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Enumerative Combinatorics

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2 March – 8 March 2014

Abstract. Enumerative Combinatorics focusses on the exact and asymptotic counting of combinatorial objects. It is strongly connected to the probabilistic analysis of large combinatorial structures and has fruitful connections to several disciplines, including statistical physics, algebraic combinatorics, graph theory and computer science. This workshop brought together experts from all these various fields, including also computer algebra, with the goal of promoting cooperation and interaction among researchers with largely varying backgrounds.

Mathematics Subject Classification (2010): Primary: 05A; Secondary: 05E, 05C80, 05E, 60C05, 60J, 68R, 82B.

Introduction by the Organisers

The workshop Enumerative Combinatorics organized by Mireille Bousquet-Mélou (Bordeaux), Michael Drmota (Vienna), Christian Krattenthaler (Vienna), and Marc Noy (Barcelona) took place on March 2-8, 2014. There were over 50 participants from the US, Canada, Australia, Japan, Korea, and various European countries. The program consisted of 13 one hour lectures, accompanied by 17 shorter contributions and the special session of 5 presentations by Oberwolfach Leibniz graduate students. There was also a lively problem session led by Svante Linusson. The lectures were intended to provide overviews of the state of the art in various areas and to present relevant new results. The lectures and short talks ranged over a wide variety of topics including classical enumerative problems, algebraic combinatorics, asymptotic and probabilistic methods, statistical physics, methods from computer algebra, among others. Special attention was paid throughout to providing a platform for younger researchers to present themselves and their results. This report contains extended abstracts of the talks and the statements of the problems that were posed during the problem session.

This was the first workshop held on Enumerative Combinatorics. The goal of the workshop was to bring together researchers from different fields with a common interest in enumeration, whether from an algebraic, analytic, probabilistic, geometric or computational angle, in order to enhance collaboration and new research projects. The organizers believe this goal was amply achieved, as demonstrated by the strong interaction among the participants and the lively discussions in and outside the lecture room during the whole week.

On behalf of all participants, the organizers would like to thank the staff and the director of the Mathematisches Forschungsinstitut Oberwolfach for providing such a stimulating and inspiring atmosphere.

Workshop: Enumerative Combinatorics

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Abstracts

Rational Associahedra

DREW ARMSTRONG
(joint work with N. Loehr, B. Rhoades, G. Warrington, N. Williams)

Given a rational number $x \in \mathbb{Q} \setminus \{-1, -\frac{1}{2}, 0\}$ there exist unique coprime integers $a, b \in \mathbb{Z}$ with $0 < a < |b|$ or $0 < b < |a|$ such that

$$x = \frac{a}{b-a}.$$

We will always make the identification $x \leftrightarrow (a, b)$. Given this, we define the Catalan number function $\mathsf{Cat} : \mathbb{Q} \cup \infty \to \mathbb{Z} \cup \infty$ by

$$\mathsf{Cat}(x) = \mathsf{Cat}(a,b) := \begin{cases} \frac{1}{a+b}\binom{a+b}{a,b} & x \in \mathbb{Q} \setminus [-1,0] \\ \frac{(-1)^b}{a+b}\binom{-a-1}{b} & x \in (-1, -\frac{1}{2}) \\ \frac{(-1)^a}{a+b}\binom{-b-1}{a} & x \in (-\frac{1}{2}, 0) \\ 1 & x \in \{-1, 0\} \\ \infty & x = -\frac{1}{2} \\ 0 & x = \infty. \end{cases}$$

Note that for $n \in \mathbb{N}$, $\mathsf{Cat}(n) = \mathsf{Cat}(n, n+1) = \frac{1}{n+1}\binom{2n}{n}$ is the usual Catalan number [8] and $\mathsf{Cat}(n, kn+1)$ is the Fuss-Catalan number [6]. Observe that the function $\mathsf{Cat} : \mathbb{Q} \cup \infty \to \mathbb{Z} \cup \infty$ is symmetric about $x = -\frac{1}{2}$ and from this fact we obtain

$$\mathsf{Cat}\left(\frac{1}{1-x}\right) = \mathsf{Cat}\left(\frac{x}{1-x}\right).$$

We call this common value the derived Catalan number:

$$\mathsf{Cat}'(x) := \mathsf{Cat}\left(\frac{1}{1-x}\right) = \mathsf{Cat}\left(\frac{x}{1-x}\right).$$

Finally, observe the following identity, which we call rational duality:

$$\mathsf{Cat}'(x) = \mathsf{Cat}'(1/x).$$

We will see that rational duality can be categorified as Alexander duality of rational associahedra.

Next let a, b be **positive** coprime integers and consider the rectangle between Euclidean coordinates $(0,0)$ and (b,a). A lattice path from $(0,0)$ to (b,a) staying above the diagonal is called a rational Dyck path. Bizley [4] proved that the number of rational Dyck paths is $\mathsf{Cat}(a,b) = \frac{1}{a+b}\binom{a+b}{a,b}$. More generally [1], one can show that the number of Dyck paths with k vertical runs are given by rational Narayana numbers

$$\mathsf{Nar}(a,b;r) = \frac{1}{a}\binom{a}{k}\binom{b-1}{k-1},$$

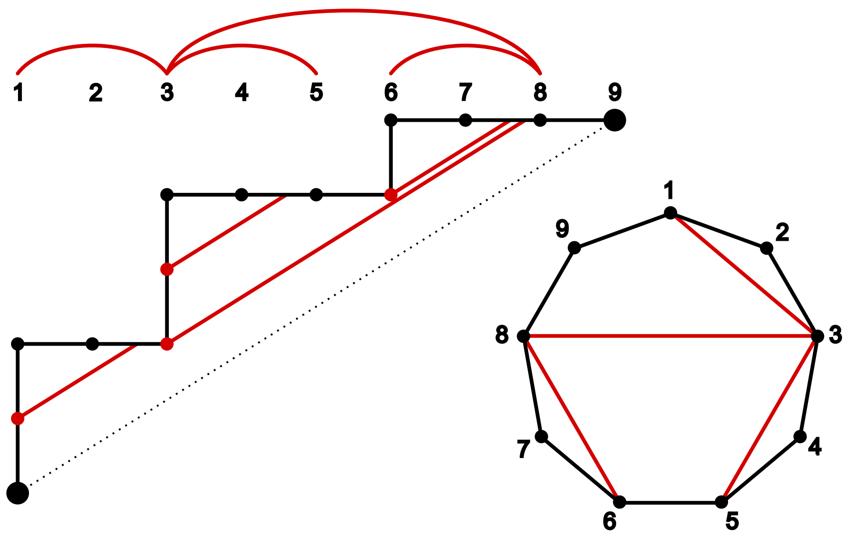

FIGURE 1. An example of a "rational triangulation"

and the number of Dyck paths with r_j vertical runs of length j are given by rational Kreweras numbers

$$\mathsf{Krew}(x;\mathbf{r}) = \frac{1}{b}\binom{b}{r_0, r_1, \ldots, r_a}.$$

For the next construction we assume that $0 < a < b$. Consider a rational Dyck path, shoot "lasers" of slope a/b from the right of each vertex of the path until it hits the path again. Send each laser to the ordered pair of "x-coordinates" where it touches the path (rounding up for the right endpoint). This defines a set of noncrossing chords in a convex $(b+1)$-gon, which we call a rational triangulation. See Figure 1. Note that each rational triangulation has exactly $a-1$ chords, one coming from each up step of the path. Let $\mathsf{Ass}(a,b)$ denote the abstract simplicial complex whose vertex set is a subset of the set of chord of a convex $(b+1)$-gon and whose maximal faces are the rational triangulations. We call this the rational associahedron.

Theorems. [2]

- $\mathsf{Ass}(n, n+1)$ is the classical associahedron.
- $\mathsf{Ass}(n, kn+1)$ is the generalized cluster complex studied by Fomin-Reading [5] and Athanasiadids-Tzanaki [3].
- $\mathsf{Ass}(x)$ has $\mathsf{Cat}(x)$ maximal faces and (reduced) Euler characteristic $\mathsf{Cat}'(x)$.
- $\mathsf{Ass}(x)$ is shellable and hence homotopy equivalent to a wedge of $\mathsf{Cat}'(x)$ many $(a-1)$-dimensional spheres.
- $\mathsf{Ass}(x)$ has h-vector given by the Narayana numbers $\mathsf{Nar}(x;k)$.
- $\mathsf{Ass}(x)$ has f-vector given by the rational Kirkman numbers

$$\mathsf{Kirk}(x;k) := \frac{1}{a}\binom{a}{k}\binom{b+k-1}{k-1}.$$

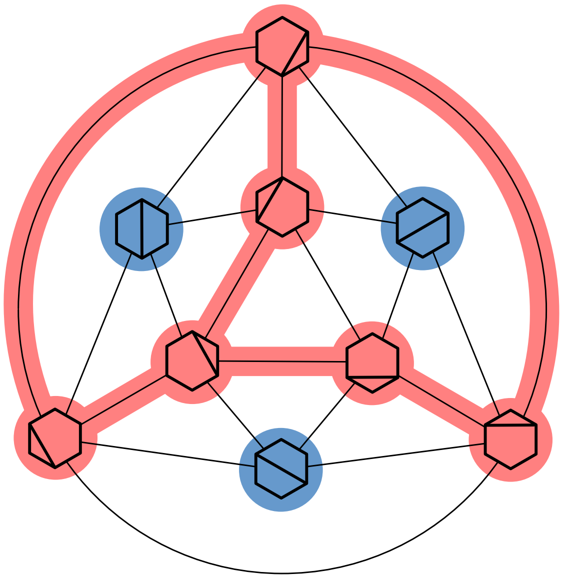

FIGURE 2. The complexes $\mathsf{Ass}(2,5)$ (blue) and $\mathsf{Ass}(3,5)$ (red) inside $\mathsf{Ass}(4)$

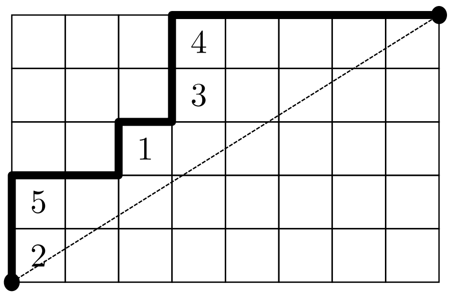

FIGURE 3. A rational parking function

Notice that $0 < a < b$ are coprime if and only if $0 < b-a < b$ are coprime. Notice also that $\mathsf{Ass}(a,b)$ and $\mathsf{Ass}(b-a,b)$ have the same Euler characteristic $\mathsf{Cat}'(a,b) = \mathsf{Cat}(b-a,b)$ (recall "rational duality"). What does this mean? Figure 2 shows the complexes $\mathsf{Ass}(2,5)$ (blue) and $\mathsf{Ass}(3,5)$ (red) as subcomplexes of the classical associahedron $\mathsf{Ass}(4,5) = \mathsf{Ass}(4)$. Note that $\mathsf{Ass}(a,b)$ and $\mathsf{Ass}(b-a,b)$ bipartition the vertices of $\mathsf{Ass}(b-1)$. We conjectured and then Brendon Rhoades proved [9] that these complexes are Alexander dual. That is, if you delete the vertices of $\mathsf{Ass}(a,b)$ from $\mathsf{Ass}(b-1)$ and all faces containing them, then what remains has a deformation retract onto $\mathsf{Ass}(b-a,a)$.

