

# ENTROPY OF SYSTOLICALLY EXTREMAL SURFACES AND ASYMPTOTIC BOUNDS

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ABSTRACT. We find an upper bound for the entropy of a systolically extremal surface, in terms of its systole. We combine the upper bound with A. Katok's lower bound in terms of the volume, to obtain a simpler alternative proof of M. Gromov's asymptotic estimate for the optimal systolic ratio of surfaces of large genus. Furthermore, we improve the multiplicative constant in Gromov's theorem. We show that every surface of genus at least 20 is Loewner. Finally, we relate, in higher dimension, the isoembolic ratio to the minimal entropy.

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## 1. ENTROPY AND SYSTOLE

We show that the volume entropy  $h$  (together with A. Katok's optimal inequality for  $h$ ) is the "right" intermediary in a transparent proof of M. Gromov's asymptotic bound for the systolic ratio of surfaces of large genus.

In this section, we review the definitions of minimal entropy and systole. The main results of this paper are presented in Section 2.

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Let  $(M, \mathcal{G})$  be an  $n$ -dimensional closed Riemannian manifold. Denote by  $(\tilde{M}, \tilde{\mathcal{G}})$  the universal Riemannian cover of  $(M, \mathcal{G})$ . Choose a point  $\tilde{x}_0 \in \tilde{M}$ . The volume entropy (or asymptotic volume)  $h(M, \mathcal{G})$  of  $(M, \mathcal{G})$  is defined as follows:

$$h(M, \mathcal{G}) = \lim_{R \rightarrow +\infty} \frac{\log(\text{vol}_{\tilde{\mathcal{G}}} B(\tilde{x}_0, R))}{R}, \quad (1.1)$$

where  $\text{vol}_{\tilde{\mathcal{G}}} B(\tilde{x}_0, R)$  is the volume of the ball of radius  $R$  centered at  $\tilde{x}_0 \in \tilde{M}$ . Since  $M$  is compact, the limit in (1.1) exists and does not depend on the point  $\tilde{x}_0 \in \tilde{M}$  [Ma79]. This asymptotic invariant describes the exponential growth rate of the volume in the universal cover.

Define the minimal volume entropy of  $M$  as the infimum of the volume entropy of metrics of unit volume on  $M$ , or equivalently

$$\text{MinEnt}(M) = \inf_{\mathcal{G}} h(M, \mathcal{G}) \cdot \text{vol}(M, \mathcal{G})^{\frac{1}{n}} \quad (1.2)$$

where  $\mathcal{G}$  runs over the space of all metrics on  $M$ .

The classical result of A. Katok [Kato83] states that every metric  $\mathcal{G}$  on a closed surface  $M$  with negative Euler characteristic  $\chi(M)$  satisfies the optimal inequality

$$h(\mathcal{G})^2 \geq \frac{2\pi|\chi(M)|}{\text{area}(\mathcal{G})}. \quad (1.3)$$

Inequality (1.3) also holds for  $\text{hom ent}(\mathcal{G})$  [Kato83], as well as the topological entropy, since the volume entropy bounds from below the topological entropy (see [Ma79]).

The systole of a nonsimply connected closed Riemannian  $n$ -manifold  $(M, \mathcal{G})$  is defined as

$$\text{sys}\pi_1(M, \mathcal{G}) = \inf_{\gamma} \{\text{length}(\gamma) \mid \gamma \text{ a noncontractible loop of } M\}.$$

We define the systolic ratio SR of  $(M, \mathcal{G})$  as

$$\text{SR}(M, \mathcal{G}) = \frac{\text{sys}\pi_1(M, \mathcal{G})^n}{\text{vol}(M, \mathcal{G})}, \quad (1.4)$$

and the optimal systolic ratio of  $M$  as

$$\text{SR}(M) = \sup_{\mathcal{G}} \text{SR}(M, \mathcal{G}), \quad (1.5)$$

where  $\mathcal{G}$  runs over the space of all metrics on  $M$ .

