Chemins confinés dans un quadrant

Kilian Raschel

Thèse effectuée sous la direction d'Irina Kurkova

Université Pierre et Marie Curie

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1. Counting the numbers of walks confined to a quadrant
   - Introduction
   - Results

2. Random walks killed at the boundary of a quadrant
   - Exact results
   - Asymptotic results: non-zero drift
   - Asymptotic results: zero drift
1. Counting the numbers of walks confined to a quadrant
   - Introduction
   - Results

2. Random walks killed at the boundary of a quadrant
   - Exact results
   - Asymptotic results: non-zero drift
   - Asymptotic results: zero drift
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   - Results

2. Random walks killed at the boundary of a quadrant
   - Exact results
   - Asymptotic results: non-zero drift
   - Asymptotic results: zero drift
Counting the numbers of paths confined to some regions of the plane

Let $\mathcal{S}$ be a step set

and let $q(i, j, k)$ be the number of paths:

- with increments in $\mathcal{S}$;
Counting the numbers of paths confined to some regions of the plane

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Let $S$ be a step set

and let $q(i, j, k)$ be the number of paths:

- with increments in $S$;
- confined to a given region of the plane;
- starting from $(i_0, j_0)$ and ending in $(i, j)$ at step $k$. 
Example

\[ q(0, 0, 0) = 1 \]
Example

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\[ q(0, 0, 1) = 0 \]
Example

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\[ q(0, 0, 1) = 0 \]

\[ q(0, 0, 2) = ? \]
Counting the numbers of walks confined to a quadrant
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Counting the numbers of walks confined to a quadrant
Random walks killed at the boundary of a quadrant

Example

\[ q(0, 0, 0) = 1 \]

\[ q(0, 0, 1) = 0 \]

\[ q(0, 0, 2) = 4 \]
Example

$q(0, 0, 0) = 1$

$q(0, 0, 1) = 0$

$q(0, 0, 2) = 4$

$q(0, 0, 3) = 0$
Counting the numbers of paths confined to some regions of the plane

Let $S$ be a step set

\[
\begin{array}{c}
\hline
& \times \times \\
\times & \times \\
\hline
\end{array}
\]

and let $q(i, j, k)$ be the number of paths:

- with increments in $S$;
- confined to a given region of the plane;
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Let

\[
Q(x, y, z) = \sum_{i, j, k} q(i, j, k) x^i y^j z^k.
\]
Counting the numbers of paths confined to some regions of the plane

Let $S$ be a step set

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$$Q(x, y, z) = \sum_{i, j, k} q(i, j, k) x^i y^j z^k.$$ 

- Explicit expression of $Q(x, y, z)$;
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- with increments in $S$;
- confined to a given region of the plane;
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Let

$$Q(x, y, z) = \sum_{i,j,k} q(i, j, k)x^i y^j z^k.$$ 

- Explicit expression of $Q(x, y, z)$;
- Dependence of $Q(x, y, z)$ on $S$, e.g. its nature.
Rational functions $Q(x, y, z)$

$$Q(x, y, z) = \frac{P(x, y, z)}{R(x, y, z)}.$$
Rational functions $Q(x, y, z)$

$$Q(x, y, z) = \frac{P(x, y, z)}{R(x, y, z)}.$$

Algebraic functions $Q(x, y, z)$

$$\sum_{0 \leq k < \infty} P_k(x, y, z)Q(x, y, z)^k = 0.$$
Rational functions \( Q(x, y, z) \)

\[
Q(x, y, z) = \frac{P(x, y, z)}{R(x, y, z)}.
\]

Algebraic functions \( Q(x, y, z) \)

\[
\sum_{0 \leq k < \infty} P_k(x, y, z) Q(x, y, z)^k = 0.
\]

Holonomic functions \( Q(x, y, z) \)

\[
\sum_{i,j,k \geq 0} P_{i,j,k}(x, y, z) \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial y^j} \frac{\partial^k}{\partial z^k} Q(x, y, z) = 0.
\]
### Rational functions $Q(x, y, z)$

$$Q(x, y, z) = \frac{P(x, y, z)}{R(x, y, z)}.$$ 

### Algebraic functions $Q(x, y, z)$

$$\sum_{0 \leq k < \infty} P_k(x, y, z)Q(x, y, z)^k = 0.$$ 

### Holonomic functions $Q(x, y, z)$

$$\sum_{i,j,k \geq 0} P_{i,j,k}(x, y, z) \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial y^j} \frac{\partial^k}{\partial z^k} Q(x, y, z) = 0.$$ 

### Hierarchy

$$\{\text{Rational functions}\} \subset \{\text{Algebraic functions}\} \subset \{\text{Holonomic functions}\}.$$
Numbers of paths in the *plane*

$Q(x, y, z)$ is *rational*.
Numbers of paths in the *plane*

$Q(x, y, z)$ is *rational*.

Numbers of paths in a *half plane*

$Q(x, y, z)$ is *algebraic*.
### Numbers of paths in the *plane*

$Q(x, y, z)$ is **rational**.

### Numbers of paths in a *half plane*

$Q(x, y, z)$ is **algebraic**.

### Numbers of paths in a *quarter plane*

Is $Q(x, y, z)$ **holonomic**?
### Numbers of paths in the *plane*

$Q(x, y, z)$ is **rational**.

### Numbers of paths in a *half plane*

$Q(x, y, z)$ is **algebraic**.

### Numbers of paths in a *quarter plane*

Is $Q(x, y, z)$ **holonomic**?

![Kreweras' walks](image)

Kreweras’ walks **algebraic**
Numbers of paths in the \textit{plane}

\[ Q(x, y, z) \text{ is rational.} \]

Numbers of paths in a \textit{half plane}

\[ Q(x, y, z) \text{ is algebraic.} \]

Numbers of paths in a \textit{quarter plane}

Is \[ Q(x, y, z) \text{ holonomic?} \]

\begin{itemize}
  \item Kreweras’ walks algebraic
  \item Knight’s walks non-holonomic
\end{itemize}
Walks with small steps in the quarter plane $\mathbb{Z}_+^2$ starting at $(0,0)$

$S \subset \{-1, 0, 1\}^2 \setminus \{(0,0)\}$

There are $2^8$ such problems.
Walks with small steps in the quarter plane $\mathbb{Z}_+^2$ starting at $(0, 0)$

$S \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$

There are $2^8$ such problems.

Some of these $2^8$ models are:

- trivial;
Walks with small steps in the quarter plane $\mathbb{Z}^2_+$ starting at $(0, 0)$

$$S \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$$

There are $2^8$ such problems.

Some of these $2^8$ models are:

- trivial;
- simple;
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Some of these $2^8$ models are:

- trivial;
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- intrinsic to a half plane;
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There are $2^8$ such problems.

Some of these $2^8$ models are:

- trivial;
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- intrinsic to a half plane;
- symmetrical.
Walks with small steps in the quarter plane $\mathbb{Z}_+^2$ starting at $(0,0)$

$S \subset \{-1, 0, 1\}^2 \setminus \{(0,0)\}$

There are $2^8$ such problems.

Some of these $2^8$ models are:

- trivial;
- simple;
- intrinsic to a half plane;
- symmetrical.

Finally, it remains 79 problems!
The functional equation

The kernel:

\[
K(x, y, z) = xyz \sum_{(k, l) \in S} \frac{x^k y^l}{z^{k+l}}.
\]

A functional equation for \( Q(x, y, z) \):

\[
K(x, y, z)Q(x, y, z) = K(x, 0, z)Q(x, 0, z) + K(0, y, z)Q(0, y, z) - K(0, 0, z)Q(0, 0, z) - xy.
\]
The functional equation

The kernel:

\[ K(x, y, z) = xyz \left[ \sum_{(k,l) \in S} x^k y^l - \frac{1}{z} \right] . \]

A functional equation for \( Q(x, y, z) \):

\[ K(x, y, z)Q(x, y, z) = K(x, 0, z)Q(x, 0, z) + K(0, y, z)Q(0, y, z) - K(0, 0, z)Q(0, 0, z) - xy. \]
The group of the walk: an example

The jump generating function $x + \frac{1}{x} + y + \frac{1}{y}$
The group of the walk: an example

The jump generating function $x + \frac{1}{x} + y + \frac{1}{y}$ is left unchanged by

$$\xi(x, y) = \left( x, \frac{1}{y} \right), \quad \eta(x, y) = \left( \frac{1}{x}, y \right)$$
The group of the walk: an example