Finally, given a Dyck path from $(0,0)$ to (b,a) we label the up steps with the numbers $\{1,2,\ldots,a\}$ such that numbers in the same column increase going up. We call the result a rational parking function. See Figure 3. Let $\mathsf{PF}(a,b)$ be the character of $\mathfrak{S}_a$ acting on parking functions by permuting labels (and reordering so labels increase in columns). We have the following results.

Theorems. [1]

- The complete homogeneous expansion is
$$\mathsf{PF}(a,b) = \sum_{\mathbf{r}\vdash a} \frac{1}{b}\binom{b}{r_0, r_1, \ldots, r_a} h_{\mathbf{r}},$$
where the sum is over $\mathbf{r} = 0^{r_0}1^{r_1}\cdots a^{r_a}$ with $\sum_i r_i = b$. Note that this is the same as
$$\mathsf{PF}(a,b) = \sum_{\mathbf{r}\vdash a} \frac{1}{b} m_{\mathbf{r}}(1^b) h_{\mathbf{r}}.$$
- Then using the Cauchy product identity gives the power sum expansion
$$\mathsf{PF}(a,b) = \sum_{\mathbf{r}\vdash a} b^{\ell(\mathbf{r})-1}\frac{p_{\mathbf{r}}}{z_{\mathbf{r}}}.$$
That is, the number of parking functions fixed by $\sigma \in \mathfrak{S}_a$ is $b^{\#\mathsf{cycles}(\sigma)-1}$. In particular, the total number of parking functions is b^{a-1}.
- Using the Cauchy product identity again gives the Schur expansion
$$\mathsf{PF}(a,b) = \sum_{\mathbf{r}\vdash a} \frac{1}{b} s_{\mathbf{r}}(1^b) s_{\mathbf{r}}.$$

As a special case, we call the multiplicities of the hook Schur functions that rational Schröder numbers
$$\mathsf{Schrö}(x;k) := \frac{1}{b} s_{[a-k,1^k]}(1^b) = \frac{1}{b}\binom{a-1}{k}\binom{b+k}{a}.$$
We remark that the Schr̈oder numbers are related to the reversed Narayana numbers $\mathsf{Nar}(x;a-k)$ as the Kirkman numbers are related to the Narayana numbers $\mathsf{Nar}(x;k)$. Thus all three sequences of numbers are equivalent. Symmetric q,t-analogues of these numbers are currently of much interest, as they are related to HOMFLY polynomials of torus knots [7].

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Multivariate juggling probabilities

Jérémie Bouttier
(joint work with Arvind Ayyer, Sylvie Corteel and François Nunzi)

This talk is based on [1], where we introduce multivariate generalizations of the models introduced in [2] (see these references for motivations and justifications regarding the relevance of these models).

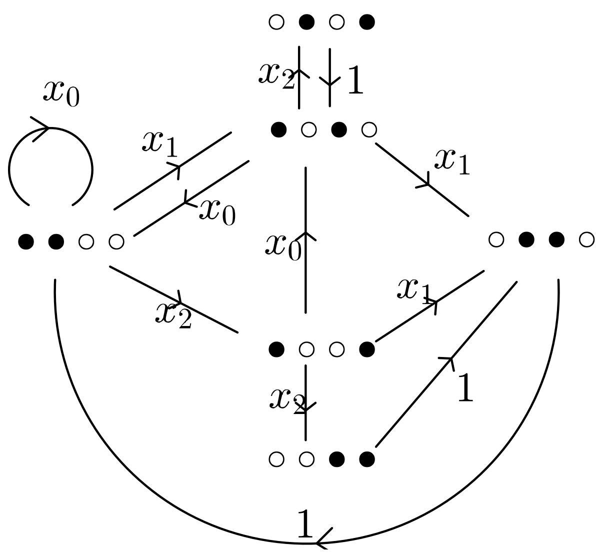

Figure 1. Transition graph for $h = 4$ and $\ell = k = 2$.

We concentrate on the simplest case, describing a juggler juggling randomly with a fixed finite number of balls. The mathematical setting is that of a Markov chain on a finite state space. Let k, ℓ be two nonnegative integers and set $h = k+\ell$. The state space is the set of words of length h on the alphabet $\{\bullet, \circ\}$ containing exactly ℓ instances of $\bullet$ (representing balls) and k instances of $\circ$ (representing vacant sites). The transitions are obtained as follows: at each time step, remove the first letter of the word, append $\circ$ at the end, then replace one of the $k + 1$ resulting instances of $\circ$ by $\bullet$. More precisely, we replace the $(i + 1)$-th occurrence of $\circ$ with probability x_i ($i = 0, \dots, k$), where $x_0, \dots, x_k$ are nonnegative real parameters summing up to 1. Figure 1 displays the transition graph for $\ell = k = 2$ and $h = 4$. In the uniform case $x_0 = \cdots = x_k = 1/(k + 1)$, we recover the model considered in [2]. It is not difficult to check that the chain is irreducible for $x_0, x_k > 0$ and that $x_0 > 0$ suffices to ensure the existence of a unique stationary distribution. Our main result is the following.

Theorem 1 ([1]). *Let $y(m) = x_m + \cdots + x_k$. The stationary probability of a state $B = b_1 \cdots b_h$ in our Markov chain is equal to*

$$\pi(B) = \frac{1}{Z} \prod_{i:b_i=\bullet} y\left(\#\{j < i, b_j = \circ\}\right) \tag{1}$$

with $Z = h_\ell(y(0), \ldots, y(k))$, h_ℓ denoting the complete homogeneous symmetric polynomial of degree ℓ.

We may prove this result by two approaches: either by a direct check (leading to an interesting reformulation in terms of integer partitions) or, more combinatorially, by introducing another Markov chain on some "enriched" state space, for which the stationary distribution will be "simple". More precisely, the stationary probabilities of the enriched states should match exactly the monomials obtained in the expansion of $\pi(B)$, as the word B varies. It turns out that the enriched states can be seen as partitions of the set $\{1, \ldots, h + 1\}$ in $k + 1$ blocks. The stationary probability of such a set partition is then obtained by attaching to each *arch* of the partition a weight x_b where b is the number of blocks that it *covers*.

We then consider several generalizations of the model. First, we may consider extensions to infinite state spaces, in which the juggler can throw balls arbitrarily high and/or keep infinitely many balls in the air. In all cases, we obtain an explicit invariant measure of the chain, as well as a necessary and sufficient condition for the model to be positive recurrent. Second, we may consider extensions where the number of balls is allowed to fluctuate: we find the multivariate generalizations of the so-called add-drop and annihilation models introduced in [2]. Again, we obtain explicit expressions for their stationary distributions. We also find that, in the annihilation model, the stationary distribution of the model is attained in h time steps regardless of initial state.

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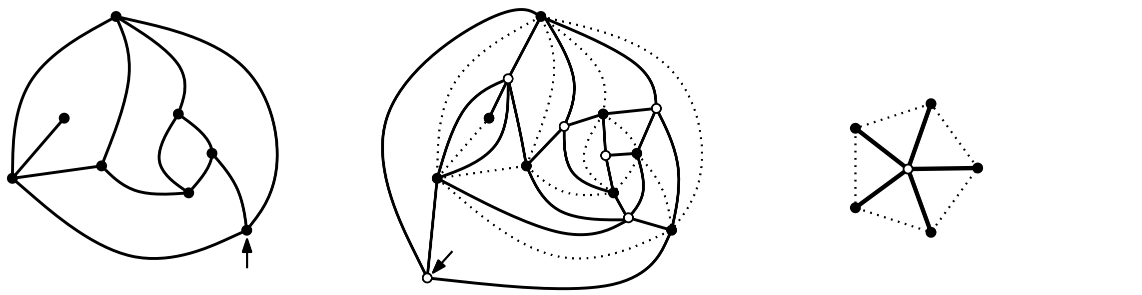
A simple recurrence formula to count maps by edges and genus.

Guillaume Chapuy

My talk at the Oberwolfach *Enumerative Combinatorics* meeting is based on a joint paper with Sean R. Carrell (University of Waterloo, Canada), in which we establish a simple recurrence formula for the number Q_g^n of rooted orientable maps counted by edges and genus [2]. This formula gives by far the fastest known way of computing these numbers, or the fixed-genus generating functions, especially for large g. Our formula is a consequence of the KP equation for the generating function of bipartite maps, coupled with a Tutte equation, and it was apparently unnoticed before. It is similar in look to a formula discovered by Goulden and

Jackson for triangulations [4], and indeed our method to go from the KP equation to the recurrence formula can be seen as a combinatorial simplification of Goulden and Jackson's approach (together with one additional combinatorial trick). These formulas have a very combinatorial flavour, but finding a bijective interpretation is currently unsolved.

Recall that a *map* is a connected graph embedded in a compact connected orientable surface in such a way that the regions delimited by the graph, called *faces*, are homeomorphic to open discs. Loops and multiple edges are allowed. A *rooted* map is a map in which an angular sector incident to a vertex is distinguished. Rooted maps are considered up to oriented homeomorphisms preserving the root sector. A *quadrangulation* is a map in which every face has degree 4. The first ingredient of our work is a classical bijection, that goes back to Tutte, between bipartite quadrangulations with n faces and genus g, and rooted maps with n edges and genus g. It is illustrated on Figure 1.



(a) A map m

(b) its associated bipartite quadrangulation q (thick edges)

(c) the local rules of the construction around a face of m

FIGURE 1. **Tutte's bijection** between general maps and bipartite quadrangulations. Root corners are indicated by arrows.

The second ingredient, that can be attributed to Goulden and Jackson [4] (although it was known before in the mathematical physics literature), is the fact that the multivariate generating function $H \equiv H(z, w; \mathbf{p})$ of bipartite maps, where z marks the number of edges, w marks the vertices, and an infinite collection of variables $\mathbf{p} = p_1, p_2, \ldots$ marks the face degrees, is a solution of the KP equation:

$$-H_{3,1} + H_{2,2} + \frac{1}{12}H_{1^4} + \frac{1}{2}(H_{1,1})^2 = 0, \tag{1}$$

where indices indicate partial derivatives with respect to the variables p_i, for example $H_{3,1} := \frac{\partial^2}{\partial p_3 \partial p_1}H$. This non trivial fact takes its roots in the deep connections between maps and the group algebra of the symmetric group, and in the fact that map generating functions can be expressed with Schur functions. No combinatorial interpretation of this statement is known in the world of maps.