C. Loewner proved the first systolic inequality. Namely, he showed that every metric  $\mathcal{G}$  on the torus  $\mathbb{T}^2$  satisfies the inequality

$$\text{sys}\pi_1(\mathcal{G})^2 \leq \frac{2}{\sqrt{3}} \text{area}(\mathcal{G}), \quad (1.6)$$

with equality if and only if the metric  $\mathcal{G}$  is flat, while the group of deck transformations of  $(\mathbb{T}^2, \mathcal{G})$  is a lattice homothetic to the lattice spanned by the cube roots of unity in  $\mathbb{C}$ . Thus, we have

$$\text{SR}(\mathbb{T}^2) = \frac{2}{\sqrt{3}}.$$

See [CK03] for a recent account on systolic inequalities. Asymptotic bounds for higher systoles are studied in [Katz03]. Systolic geometry has recently seen a period of rapid growth. The papers [Ba03, Ba04, BB04] study the 1-systolic constant of higher-dimensional manifolds. Optimal generalisations of the Loewner inequality are studied in [Am04, BK03, BK04, BCIK04, IK04, KL04]. A generalisation of Pu's inequality is studied in [BCIK05]. A general framework for systolic geometry in a topological context is proposed in [KR04]. The most recent developments may be found in [Sa04, KS04, KS05, Sa05, Sa06].

In Section 2, we present the main results of the paper. In Section 3, we describe the basic estimate, based on a maximal packing argument combined with area lower bounds for systolically optimal surfaces. In Section 4, we combine the basic estimate with Katok's inequality to prove one of our main results, Theorem 2.2. In Section 5, we prove that every surface of genus at least 20 is Loewner. The last section contains higher dimensional generalisations and a proof of Theorem 2.3.

## 2. THE RESULTS

We will relate the minimal entropy of a closed surface to its optimal systolic ratio. Namely, we find an upper bound for the entropy of a systolically extremal surface, in terms of its systole.

**Proposition 2.1.** *Every extremal metric  $\mathcal{G}$  on a surface  $M$  satisfies*

$$h(\mathcal{G}) \leq -\frac{1}{\beta \text{sys}\pi_1(\mathcal{G})} \log(2\alpha^2 \text{SR}(M)), \quad (2.1)$$

whenever  $\alpha, \beta > 0$  and  $4\alpha + \beta < \frac{1}{2}$ .

We combine this upper bound with A. Katok's optimal lower bound in terms of the volume, see (1.3), to obtain a simpler alternative proof of M. Gromov's asymptotic estimate for the optimal systolic ratio of surfaces of large genus. Furthermore, we improve the multiplicative constant in Gromov's theorem (see Section 4 for further details).

**Theorem 2.2.** *A surface  $M_g$  of genus  $g$  satisfies the bound*

$$\mathrm{SR}(M_g) \leq \frac{\log^2 g}{\pi g} (1 + o(1)) \text{ when } g \rightarrow \infty. \quad (2.2)$$

An alternative approach is taken by F. Balacheff [Bal04], but his constant is not as good as Gromov's, see Section 4. Furthermore, the approach of the present paper lends itself to higher-dimensional generalisations [Sa06].

As an application, we show that every metric on a surface of genus at least 20 satisfies the Loewner inequality (1.6) for the torus. This improves the best earlier estimate of 50.

Finally, we relate, in higher dimension, the optimal isoembolic ratio to the minimal entropy (1.2). Recall that the optimal isoembolic ratio of an  $n$ -manifold  $M$  is defined as

$$\mathrm{Emb}(M) = \inf_{\mathcal{G}} \frac{\mathrm{vol}(M, \mathcal{G})}{\mathrm{inj}(M, \mathcal{G})^n} \quad (2.3)$$

where  $\mathcal{G}$  runs over the space of all metrics on  $M$  and  $\mathrm{inj}(M, \mathcal{G})$  is the injectivity radius of  $(M, \mathcal{G})$ . We show the following (see Section 6 for further details).