The jump generating function $x + \frac{1}{x} + y + \frac{1}{y}$ is left unchanged by

$$\xi(x, y) = \left( x, \frac{1}{y} \right), \quad \eta(x, y) = \left( \frac{1}{x}, y \right)$$

and thus by any element of the group

$$\langle \xi, \eta \rangle = \left\{ (x, y), \left( x, \frac{1}{y} \right), \left( \frac{1}{x}, \frac{1}{y} \right), \left( \frac{1}{x}, y \right) \right\}.$$
The group of the walk: the general case

\[ \sum_{(k,l) \in S} x^k y^l = \begin{cases} B_{-1}(y)x^{-1} + B_0(y) + B_{+1}(y)x^1 \\ A_{-1}(x)y^{-1} + A_0(x) + A_{+1}(x)y^1 \end{cases} \]
The group of the walk: the general case

\[\sum_{(k,l)\in S} x^k y^l = \begin{cases} B_{-1}(y)x^{-1} + B_0(y) + B_{+1}(y)x^{+1} \\ A_{-1}(x)y^{-1} + A_0(x) + A_{+1}(x)y^{+1} \end{cases} \]

is left unchanged by

\[\xi(x, y) = \left( x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y} \right), \quad \eta(x, y) = \left( \frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y \right)\]
The group of the walk: the general case

\[
\sum_{(k,l) \in S} x^k y^l = \begin{cases} 
B_{-1}(y)x^{-1} + B_0(y) + B_{+1}(y)x^{+1} \\
A_{-1}(x)y^{-1} + A_0(x) + A_{+1}(x)y^{+1}
\end{cases}
\]

is left unchanged by

\[
\xi(x, y) = \left( x, \frac{A_{-1}(x) 1}{A_{+1}(x) y} \right), \quad \eta(x, y) = \left( \frac{B_{-1}(y) 1}{B_{+1}(y)} x, y \right)
\]

and thus by any element of the group

\[\langle \xi, \eta \rangle.\]
Examples

Order 4;
Examples

Order 4;  
order 6;
Examples

Order 4;  
order 6;  
order 8;
Examples

Order 4;  
order 6;  
order 8;  
order ∞.
Classification of the 79 problems

79 problems
Classification of the 79 problems

79 problems

23 admit a finite group

56 have an infinite group
Classification of the 79 problems

- 79 problems
- 23 admit a finite group
- 56 have an infinite group
- 22 are solved
- 1 is not solved
Classification of the 79 problems

79 problems

- 23 admit a finite group
- 56 have an infinite group

- 22 are solved
- 1 is not solved
- 2 are solved
- 54 are not solved
The functional equation

The kernel:

\[ K(x, y, z) = xyz \left[ \sum_{(k, l) \in S} x^k y^l - 1/z \right]. \]

A functional equation for \( Q(x, y, z) \):

\[
K(x, y, z)Q(x, y, z) = K(x, 0, z)Q(x, 0, z) + K(0, y, z)Q(0, y, z) - K(0, 0, z)Q(0, 0, z) - xy.
\]
An example of resolution in the case of a finite group

The jump function $\sum_{(k,l) \in S} x^k y^l = x + \frac{1}{x} + y + \frac{1}{y}$ is left unchanged by the 4 elements $(x, y), \left(\frac{1}{x}, y\right), \left(\frac{1}{x}, \frac{1}{y}\right), (x, \frac{1}{y})$. 
An example of resolution in the case of a finite group

The jump function \( \sum_{(k,l) \in S} x^k y^l = x + \frac{1}{x} + y + \frac{1}{y} \) is left unchanged by the 4 elements \((x, y), (\frac{1}{x}, y), (\frac{1}{x}, \frac{1}{y}), (x, \frac{1}{y})\).

\[
\left[ \sum_{(k,l) \in S} x^k y^l - \frac{1}{z} \right] x y z Q(x, y, z) = z x Q(x, 0, z) + z y Q(0, y, z) - x y
\]

Making the alternating sum, we obtain:

\[
\sum_{(k,l) \in S} x^k y^l - \frac{1}{z}
\]
An example of resolution in the case of a finite group

The jump function $\sum_{(k,l) \in S} x^k y^l = x + \frac{1}{x} + y + \frac{1}{y}$ is left unchanged by the 4 elements $(x, y), \left(\frac{1}{x}, y\right), \left(\frac{1}{x}, \frac{1}{y}\right), (x, \frac{1}{y})$.

\[
\sum_{(k,l) \in S} x^k y^l - 1/z \left(\begin{array}{c}
x \\
y \\
z \\
\end{array}\right) = z x Q(x, 0, z) + z y Q(0, y, z) - x y
\]

\[
\frac{1}{x} y z Q\left(\frac{1}{x}, y, z\right) = z \frac{1}{x} Q\left(\frac{1}{x}, 0, z\right) + z y Q(0, y, z) - \frac{1}{x} y
\]

Making the alternating sum, we obtain:

\[
\sum_{(k,l) \in S} x^k y^l - 1/z \left(\begin{array}{c}
x \\
y \\
z \\
\end{array}\right)
\]
An example of resolution in the case of a finite group

The jump function $\sum_{(k,l) \in S} x^k y^l = x + \frac{1}{x} + y + \frac{1}{y}$ is left unchanged by the 4 elements $(x, y), (\frac{1}{x}, y), (\frac{1}{x}, \frac{1}{y}), (x, \frac{1}{y})$.

$\left[ \sum_{(k,l) \in S} x^k y^l - 1/z \right] x y z Q(x, y, z) = z x Q(x, 0, z) + z y Q(0, y, z) - x y$

$\left[ \sum_{(k,l) \in S} x^k y^l - 1/z \right] \frac{1}{x} y z Q(\frac{1}{x}, y, z) = z \frac{1}{x} Q(\frac{1}{x}, 0, z) + z y Q(0, y, z) - \frac{1}{x} y$

$\left[ \sum_{(k,l) \in S} x^k y^l - 1/z \right] \frac{1}{x} \frac{1}{y} z Q(\frac{1}{x}, \frac{1}{y}, z) = z \frac{1}{x} Q(\frac{1}{x}, 0, z) + z \frac{1}{y} Q(0, \frac{1}{y}, z) - \frac{1}{x} \frac{1}{y}$

$\sum_{(k,l) \in S} x^k y^l - 1/z = 0$ if $(x, y, z) = (1, 0, 0)$ or $(0, 0, 1)$ or $(1, 1, 1)$ or $(0, 1, 1)$.

Making the alternating sum, we obtain:

$\sum_{(k,l) \in S} x^k y^l = \frac{1}{1-x} + \frac{1}{1-y} \frac{1}{1-\frac{1}{x}} + \frac{1}{1-\frac{1}{y}} \frac{1}{1-\frac{1}{x}} + \frac{1}{1-\frac{1}{x}} \frac{1}{1-\frac{1}{y}}$
An example of resolution in the case of a finite group

The jump function \( \sum_{(k,l) \in S} x^k y^l = x + \frac{1}{x} + y + \frac{1}{y} \) is left unchanged by the 4 elements \((x, y), (\frac{1}{x}, y), (\frac{1}{x}, \frac{1}{y}), (x, \frac{1}{y})\).

\[
\sum_{(k,l) \in S} x^k y^l - 1/z \cdot xyzQ(x, y, z) = zx Q(x, 0, z) + zy Q(0, y, z) - xy \\
\sum_{(k,l) \in S} x^k y^l - 1/z \cdot \frac{1}{x} yzQ(\frac{1}{x}, y, z) = z\frac{1}{x} Q(\frac{1}{x}, 0, z) + zy Q(0, y, z) - \frac{1}{x} y \\
\sum_{(k,l) \in S} x^k y^l - 1/z \cdot \frac{1}{x} yzQ(\frac{1}{x}, \frac{1}{y}, z) = z\frac{1}{x} Q(\frac{1}{x}, 0, z) + z\frac{1}{y} Q(0, \frac{1}{y}, z) - \frac{1}{x} \frac{1}{y} \\
\sum_{(k,l) \in S} x^k y^l - 1/z \cdot \frac{1}{y} xzQ(x, \frac{1}{y}, z) = zx Q(x, 0, z) + z\frac{1}{y} Q(0, \frac{1}{y}, z) - x \frac{1}{y}
\]

Making the alternating sum, we obtain:

\[
\sum_{\theta \in \langle \xi, \eta \rangle} (-1)^\theta \cdot \theta \theta \theta = -xy + 1 + x y - 1/x + x 1/y + x 1 + y - 1/y + 1/x - 1/z Q(x, 0, z) - 1/y Q(0, \frac{1}{y}, z) - x 1/y - x 1/x
\]
An example of resolution in the case of a finite group

The jump function \( \sum_{(k,l) \in S} x^k y^l = x + \frac{1}{x} + y + \frac{1}{y} \) is left unchanged by the 4 elements \((x, y), (\frac{1}{x}, y), (\frac{1}{x}, \frac{1}{y}), (x, \frac{1}{y})\).