From there, using only simple combinatorial arguments, we were able to get a closed recurrence formula for the number Q_g^n of rooted maps of genus g with n

edges (which is also the number of rooted bipartite quadrangulations of genus g with n faces). More precisely our main result is the following recurrence relation:

$$\frac{n+1}{6}Q_g^n = \frac{4n-2}{3}Q_g^{n-1} + \frac{(2n-3)(2n-2)(2n-1)}{12}Q_{g-1}^{n-2} + \frac{1}{2}\sum_{\substack{k+\ell=n\\k,\ell\geq 1}}\sum_{\substack{i+j=g\\i,j\geq 0}}(2k-1)(2\ell-1)Q_i^{k-1}Q_j^{\ell-1},$$

for $n \geq 1$, with the initial conditions $Q_g^0 = \mathbf{1}_{\{g=0\}}$, and $Q_g^n = 0$ if $g < 0$ or $n < 0$.

This recurrence formula is extremely simple compared to other existing methods to count maps. In particular, approaches based on Tutte equations (such as [1, 3]) require to add $O(g)$ additional variables in order to obtain closed recurrence formulas. With the help of our formula, it is very easy to get, for each $g \geq 0$, a closed form for the generating function $Q_g(t) = \sum_{n\geq 0} Q_g^n t^n$.

The Oberwolfach meeting and a crucial improvement

During the conference a participant (Éric Fusy) asked me the following simple question: could you also keep track of the face number in your recursion? We had not thought of this question before, and it turned out that the answer was (almost straightforwardly) yes. This led to the following result. Let $M_g^{i,j}$ be the number of rooted maps of genus g with i vertices and j faces. Then we have the following recurrence relation:

$$\frac{n+1}{6}M_g^{i,j} = \frac{(2n-1)}{3}\Big(M_g^{i-1,j} + M_g^{i,j-1} + \frac{(2n-3)(2n-2)}{4}M_{g-1}^{i,j}\Big) + \frac{1}{2}\sum_{\substack{i_1+i_2=i\\i_1,i_2\geq 1}}\sum_{\substack{j_1+j_2=j,\\j_1,j_2\geq 1}}\sum_{\substack{g_1+g_2=g\\g_1,g_2\geq 0}}(2n_1-1)(2n_2-1)M_{g_1}^{i_1,j_1}M_{g_2}^{i_2,j_2},$$

for $i,j \geq 1$, with the initial conditions that $M_g^{i,j} = 0$ if $i+j+2g < 2$, that if $i+j+2g = 2$ then $M_g^{i,j} = \mathbf{1}_{\{(i,j)=(1,1)\}}$, and where we use the notation $n = i+j+2g-2$, $n_1 = i_1+j_1+2g_1-1$, and $n_2 = i_2+j_2+2g_2-1$.

This last result is remarkable, in particular because it contains (as the very special case of one-face maps, i.e. $j=1$), the Harer-Zagier recurrence formula [5]. It is surprising that the Harer-Zagier formula, that has been known for years (and whose all other proofs I know *really* rely on the very particular nature of one-face maps) has such a simple generalization to arbitrarily many faces. This formula and the one for Q_g^n seem "too simple" compared to our present understanding of maps, and we hope to learn much looking for bijective interpretation of them.

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A triangular gap of size two in a sea of dimers on a 60° angle

Mihai Ciucu
(joint work with Ilse Fischer)

In their paper [10] from 1963, Fisher and Stephenson have introduced the concept of the correlation of two monomers in a sea of dimers, and based on their precise numerical findings conjectured that this correlation is rotationally invariant in the scaling limit. In a series of articles (see [3][4][5][9]), the first author has extended the problem of Fisher and Stephenson to the situation when one is allowed to have any finite number of gaps, each of an arbitrary size, and has shown that a close parallel to electrostatics emerges: As the distances between the gaps approach infinity, their correlation is given by the exponential of the electrostatic energy of a two dimensional system of charges that correspond to the gaps in a natural way.

This parallel to electrostatics has been extended in [6] and [7], where it was shown that the discrete field of the average tile orientations approaches, in the scaling limit, the electric field.

One particular aspect of this analogy is the behavior of the correlation of gaps near the boundary of lattice regions, which turns out to be in close connection with the behavior of charges near conductors. In [4] it was shown that the asymptotics of the correlation of gaps on the triangular lattice near a constrained zig-zag boundary is given by a variant of the method of images from electrostatics, in which the image charges have the same signs as the original ones. The case of a free boundary was considered in [8], where it was shown that the correlation of a single gap of size two with a free lattice line boundary on the triangular lattice is given, in the scaling limit, precisely by the method of images from electrostatics.

In this paper the analogy to the method of images is given more substance by establishing it in a more complex setting, in which the gap has not just one image (as it was the case in [4] and [8]), but five. Indeed, we consider a triangular gap of size two in a 60° degree angular region on the triangular lattice whose sides are zig-zags. The gap has two direct images in the two sides of the angular region, which generate further images in the sides, to a total of five images of the original gap. The main result of this paper is that the asymptotics of the correlation of the gap with the corner of the angular region, as the distances between the gap and the sides grow large, is given by a numerical constant times the exponential of one sixth of the electrostatic energy of the 2D system of charges consisting of the gap viewed as a charge, together with its above five images. The proof is based on Kuo's method of graphical condensation [11].

We note that, from the point of view of the literature on plane partitions and their symmetry classes (see for instance [13], [1], [14] and [12]), two other results of this paper represent generalizations of the cyclically symmetric, self-complementary case, first solved by Kuperberg in [12] (see [2] for a simple proof).

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Steep tilings: from Aztec diamonds and pyramids to general enumeration

Sylvie Corteel

(joint work with Jérémie Bouttier, Guillaume Chapuy)

An Aztec diamond of order ℓ consists of all squares of a square lattice whose centers (x, y) satisfy $|x| + |y| \leq \ell$. Here ℓ is a fixed integer, and the square lattice consists of unit squares with integer coordinates, so that both x and y are half-integers [2]. See Figure 1 for $\ell = 4$. The Aztec diamond theorem states that the number of domino tilings of the Aztec diamond of order ℓ is $2^{\ell(\ell+1)/2}$ [2]. A more precise result [3] states that the generating polynomial of these tilings with respect to the minimum number of flips needed to obtain a tiling from the one with all horizontal tiles is $\prod_{i=1}^{\ell}(1 + q^{2i-1})^{\ell+1-i}$. See below for the definition of a flip.

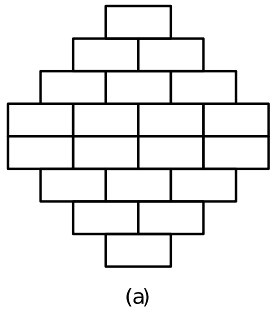

FIGURE 1. (a) The Aztec diamond of size 4, with its minimal tiling consisting only of horizontal dominos; (b) Another tiling of the same region.

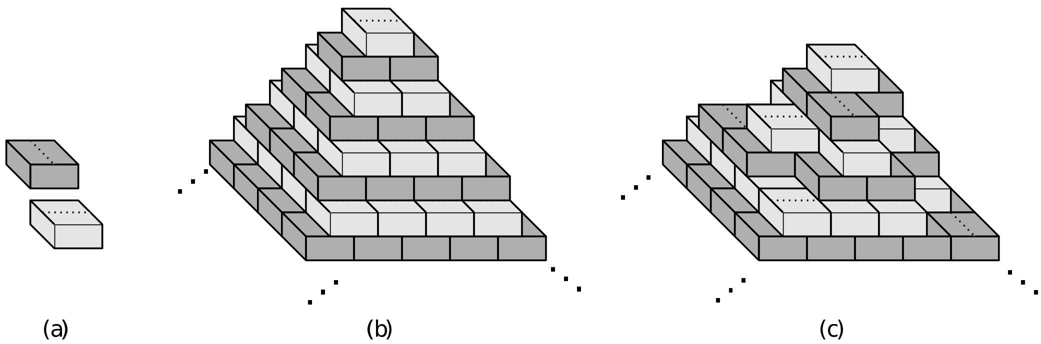

FIGURE 2. (a) The two types of bricks used in the construction of pyramid partitions; (b) The "minimal" pyramid partition, from which all others are obtained by removing some bricks; (c) A pyramid partition.

A pyramid partition is an infinite heap of bricks of size $2 \times 2 \times 1$ in $\mathbb{R}^3$, as shown on Figure 2. A pyramid partition has a finite number of maximal bricks and each brick rests upon two side-by-side bricks, and is rotated 90 degrees from the bricks immediately below it. The *empty pyramid partition* is the pyramid partition with a unique maximal brick. We denote by a_n the number of pyramid partitions obtained from the empty pyramid partition after the removal of n bricks. The generating function $P(q) = \sum_n a_n q^n$ was conjectured by Kenyon [4] and Szendrői [5]. It was computed by Ben Young [6] using a generalization of the domino shuffling algorithm [3]:

$$P(q) = \prod_{k \geq 0} \frac{(1+q^{2k-1})^{2k-1}}{(1-q^{2k})^{2k}}.$$

Aztec diamond and pyramid partitions are closely related. Indeed a pyramid partition can be seen as a tiling of the whole plane. In this setting the removal of a brick corresponds to the flip of two dominos. The goal of this paper is to show that these are indeed part of the same family of tilings that we call *steep tilings*.

Recall that a *domino* is a 2×1 (horizontal domino) or 1×2 (vertical domino) rectangle whose corners have integer coordinates. Fix a positive integer ℓ, and consider the *oblique strip* (of width 2ℓ) which is the region of the xy plane comprised between the lines $y = x$ and $y = x - 2\ell$. A *tiling* of the oblique strip is a set of dominos whose interiors are disjoint, and whose union R is "almost" the oblique strip in the sense that

$$\{(x, y) \in \mathbb{R}^2, |x - y - \ell| \leq \ell - 1\} \subset R \subset \{(x, y) \in \mathbb{R}^2, |x - y - \ell| \leq \ell + 1\}. \tag{1}$$

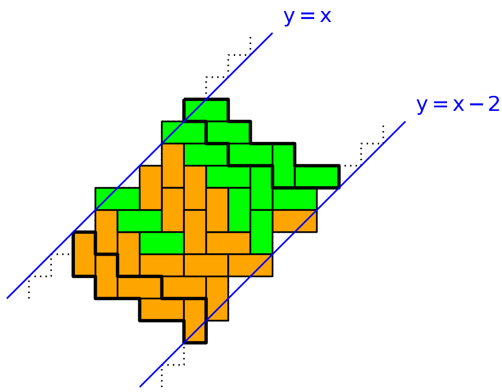

Figure 3. Left: a steep tiling of the oblique strip of width $2\ell = 10$. North- and east-going (resp. south- and west-going) dominos are represented in green (resp. orange). Outside of the displayed region, the tiling is obtained by repeating the "fundamental patterns" surrounded by thick lines.