**Theorem 2.3.** *There exists a positive constant  $\lambda_n$  such that every  $n$ -manifold  $M$  satisfies*

$$\mathrm{Emb}(M) \geq \lambda_n \frac{\mathrm{MinEnt}(M)^n}{\log^n(1 + \mathrm{MinEnt}(M))}. \quad (2.4)$$

### 3. BASIC ESTIMATE

In Lemma 3.6 we recall the following well-known fact: the volume entropy agrees with the exponential growth rate of orbits under the action of the fundamental group in the universal cover, sometimes called the *critical exponent*.

We will also need the following estimate, *cf.* (3.5). M. Gromov showed in [Gr83] that every aspherical closed surface  $M$  satisfies the inequality

$$\mathrm{SR}(M) \leq \frac{4}{3}. \quad (3.1)$$

He also showed that every nonsimply connected closed surface admits an extremal metric in a suitable generalized sense, namely a metric  $\mathcal{G}_{\mathrm{ex}}$  with optimal systolic ratio  $\mathrm{SR}(M, \mathcal{G}_{\mathrm{ex}}) = \mathrm{SR}(M)$ . Furthermore, the disks  $D(x, r)$  of radius  $r \leq \frac{1}{2} \mathrm{sys}\pi_1(\mathcal{G}_{\mathrm{ex}})$  of extremal surfaces satisfy the bound

$$\mathrm{area} D(x, r) \geq 2r^2. \quad (3.2)$$

We will use arguments developed in [Kato83, p. 357] to prove Proposition 3.1 below. Related arguments have been exploited in [BM02, Ge04].

**Proposition 3.1.** *Every extremal metric  $\mathcal{G}$  on a surface  $M$  satisfies*

$$h(\mathcal{G}) \leq -\frac{1}{\beta \operatorname{sys}\pi_1(\mathcal{G})} \log(2\alpha^2 \operatorname{SR}(M)), \quad (3.3)$$

whenever  $\alpha, \beta > 0$  and  $4\alpha + \beta < \frac{1}{2}$ .

**Remark 3.2.** As shown in [Sa05], the volume entropy of surfaces with unit systole is bounded from above by a constant which does not depend on the metric. However, the constant found is not as good as in inequality (3.3).

*Proof of Proposition 3.1.* The idea is to bound from above the number of homotopy classes of based loops in  $M$ , by deforming such a loop into a fixed subgraph in  $M$ . The subgraph can be allowed to be as “coarse” as the size of the systole of  $M$ . More precisely, let  $x_0 \in M$  be a fixed basepoint. Consider a maximal system of disjoint disks

$$D_i = D(x_i, R) \subset M \quad (3.4)$$

of radius  $R = \alpha \operatorname{sys}\pi_1(\mathcal{G})$  and centers  $x_i$  with  $i \in I$ , including  $x_0$ . Since the metric  $\mathcal{G}$  is assumed extremal, inequality (3.2) implies

$$\operatorname{area} D_i \geq 2\alpha^2 \operatorname{sys}\pi_1(\mathcal{G})^2 \quad \forall i \in I. \quad (3.5)$$

Therefore, this system admits at most  $\frac{\operatorname{area}(\mathcal{G})}{2\alpha^2 \operatorname{sys}\pi_1(\mathcal{G})^2}$  disks. Thus,

$$|I| \leq (2\alpha^2 \operatorname{SR}(M))^{-1}. \quad (3.6)$$

Let  $c : [0, T] \rightarrow M$  be a geodesic loop of length  $T$  based at  $x_0$ . Let

$$m = \left\lceil \frac{T}{\beta \operatorname{sys}\pi_1(\mathcal{G})} \right\rceil \quad (3.7)$$

be the integer part. The point  $p_0 = x_0$ , together with the points

$$p_k = c(k\beta \operatorname{sys}\pi_1(\mathcal{G})), \quad k = 1, \dots, m$$

and the point  $p_{m+1} = x_0$ , partition the loop  $c$  into  $m + 1$  segments of length at most  $\beta \operatorname{sys}\pi_1(\mathcal{G})$ . Since the system of disks  $D_i$  is maximal, the disks of radius  $2R = 2\alpha \operatorname{sys}\pi_1(\mathcal{G})$  centered at  $x_i$  cover  $M$ . Therefore, for every  $p_k$ , a nearest point  $q_k$  among the centers  $x_i$ , is at distance at most  $2R$  from  $p_k$ . Consider the loop