\[
\begin{align*}
\sum_{(k,l) \in S} x^k y^l - 1/z &\cdot x y z Q(x, y, z) = z x Q(x, 0, z) + z y Q(0, y, z) - x y \\
\sum_{(k,l) \in S} x^k y^l - 1/z &\cdot \frac{1}{x} y z Q(\frac{1}{x}, y, z) = z \frac{1}{x} Q(\frac{1}{x}, 0, z) + z y Q(0, y, z) - \frac{1}{x} y \\
\sum_{(k,l) \in S} x^k y^l - 1/z &\cdot \frac{1}{x} y z Q(\frac{1}{x}, \frac{1}{y}, z) = z \frac{1}{x} Q(\frac{1}{x}, 0, z) + z \frac{1}{y} Q(0, \frac{1}{y}, z) - \frac{1}{x} \frac{1}{y} \\
\sum_{(k,l) \in S} x^k y^l - 1/z &\cdot \frac{1}{y} z Q(x, \frac{1}{y}, z) = z x Q(x, 0, z) + z \frac{1}{y} Q(0, \frac{1}{y}, z) - x \frac{1}{y}
\end{align*}
\]

Making the alternating sum, we obtain:

\[
\sum_{\theta \in \langle \xi, \eta \rangle} (-1)^\theta \theta [xyz Q(x, y, z)] = \frac{-xy + \frac{1}{x} y - \frac{1}{y} \frac{1}{y} + x \frac{1}{y}}{\sum_{(k,l) \in S} x^k y^l - 1/z}
\]
An example of resolution in the case of a finite group

The jump function $\sum_{(k,l) \in S} x^k y^l = x + \frac{1}{x} + y + \frac{1}{y}$ is left unchanged by the 4 elements $(x, y), (\frac{1}{x}, y), (\frac{1}{x}, \frac{1}{y}), (x, \frac{1}{y})$.

\[
\sum_{(k,l) \in S} x^k y^l - \frac{1}{z} x y z Q(x, y, z) = z x Q(x, 0, z) + z y Q(0, y, z) - x y
\]
\[
\sum_{(k,l) \in S} x^k y^l - \frac{1}{z} \frac{1}{x} y z Q(\frac{1}{x}, y, z) = z \frac{1}{x} Q(\frac{1}{x}, 0, z) + z y Q(0, y, z) - \frac{1}{x} y
\]
\[
\sum_{(k,l) \in S} x^k y^l - \frac{1}{z} \frac{1}{x} y z Q(\frac{1}{x}, \frac{1}{y}, z) = z \frac{1}{x} Q(\frac{1}{x}, 0, z) + z \frac{1}{y} Q(0, \frac{1}{y}, z) - \frac{1}{x} \frac{1}{y}
\]
\[
\sum_{(k,l) \in S} x^k y^l - \frac{1}{z} \frac{1}{y} z Q(x, \frac{1}{y}, z) = z x Q(x, 0, z) + z \frac{1}{y} Q(0, \frac{1}{y}, z) - x \frac{1}{y}
\]

Making the alternating sum, we obtain:

\[
[x^>] [y^>] \sum_{\theta \in \langle \xi, \eta \rangle} (-1)^{\theta} \theta [xyz Q(x, y, z)] = [x^>] [y^>] \frac{\frac{1}{x} y - \frac{1}{x} \frac{1}{y} + \frac{1}{x} \frac{1}{y}}{\sum_{(k,l) \in S} x^k y^l - \frac{1}{z}} - x y
\]
An example of resolution in the case of a finite group

The jump function $\sum_{(k,l) \in S} x^k y^l = x + \frac{1}{x} + y + \frac{1}{y}$ is left unchanged by the 4 elements $(x, y), \left(\frac{1}{x}, y\right), \left(\frac{1}{x}, \frac{1}{y}\right), (x, \frac{1}{y})$.

\[
\left[ \sum_{(k,l) \in S} x^k y^l - 1/z \right] x y z Q(x, y, z) = z x Q(x, 0, z) + z y Q(0, y, z) - x y
\]
\[
\left[ \sum_{(k,l) \in S} x^k y^l - 1/z \right] \frac{1}{x} y z Q\left(\frac{1}{x}, y, z\right) = z \frac{1}{x} Q\left(\frac{1}{x}, 0, z\right) + z y Q(0, y, z) - \frac{1}{x} y
\]
\[
\left[ \sum_{(k,l) \in S} x^k y^l - 1/z \right] \frac{1}{x} \frac{1}{y} z Q\left(\frac{1}{x}, \frac{1}{y}, z\right) = z \frac{1}{x} Q\left(\frac{1}{x}, 0, z\right) + z \frac{1}{y} Q(0, \frac{1}{y}, z) - \frac{1}{x} \frac{1}{y}
\]
\[
\left[ \sum_{(k,l) \in S} x^k y^l - 1/z \right] x \frac{1}{y} z Q(x, \frac{1}{y}, z) = z x Q(x, 0, z) + z \frac{1}{y} Q(0, \frac{1}{y}, z) - x \frac{1}{y}
\]

Making the alternating sum, we obtain:

\[xyzQ(x, y, z) = [x^>]\left[ y^>] \frac{-xy + \frac{1}{x} y - \frac{1}{x} \frac{1}{y} + x \frac{1}{y}}{\sum_{(k,l) \in S} x^k y^l - 1/z} \right]\]
Counting the numbers of walks confined to a quadrant
Random walks killed at the boundary of a quadrant

Introduction
Results

Classification of the 79 problems

79 problems

23 admit a finite group

56 have an infinite group

19 are solved via “orbit sums”

3 are solved via “half orbit sums”

1 is not solved

2 are solved

54 are not solved
Gessel’s walks
Gessel’s walks

2001: conjecture of I. Gessel

\[ q(0, 0, 2k) = 16^k \frac{(5/6)_k (1/2)_k}{(2)_k (5/3)_k}, \quad (a)_k = a(a + 1) \cdots (a + k - 1) \]
Gessel’s walks

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2001: conjecture of I. Gessel

2009: proof of the conjecture

M. Kauers, C. Koutschan, D. Zeilberger
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2009: proof of the algebraicity of \( Q(x, y, z) \)
A. Bostan, M. Kauers [explicit expression: M. van Hoeij]
1. Counting the numbers of walks confined to a quadrant
   - Introduction
   - Results

2. Random walks killed at the boundary of a quadrant
   - Exact results
   - Asymptotic results: non-zero drift
   - Asymptotic results: zero drift
For all walks,

\[ K(x, 0, z) Q(x, 0, z) - K(0, 0, z) Q(0, 0, z) = \]
Results (1/3)

For all walks,

\[ K(x,0,z)Q(x,0,z) - K(0,0,z)Q(0,0,z) = x Y_0(x,z) + \]
\[ \frac{1}{2\pi i} \int_{x_1(z)}^{x_2(z)} t [Y_0(t,z) - Y_1(t,z)] \left[ \frac{\partial_t w(t,z)}{w(t,z) - w(x,z)} - \frac{\partial_t w(t,z)}{w(t,z) - w(0,z)} \right] dt, \]

where:

- \( Y_0(x,z) \) and \( Y_1(x,z) \) are the two \( y \)-roots of the kernel \( K(x,y,z) \).
- \( x_1(z) \) and \( x_2(z) \) are branch points of the algebraic function \( Y(x,z) \).
- \( w \) will be defined soon.
For all walks,

\[ K(x,0,z)Q(x,0,z) - K(0,0,z)Q(0,0,z) = xY_0(x,z) + \]

\[ \frac{1}{2\pi i} \int_{x_1(z)}^{x_2(z)} t \left[ Y_0(t,z) - Y_1(t,z) \right] \left[ \frac{\partial_t w(t,z)}{w(t,z) - w(x,z)} - \frac{\partial_t w(t,z)}{w(t,z) - w(0,z)} \right] dt, \]

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Results (1/3)

For all walks, 

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where:

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- $x_1(z)$ and $x_2(z)$ are branch points of the algebraic function $Y(x,z)$;
- $w$ will be defined soon.
A similar identity holds for

\[ K(0,y,z)Q(0,y,z) - K(0,0,z)Q(0,0,z); \]
A similar identity holds for

\[ K(0, y, z)Q(0, y, z) - K(0,0, z)Q(0,0, z); \]

\[ Q(0,0, z) \] is deduced by evaluation;
A similar identity holds for

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\[ Q(0, 0, z) \] is deduced by evaluation;

\[ Q(x, y, z) \] is obtained thanks to the functional equation:

\[ K(x, y, z)Q(x, y, z) = K(x, 0, z)Q(x, 0, z) + K(0, y, z)Q(0, y, z) - K(0, 0, z)Q(0, 0, z) - xy. \]
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- \( Q(0, 0, z) \) is deduced by evaluation;
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\[
K(x, y, z)Q(x, y, z) = K(x, 0, z)Q(x, 0, z) + K(0, y, z)Q(0, y, z) - K(0, 0, z)Q(0, 0, z) - xy. 
\]

- It remains to find \( w! \).
A 3-steps method

- Continuation of the generating functions $Q(x, 0, z)$ and $Q(0, y, z)$

- Boundary value problems

\[ KQ(x, 0, z) - KQ(\overline{x}, 0, z) = \ldots \]