Following a classical terminology [1], we say that a horizontal (resp. vertical) domino is *north-going* (resp. *east-going*) if the sum of the coordinates of its top left corner is odd, and *south-going* (resp. *west-going*) otherwise. We are interested in tilings of the oblique strip which are *steep* in the following sense: going towards infinity in the north-east (resp. south-west) direction, we eventually encounter only north- or east-going (resp. south- or west-going) dominos. Figure 3 displays an example of such a tiling.

Proposition 1. *Given a steep tiling of the oblique strip of width 2ℓ, there exists a unique word $w = (w_1, \ldots, w_{2\ell})$ on the alphabet $\{+, -\}$ and an integer A such that, for all $k \in \{1, \ldots, \ell\}$, the following hold:*

- *for all $x > A$, $(x, x - 2k)$ is the bottom right corner of a domino which is north-going if $w_{2k-1} = +$ and east-going if $w_{2k-1} = -$,*
- *for all $x < -A$, $(x, x - 2k + 2)$ is the top left corner of a domino which is west-going if $w_{2k} = +$ and south-going if $w_{2k} = -$.*

Example 1. The steep tiling of Figure 3 corresponds to the word $w = (+ + + + + - - - ++)$.

A steep tiling is called *pure* if there is no "gap" between covered and uncovered squares on each of the two lines $y = x$ and $y = x - 2\ell$, *i.e.* if there exists two half integers a, b such that the following two conditions hold:

(1) the unit square centered at (x, x) is covered if $x \geq a$ and uncovered if $x < a$.
(2) the unit square centered at $(x, x - 2\ell)$ is covered if $x \leq b$ and uncovered if $x > b$.

We denote by $\mathcal{T}_w$ the set of pure steep tilings of asymptotic data w, considered up to translation along the direction $(1, 1)$.

In order to state our main result, we need to introduce the notion of flip. A *flip* is the operation consisting of replacing a pair of horizontal dominos forming a 2×2 block by a pair of vertical dominos, or vice-versa. A flip can be horizontal-to-vertical or vertical-to-horizontal with obvious definitions. We say that the flip is centered on the k-th diagonal if the center of the 2×2 block lies on the diagonal $y = x - k$, for $0 < k < 2\ell$. For each word $w \in \{+, -\}^{2\ell}$, there exists a unique element of $\mathcal{T}_w$, called the *minimal tiling*, such that every element of $\mathcal{T}_w$ can be obtained from it using only flips. Our main results are the two following theorems:

Theorem 2. *Let $w \in \{+, -\}^{2\ell}$ be a word. Let $T_w(q)$ be the generating function of pure steep tilings of asymptotic data w, where the exponent of q records the minimal number of flips needed to obtain a tiling from the minimal one. Then one has:*

$$T_w(q) = \prod_{\substack{i<j \\ w_i=+, w_j=- \\ i-j \text{ odd}}} \left(1 + q^{j-i}\right) \prod_{\substack{i<j \\ w_i=+, w_j=- \\ i-j \text{ even}}} \frac{1}{1 - q^{j-i}}.$$

Theorem 3. *Let $w \in \{+, -\}^{2\ell}$ be a word. Let $T_w \equiv T_w(x_1, \ldots, x_{2\ell-1})$ be the generating function of pure steep tilings of asymptotic data w, where the exponent of the variable x_i records the number of flips centered on the i-th diagonal in a shortest sequence of flips from the minimal tiling. Then one has:*

$$T_w = \prod_{\substack{i<j \\ w_i=+, w_j=- \\ i-j \text{ odd}}} (1 + x_i x_{i+1} \ldots x_{j-1}) \prod_{\substack{i<j \\ w_i=+, w_j=- \\ i-j \text{ even}}} \frac{1}{1 - x_i x_{i+1} \ldots x_{j-1}}.$$

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The Tutte polynomial and planar maps

Julien Courtiel

Intended as a generalization of the chromatic polynomial [1, 2], the Tutte polynomial is a graph invariant playing a fundamental role in graph theory. It has strong connections with the Potts model in statistical physics. This model consists in colouring the vertices of a graph and counting all such configurations according to the number of monochromatic edges (edges whose endpoints have the same colour).

Given a connected graph G, the Tutte polynomial of G, denoted by T_G, is the generating function of spanning subgraphs of G counted by the number of components and the cyclomatic number (i.e. the minimal number of edges one needs to remove from the graph to make it acyclic), respectively denoted by $\mathrm{cc}(S)$ and $\mathrm{cycl}(S)$ for a subgraph S:

$$T_G(x, y) = \sum_{S \text{ subgraph of } G} (x-1)^{\mathrm{cc}(S)-1}(y-1)^{\mathrm{cycl}(S)}. \tag{1}$$

This talk introduces two parts of my PhD work dealing with this polynomial.

Spanning forests in planar maps

(joint work with Mireille Bousquet Mélou)

A planar map is a connected graph properly embedded in the sphere. The enumerative and asymptotic properties of planar maps are well understood nowadays, thanks to a broad range of enumerative methods: Tutte's recursive approach, Schaeffer's bijections, matrix integrals... Many physicists, combinatorists and probabilists now devote their work to planar maps equipped with an additional structure. As such, a lot of current research is oriented towards the Potts model on several families of planar maps [3, 4, 5, 6, 7]. As mentioned in the introduction, this also means counting maps weighted by their Tutte polynomial.

We study a one-variable specialization of the Tutte polynomial. This specialization is obtained by setting to 1 the variable that counts the number of cycles, and can be also seen as a certain limit $q \to 0$ in the Potts model. Combinatorially, we simply count planar maps equipped with a *spanning forest*, also named *forested maps*.

The resulting generating function $F(z, u)$ keeps track of the size of the map (the number of faces or edges; variable z) and of the number of trees in the forest (minus one; variable u). In particular, the specialization $u = 0$ counts maps equipped with a spanning tree, a case which had been solved by Mullin a long time ago [8]. We have characterized in a purely combinatorial way the generating function $F(z, u)$ as the solution of a system of functional equations.

The first question we asked is the nature of the series F. We have shown that $F(z, u)$ is D-algebraic, meaning that it satisfies a differential equation in z with polynomial coefficients. Furthermore, we have been able to compute the explicit differential equations in many cases. These equations are huge, but we have good reasons to believe that they can not be reduced. We have also proved that these generating functions are not *D-finite*, which means that they do not satisfy any linear differential equation in z for a generic value of u.

The series $F(z,u)$ has a natural combinatorial interpretation where the natural domain for u is $[-1,+\infty)$ rather than $[0,+\infty)$. Indeed, one can see $F(z,v-1)$ as the generating function of maps equipped with a spanning tree, where v counts now the number of *internally active* edges. (This notion will be defined in the next section.) We have thereby studied the asymptotic behaviour of the coefficients of z^n in $F(z,u)$, when u is a fixed number in $[-1,+\infty)$. A phase transition occurs at $u=0$ (where one enumerates maps equipped with a spanning tree): When $u>0$, the asymptotic regime is standard for planar maps ($\mu^n\, n^{-\frac{5}{2}}$) but when $u<0$ we have established a very unusual asymptotic behaviour in $\mu^n\, n^{-3}\,(\ln n)^{-2}$. It is the first time to our knowledge that such a regime is seen in the world of maps.

A general framework for edge activities

My work also deals with the Tutte polynomial *per se*, in a different (but connected) perspective. As already mentioned, the Tutte polynomial $T_G(x,y)$ has non-negative coefficients in x and y. Tutte found in 1954 a combinatorial interpretation for these coefficients, proving that $T_G(x,y)$ counts spanning trees of G by the number of internal and external "active" edges [2]:

$$T_G(x,y) = \sum_{T \text{ spanning tree of } G} x^{\mathrm{intact}(T)}\, y^{\mathrm{extact}(T)}. \tag{2}$$

The notion of activity involves a linear order on the edges. Some decades later, Bernardi gave a similar characterization with a notion of activity involving this time an embedding of the graph [9]. One also finds in the literature notions of partial edge activity. For example, Gessel and Sagan have defined a so-called DFS-activity for external edges [10].

I have found a general framework that unifies these (non equivalent) notions of activity. I thus defined a notion of Δ-activity which includes as special cases all these definitions. I gave several proofs of the equivalence between (1) and (2). Also, I clarified the underlying combinatorics, already discussed in a former paper of Gordon and Traldi [11].

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T-system combinatorics: from friezes to arctic curves

Philippe Di Francesco
(joint work with Rodrigo Soto Garrido)

The octahedron relation is the following system of recursion relations for a quantity $T_{i,j,k}$, $i,j,k \in \mathbb{Z}$:

$$T_{i,j,k+1}T_{i,j,k-1} = T_{i+1,j,k}T_{i-1,j,k} + T_{i,j+1,k}T_{i,j-1,k} \tag{1}$$

It can be understood as a 2+1-dimensional dynamical system, in which k plays the role of discrete time. This equation must be supplemented with some initial condition which consists of a pair $(\mathbf{k}, \mathbf{t})$ where $\mathbf{k}$ is a stepped surface $\mathbf{k} = \{(i,j,k_{i,j})|i,j \in \mathbb{Z}\}$ where $|k_{i+1,j} - k_{i,j}| = |k_{i,j+1} - k_{i,j}| = 1$ for all $i,j \in \mathbb{Z}$., and $\mathbf{t} = (t_{i,j})_{i,j\in\mathbb{Z}}$ are initial values to be assigned along the stepped surface $\mathbf{k}$, namely we have to impose the intial condition $T_{i,j,k_{i,j}} = t_{i,j}$ for all $i,j \in \mathbb{Z}$.

Various restrictions of this equation have first appeared in the physics literature under the name of T-systems, originally introduced in the context of integrable quantum spin chains with Lie algebra symmetry [11]. The simplest of these is the so-called A_1 T-system, and consists of imposing the extra boundary condition $T_{0,j,k} = T_{2,j,k} = 1$ and restricting to indices $i = 0, 1, 2$. Denoting by $T_{j,k} = T_{1,j,k}$, the A_1 T-system reads for $j,k \in \mathbb{Z}$ with $j + k = 0 \bmod 2$:

$$T_{j,k+1}T_{j,k-1} = 1 + T_{j+1,k}T_{j-1,k} \tag{2}$$

Equivalently, we may picture $T_{j,k}$ as variables defined at the vertices of the square lattice $\{(j,k) \in \mathbb{Z}^2 | j + k = 1 \bmod 2\}$, and such that the determinant for each elementary square is 1. This is nothing but the Coxeter-Conway frieze condition [2]. Initial data for the latter are of the form $(\mathbf{k}, \mathbf{t})$ where $\mathbf{k}$ is a path $\mathbf{k} = \{(j, k_j), j \in \mathbb{Z}\}$ with $|k_{j+1} - k_j| = 1$ for all j, and $\mathbf{t} = (t_j)_{j\in Z}$ the initial value assignment $T_{j,k_j} = t_j$ for all j. A frieze corresponds usually to taking $t_j = 1$ for all j, and gives rise to only non-negative integers $T_{j,k}$ for $j,k \in \mathbb{Z}$.