$$\alpha_k = c_k \cup [p_{k+1}, q_{k+1}] \cup [q_{k+1}, q_k] \cup [q_k, p_k],$$

where  $c_k$  is the arc of  $c$  joining  $p_k$  to  $p_{k+1}$ , while  $[x, y]$  denotes a minimizing path joining  $x$  to  $y$ . Then

$$\text{length}(\alpha_k) \leq 2(4\alpha + \beta) \text{sys}\pi_1(\mathcal{G}) < \text{sys}\pi_1(\mathcal{G}),$$

by our hypothesis on  $\alpha, \beta$ . Thus the loop  $\alpha_k$  is contractible. The same is true for the loops  $c_0 \cup [p_1, q_1] \cup [q_1, x_0]$  and  $c_m \cup [x_m, q_m] \cup [q_m, p_m]$ .

Therefore, the geodesic loop  $c$  is homotopic to a piecewise geodesic loop

$$c' = (x_0, q_1, \dots, q_m, x_0). \quad (3.8)$$

Note that the minimizing path from  $p_k$  to  $q_k$  may not be unique, but we choose one, being careful that the same choice is used on “both sides”, *i.e.* for both loops  $\alpha_{k-1}$  and  $\alpha_k$ .

Thus, two nonhomotopic closed geodesic loops  $c_1$  and  $c_2$  based at  $x_0$ , give rise to two distinct loops  $c'_1$  and  $c'_2$  as in (3.8). In constructing the loops  $c'$ , we always choose the same minimizing path between a given pair of points  $q_k$ . Thus, the number  $P'(T)$  of homotopy classes which can be represented by loops of length  $T$  based at  $x_0$  satisfies

$$\begin{aligned} P'(T) &\leq |I|^m \\ &\leq |I|^{\frac{T}{\beta \text{sys}\pi_1(\mathcal{G})}} \\ &\leq (2\alpha^2 \text{SR}(M))^{-\frac{T}{\beta \text{sys}\pi_1(\mathcal{G})}}, \end{aligned} \quad (3.9)$$

and the proposition now follows from Lemma 3.6.  $\square$

**Remark 3.3.** Instead of relying on the existence of systolically extremal surfaces, we could have exploited instead  $\epsilon$ -regular surfaces satisfying (3.2) for  $r \geq \epsilon$ , whose existence is considerably easier to establish [Gr83, 5.6.C’]. Since the choice of  $\alpha$  in (4.5) entails exploiting packings by arbitrarily small disks in (3.4), we have to be careful to choose  $\epsilon < \alpha \text{sys}\pi_1(M)$ .

**Remark 3.4.** A similar technique can be used to bound the number of free homotopy classes of loops in  $X$ . In higher dimensions, the latter bound seems to be less useful, therefore we chose to bound the number of based loops.

Combining inequalities (1.3) and (3.3), we obtain the following corollary.

**Corollary 3.5.** *Let  $M_g$  be a closed orientable surface of genus  $g$ . Whenever  $4\alpha + \beta < \frac{1}{2}$ , we have*

$$\frac{\log^2(2\alpha^2 \text{SR}(M_g))}{\text{SR}(M_g)} \geq 4\pi\beta^2(g-1). \quad (3.10)$$

We conclude this section by recalling the following well-known fact, cf. [KH95, Proposition 9.6.6, p. 374].

**Lemma 3.6.** *Let  $(M, \mathcal{G})$  be a closed Riemannian manifold. Then,*

$$h(M, \mathcal{G}) = \lim_{T \rightarrow +\infty} \frac{\log(P'(T))}{T} \quad (3.11)$$

where  $P'(T)$  is the number of homotopy classes of loops based at some fixed point  $x_0$  which can be represented by loops of length at most  $T$ .