- Analysis of $w(x, z)$ [uniformization]

\[ KQ(w^{-1})^{-1}(x, 0, z) = \ldots \]
A 3-steps method

- Continuation of the generating functions $Q(x, 0, z)$ and $Q(0, y, z)$
- Boundary value problems

\[
KQ(x, 0, z) - KQ(w^{-1} - (x), 0, z) = \ldots
\]

- Analysis of $w(x, z)$ [uniformization]
A 3-steps method

- Continuation of the generating functions $Q(x, 0, z)$ and $Q(0, y, z)$
- Boundary value problems

$$KQ(x, 0, z) - KQ([w^{-1}]^+(u), 0, z) = \ldots$$
$$KQ([w^{-1}]^-(u), 0, z) = \ldots$$
A 3-steps method

- Continuation of the generating functions $Q(x,0,z)$ and $Q(0,y,z)$
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$$KQ(x,0,z) - KQ(\overline{x},0,z) = [\ldots]$$
$$KQ([w^{-1}]^+(u),0,z) - KQ([w^{-1}]^-(u),0,z) = [\ldots]$$

- Analysis of $w(x,z)$ [uniformization]
Results (3/3)

\[ \langle \xi, \eta \rangle \text{ finite} \quad \sum_{(k,l) \in S} kl \leq 0 \right\} \Rightarrow w \text{ rational} \]
\begin{align*}
\langle \xi, \eta \rangle \text{ finite } & \sum_{(k,l) \in S} kl \leq 0 \Rightarrow w \text{ rational} \\
\end{align*}
Results (3/3)

\[
\begin{align*}
\{\xi, \eta\} \text{ finite } & \quad \sum_{(k,l) \in S} kl \leq 0 \} \Rightarrow w \text{ rational} \\
\{\xi, \eta\} \text{ finite } & \quad \sum_{(k,l) \in S} kl > 0 \} \Rightarrow w \begin{cases} \text{algebraic} \\ \text{non-rational} \end{cases}
\end{align*}
\]
Results (3/3)

\[ \langle \xi, \eta \rangle \text{ finite } \sum_{(k,l) \in S} kl \leq 0 \Rightarrow w \text{ rational} \]

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\[ \sum_{(k,l) \in S} kl > 0 \} \Rightarrow w \text{ non-holonomic} \]

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\[ w \text{ explicit} [\wp\text{-Weierstrass functions}] \]
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Some motivations

- Non-colliding random walks [P. Eichelsbacher, W. König, P. Schmid]
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    ((Minimal) Martin boundary and asymptotic of the Green functions for the killed walk)
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### Some motivations

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- **Examples of non-homogeneous processes**
Some motivations

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- Examples of non-homogeneous processes

Homogeneous processes in $\mathbb{Z}^d$

- Minimal Martin boundary [J. Doob, J. Snell, R. Williamson (1960)]

- Asymptotic of Green functions, Martin compactification
  - Non-zero-drift case [P. Ney, F. Spitzer (1966)]
  - Zero drift case [F. Spitzer (1964)]

Random walks of $\mathbb{Z}_+^2$
- killed at the boundary;
- with a non-zero drift.

$$\lim_{j/i \to \tan(\gamma)} \frac{G_{i,j}^{i_0,j_0}}{G_{1,1}^{i,j}}$$

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Results of I. Ignatiouk-Robert (2009)

Random walks of $\mathbb{Z}_+^d$

- killed at the boundary;
- with a drift having at the most one zero coordinate.

$$\lim \frac{G_{i_0,j_0,k_0}^{i,j,k}}{G_{1,1,1}^{1,1,1}}$$
Results of I. Ignatiouk-Robert (2009)

Random walks of $\mathbb{Z}^d_+$
- killed at the boundary;
- with a drift having $k \geq 2$
  zero coordinates.

Need results on harmonic functions for killed random walks having a zero drift in dimension $2, \ldots, k$. 
Results of P. Biane (1991-1992)
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Doob $h$-transform
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Doob $h$-transform

Minimal Martin boundary
Counting the numbers of walks confined to a quadrant
Random walks killed at the boundary of a quadrant

Processes considered and exact results
Asymptotic results: non-zero drift
Asymptotic results: zero drift

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$G_{i_0,j_0}^{i,j}$
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Processes considered

Random walks of the quarter plane $\mathbb{Z}^2_+$:

- homogeneous inside of $\mathbb{Z}^2_+$;
Processes considered

Random walks of the quarter plane $\mathbb{Z}^2_+$:

- homogeneous inside of $\mathbb{Z}^2_+$;

- killed at the boundary of $\mathbb{Z}^2_+$. 

\[\begin{array}{c|c|c}
 p_{-1,1} & p_{0,1} & p_{1,1} \\
 p_{-1,0} & p_{0,0} & p_{1,0} \\
 p_{-1,-1} & p_{0,-1} & p_{1,-1} \\
\end{array}\]
Counting the numbers of walks confined to a quadrant
Random walks killed at the boundary of a quadrant

Processes considered and exact results
Asymptotic results: non-zero drift
Asymptotic results: zero drift

Explicitly calculated quantities

\[
\tilde{h}^{i_0,j_0}(y) = \sum_{i \geq 1} P_{i_0,j_0}[(X, Y) \text{ is killed at } (i, 0)] x^{i-1}, \quad \bar{h}^{i_0,j_0}(y), \quad h^{i_0,j_0}
\]
Explicitly calculated quantities

\[ h_{i_0,j_0}^i(x) = \sum_{i \geq 1} P_{(i_0,j_0)}[(X, Y) \text{ is killed at } (i,0)] x^{i-1}, \quad h_{0,0}^{i_0,j_0}; \]

\[ G_{i_0,j_0}^i(x, y) = \sum_{i,j \geq 1} E_{(i_0,j_0)} \left[ \sum_{k \geq 0} 1_{\{(X(k), Y(k))=(i,j)\}} \right] x^{i-1} y^{j-1}; \]
Explicitly calculated quantities

\[ h_{i_0,j_0}^i (x) = \sum_{i \geq 1} P_{(i_0,j_0)}[(X, Y) \text{ is killed at } (i, 0)] x^{i-1}, \quad \tilde{h}_{i_0,j_0}^i (y), \quad h_{0,0}^{i_0,j_0}; \]

\[ G_{i_0,j_0}^{i,j} (x, y) = \sum_{i,j \geq 1} E_{(i_0,j_0)} \left[ \sum_{k \geq 0} 1 \{(X(k), Y(k))=(i,j)\} \right] x^{i-1} y^{j-1}; \]

\[ xy \left[ \sum_{-1 \leq k, l \leq 1} p_{k,l} x^k y^l - 1 \right] G_{i_0,j_0}^{i,j} (x, y) = h_{i_0,j_0}^i (x) + \tilde{h}_{i_0,j_0}^i (y) + h_{0,0}^{i_0,j_0} - x^{i_0} y^{j_0}; \]
Explicitly calculated quantities

\[ h^{i_0,j_0}(x) = \sum_{i \geq 1} \mathbb{P}_{(i_0,j_0)}[(X,Y) \text{ is killed at } (i,0)]x^{i-1}, \quad \tilde{h}^{i_0,j_0}(y), \quad h^{i_0,j_0}_{0,0}; \]

\[ G^{i_0,j_0}(x,y) = \sum_{i,j \geq 1} \mathbb{E}_{(i_0,j_0)} \left[ \sum_{k \geq 0} 1\{(X(k),Y(k))=(i,j)\} \right] x^{i-1} y^{j-1}; \]

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Functional equation & boundary value problems & conformal gluing.
Green functions

The functional equation

\[
xy \left[ \sum_{-1 \leq k, l \leq 1} p_{k, l} x^k y^l - 1 \right] G_{i_0, j_0}^{i, j}(x, y) = h_{i_0, j_0}^{i, j}(x) + \tilde{h}_{i_0, j_0}^{i, j}(y) + h_{0, 0}^{i_0, j_0} - x^{i_0} y^{j_0}
\]

and Cauchy's formulas yield:

\[
G_{i, j}^{i_0, j_0} = \frac{1}{[2\pi i]^2} \int \int_{\{|x|=|y|=1\}} \frac{h_{i_0, j_0}^{i, j}(x) + \tilde{h}_{i_0, j_0}^{i, j}(y) + h_{0, 0}^{i_0, j_0} - x^{i_0} y^{j_0}}{xy \left[ \sum_{-1 \leq k, l \leq 1} p_{k, l} x^k y^l - 1 \right]} x^i y^j \, dx \, dy.
\]
Counting the numbers of walks confined to a quadrant
Random walks killed at the boundary of a quadrant

Asymptotic results: zero drift

Green functions

The functional equation

\[
xy \left[ \sum_{-1 \leq k, l \leq 1} p_{k,l} x^k y^l - 1 \right] G_{i,j}^{0,0}(x, y) = h_{i,j}^{0,0}(x) + \tilde{h}_{i,j}^{0,0}(y) + h_{0,0}^{i,j} - x^i y^j
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\]

Aim

We look for the asymptotic of the Green functions along \( i + j \to \infty \) and \( j/i \to \tan(\gamma), \ \gamma \in [0, \pi/2] \).
Green functions

The functional equation

\[
xy \left[ \sum_{-1 \leq k, l \leq 1} p_{k, l} x^k y^l - 1 \right] G_{i_0, j_0}^i (x, y) = h_{i_0, j_0}^i (x) + \tilde{h}_{i_0, j_0}^i (y) + h_{0, 0}^i - x_{i_0} y_{j_0}
\]

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G_{i, j}^{i_0, j_0} = \frac{1}{[2\pi i]^2} \int \int_{\{ |x| = |y| = 1 \}} \frac{h_{i_0, j_0}^i (x) + \tilde{h}_{i_0, j_0}^i (y) + h_{0, 0}^i - x_{i_0} y_{j_0}}{xy \left[ \sum_{-1 \leq k, l \leq 1} p_{k, l} x^k y^l - 1 \right]} x^i y^j \, dx \, dy.
\]

Aim

We look for the asymptotic of the Green functions along \( i + j \to \infty \) and \( j/i \to \tan(\gamma) \), \( \gamma \in [0, \pi/2] \).