The octahedron equation and the various T-systems based on Lie algebras were shown [4] to be particular mutations in suitable, possibly infinite rank cluster algebras [6]. As such, they enjoy the Laurent property, which implies that the solution $T_{i,j,k}$ is a Laurent polynomial of the initial data $t_{i,j}$ for all k. This explains the integrality of the friezes, however the positivity relies on a stronger conjecture and partial result that the Laurent polynomials have always non-negative integer coefficients. For the octahedron and T-system equations (1-2), this can be proved by formulating the general solution as the partition function for a statistical model with Boltzmann weights that are positive Laurent monomials of the initial data $t_{i,j}$ [13, 3].

The T-systems are discrete integrable systems, namely there exist quantities that are conserved modulo the equation of evolution. Alternatively, these systems may all be solved exactly using the existence of a flat connection on their solutions [1, 3]. By interpreting these in terms of weighted directed graphs or networks, and configurations of non-intersecting paths on them, it is possible to express the solution $T_{i,j,k}$ as a sum over dimer configurations (or matchings) of a graph coded by the initial data surface $\mathbf{k}$, with weights expressed in terms of the initial values $t_{i,j}$. In the case of the T-system with "flat" initial data surface $\mathbf{k}^{(0)}$, with $k_{i,j}^{(0)} = 1 + i + j \bmod 2$, the corresponding graph is known to be the Aztec graph, dual to the so-called Aztec diamond. The latter may be cut out of the square lattice $\mathbb{Z}^2$ with edges connecting nearest neighbor vertices, by isolating the domain inside the tilted square of size k $\{(x,y) \in \mathbb{Z}^2 || x - i - \frac{1}{2}| + |y - j - \frac{1}{2}| \leq k\}$. The configurations of the model are obtained by occupying edges with dimers in such a way that each vertex of the Aztec graph belongs to exactly one dimer, and the face centered at $(a + \frac{1}{2}, b + \frac{1}{2})$ receives the weight $(t_{a,b})^{1-D_{a,b}}$, where $D_{a,b}$ is the total number of dimers covering the edges of the face (a, b), while the total weight of the configuration is the product of all face weights. Summing over configurations produces clearly a Laurent polynomial of the initial values $t_{a,b}$ with non-negative integer coefficients.

Dimer models have been the subject of intensive study, in particular their thermodynamic limit of large size and small mesh, in which the dimer configurations display very different qualitative phases depending on the shape of the covered graphs. For the Aztec diamond with uniform weights ($t_{i,j} = 1$ for all i, j, so that $T_{i,j,k} = 2^{k(k+1)/2}$), one can show that each corner induces a crystalline phase in which each square is adjacent to exactly one dimer, whereas the dimer configurations gain entropy as one goes away from the corners. The separation of phases is along the famous "arctic circle" [7].

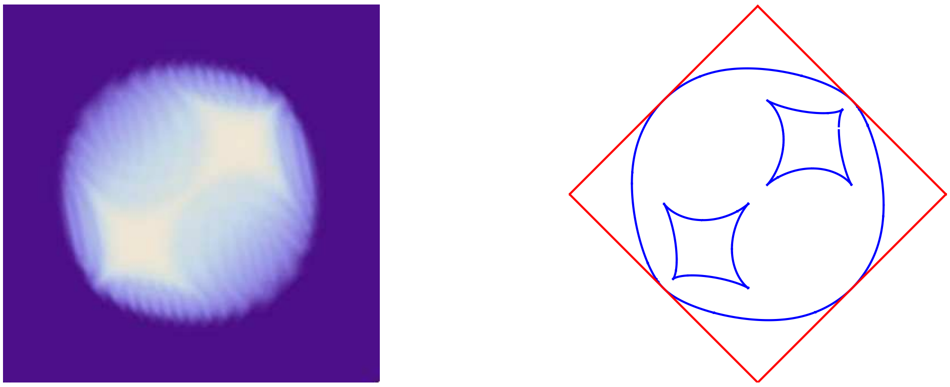
This is easily explained by considering the *density* function [10]:

$$\rho_{i,j,k} = \partial_{t_{0,0}} \mathrm{Log}\, T_{i,j,k} \Big|_{t_{0,0}=1} = \langle 1 - D_{0,0} \rangle \tag{3}$$

which measures the average over dimer configurations of the quantity $1 - D_{0,0}$ around the square labelled $(0, 0)$. We may derive a simple linear recursion relation for ρ by simply differentiating the octahedron equation (1):

$$2(\rho_{i,j,k+1} + \rho_{i,j,k-1}) = \rho_{i+1,j,k} + \rho_{i-1,j,k} + \rho_{i,j+1,k} + \rho_{i,j-1,k}$$

with the initial condition that $\rho_{i,j,0} = 0$ and $\rho_{i,j,1} = \delta_{i,0}\delta_{j,0}$. Solving this recursion gives access to the large k behavior of $\rho_{uk,vk,k}$ for fixed u, v with $|u| + |v| \leq 1$. The generating series $\rho(x, y, z) = \sum_{i,j,\in\mathbb{Z};k\in\mathbb{Z}_+} x^i y^j z^k \rho_{i,j,k}$ is a simple rational fraction, and the singularity locus from its denominator immediately give the phase separation, along the circle $2(u^2 + v^2) = 1$, using the theory of multivariate



(a) (b)

FIGURE 1. Contour plot (a) of the actual density $\rho_{i,j,k}$ for $k = 77$ as a function of (i, j) for some sample doubly periodic initial data $t_{i,j} = t_{i+2,j+2} = t_{i+m,j-m}$ for $m = 3$, and the corresponding asymptotic arctic curve (b).

asymptotics [12]. The above simple argument relies on the existence of a simple solution for the T-system for uniform initial values on the flat stepped surface $\mathbf{k}^{(0)}$.

However there are other simple solutions [5] for the surface $\mathbf{k}^{(0)}$ with non-uniform values of $t_{i,j}$, namely those that are doubly periodic with $t_{i+2,j+2} = t_{i,j}$ and $t_{i+m,j-m} = t_{i,j}$ for some fixed $m > 0$. In this case, we may write the solution $T_{i,j,k}$ of (1) explicitly as a Laurent monomial of the $T_{i,j,\ell}$, $\ell = 0, 1, 2, 3$, that is also a Laurent polynomial of the $t_{i,j}$ with $2^{k(k+1)/2}$ terms. Differentiating again the octahedron relation we get a linear recursion for the density in the form:

$$(4)\quad \rho_{i,j,k+1} + \rho_{i,j,k-1} = \lambda_{i,j,k}(\rho_{i+1,j,k} + \rho_{i-1,j,k}) + (1 - \lambda_{i,j,k})(\rho_{i,j+1,k} + \rho_{i,j-1,k})$$

with $\lambda_{i,j,k}$ an explicit function of the $t_{i,j}$'s which turns out to be triply periodic: $\lambda_{i+2,j+2,k} = \lambda_{i+m,j-m,k} = \lambda_{i+1,j+1,k+2}$. As a consequence, the generating function $\rho(x, y, z)$ is again a rational fraction, and we may again derive its singularity locus explicitly. For $m \geq 2$, the resulting algebraic arctic curve has generically m real connected components: an exterior higher degree curve tangent to the bounding square replacing the arctic circle, still separating the frozen crystalline corners from the disordered phase, but there are also $(m - 1)$ 4-cusp inner curves each corresponding to a new facet-like (or pseudo-crystalline) phase within which the dimer configurations are pinned to some sublattice, albeit retaining some entropy. This is illustrated in Fig.1 for $m = 3$, where we have represented (a) the contour plot of some sample solution $\rho_{i,j,k}$ of (4) and (b) the corresponding degree 14 arctic curve. This exact result illustrates the general theory of [8, 9].

More generally, we may consider for a general cluster algebra say of geometric type, with skew-symmetric exchange matrix $B_{i,j}$ coding a quiver Q, and a mutation relation of the form:

$$x_k x'_k = \prod_{\substack{\text{tails } i \to k \\ in Q}} x_i + \prod_{\substack{\text{heads } k \to j \\ in Q}} x_j \ ,$$

the analogue of the density $\rho_i = \partial_t \mathrm{Log}\, x_i\big|_{t=1}$, defined as the first order variation of any cluster variable x_i w.r.t. some initial cluster data t. The latter obeys the linearized mutation equation:

$$\rho_k + \rho'_k = \lambda_k \sum_{\text{tails } i\to k} \rho_i + (1-\lambda_k) \sum_{\text{heads } k\to j} \rho_j$$

with $\lambda_k = y_k/(1+y_k)$, where $y_k = \prod_j x_i^{B_{j,i}}$. In analogy with the above results, it would be interesting to classify the "B- and y-finite" cluster algebras, namely those for which the equations for ρ will be periodic, possibly by restricting the set of allowed mutations. Our octahedron equation with particular periodic initial conditions gives an example of these, by noting that the corresponding infinite quiver is *de facto* folded into a finite quiver with the B- and y-finiteness property, when allowing only mutations at vertices with two tails and two heads.

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Generalized Quandrangulation Relation for Constellations and Hypermaps

Wenjie Fang

Quandrangulations, constellations and hypermaps are all classes of *combinatorial maps*. A combinatorial map is simply a graph embedded onto a surface, with the condition that it cuts the surface into topological disks called faces.

Quadrangulations are combinatorial maps with all faces of degree 4. In 1999, Jackson and Visentin discovered in [1] the following enumerative relation between ordinary quadrangulations in genus g and bipartite quadrangulations, *i.e.* with vertices colored black and white, with marked vertices in genus at most g.

$$Q_n^{(g)} = 2^{2g} B_n^{(g,0)} + 4^{2g-2} B_n^{(g-1,2)} + \ldots + B_n^{(0,2g)} = \sum_{g'=0}^{g} 2^{2g'} B_n^{(g',2(g-g'))}$$

Here, n is the number of edges, and g the genus. $Q_n^{(g)}$ counts *ordinary* quandrangulations, and $B_n^{(g,k)}$ counts *bipartite* quadrangulations with k marked black vertices. This relation expresses the number of colored objects on higher genus as a weighted sum of numbers of colorless objects on lower genera. This relation is later extended by the same authors in [2] to any maps with faces of even degree, possibly with arbitrary restriction on face degrees.