*Proof.* Fix  $x_0$  in  $M$  and a lift  $\tilde{x}_0$  in the universal cover  $\tilde{M}$ . The group

$$\Gamma := \pi_1(M, x_0)$$

acts on  $\tilde{M}$  by isometries. The orbit of  $\tilde{x}_0$  by  $\Gamma$  is denoted  $\Gamma.\tilde{x}_0$ . Consider a fundamental domain  $\Delta$  for the action of  $\Gamma$ , containing  $\tilde{x}_0$ . Denote by  $D$  the diameter of  $\Delta$ . The cardinal of  $\Gamma.\tilde{x}_0 \cap B(\tilde{x}_0, R)$  is bounded from above by the number of translated fundamental domains  $\gamma.\Delta$ , where  $\gamma \in \Gamma$ , contained in  $B(\tilde{x}_0, R + D)$ . It is also bounded from below by the number of translated fundamental domains  $\gamma.\Delta$  contained in  $B(\tilde{x}_0, R)$ . Therefore, we have

$$\frac{\text{vol}(B(\tilde{x}_0, R))}{\text{vol}(M, \mathcal{G})} \leq \text{card}(\Gamma.\tilde{x}_0 \cap B(\tilde{x}_0, R)) \leq \frac{\text{vol}(B(\tilde{x}_0, R + D))}{\text{vol}(M, \mathcal{G})}. \quad (3.12)$$

Take the log of these terms and divide by  $R$ . The lower bound becomes

$$\begin{aligned} \frac{1}{R} \log \left( \frac{\text{vol}(B(R))}{\text{vol}(\mathcal{G})} \right) &= \\ &= \frac{1}{R} \log(\text{vol}(B(R))) - \frac{1}{R} \log(\text{vol}(\mathcal{G})), \end{aligned} \quad (3.13)$$

and the upper bound becomes

$$\begin{aligned} \frac{1}{R} \log \left( \frac{\text{vol}(B(R + D))}{\text{vol}(\mathcal{G})} \right) &= \\ &= \frac{R + D}{R} \frac{1}{R + D} \log(\text{vol}(B(R + D))) - \frac{1}{R} \log(\text{vol}(\mathcal{G})). \end{aligned} \quad (3.14)$$

Hence both bounds tend to  $h(\mathcal{G})$  when  $R$  goes to infinity. Therefore,

$$h(\mathcal{G}) = \lim_{R \rightarrow +\infty} \frac{1}{R} \log(\text{card}(\Gamma.\tilde{x}_0 \cap B(\tilde{x}_0, R))). \quad (3.15)$$

This yields the result since  $P'(R) = \text{card}(\Gamma.\tilde{x}_0 \cap B(\tilde{x}_0, R))$ .  $\square$

## 4. ASYMPTOTIC BEHAVIOR OF SYSTOLIC RATIO FOR LARGE GENUS

We now consider the asymptotic behavior of the optimal systolic ratio of surfaces. M. Gromov [Gr83, p. 74] established a bound for the ratio  $\text{SR}(M_g)$  by using a technique known as “diffusion of chains”. The multiplicative constant  $\frac{1}{\pi}$  in (4.2) below improves the constant 4 which could be obtained from the techniques in [Gr83]. F. Balacheff [Bal04] found another proof of a similar inequality, by combining the works of S. Kodani [Ko87], and B. Bollobás and E. Szemerédi [BoS02] on systolic inequalities of graphs, obtaining a multiplicative constant of  $\frac{8}{3(\log 2)^2}$ .