Method

Adaptation of the saddle point and/or the steepest descent method.
Green functions

\[ G_{i_0,j_0}^{i,j} = \frac{1}{[2\pi i]^2} \int\int_{\{|x| = |y| = 1\}} \frac{h_{i_0,j_0}(x) + \tilde{h}_{i_0,j_0}(y) + h_{0,0}^{i,j} - x_0 y_0}{xy \left[ \sum_{-1 \leq k, l \leq 1} p_k, l x^k y^l - 1 \right]} \, dx \, dy. \]
### Green functions

\[
G_{i,j}^{i_0,j_0} = \frac{1}{[2\pi i]^2} \int \int_{\{|x|=|y|=1\}} \frac{h_{i_0,j_0}(x) + \tilde{h}_{i_0,j_0}(y) + h_{0,0}^{i_0,j_0} - x^{i_0}y^{j_0}}{xy \left[ \sum_{-1 \leq k,l \leq 1} p_{k,l} x^k y^l - 1 \right]} x^i y^j \, dx \, dy.
\]

### Problem

Behavior of \( h_{i_0,j_0}(x) + \tilde{h}_{i_0,j_0}(y) + h_{0,0}^{i_0,j_0} - x^{i_0}y^{j_0} \) near the saddle point?
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Green functions

\[ G_{i,j}^{i_0,j_0} = \frac{1}{[2\pi \nu]^2} \int \int \frac{h_{i_0,j_0}(x) + \tilde{h}_{i_0,j_0}(y) + h_{0,0} - x^i y^j}{x y \left( \sum_{-1 \leq k, l \leq 1} p_{k,l} x^k y^l - 1 \right)} \, dx \, dy. \]

Problem

Behavior of \( h_{i_0,j_0}(x) + \tilde{h}_{i_0,j_0}(y) + h_{0,0} - x^i y^j \) near the saddle point?

Non-zero drift

Suppose that \( j/i \to \tan(\gamma) \) and that \((x,y)\) is the underlying saddle point (depending on \(\gamma\)).

- \( \gamma \in ]0, \pi/2[ \): \( h_{i_0,j_0}(x) + \tilde{h}_{i_0,j_0}(y) + h_{0,0} - x^i y^j \neq 0. \)
### Green functions

\[
G_{i,j}^{i_0,j_0} = \frac{1}{[2\pi i]^2} \int \int \frac{h_{i_0,j_0}(x) + \tilde{h}_{i_0,j_0}(y) + h_{i_0,j_0}^{i_0,j_0} - x_i^0 y_j^0}{xy \left[ \sum_{-1 \leq k, l \leq 1} p_{k,l} x^k y^l - 1 \right]} \ dx \ dy.
\]

### Problem

Behavior of \(h_{i_0,j_0}(x) + \tilde{h}_{i_0,j_0}(y) + h_{0,0}^{i_0,j_0} - x_i^0 y_j^0\) near the saddle point?

### Non-zero drift

Suppose that \(j/i \to \tan(\gamma)\) and that \((x, y)\) is the underlying saddle point (depending on \(\gamma\)).

- \(\gamma \in ]0, \pi/2[\): \(h_{i_0,j_0}^{i_0,j_0}(x) + \tilde{h}_{i_0,j_0}(y) + h_{0,0}^{i_0,j_0} - x_i^0 y_j^0 \neq 0\).
- \(\gamma = 0, \pi/2\), two difficulties:
  - the saddle point is a branch point;
  - \(h_{i_0,j_0}^{i_0,j_0}(x) + \tilde{h}_{i_0,j_0}(y) + h_{0,0}^{i_0,j_0} - x_i^0 y_j^0 = 0\).
Green functions in the case $\sum_{k,l} kp_{k,l} > 0$ and $\sum_{k,l} lp_{k,l} > 0$

\[
G_{i,j}^{i_0,j_0} \sim j/i \to \tan(\gamma)
\]

\[
C \frac{s_x(j/i)^{i_0} s_y(j/i)^{j_0} - h^{i_0,j_0}(s_x(j/i)) - \tilde{h}^{i_0,j_0}(s_y(j/i)) - h^{i_0,j_0}_{0,0}}{i^{1/2}s_x(j/i)^i s_y(j/i)^j}
\]
Green functions in the case $\sum_{k,l} kp_{k,l} > 0$ and $\sum_{k,l} lp_{k,l} > 0$
Green functions in the case $\sum_{k,l} kp_{k,l} > 0$ and $\sum_{k,l} lp_{k,l} > 0$

\[
G^{i_0,j_0}_{i,j} \sim j/i \to \tan(\gamma)
\]

\[
C \frac{s_x(j/i)^i_0 s_y(j/i)^j_0 - h^{i_0,j_0}(s_x(j/i)) - \tilde{h}^{i_0,j_0}(s_y(j/i)) - h^{i_0,j_0}_{0,0}}{i^{1/2}s_x(j/i)^i s_y(j/i)^j}
\]

Kilian Raschel

Chemins confinés dans un quadrant
Green functions in the case $\sum_{k,l} kp_{k,l} > 0$ and $\sum_{k,l} lp_{k,l} > 0$

\[
G_{i_0,j_0}^{i,j} \sim j/i \to \tan(\gamma)
\]

\[
C \frac{s_x(j/i)^{i_0} s_y(j/i)^{j_0} - h^{i_0,j_0} (s_x(j/i)) - \tilde{h}^{i_0,j_0} (s_y(j/i)) - h^{i_0,j_0}_{0,0}}{j^{1/2} s_x(j/i)^i s_y(j/i)^j}
\]
Green functions in the case $\sum_{k,l} kp_{k,l} > 0$ and $\sum_{k,l} lp_{k,l} > 0$

$$\phi(u,v) = \sum_{-1 \leq k,l \leq 1} p_{k,l} e^{ku} e^{lv} = 1$$

$$\nabla \phi(u,v) = \begin{pmatrix} \partial_u \phi(u,v) \\ \partial_v \phi(u,v) \end{pmatrix}$$

$$(s_x(j/i), s_y(j/i)) = (\phi_x(j/i), \phi_y(j/i))$$

$$G_{i,j} \sim j/i \to \tan(\gamma)$$

$$C \frac{s_x(j/i)^{i_0} s_y(j/i)^{j_0} - h^{i_0,j_0} (s_x(j/i)) - \tilde{h}^{i_0,j_0} (s_y(j/i)) - h_{0,0}^{i_0,j_0}}{i^{1/2} s_x(j/i)^i s_y(j/i)^j}.$$
Green functions in the case $\sum_{k,l} kp_{k,l} > 0$ and $\sum_{k,l} lp_{k,l} > 0$

\[
\phi(u, v) = \sum_{-1 \leq k, l \leq 1} p_{k,l} e^{ku} e^{lv} = 1
\]

\[
(u, v) \mapsto \frac{\text{grad } \phi(u, v)}{|\text{grad } \phi(u, v)|}
\]

\[
G_{i,j}^{i_0,j_0} \sim j/i \to \tan(\gamma)
\]

\[
C \frac{s_x(j/i)^i_0 s_y(j/i)^j_0 - h^{i_0,j_0}(s_x(j/i)) - \tilde{h}^{i_0,j_0}(s_y(j/i)) - h^{i_0,j_0}_{0,0}}{i^{1/2}s_x(j/i)^i s_y(j/i)^j}.
\]
Green functions in the case $\sum_{k,l} kp_{k,l} > 0$ and $\sum_{k,l} lp_{k,l} > 0$

\[
\phi(u, v) = \sum_{-1 \leq k,l \leq 1} p_{k,l} e^{ku} e^{lv} = 1
\]

\[
\frac{\text{grad } \phi(u, v)}{|\text{grad } \phi(u, v)|} = (\cos(\gamma), \sin(\gamma))
\]

\[
(s_x(j/i), s_y(j/i)) = (\tan(\gamma), \tan(\gamma))
\]