In our work, we present a generalization of this simple enumerative relation to constellations and hypermaps. Constellations extend the notion of bipartiteness on combinatorial maps to an arbitrary number m of colors. On the other hand, a hypermap can be seen as a "colorless" constellation. We obtain the following simple relation between m-constellations and m-hypermaps with m colors.

$$H_{n,m,D}^{(g)} = \sum_{i=0}^{g} m^{2g-2i} \sum_{\substack{k_1,\ldots,k_{m-1}\geq 0 \\ k_1+\cdots+k_{m-1}=2i}} c_{k_1,\ldots,k_{m-1}}^{(m)} C_{n,m,D}^{(g-i,k_1,\ldots,k_{m-1})}$$

Here, n is the number of hyperedges, g is the genus, and D denotes an arbitrary restriction on hyperface degree. $H_{n,m,D}^{(g)}$ counts m-hypermaps, and $C_{n,m,D}^{(g-i,k_1,\ldots,k_{m-1})}$ counts m-constellations with k_i marked vertices for each color i. We also proved that coefficients $c_{k_1,\ldots,k_{m-1}}^{(m)}$ are all positive integers, hinting the possibility of a combinatorial interpretation. This result is obtained using algebraic method, with the interpretation of combinatorial maps as *factorization of the identity element in S_n*, and a theorem on characters by Littlewood in [3].

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Vertically symmetric alternating sign matrices and a multivariate Laurent polynomial identity

Ilse Fischer
(joint work with Lukas Riegler)

Consider the following rational function P

$$\prod_{1 \le i < j \le n} \frac{z_i^{-1} + z_j - 1}{1 - z_i z_j^{-1}}$$

and let R denote the function we obtain after symmetrizing it, that is $R = \mathbf{Sym}\, P$ with $\mathbf{Sym}\, f(z_1, \ldots, z_n) = \sum_{\sigma \in \mathcal{S}_n} f(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$. Since $P(z_1, \ldots, z_n) = P(z_n^{-1}, \ldots, z_1^{-1})$, it is obvious that $R(z_1, \ldots, z_n) = R(z_1^{-1}, \ldots, z_n^{-1})$, however, computer experiment suggest that also

$$R(z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_n) = R(z_1, \ldots, z_{i-1}, z_i^{-1}, z_{i+1}, \ldots, z_n).$$

This is the special case $s = 0$ of the following conjecture.

Conjecture 1 (Fischer, Riegler). *For integers $s, t \ge 0$, consider the following rational function $P_{s,t}$*

$$\prod_{i=1}^{s} z_i^{2s-2i-t+1} (1 - z_i^{-1})^{i-1} \prod_{i=s+1}^{s+t-1} z_i^{2i-2s-t} (1 - z_i^{-1})^{s} \prod_{1 \le p < q \le s+t-1} \frac{1 - z_p + z_p z_q}{z_q - z_p}$$

and let $R_{s,t} = \mathbf{Sym}\, P_{s,t}$. If $s \le t$ then

$$R_{s,t}(z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_{s+t-1}) = R_{s,t}(z_1, \ldots, z_{i-1}, z_i^{-1}, z_{i+1}, \ldots, z_{s+t-1})$$

for all $i \in \{1, 2, \ldots, s + t - 1\}$.

In the talk I first explained how we came up with this conjecture in an attempt to prove a conjecture on a refined enumeration of *vertically symmetric alternating sign matrices*. An *alternating sign matrix* is a quadratic $0, 1, -1$ matrix such that the non-zero entries alternate and sum up to 1 in each row and column. Next we give an example of such an object

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

which is in fact symmetric with respect to the vertically axis. Vertically symmetric alternating sign matrices have been enumerated by Kuperberg [3]. In [1], I presented the following conjecture on a refined enumeration of vertically symmetric alternating sign matrices.

Conjecture 2. *The number of* $(2n+1)\times(2n+1)$ *vertically symmetric alternating sign matrices where the first* 1 *in the second row is in column* i *is*

$$\frac{\binom{2n+i-2}{2n-1}\binom{4n-i-1}{2n-1}}{\binom{4n-2}{2n-1}} \prod_{j=1}^{n-1} \frac{(3j-1)(2j-1)!(6j-3)!}{(4j-2)!(4j-1)!}.$$

In [2], this was shown that a consequence of Conjecture 1 implies Conjecture 2.

Theorem 1. *If* $R_{s,t}(z_1,\ldots,z_{s+t-1}) = R_{s,t}(z_1^{-1},\ldots,z_{s+t-1}^{-1})$ *for all* $1 \le s \le t$ *then Conjecture 2 is true.*

In the talk, I have also sketched the proof of the following partial result towards proving Conjecture 1:

Theorem 2. *Suppose*

$$R_{s,t}(z_1,\ldots,z_{s+t-1}) = R_{s,t}(z_1^{-1},\ldots,z_{s+t-1}^{-1}) \tag{1}$$

if $t=s$ *and* $t=s+1$, $s\ge 1$. *Then* (1) *holds for all* s,t *with* $1\le s\le t$.

Coming back to the special case mentioned in the beginning: another result we have obtained is the following.

Theorem 3. *The coefficient of* z^i *in* $R(z,1,\ldots,1)$ *is the number of* $(2n+1)\times(2n+1)$ *vertically symmetric alternating sign matrices where the unique* 1 *in the first column is in row* $n+i+1$.

Conjecture 1 implies $R(z,1,\ldots,1) = R(z^{-1},1,\ldots,1)$, which has from the point of view of Theorem 3 the explanation that reflecting a $(2n+1)\times(2n+1)$ vertically symmetric alternating sign matrix $A=(a_{i,j})$ with $a_{n+i+1,1}=1$ along the vertically axis transforms it into a matrix with $a_{n+1-i,1}=1$. This makes it plausible that $R(z_1,\ldots,z_n)$ is a certain generating function of vertically symmetric alternating sign matrices, which, once the weight is identified, could also imply the fact that R is invariant under replacing z_i by z_i^{-1}.

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Pulling self-avoiding walks from a surface.

Anthony J. Guttmann
(joint work with Stu Whittington and Iwan Jensen)

In recent years a mixture of analytic and probabilistic techniques have been used to prove previously conjectured values of the critical point of self-avoiding walks (SAWs) in the bulk and self-avoiding walks attracted to a surface, in both cases on the hexagonal lattice. We now consider the more general problem of SAWs originating in and attracted to a surface of the square lattice, but with their end-point vertex pulled away from the surface, in a direction normal to the surface. This models a number of important experiments on DNA and other bio-polymers. A number of results can be proved analytically, and careful numerical work based on new and more efficient enumeration algorithms allows the phase diagram to be constructed very accurately.

The q, t-Schröder Polynomial and the Superpolynomial of Torus Knots

Jim Haglund

The q, t-Schröder polynomial $C_n(q, t, z)$ was originally defined by the speaker as a weighted sum over lattice paths, and shown [8] to have an interpretation in terms of the representation theory of diagonal harmonics. We overview this result and also discuss another interpretation for $C_n(q, t, z)$, first noticed by E. Gorsky, as the superpolynomial of a $(n, n + 1)$ torus knot. It follows from the diagonal harmonic interpretation that $C_n(q, t, z) = C_n(t, q, z)$, and we introduce a more general function $C_n(q, t, w, z)$, which conjecturally satisfies the relations

$$C_n(q, t, w, z) = C_n(t, q, w, z) = C_n(q, t, z, w). \tag{1}$$

We also discuss a more general conjecture of the speaker which expresses $C_n(q, t, w, z)$ in terms of Macdonald polynomials, which implies (1).

A Dyck path is a lattice path from $(0, 0)$ to (n, n) consisting of unit North and East steps which never goes below the line $y = x$. Given a Dyck path π as in Figure 1, let $a_i = a_i(\pi)$ denote the number of squares in the ith row (from the bottom) which are to the right of π and to the left of the diagonal $x = y$, where $1 \le i \le n$. We let $\text{area}(\pi)$ denote the sum of the a_i. Furthermore let $\text{dinv}(\pi)$ denote the number of pairs (i, j), $1 \le i < j \le n$, with either

$$a_i = a_j \text{ or } a_i = a_j + 1. \tag{2}$$

Next define the *reading order* of the rows of π to be the order in which the rows are listed by decreasing value of a_i, where if two rows have the same a_i-value, the row above is listed first. For the path of Figure 1 the reading order is

$$\text{row 6, row 4, row 5, row 3, row 2, row 7, row 1} \tag{3}$$

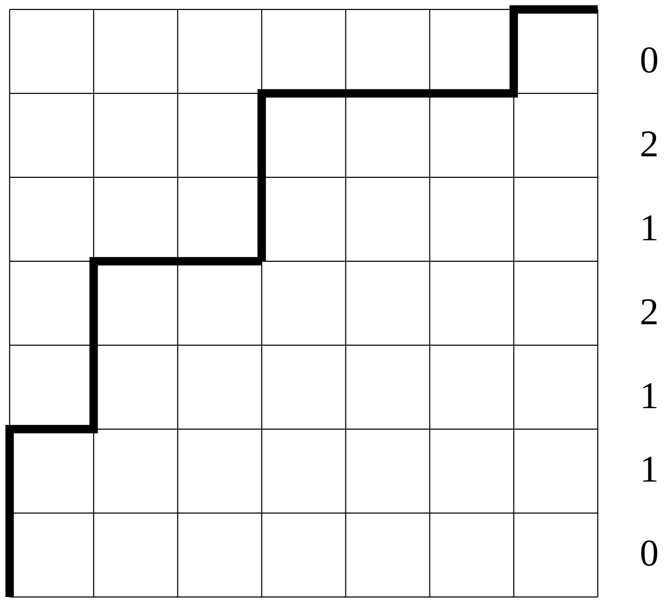

Figure 1. A Dyck path, with row lengths on the right. The area statistic is $1 + 1 + 2 + 1 + 2 = 7$.

Finally let $b_k = b_k(\pi)$ be the number of inversion pairs as in (2) which involve the kth row in the reading order and rows before it in the reading order. For the path of Figure 1, we have

$$b_1 = 0, b_2 = 1, b_3 = 1, b_4 = 1, b_5 = 2, b_6 = 3, b_7 = 1. \tag{4}$$

Note dinv is the sum of the b_k, and that values of i for which $b_i > b_{i-1}$ correspond to tops of columns in π (where we define $b_0 = -1$ so that $b_1 > b_0$).

One way of defining the q, t-Schröder polynomial from [8] is

$$C_n(q, t, z) = \sum_{\pi} t^{\text{area}} q^{\text{dinv}} \prod_{b_i > b_{i-1}} (1 + z/q^{b_i}). \tag{5}$$

Here the sum is over all Dyck paths π from $(0, 0)$ to (n, n). For the path π in Figure 1, the weight assigned to π in the right-hand-side of (5) is

$$t^7 q^9 (1 + z)(1 + z/q)(1 + z/q^2)(1 + z/q^3). \tag{6}$$

We note that $C_n(q, t, 0)$ is Garsia and Haiman's q, t-Catalan sequence [5].

