points, where it has a conical singularity with total angle π around each. Consider a Loewner loop

$$LL \subset \mathbb{T}^2(a, b, c, d)$$

on this torus, *e.g.* a systolic loop realizing (2.2). Let L be the projection of LL to S^2. The simple loop $L \subset S^2$ separates the four points a, b, c, d into two pairs, say a, b on one side and c, d, on the other. If the lift of L to X closes up, we obtain a Loewner loop on X and the theorem is proved. Thus, we may assume that the following three equivalent conditions are satisfied:

(1) the lift of L to X does not close up;
(2) the inverse image $Q^{-1}(L) \subset X$ is connected;
(3) the loop L surrounds precisely 3 ramification points of Q.

Definition 4.3. The simple loop L partitions the sphere into two hemispheres, H_+ and H_-, with $a, b, e \in H_+$ and $c, d, f \in H_-$ where a, b, c, d, e, f are the 6 ramification points of Q.

Using a pair of companion tori, we will construct two loops on the sphere, defining two distinct partitions of the ramification locus into a pair of triples. The basic example to think of is the case of a centrally symmetric 6-tuple of points, *e.g.*, corresponding to the curve

$$y^2 = x^5 - x,$$

and a pair of generic great circles, such that each of the four digons contains at least one ramification point. We now construct a companion torus $\mathbb{T}(a, b, e, f)$.

Consider a Loewner loop $LL' \subset \mathbb{T}^2(a, b, e, f)$, and its projection $L' \subset S^2$. If its lift to X closes up, the theorem is proved. Therefore assume that the lift of L' to X does not close up, *i.e.* L' surrounds exactly 3 ramification points. Now L' separates the four points a, b, e, f into two pairs. Hence it defines a different splitting of the six points into two triples. The connected components of $L' \cap H_+$ form a nonempty finite collection of disjoint nonselfintersecting arcs α.

Each arc α divides H_+ into a pair of regions homeomorphic to disks. Such regions are partially ordered by inclusion. A minimal element for the partial order is necessarily a digon. Such a digon must contain at least one ramification point of Q (otherwise exchange the two sides of the digon between the loops L and L', so as to decrease the total number of intersections, or else argue as in Lemma 4.1). It is clear that there are at least two such digons in H_+.

Hence one of them, denoted $D \subset H_+$, must contain precisely one of the 3 ramification points of H_+. We now exchange the two sides of D between the loops L, L', obtaining two new loops M, M'. Each of the new loops surrounds a nonzero even number of ramification points. Since

$$\text{length}(M) + \text{length}(M') = \text{length}(L) + \text{length}(L'),$$

one of the loops M or M' is no longer than Loewner. Moreover, its lift to X closes up, producing a Loewner loop on X, as required.

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