\[
G_{i,j}^{i_0,j_0} \sim j/i \rightarrow \tan(\gamma)
\]

\[
C = \frac{s_x(j/i) s_y(j/i) - h^{i_0,j_0}(s_x(j/i)) - \tilde{h}^{i_0,j_0}(s_y(j/i)) - h^{i_0,j_0}_{0,0}}{i^{1/2} s_x(j/i) s_y(j/i)}
\]

\[
\phi(v, w) = \sum_{-1 \leq k,l \leq 1} p_{k,l} e^{ku} e^{lv} = 1
\]
Green functions in the case $\sum_{k,l} kp_{k,l} > 0$ and $\sum_{k,l} lp_{k,l} > 0$

$$\phi(u, v) = \sum_{-1 \leq k, l \leq 1} p_{k,l} e^{ku} e^{lv} = 1$$

$$\frac{\text{grad } \phi(u_\gamma, v_\gamma)}{|\text{grad } \phi(u_\gamma, v_\gamma)|} = (\cos(\gamma), \sin(\gamma))$$

$$\left(s_x\left(j/i\right), s_y\left(j/i\right)\right) = \left(e^{u_{\text{arctan}(j/i)}}, e^{v_{\text{arctan}(j/i)}}\right)$$

$$G_{i,j}^{i_0,j_0} \sim j/i \rightarrow \tan(\gamma)$$

$$C \frac{s_x(j/i)^{i_0} s_y(j/i)^{j_0} - h_{i_0,j_0}(s_x(j/i)) - \tilde{h}_{i_0,j_0}(s_y(j/i)) - h_{0,0}^{i_0,j_0}}{i^{1/2} s_x(j/i)^j s_y(j/i)^j}$$
Green functions in the case $\sum_{k,l} kp_{k,l} > 0$ and $\sum_{k,l} lp_{k,l} > 0$

\[ \phi(u, v) = \sum_{-1 \leq k, l \leq 1} p_{k,l} e^{ku} e^{lv} = 1 \]

\[ \frac{\text{grad } \phi(u_\gamma, v_\gamma)}{|\text{grad } \phi(u_\gamma, v_\gamma)|} = (\cos(\gamma), \sin(\gamma)) \]

\[ (s_x(j/i), s_y(j/i)) = (e^{u\arctan(j/i)}, e^{v\arctan(j/i)}) \]

\[ G_{i_0,j_0}^{i,j} \sim j/i \to \tan(\gamma) \]

\[ C \frac{s_x(j/i)^{i_0} s_y(j/i)^{j_0} - h_{0,j_0}^{i_0}(s_x(j/i)) - \tilde{h}_{0,j_0}^{i_0}(s_y(j/i)) - h_{0,0}^{i_0,j_0}}{i^{1/2}s_x(j/i)^{i} s_y(j/i)^{j}}. \]

\[ G_{i_0,j_0}^{i,j} \sim j/i \to 0 \quad C_0 \frac{j_{0} s_x(j/i)^{i_0} s_y(j/i)^{j_0} - 1 - \partial_y \tilde{h}_{0,j_0}^{i_0}(s_y(j/i))}{i^{1/2}s_x(j/i)^{i} s_y(j/i)^{j}}. \]
1 Counting the numbers of walks confined to a quadrant
   - Introduction
   - Results

2 Random walks killed at the boundary of a quadrant
   - Exact results
   - Asymptotic results: non-zero drift
   - Asymptotic results: zero drift
Green functions

\[ G_{i,j}^{i_0,j_0} = \frac{1}{[2\pi \iota]^2} \int \int_{\{|x|=|y|=1\}} h_{i_0,j_0}^{i,j}(x) + \tilde{h}_{i_0,j_0}^{i,j}(y) + h_{0,0}^{i,j} - x^{i_0}y^{j_0} \]

\[ \frac{x^{i}y^{j}}{\sum_{-1 \leq k, l \leq 1} p_{k,l}x^{k}y^{l} - 1} \text{d}x\text{d}y. \]

Problem

Behavior of \( h_{i_0,j_0}^{i,j}(x) + \tilde{h}_{i_0,j_0}^{i,j}(y) + h_{0,0}^{i,j} - x^{i_0}y^{j_0} \) near the saddle point?
Counting the numbers of walks confined to a quadrant
Random walks killed at the boundary of a quadrant

Processes considered and exact results
Asymptotic results: non-zero drift
Asymptotic results: zero drift

Green functions

\[ G_{i,j}^{i_0,j_0} = \frac{1}{[2\pi \gamma]^2} \int \int_{\{|x|=|y|=1\}} \frac{h_{i_0,j_0}(x) + \tilde{h}_{i_0,j_0}(y) + h_{0,0}^{i_0,j_0} - x^{i_0}y^{j_0}}{xy \left[ \sum_{-1 \leq k,l \leq 1} p_{k,l} x^k y^l - 1 \right]} \, dx \, dy. \]

Problem

Behavior of \( h_{i_0,j_0}(x) + \tilde{h}_{i_0,j_0}(y) + h_{0,0}^{i_0,j_0} - x^{i_0}y^{j_0} \) near the saddle point?

Zero drift

The saddle point is equal to \((1, 1)\) for any \(\gamma \in [0, \pi/2]\). Encouraging!

Kilian Raschel

Chemins confinés dans un quadrant
Green functions

\[ G_{i,j}^{i_0,j_0} = \frac{1}{[2\pi i]^2} \iint_{\{|x|=|y|=1\}} \frac{h_{i,j}^{i_0,j_0}(x) + \tilde{h}_{i,j}^{i_0,j_0}(y) + h_{0,0}^{i_0,j_0} - x^{i_0}y^{j_0}}{xy \left[ \sum_{-1 \leq k,l \leq 1} p_{k,l} x^k y^l - 1 \right] x^i y^j} \, dx \, dy. \]

Problem

Behavior of \( h_{i,j}^{i_0,j_0}(x) + \tilde{h}_{i,j}^{i_0,j_0}(y) + h_{0,0}^{i_0,j_0} - x^{i_0}y^{j_0} \) near the saddle point?

Zero drift

The saddle point is equal to \((1, 1)\) for any \(\gamma \in [0, \pi/2]\). Encouraging!

[Only a few results exist and they are necessary for higher dimension]
Green functions

\[ G_{i_0,j_0} = \frac{1}{[2\pi i]^2} \int \int_{\{|x|=|y|=1\}} \frac{h_{i_0,j_0}(x) + \tilde{h}_{i_0,j_0}(y) + h_{0,0}^i - x^i y^j}{xy \left[ \sum_{-1 \leq k, l \leq 1} p_{k,l} x^k y^l - 1 \right] x^i y^j} \, dx \, dy. \]

Problem

Behavior of \( h_{i_0,j_0}(x) + \tilde{h}_{i_0,j_0}(y) + h_{0,0}^i - x^i y^j \) near the saddle point?

Zero drift

The saddle point is equal to \((1, 1)\) for any \(\gamma \in [0, \pi/2]\). Encouraging! [Only a few results exists and they are necessary for higher dimension]

- \( h_{i_0,j_0}(1) + \tilde{h}_{i_0,j_0}(1) + h_{0,0}^i - 1 = 0 \). But more precisely?
Green functions

\[ G_{i_0,j_0}^{i,j} = \frac{1}{[2\pi i]^2} \iint_{\{|x|=|y|=1\}} \frac{h_{i_0,j_0}(x) + \tilde{h}_{i_0,j_0}(y) + h_{0,0}^{i_0,j_0} - i_0 y_{j_0} x_{i_0} y_{j_0}}{x y \left[ \sum_{-1 \leq k,l \leq 1} p_{k,l} x^k y^l - 1 \right]} dxdy. \]

Problem

Behavior of \( h_{i_0,j_0}(x) + \tilde{h}_{i_0,j_0}(y) + h_{0,0}^{i_0,j_0} - x_{i_0} y_{j_0} \) near the saddle point?

Zero drift

The saddle point is equal to \((1, 1)\) for any \( \gamma \in [0, \pi/2] \). Encouraging! [Only a few results exists and they are necessary for higher dimension]

- \( h_{i_0,j_0}(1) + \tilde{h}_{i_0,j_0}(1) + h_{0,0}^{i_0,j_0} - 1 = 0 \). But more precisely?
- Using the explicit formulations given by the Riemann-Carleman problems seems too complicated.
Green functions

\[ G_{i,j}^{i_0,j_0} = \frac{1}{[2\pi \nu]^2} \int \int_{\{|x|=|y|=1\}} \frac{h_{i_0,j_0}^i(x) + \tilde{h}_{i_0,j_0}^i(y) + h_{0,0}^{i_0,j_0} - x_{i_0}^i y_{j_0}^j}{x y \left[ \sum_{-1 \leq k, l \leq 1} p_{k,l} x^k y^l - 1 \right] x^i y^j} \text{d}x \text{d}y. \]

Problem

Behavior of \( h_{i_0,j_0}^i(x) + \tilde{h}_{i_0,j_0}^i(y) + h_{0,0}^{i_0,j_0} - x_{i_0}^i y_{j_0}^j \) near the saddle point?