Dunfield, Gukov, and Rasmussen [4] hypothesized the existence of a *superpolynomial* knot invariant $P_K(a, q, t)$ of a knot K which would contain the HOMFLY and Jones polynomials as limiting cases, as well as having other desirable properties. Possible definitions of the superpolynomial for torus knots $T_{(m,n)}$ have recently been suggested by Angnanovic and Shakirov [2] (see also [1]), Cherdnik [3], and Oblomkov, Rasmussen, and Shende [11]. All three methods seem to give the same superpolynomial, and in fact Gorsky and Negut [6] have proved the descriptions in [2] and [3] do in fact give the same superpolynomial. The description in [2] is in terms of Macdonald polynomials, and Gorsky and Negut note this is the same Macdonald polynomial expression for $C_n(q,t,-a)$ in [8], using the Cherednik parameterization. Gorsky and Negut also give a constant-term expression for $C_n(q,t,z)$, or equivalently an expression as a weighted sum over Tesler matrices. Their results also apply to general $T_{(m,n)}$ torus knots, where m,n is any pair of relatively prime positive integers. See [7] for further background on the superpolynomial.

Let

$$C_n(q,t,w,z) = \sum_{\pi} t^{\mathrm{area}} q^{\mathrm{dinv}} \prod_{b_i > b_{i-1}} (1 + z/q^{b_i}) \prod_{a_i > a_{i-1}} (1 + w/t^{a_i}), \tag{7}$$

where $a_0 = -1$.

Conjecture 1

$$C_n(q,t,w,z) = C_n(t,q,w,z) = C_n(q,t,z,w). \tag{8}$$

We now describe some more technical conjectures which embed the conjectured symmetry relations (8) in the theory of Macdonald polynomials and diagonal harmonics. For any symmetric function $f(X)$, let Δ_f be the linear operator defined on the Macdonald basis $\tilde{H}_\mu$ via

$$\Delta_f \tilde{H}_\mu(X;q,t) = f[B_\mu]\tilde{H}_\mu(X;q,t), \tag{9}$$

where $B_\mu(q,t) = \sum_i t^{i-1}(1-q^{\mu_i})/(1-q)$.

Conjecture 2

$$C_n(q,t,w,z)|_{z^a w^b} = \Delta_{e_{n-b}} \langle e_n, h_a e_{n-a} \rangle \tag{10}$$

Let P be a parking function, viewed as a placement of the integers 1 through n just to the right of the North steps of a Dyck path, with strict decrease down columns.

Conjecture 3 For any integer k, $0 \le k \le n$,

$$\Delta_{e_{n-k}} e_n = \sum_{\pi} \sum_{P \in \mathrm{PF}(\pi)} t^{\mathrm{area}} q^{\mathrm{dinv}(P)} F_{\mathrm{des}(\mathrm{read}(\mathrm{P})^{-1})} \prod_{a_i > a_{i-1}} (1 + w/t^{a_i})|_{w^k}, \tag{11}$$

where $\mathrm{PF}(\pi)$ is the set of parking functions for π, and F is the quasisymmetric-function weight attached to P. (See Chapters five and seven of [9] for a definition of $\mathrm{dinv}(P)$, and also the quasisymmetric function weight attached to P.)

The case $k = 0$ of (11) is the "shuffle conjecture" from [10]. By taking the scalar product of (11) with $h_a e_{n-a}$, one can show (11) implies (10), which in turn implies (8).

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Restricted Lattice Walks in Three Dimensions

Manuel Kauers
(joint work with Alin Bostan, Mireille Bousquet-Mélou, Stephen Melczer)

Nearest neighbour walks in the quarter plane have been studied intensively during the past few years in the combinatorial community. Thanks to the efforts of a variety of authors using a variety of quite different techniques, we now know for every step set $S \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$ whether the generating function

$$a(x,y,t) = \sum_{n=0}^{\infty} \sum_{i,j} a_{n,i,j} x^i y^j t^n \in \mathbb{Q}[x,y][[t]]$$

for the number $a_{n,i,j}$ of all lattice walks in $\mathbb{N}^2$ that start at $(0,0)$, consist of n steps, each step taken from S, and which end at (i,j) is D-finite or not. To be D-finite means to satisfy a linear differential equation with polynomial coefficients. For details on this classification, see [2, 3, 4, 5, 6, 7, 8] and the references given there.

With the classification of walks in the quarter plane being complete, we turn the attention to the analogous problem in three dimensions: for every step set $S \subseteq \{-1,0,1\}^3 \setminus \{(0,0,0)\}$ we consider the generating function

$$a(x,y,z,t) = \sum_{n=0}^{\infty} \sum_{i,j,k} a_{n,i,j,k} x^i y^j z^k t^n \in \mathbb{Q}[x,y,z][[t]]$$

for the number $a_{n,i,j,k}$ of all lattice walks in $\mathbb{N}^3$ that start at $(0,0,0)$, consist of n steps, each step taken from S, and which end at (i,j,k). The question is again for which step sets S this generating function $a(x,y,z,t)$ is D-finite, and for which it is not.

Compared to the 2D case, the 3D case offers new technical difficulties as well as new combinatorial phenomena. In the talk at MFO, we have given an overview over some first results that we obtained. The first difficulty in 3D is the number of cases: while there are only 256 different models in 2D, there are more than 67 million different models in 3D. Even after applying obvious symmetries and discarding trivial cases, we are left with more than 11 million cases, way too many for an exhaustive investigation. We decided to focus on the models $S \subseteq \{-1,0,1\}^3 \setminus \{(0,0,0)\}$ with at most six different directions, i.e., $|S| \leq 6$. Up to symmetries, these are some 35000 cases.

For these 35000 step sets we attempted to determine whether the corresponding generating functions are D-finite. We applied the following three techniques.

(1) The algebraic kernel method. This method is based on determining a certain group associated to the step set and forming the so-called orbit sum in order to obtain an expression for the generating function as the positive part of a rational function, thus implying D-finiteness. This method is a direct generalization of the main technique used for 2D models [5]. It applies when the group is finite and the orbit sum is nonzero.

(2) Projection to lower dimensions. This method exploits the fact that there is a significant number of step sets in 3D for which at least one of the dimensions is redundant. Such models are in a natural bijection with the models in 2D, and these have already been classified. Note however that the projection does not in all cases lead to one of the known step sets $S \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$. In addition, there also arise models with certain distinguishable steps in the same direction. Two particularly annoying instances are the models $\{(-1,0),(-1,0)',(-1,-1),(-1,1),(1,1),(1,0)\}$ and $\{(1,0),(1,0)',(1,1),(1,-1),(-1,-1),(-1,0)\}$, which are D-finite and algebraic, respectively. Altogether we encountered 527 inherently different 2D models, 118 of which were proved to have D-finite (or algebraic) generating functions. The remaining 409 models are believed to be non-D-finite.

(3) Decomposition. Some step sets can be written as a direct product of lower dimensional step sets. For example, the 3D step set

$$S = \{(-1,-1,-1),(1,0,-1),(0,1,-1),(-1,-1,1),(1,0,1),(0,1,1)\}$$

can be written as $\{(-1,-1),(1,0),(0,1)\} \times \{-1,1\}$. A 3D lattice walk with the model S can thus be interpreted as a pair of lattice walks, one in 2D by the model $\{(-1,-1),(1,0),(0,1)\}$, and one in 1D by the model $\{-1,1\}$. The generating function for the 3D model can be expressed as a Hadamard-product of the two generating functions of the lower dimensional models. Since D-finiteness is preserved under Hadamard-product, decompositions can sometimes be used to recognize that a 3D model has a D-finite generating function.

Each step set may or may not have a finite group, may or may not be projectable, and may or may not be decomposable. For seven of the eight possible combinations, there are examples. Only for finite group / projectable / not decomposable, there are no step sets with at most six steps. Some 20000 cases don't seem to have a finite group, and are not projectable, and are not decomposible. We believe that the generating function for these models are not D-finite. The smallest example of this class is $\{(-1,-1,1),(0,1,-1),(1,1,0)\}$.

There are 23 models which have a finite group but are neither projectable nor decomposable. Four of them have a non-zero orbit sum, so the algebraic kernel method routinely implies that their generating functions are D-finite. The remaining 19 models are mysterious. Despite intensive calculations on a supercomputer we were not able to find conjectured D-finite equations for any of these models using the first 5000 terms of the expansions of $a(1,1,1,t)$, $a(0,1,1,t)$, $a(1,0,1,t)$, $a(1,1,0,t)$, $a(0,0,1,t)$, $a(0,1,0,t)$, $a(1,0,0,t)$, or $a(0,0,0,t)$. It may still be that some of the 19 models satisfy an equation that is too big to be recovered from 5000 terms, but the equipment available to us does not permit the computation of a significant number of additional terms at a reasonable cost. Also, although there do exist quite large equations for some step sets that we recognized as D-finite (the largest equation we found required 20000 terms), it is noteworthy that even when one series $a(x_0, y_0, z_0, t)$ for some choice $x_0, y_0, z_0 \in \{0,1\}$ has only a very large equation, there was always at least one other choice $x_1, y_1, z_1 \in \{0,1\}$ for which $a(x_1, y_1, z_1, t)$ satisfies a much smaller equation. The fact that for the 19 mysterious models we did not find any equation for any choice may be an indication that their generating functions are perhaps not D-finite. If so, this would be a remarkable mismatch to the situation in 2D, where it has been observed that a generating function is D-finite if and only if the model has a finite group.

Full details on our results will be made available in [1].

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The Selberg integral and Young books

Jang Soo Kim
(joint work with Suho Oh)

The Selberg integral is the following integral first evaluated by Selberg [4] in 1944:

$$(1)\quad S_n(\alpha,\beta,\gamma)=\int_0^1\cdots\int_0^1\prod_{i=1}^n x_i^{\alpha-1}(1-x_i)^{\beta-1}\prod_{1\le i<j\le n}|x_i-x_j|^{2\gamma}dx_1\cdots dx_n$$
$$=\prod_{j=1}^n\frac{\Gamma(\alpha+(j-1)\gamma)\Gamma(\beta+(j-1)\gamma)\Gamma(1+j\gamma)}{\Gamma(\alpha+\beta+(n+j-2)\gamma)\Gamma(1+\gamma)},$$

where n is a positive integer and α,β,γ are complex numbers such that $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$, and $\mathrm{Re}(\gamma)>-\min\{1/n,\mathrm{Re}(\alpha)/(n-1),\mathrm{Re}(\beta)/(n-1)\}$. We refer the reader to Forrester and Warnaar's exposition [1] for the history and importance of the Selberg integral.

In this talk we review Stanley's combinatorial interpretation of the Selberg integral when $\alpha=\beta=1$ and $2\gamma=m$ is a nonnegative integer by introducing certain permutations called Selberg permutations. Then we define "Selberg books" which are in natural bijection with special Selberg permutations. We also define "Young books" and show that there is a very simple relation between the number of Selberg books and that of Young books. Young books are a generalization of both of shifted Young tableaux of staircase shape and standard Young tableaux of square shape. Using the relation between Selberg books and Young books and the Selberg integral formula we get a formula for the number of Young books.