**Theorem 4.1.** *Given a real number  $\lambda < \pi$ , every surface  $M_g$  of genus  $g$  satisfies*

$$\text{SR}(M_g) \leq \frac{\log^2 g}{\lambda g} \quad (4.1)$$

for  $g$  large enough. Thus,

$$\text{SR}(M_g) \lesssim \frac{\log^2 g}{\pi g} \quad \text{as } g \rightarrow \infty. \quad (4.2)$$

**Remark 4.2.** A lower bound for  $\text{SR}(M_g)$  was found by P. Buser and P. Sarnak [BS94]. Namely, they construct hyperbolic surfaces  $(M_g, \mathcal{G}_g)$  of arbitrarily large genus  $g$  obtained as congruence coverings of an arithmetic Riemann surface, such that

$$\text{SR}(M_g, \mathcal{G}_g) \gtrsim \frac{4}{9\pi} \frac{\log^2 g}{g}. \quad (4.3)$$

Therefore, we have

$$\frac{4}{9\pi} \leq \limsup_{g \rightarrow \infty} \text{SR}(M_g) \frac{g}{\log^2 g} \leq \frac{1}{\pi}. \quad (4.4)$$

*Proof of Theorem 4.1.* Let  $\mathcal{G}$  be an extremal metric on  $M_g$ . We now apply Corollary 3.5. Note that  $\text{SR}(M_g, \mathcal{G})$  tends to zero as the genus  $g$  becomes unbounded, cf. (5.4). For a right choice of  $\alpha$  and  $\beta$ , inequality (3.10) leads to the asymptotic implicit upper bound (4.6) on  $\text{SR}(M_g)$  below. Indeed, given  $\lambda < \lambda_+ < \pi$ , we set

$$\beta = \sqrt{\frac{\lambda_+}{4\pi}} < \frac{1}{2},$$

and choose

$$\alpha < \frac{1}{4} \left( \frac{1}{2} - \beta \right). \quad (4.5)$$

Inequality (3.10) implies

$$\frac{\log^2(\text{SR}(M_g))}{\text{SR}(M_g)} \geq \lambda g \quad (4.6)$$

if  $g$  is large enough. Now we want to invert this relation in order to get an asymptotic upper bound on  $\text{SR}(M_g)$ . Let  $\rho = \text{SR}(M_g)^{-\frac{1}{2}}$  and  $\delta = \frac{1}{2}\sqrt{\lambda g}$ . Then inequality (4.6) yields the following estimate:

$$\begin{aligned} \rho \log \rho &\geq \delta \\ &\geq \delta - \frac{\delta \log \log \delta}{\log \delta} \\ &= \frac{\delta}{\log \delta} \log \left( \frac{\delta}{\log \delta} \right). \end{aligned}$$

Since the function  $x \log x$  is increasing for  $x$  large enough, we deduce that  $\rho \geq \frac{\delta}{\log \delta}$ , and the latter inequality translates back into (4.1).  $\square$

## 5. WHEN IS A SURFACE LOEWNER?

We now extend the classical Loewner inequality (1.6) on the torus to surfaces of higher genus. We will say that a surface  $M$  is *Loewner* if  $\text{SR}(M) \leq \frac{2}{\sqrt{3}}$ .

**Theorem 5.1.** *Every surface of genus at least 20 is Loewner.*

*Proof.* Let  $\mathcal{G}$  be an extremal metric on  $M$ . If  $\beta = \frac{1}{2} - 4\alpha$ , inequality (3.10) yields

$$\frac{\log^2(2\alpha^2 \text{SR}(M))}{\text{SR}(M)} \geq 4\pi \left(\frac{1}{2} - 4\alpha\right)^2 (g-1) \quad (5.1)$$

for every  $\alpha \leq \frac{1}{8}$ .

Suppose now that  $\text{SR}(M) > \frac{2}{\sqrt{3}}$ . Since  $\text{SR}(M) \leq \frac{4}{3}$  by (3.1), we have  $2\alpha^2 \text{SR}(M) \leq 1$  for every  $\alpha \leq \frac{1}{8}$ . Therefore,

$$\frac{\sqrt{3}}{2} \log^2 \left( \frac{\sqrt{3}}{4\alpha^2} \right) \geq 4\pi \left(\frac{1}{2} - 4\alpha\right)^2 (g-1) \quad (5.2)$$

for every  $\alpha \leq \frac{1}{8}$ . Hence,

$$\min_{0 < \alpha \leq \frac{1}{8}} \frac{\sqrt{3}}{8\pi} \left( \frac{\log \left( \frac{\sqrt{3}}{4\alpha^2} \right)}{\frac{1}{2} - 4\alpha} \right)^2 \geq g-1 \quad (5.3)$$