Zero drift

The saddle point is equal to \((1, 1)\) for any \(\gamma \in [0, \pi/2]\). Encouraging!

[Only a few results exists and they are necessary for higher dimension]

- \( h_{i_0,j_0}^i(1) + \tilde{h}_{i_0,j_0}^i(1) + h_{0,0}^{i_0,j_0} - 1 = 0 \). But more precisely?

- Using the explicit formulations given by the Riemann-Carleman problems seems too complicated.

- Using the finiteness of the group \( \langle \xi, \eta \rangle \) may yield closed formulas for \( h_{i_0,j_0}^i(x) + \tilde{h}_{i_0,j_0}^i(y) + h_{0,0}^{i_0,j_0} - x_{i_0}^i y_{j_0}^j \), typically \( \sum_{\theta \in \langle \xi, \eta \rangle} (-1)^{\theta} \theta [x_{i_0}^i y_{j_0}^j] \).
Green functions in the case \( \sum_{k,l} kp_{k,l} = 0 \) and \( \sum_{k,l} lp_{k,l} = 0 \) (1/3)

Example
Green functions in the case $\sum_{k,l} kp_{k,l} = 0$ and $\sum_{k,l} lp_{k,l} = 0$ \(1/3\)

Example

\[
\begin{array}{ccc}
 p & & \\
 q & \text{walks with zero drift} & q \\
p & & \\
\end{array}
\]

\{ \text{walks with zero drift} \}

\{ \text{and group of order 4} \}
Green functions in the case $\sum_{k,l} kp_{k,l} = 0$ and $\sum_{k,l} lp_{k,l} = 0$ (1/3)

Example

```
p
q
q
p
```

\begin{align*}
&\begin{cases}
\text{walks with zero drift} \\
\text{and group of order 4}
\end{cases} \\
\quad \quad \rightarrow

&\begin{cases}
\text{walks with zero drift} \\
\text{and a $\geq 0$ degree 2 harmonic polynomial}
\end{cases} \\
&\forall_{i_0,j_0}^p = i_0 j_0
\end{align*}
Green functions in the case $\sum_{k,l} kp_{k,l} = 0$ and $\sum_{k,l} lp_{k,l} = 0$ (1/3)

Example

$$G_{i_0,j_0}^{i,j} \sim j/i \rightarrow \tan(\gamma) \quad V_p^{i_0,j_0} f_p(i,j).$$

walks with zero drift and a $\geq 0$ degree 2 harmonic polynomial $V_p^{i_0,j_0} = i_0 j_0$
Green functions in the case $\sum_{k,l} kp_{k,l} = 0$ and $\sum_{k,l} lp_{k,l} = 0$ (2/3)

Example

\[
\begin{cases}
\text{walks with zero drift} \\
\text{and a $\geq 0$ degree 3}
\end{cases}
\]

\[
\text{harmonic polynomial}
\]

\[
V^{i_0,j_0}_{\alpha,\beta} = i_0j_0(i_0 + \alpha j_0 + \beta)
\]
Green functions in the case $\sum_{k,l} kp_{k,l} = 0$ and $\sum_{k,l} lp_{k,l} = 0 \quad (2/3)$

Example

Walks with zero drift and a $\geq 0$ degree 3 harmonic polynomial

$V_{i_0,j_0}^{i_0,j_0} = i_0 j_0 (i_0 + \alpha j_0 + \beta)$
Green functions in the case $\sum_{k,l} kp_{k,l} = 0$ and $\sum_{k,l} lp_{k,l} = 0$ \( (2/3) \)

Example

walks with zero drift and a $\geq 0$ degree 3 harmonic polynomial

$$V_{i_0,j_0}^{i_0,j_0} = i_0 j_0 (i_0 + \alpha j_0 + \beta)$$

walks with zero drift and group of order 6
Green functions in the case $\sum_{k,l} kp_{k,l} = 0$ and $\sum_{k,l} lp_{k,l} = 0$ (2/3)

Example

\[
\begin{cases}
\text{walks with zero drift} \\
\text{and a } \geq 0 \text{ degree 3 harmonic polynomial}
\end{cases}
\]

\[V_{\alpha,\beta}^{i_0,j_0} = i_0 j_0 (i_0 + \alpha j_0 + \beta)\]

\[G_{i,j}^{i_0,j_0} \sim \frac{j}{i} \tan(\gamma) V_{\alpha,\beta}^{i_0,j_0} f_{\alpha,\beta}(i,j)\]
Green functions in the case $\sum_{k,l} kp_{k,l} = 0$ and $\sum_{k,l} lp_{k,l} = 0$ (3/3)

Example

\[ a_n = \sin\left(\frac{\pi}{n}\right)^2 / 2 \]
\[ b_n = \cos\left(\frac{\pi}{n}\right)^2 / 2 \]
Green functions in the case $\sum_{k,l} kp_{k,l} = 0$ and $\sum_{k,l} lp_{k,l} = 0$ (3/3)

Example

$\begin{align*}
a_n &= \sin(\pi/n)^2/2 \\
bn &= \cos(\pi/n)^2/2
\end{align*}$
Green functions in the case $\sum_{k,l} kp_{k,l} = 0$ and $\sum_{k,l} lp_{k,l} = 0$ (3/3)

Example

$$a_n = \sin\left(\frac{\pi}{n}\right)^2 / 2$$
$$b_n = \cos\left(\frac{\pi}{n}\right)^2 / 2$$
Green functions in the case $\sum_{k,l} kp_k,l = 0$ and $\sum_{k,l} lp_k,l = 0$ (3/3)

Example

\[ a_n = \sin(\pi/n)^2/2, \quad b_n = \cos(\pi/n)^2/2 \]

walks with zero drift and a $\geq 0$ degree $n$ harmonic polynomial $V_n^{i_0,j_0}$

\[ G_{i,j}^{i_0,j_0} \sim j/i \to \tan(\gamma) V_n^{i_0,j_0} f_n(i,j). \]
Green functions in the case $\sum_{k,l} kp_{k,l} = 0$ and $\sum_{k,l} lp_{k,l} = 0$ (3/3)
Green functions in the case $\sum_{k,l} kp_{k,l} = 0$ and $\sum_{k,l} lp_{k,l} = 0$ (3/3)

$n = 3, 4$: $V_n \phi(i_0, j_0) \left\{ \begin{array}{l}
\text{is homogeneous} \\
\text{is harmonic for BM}
\end{array} \right.$
Green functions in the case $\sum_{k,l} kp_{k,l} = 0$ and $\sum_{k,l} lp_{k,l} = 0$ (3/3)

$n = 3, 4$: $V_n \phi(i_0,j_0)$
- is homogeneous
- is harmonic for BM

$n \geq 5$: $V_n \phi(i_0,j_0)$
- is non-homogeneous
- has its dominant coefficient harmonic for BM
Counting the numbers of walks confined to a quadrant
Random walks killed at the boundary of a quadrant
Processes considered and exact results
Asymptotic results: non-zero drift
Asymptotic results: zero drift

Green functions in the case $\sum_{k,l} kp_{k,l} = 0$ and $\sum_{k,l} lp_{k,l} = 0$: summary

$$a_n = \sin\left(\frac{\pi}{n}\right)^2/2$$
$$b_n = \cos\left(\frac{\pi}{n}\right)^2/2$$

- Harmonic polynomial $V_{p_{i_0,j_0}}$ of degree 2
- Harmonic polynomial $V_{\alpha,\beta_{i_0,j_0}}$ of degree 3
- Harmonic polynomial $V_{n_{i_0,j_0}}$ of degree $n$

- Group of order 4
- Group of order 6
- Group of order $2n$

$$G_{i,j_{i_0,j_0}} \sim j/i \rightarrow \tan(\gamma) V_{i_0,j_0} f(i,j).$$
### Hitting time of the boundary for SU(2) × SU(2), SU(3) and Sp(4)

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Kilian Raschel

Chemins confinés dans un quadrant
Hitting time of the boundary for $SU(2) \times SU(2)$, $SU(3)$ and $Sp(4)$

$\mathbb{P}_{(i_0, j_0)} [(X, Y) \text{ is absorbed on the x-axis after time } k] \sim_{k \to \infty} \frac{K \cdot V^{i_0, j_0}}{k \# \langle \xi, \eta \rangle / 4}$. 

\[
\begin{array}{c|c|c|c}
 & p & q & p \\
\hline
q & & & \\
\hline
p & & & \\
\end{array}
\quad
\begin{array}{c|c|c|c}
1/3 & & 1/3 \\
\hline
1/3 & & \\
\end{array}
\quad
\begin{array}{c|c|c|c}
1/4 & 1/4 & 1/4 & 1/4 \\
\hline
1/4 & & & \\
\end{array}
\]
Hitting time of the boundary for $\text{SU}(2) \times \text{SU}(2)$, $\text{SU}(3)$ and $\text{Sp}(4)$

\[
\begin{align*}
\mathcal{V}^{i_0,j_0} &= i_0 j_0 \\
\mathcal{V}^{i_0,j_0} &= i_0 j_0 (i_0 + j_0) \\
\mathcal{V}^{i_0,j_0} &= i_0 j_0 (i_0 + j_0)(i_0 + 2j_0)
\end{align*}
\]

\[\mathbb{P}_{(i_0,j_0)}[(X, Y) \text{ is absorbed on the } x\text{-axis after time } k] \sim_{k \to \infty} \frac{K \cdot \mathcal{V}^{i_0,j_0}}{k \# \langle \xi, \eta \rangle / 4}.\]
Merci !
Group, covariance, nature of $Q$ and nature of $w$
Observations *a posteriori*

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<th>Group</th>
<th>Covariance</th>
<th>Nature of $Q$</th>
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Chemins confinés dans un quadrant
Observations \textit{a posteriori}

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<td>?</td>
<td>non-holonomic</td>
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\[
K(x,0,z)Q(x,0,z) - K(0,0,z)Q(0,0,z) = xY_0(x,z) + \frac{1}{2\pi i} \int_{x_1(z)}^{x_2(z)} t [Y_0(t,z) - Y_1(t,z)] \left[ \frac{\partial_t w(t,z)}{w(t,z) - w(x,z)} - \frac{\partial_t w(t,z)}{w(t,z) - w(0,z)} \right] dt.
\]
How to obtain the Riemann-Carleman problem?
Riemann-Carleman problem

\[ \forall t \in C : \quad h^{i_0,j_0}(t) - h^{i_0,j_0}(\bar{t}) = t^{i_0}X_0^{-1}(t)^{j_0} - \bar{t}^{i_0}X_0^{-1}(\bar{t})^{j_0}, \]

where

\[ \sum_{-1 \leq k,l \leq 1} p_{k,l}X_0(y)^k y^l = 0. \]
Riemann-Carleman problem

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Remark

In the general case, we have to continue the function \( h^{i_0,j_0} \) before obtaining the boundary condition above.
How to obtain this Riemann-Carleman problem? (1/2)

The functional equation:

\[ xy \left[ \sum_{-1 \leq k, l \leq 1} p_{k,l} x^k y^l - 1 \right] G_{i^0,j^0}(x, y) = h_{i^0,j^0}(x) + \tilde{h}_{i^0,j^0}(y) + h_{0,0} - x^{i_0} y^{j_0}. \]
How to obtain this Riemann–Carleman problem? (1/2)

The functional equation:

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xy \left[ \sum_{-1 \leq k, l \leq 1} p_{k, l} x^k y^l - 1 \right] G_{i_0, j_0}^{i_0, j_0}(x, y) = h_{i_0, j_0}^{i_0, j_0}(x) + \tilde{h}_{i_0, j_0}^{i_0, j_0}(y) + h_{0, 0}^{i_0, j_0} - x^{i_0} y^{j_0}.
\]

Roots of the kernel:

\[
xy \left[ \sum_{-1 \leq k, l \leq 1} p_{k, l} x^k y^l - 1 \right] = 0 \Leftrightarrow x = X_0(y) \text{ or } X_1(y).
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The functional equation:

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\]

Two new functional equations:

\[
0 = h_{i_0,j_0}^{i_0,j_0}(X_i(y)) + \tilde{h}_{i_0,j_0}^{i_0,j_0}(y) + h_{0,0}^{i_0,j_0} - X_i(y)^{i_0} y^{j_0}.
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How to obtain this Riemann-Carleman problem? (1/2)

The functional equation:

\[
xy \left[ \sum_{-1 \leq k, l \leq 1} p_{k,l} x^k y^l - 1 \right] G_{i_0,j_0}^i(x, y) = h_{i_0,j_0}^{i_0,j_0}(x) + \tilde{h}_{i_0,j_0}^{i_0,j_0}(y) + h_{0,0}^{i_0,j_0} - x^{i_0} y^{j_0}.
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Roots of the kernel:

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xy \left[ \sum_{-1 \leq k, l \leq 1} p_{k,l} x^k y^l - 1 \right] = 0 \iff x = X_0(y) \text{ or } X_1(y).
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Two new functional equations:

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\]

We obtain:

\[
h_{i_0,j_0}^{i_0,j_0}(X_0(y)) - h_{i_0,j_0}^{i_0,j_0}(X_1(y)) = X_0(y)^{i_0} y^{j_0} - X_1(y)^{i_0} y^{j_0}.
\]
How to obtain this Riemann-Carleman problem? (2/2)

\[ h^{i_0,j_0}(X_0(y)) - h^{i_0,j_0}(X_1(y)) = X_0(y)^{i_0} y^{j_0} - X_1(y)^{i_0} y^{j_0}. \]

\( X_0(y) \) and \( X_1(y) \) are complex conjugate

\( X_0(y) \) and \( X_1(y) \) are real

\( X_0([y_1, y_2]) \)

\( X_1([y_1, y_2]) \)
∀ \( t \in \mathbb{C} \): \( h_{i_0,j_0}^0(t) - h_{i_0,j_0}^0(\bar{t}) = t_0^{-1} X_0^{-1}(t)^{j_0} - \bar{t}_0^{-1} X_0^{-1}(\bar{t})^{j_0} \),

where

\[
\sum_{-1 \leq k,l \leq 1} p_{k,l} X_0(y)^k y^l = 0.
\]
Resolution of the Riemann-Carleman problem

\[ h^{i_0,j_0}(x) = \frac{1}{2\pi i} \int_C t^{i_0} X_0^{-1}(t)^{j_0} \left[ \frac{w'(t)}{w(t) - w(x)} - \frac{w'(t)}{w(t) - w(0)} \right] dt, \]

where \( w \) is a conformal gluing function for the set bounded by \( C \).
Resolution of the Riemann-Carleman problem

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Conformal gluing function

\[ w \]

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Resolution of the Riemann-Carleman problem

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Continuation of the generating functions
Continuation of the generating functions (1/3)

\[ h_{i_0,j_0}(x) = \]
Continuation of the generating functions (2/3)

\[
\hat{h}^{i_0,j_0}(x, y)
\]

\[
h^{i_0,j_0}(x)
\]

\[
h^{i_0,j_0}(x) = ?
\]
Continuation of the generating functions (3/3)

\[ h^{i_0,j_0}(x) = \hat{h}^{i_0,j_0}(x, y) \text{ if the kernel } = 0 \]
Different notions for the group
Group of the random walk in the general case

Generating function of the jump probabilities:

\[ \sum_{-1 \leq k, l \leq 1} p_{k,l} x^k y^l. \]

\[ \xi(x, y) = \left( x, \frac{\sum_{-1 \leq k \leq 1} p_{k,-1} x^k}{\sum_{-1 \leq k \leq 1} p_{k,+1} x^k} \frac{1}{y} \right) , \]
\[ \eta(x, y) = \left( \frac{\sum_{-1 \leq l \leq 1} p_{-1,l} y^l}{\sum_{-1 \leq l \leq 1} p_{1,l} y^l} \frac{1}{x}, y \right). \]

Group of the random walk \( \langle \xi, \eta \rangle \): dihedral group of even order \( \geq 4 \).
Group of the random walk in the general case

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- Group of birational transformations on \( \mathbb{C}^2 \);
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- Group of birational transformations on \( \mathbb{C}^2 \);
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Principle: “finite group \( \Rightarrow \) nice calculations”.

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Group of the random walk in the general case

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Group of the random walk $\langle \xi, \eta \rangle$: dihedral group of even order $\geq 4$.

- Group of birational transformations on $\mathbb{C}^2$;
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Principle: “finite group $\Rightarrow$ nice calculations”.

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Perspectives
Perspectives for the first part

- Nature of the generating functions $Q(x,0,z)$ and $Q(0,y,z)$ in the case of an infinite group;
Perspectives for the first part

- Nature of the generating functions $Q(x,0,z)$ and $Q(0,y,z)$ in the case of an infinite group;

- Slight extensions of the model (like weighted paths or more general behavior on the boundary);
Perspectives for the first part

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- More general jumps;
Perspectives for the first part

- Nature of the generating functions $Q(x,0,z)$ and $Q(0,y,z)$ in the case of an infinite group;
- Slight extensions of the model (like weighted paths or more general behavior on the boundary);
- More general jumps;
- Higher dimension.
Perspectives for the second part

• Asymptotic of the Green functions and Martin compactification for any walk with drift zero;
Perspectives for the second part

- Asymptotic of the Green functions and Martin compactification for any walk with drift zero;

- Asymptotic tail distribution of the hitting time of the boundary for any walk;
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Perspectives for the second part

- Asymptotic of the Green functions and Martin compactification for any walk with drift zero;
- Asymptotic tail distribution of the hitting time of the boundary for any walk;
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- **Higher dimension.**