For  $\alpha = .031$ , the expression to minimize is about 18.201. Thus,  $g \leq 19$ . Therefore, every surface of genus greater or equal to 20 is Loewner.  $\square$

Note that M. Gromov [Gr83, p. 50] (*cf.* [Ko87, Theorem 4, part (1)]) proved a general estimate which implies that

$$\text{SR}(M_g) < \frac{64}{4\sqrt{g} + 27}. \quad (5.4)$$

It follows from Gromov's estimate (5.4) that orientable surfaces satisfy Loewner inequality (1.6) if the genus  $g$  is bigger than 50. Our theorem brings this bound down to 20. It has recently been shown [KS04] that the genus 2 surface is Loewner. The remaining open cases are therefore  $g = 3, \dots, 19$ .

**Remark 5.2.** Let  $\alpha = \frac{1}{30}$ . Instead of taking an arbitrary packing, we start with a systolic loop, and choose 15 disjoint disks centered at equally spaced points of the loop. By an averaging procedure [Gr83], we can get the combined area of these disks to be, not  $15(2r^2)$ , but rather  $15(3r^2)$ . Now we complete these 15 disks to a maximal packing, and argue as before. The only difference is that we have a better lower bound for the area of the disks.

If  $\alpha = \frac{1}{30}$ , then there are at most 382 balls in the packing of an unloewner surface. Hence the area of the packing is at least  $2.039\alpha^2$ , instead of  $2\alpha^2$ . Calculating the resulting expression in (5.3) yields about 18.12, which is better than 18.20 but not enough to dip under 18.

## 6. ISOEMBOLIC RATIO, MINIMAL ENTROPY, AND SIMPLICIAL NORM

Analogous estimates can be proved in higher dimension. However, the results we obtain are weaker than in the previous sections, in the absence of similar results on the existence of systolically extremal metrics.

M. Berger proved in [Be80] that the isoembolic ratio  $\text{Emb}(M, \mathcal{G})$  of every Riemannian  $n$ -manifold  $(M, \mathcal{G})$ , defined as

$$\text{Emb}(M, \mathcal{G}) = \frac{\text{vol}(M, \mathcal{G})}{\text{inj}(M, \mathcal{G})^n}, \quad (6.1)$$

satisfies

$$\text{Emb}(M, \mathcal{G}) \geq \text{Emb}(S^n, \text{can}) \quad (6.2)$$

with equality if and only if  $M$  is a sphere, while  $\mathcal{G}$  has constant sectional curvature. In particular,  $\text{Emb}(M) \geq \text{Emb}(S^n)$ , where  $\text{Emb}(M)$  is the optimal isoembolic ratio defined in (2.3).

C. Croke showed in [Cr80] that for every Riemannian manifold  $(M, \mathcal{G})$ ,

$$\text{vol } B(R) \geq c_n R^n \quad (6.3)$$

for every  $R \leq \frac{1}{2} \text{inj}(\mathcal{G})$ , where  $\text{vol} B(R)$  is the volume of a ball of radius  $R$  in  $M$ . Furthermore, he showed in [Cr88] that

$$\text{Emb}(M) \geq C_n \text{ballcat}(M), \quad (6.4)$$

where  $\text{ballcat}(M)$  is the minimum number  $n$  such that  $M$  can be covered by  $n + 1$  topological balls. If  $M$  is a closed  $n$ -dimensional manifold, then  $\text{ballcat}(M) \leq n$  (see [CT03, p. 77]). O. Durumeric showed in [Du01] that  $\text{Emb}(M)$  can be bounded from below in terms of the minimal number of generators of  $\pi_1(M)$ . Note also that T. Yamaguchi [Ya88] proved that, for every real  $C$ , the class of closed  $n$ -dimensional manifolds  $M$  satisfying  $\text{Emb}(M) \leq C$  contains finitely many homotopy types. Furthermore, I. Babenko [Ba93, Theorem 1.2, p. 6] showed that the minimal volume entropy is a homotopy invariant of  $M$ .

The following analogue of Proposition 3.1 holds.

**Proposition 6.1.** *Every Riemannian  $n$ -manifold  $(M, \mathcal{G})$  satisfies*

$$h(\mathcal{G}) \leq \frac{1}{\beta \text{inj}(\mathcal{G})} \log \left( \frac{\text{Emb}(\mathcal{G})}{c_n \alpha^n} \right) \quad (6.5)$$

whenever  $\alpha, \beta > 0$  and  $4\alpha + \beta < \frac{1}{2}$ .

The proof is identical to that of Proposition 3.1, with  $c_n$  from (6.3) replacing the coefficient 2 in (3.5), (3.6) and (3.9), and  $n$  replacing the dimension 2. A very similar result also appears in [Du01].

Arguing as in Theorem 4.1, we obtain the following theorem (compare with (6.4)).

**Theorem 6.2.** *There exists a positive constant  $\lambda_n$  such that every  $n$ -manifold  $M$  satisfies*

$$\text{Emb}(M) \geq \lambda_n \frac{\text{MinEnt}(M)^n}{\log^n(1 + \text{MinEnt}(M))} \quad (6.6)$$

G. Besson, G. Courtois, and S. Gallot [BCG95] proved that the minimal entropy of a closed negatively curved locally symmetric  $n$ -manifold  $(M, \mathcal{G}_0)$  satisfies

$$\text{MinEnt}(M)^n = \text{vol}(\mathcal{G}_0) \cdot h(\mathcal{G}_0)^n. \quad (6.7)$$

This theorem generalizes (1.3).

**Remark 6.3.** The relation (6.7) is a sharp version, for negatively curved locally symmetric manifolds, of the following general result of

M. Gromov (see [Gr81, p. 37]). Every  $n$ -manifold  $M$  with simplicial volume  $\|M\|$  satisfies

$$\text{MinEnt}(M)^n \geq C_n \|M\|. \quad (6.8)$$

Therefore, inequalities (6.6) and (6.8) show that there exists a positive constant  $\lambda'_n$  such that

$$\text{Emb}(M) \geq \lambda'_n \frac{\|M\|}{\log^n(1 + \|M\|)}. \quad (6.9)$$

M. Gromov proved in [Gr83, p. 74] that there exists a positive constant  $\lambda''_n$  such that

$$\text{SR}(M) \leq \lambda''_n \frac{\log^n(1 + \|M\|)}{\|M\|}. \quad (6.10)$$

Thus, inequality (6.9) appears as a particular case of (6.10).

**Question 6.4.** Are there manifolds with large minimal entropy and small simplicial volume? Such manifolds (if they exist) would provide examples where (6.6) yields a better estimate than (6.10).

More generally, in the presence of a lower bound for volumes of balls, our argument yields the following result.

**Proposition 6.5.** *Let  $\mathcal{G}$  be a metric on a closed  $n$ -manifold  $M$  such that for some  $c > 0$ , every ball of radius  $r$  with  $0 < r < \frac{1}{2} \text{sys}\pi_1(\mathcal{G})$  satisfies*

$$\text{area } B(r) \geq cr^n. \quad (6.11)$$

*Then, we have*

$$h(\mathcal{G}) \leq -\frac{1}{\beta \text{sys}\pi_1(\mathcal{G})} \log(c\alpha^n \text{SR}(M, \mathcal{G})), \quad (6.12)$$

*whenever  $\alpha, \beta > 0$  and  $4\alpha + \beta < \frac{1}{2}$ . Therefore, there exists a positive constant  $\lambda = \lambda(n, c)$  such that*

$$\text{SR}(M, \mathcal{G}) \geq \lambda(n, c) \frac{\text{MinEnt}(M)^n}{\log^n(1 + \text{MinEnt}(M))}. \quad (6.13)$$

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