Let $A(n,m)$ be the following set of letters

$$A(n,m)=\{x_i:1\le i\le n\}\cup\{a_{ij}^{(k)}:1\le i<j\le n,1\le k\le m\}.$$

A permutation of $A(n,m)$ is called a *Selberg permutation* if the following conditions hold:

- $x_1,x_2,\dots,x_n$ are in this order,
- $a_{ij}^{(k)}$ is between x_i and x_j for $1\le i<j\le n$ and $1\le k\le m$.

Let $SP(n,m)$ denote the set of Selberg permutations of $A(n,m)$.

Stanley [6] showed that

$$\int_0^1\cdots\int_0^1\prod_{i=1}^n\prod_{1\le i<j\le n}|x_i-x_j|^m dx_1\cdots dx_n=\frac{n!|SP(n,m)|}{(n+mn(n-1)/2)!}.$$

By (1) we have

$$(2)\qquad |SP(n,m)|=\frac{2^n(n+mn(n-1)/2)!}{n!}\prod_{j=1}^n\frac{(jm)!!((j-1)m)!!^2}{m!!(2+(n+j-2)m)!!},$$

where

$$(2k)!!=(2k)(2k-2)\cdots 2,\qquad (2k-1)!!=(2k-1)(2k-3)\cdots 1.$$

We now define Selberg books which are in natural bijection with Selberg permutations. We then define Young books which are defined similarly as Selberg books.

The *shifted staircase of size* n is the shifted partition $(n, n-1, \ldots, 1)$. We will identify the shifted staircase of size n with its shifted Ferrers diagram.

Let $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m)}$ be shifted staircases of size n. We identify the cells in the ith row and ith column of $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m)}$ to a single cell and call it the *ith diagonal cell.* An *(n,m)-Selberg book* is a filling of the m-tuple $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m)})$ with integers $1, 2, \ldots, n + m\binom{n}{2}$ such that for each shifted staircase the integer in the ith row and jth column is bigger than the integer in the ith diagonal cell and smaller than the integer in the jth diagonal cell and smaller. Let $SB(n,m)$ be the set of (n,m)-Selberg books.

There is a natural bijection between $SB(n,m)$ and $SP(n,m)$ as follows. For $B \in SB(n,m)$, define the corresponding permutation $\pi = \pi_1\pi_2 \ldots \pi_{n+mn(n-1)/2}$ by

$$\pi_\ell = \begin{cases} x_i, & \text{if } B \text{ has the integer } \ell \text{ in the } i\text{th diagonal cell,} \\ a_{ij}^{(k)}, & \text{if } B \text{ has the integer } \ell \text{ in the } i\text{th row and } j\text{th column} \\ & \text{of the } k\text{th shifted staircase.} \end{cases}$$

By (2) we have

$$|SB(n,m)| = \frac{2^n(n+mn(n-1)/2)!}{n!m!!^n} \prod_{j=1}^{n} \frac{((j-1)m)!!^2(jm)!!}{(2+(n+j-2)m)!!}. \tag{3}$$

An (n,m)-*Young book* is an (n,m)-Selberg book with the additional condition that for each shifted staircase the integers are increasing along each row and column. Let $YB(n,m)$ be the set of (n,m)-Young books.

For $B \in SB(n,m)$ or $B \in YB(n,m)$, let a_i be the integer in the ith diagonal cell for $1 \le i \le n$. We define the *diagonal gap* of B to be the sequence $(d_1, \ldots, d_{n-1})$ where $d_i = a_{i+1} - a_i - 1$ is the number of integers between a_i and a_{i+1}. We denote by $SB(n,m;d_1,\ldots,d_{n-1})$ and the set of (n,m)-Selberg books with diagonal gap $(d_1,\ldots,d_{n-1})$. We define $YB(n,m;d_1,\ldots,d_{n-1})$ similarly.

Kim and Oh [2] showed that

$$\sum_{d_1,\ldots,d_{n-1}\ge 0} |SB(n,m;d_1,\ldots,d_{n-1})| \frac{t_1^{d_1}\ldots t_{n-1}^{d_{n-1}}}{d_1!\ldots d_{n-1}!} = \prod_{i<j}(t_i+t_{i+1}+\cdots+t_{j-1})^m. \tag{4}$$

Postnikov [3] showed that

$$\sum_{d_1,\ldots,d_{n-1}\ge 0} |YB(n,1;d_1,\ldots,d_{n-1})| \frac{t_1^{d_1}\ldots t_{n-1}^{d_{n-1}}}{d_1!\ldots d_{n-1}!} = \prod_{i<j} \frac{t_i+t_{i+1}+\cdots+t_{j-1}}{j-i}. \tag{5}$$

Using (4) and (5) one can show that

$$|SB(n,m;d_1,\ldots,d_{n-1})| = (1!2!\cdots(n-1)!)^m \cdot |YB(n,m;d_1,\ldots,d_{n-1})|, \tag{6}$$

$$|SB(n,m)| = (1!2!\cdots(n-1)!)^m \cdot |YB(n,m)|. \tag{7}$$

By (3) and (7) we get the number of (n, m)-Young books:

$$|YB(n,m)| = \frac{2^n(n + mn(n-1)/2)!}{n!m!!^n} \prod_{j=1}^{n} \frac{((j-1)m)!!^2(jm)!!}{(j-1)!^m(2+(n+j-2)m)!!}. \tag{8}$$

If $m = 1$ in (8), then we get the hook length formula for the number of shifted standard Young tableaux of staircase shape $(n, n-1, \ldots, 1)$. If $m = 2$ in (8), the we get the hook length formula for the number of standard Young tableaux of square shape $(n, n, \ldots, n)$. This gives an algebro-combinatorial proof of the Selberg integral for $\alpha = \beta = 1$ and $\gamma \in \{1/2, 1\}$.

The ultimate goal of this research is to find a combinatorial proof of the Selberg integral when $r = \alpha - 1, s = \beta - 1$ and $m = 2\gamma$ are nonnegative integers. We can achieve this goal for $r = s = 0$ and $m \in \{1, 2\}$ if we solve one of the following two problems because there is a combinatorial proof for the (regular and shifted) hook lengh formula.

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Counting and typical parameters for phylogenetic networks

Colin McDiarmid

(joint work with Charles Semple, Dominic Welsh)

It is well known how to count phylogenetic trees (rooted binary trees, either fully labelled or with just the leaves labelled). But what about phylogenetic networks, where we allow reticulation nodes, corresponding to evolutionary processes such as recombination and hybridisation?

For a finite set X, a *phylogenetic network on* X (see for example [3]) is a rooted acyclic directed graph with the following properties:

(i) the (unique) root is a vertex with in-degree 0 and out-degree two;

(ii) a vertex (leaf) with out-degree zero has in-degree one, and the set of vertices with out-degree zero is X; and

(iii) all other vertices either have in-degree one and out-degree two (tree vertices), or in-degree two and out-degree one (reticulation vertices).

Tree-child and *normal* networks are subclasses of networks of particular interest.

We discuss approximate counting formulae for the numbers of labelled general, tree-child, and normal phylogenetic networks on n vertices. These formulae are of the form $2^{\gamma n \log n + O(n)}$, where the constant γ is $\frac{3}{2}$ for general networks, and $\frac{5}{4}$ for tree-child and normal networks. We shall also see that the numbers of leaf-labelled tree-child and normal networks with ℓ leaves are both $2^{2\ell \log \ell + O(\ell)}$.

We work with a configuration model much as for random cubic graphs, and find that upper bounds are quite straightforward, depending on the numbers of leaves. For the lower bound for general networks, we use the result of Robinson and Wormald [2] that almost all cubic graphs have a Hamilton circuit. For tree-child and normal networks we use an explicit construction.

Further we find the typical proportions of leaves, tree vertices, and reticulation vertices for each of these classes of networks.

(i) Almost all n-vertex general networks have $o(n)$ leaves and $(\frac{1}{2} + o(1))n$ reticulation vertices.

(ii) Almost all n-vertex tree-child and normal networks have $(\frac{1}{4} + o(1))n$ leaves and $(\frac{1}{4} + o(1))n$ reticulation vertices.

(iii) Almost all ℓ-leaf leaf-labelled tree-child and normal networks have $(1 + o(1))\ell$ reticulation vertices and $(4 + o(1))\ell$ vertices in total.

These results are taken from [1].

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A survey of the switching method for combinatorial estimation

Brendan D. McKay

(joint work with Verle Fack, Catherine Greenhill, Mahdieh Hasheminezhad)

The switching method, also called the method of perturbations and other names, is a family of techniques for approximate counting of combinatorial objects. The basic idea is to partition the space into parts, and provide one or more operations ("switchings") on objects that sometimes move them between parts. By counting how often the switchings and their inverses apply, we obtain information about the relative sizes of the parts.

One application of the switching method is in the bounding of tails of distributions, which in the model we have described refers to bounding the sizes of minor parts. The process is described in terms of a directed graph whose vertices are the parts and whose edges show possible trajectories of switchings. Bounds on the statistics of switchings gives a linear program for the part sizes, which has a special form often allowing explicit solution. The case where the graph is acyclic apart from loops was solved in [1], while the general case was solved in [3]. In [2] a good approximation was given to the generalization where multiple types of switchings are employed simultaneously.

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The scaling limit fo uniform random plane maps, *via* the Ambjørn-Budd bijection

Grégory Miermont

(joint work with Jérémie Bettinelli, Emmanuel Jacob)

The topic of limits of random maps has met an increasing interest over the last two decades, as it is recognized that such objects provide natural model of discrete and continuous 2-dimensional geometries [4, 5]. Recall that a plane map is a cellular embedding of a finite graph (possibly with multiple edges and loops) into the sphere, considered up to orientation-preserving homeomorphisms. By *cellular*, we mean that the faces of the map (the connected components of the complement of edges) are homeomorphic to 2-dimensional open disks. A popular setting for studying scaling limits of random maps is the following. We see a map $\mathbf{m}$ as a metric space by endowing the set $V(\mathbf{m})$ of its vertices with its natural graph metric $d_{\mathbf{m}}$: the graph distance between two vertices is the minimal number of edges of a path linking them. We then choose at random a map of "size" n in a given class and look at the limit as $n \to \infty$ in the sense of the Gromov–Hausdorff topology [13] of the corresponding metric space, once rescaled by the proper factor.

This question first arose in [11], focusing on the class of plane quadrangulations, that is, maps whose faces are of degree 4, and where the size is defined as the number of faces. A series of papers, including [19, 14, 21, 15, 10], have been motivated by this question and contributed to its solution, which was completed in [16, 22] by different approaches. Specifically, there exists a random compact metric space $\mathbf{S}$ called *the Brownian map* such that, if Q_n denotes a uniform random (rooted) quadrangulation with n faces, then the following convergence holds in distribution for the Gromov–Hausdorff topology on the set of isometry classes of compact metric spaces:

$$\left(\frac{9}{8n}\right)^{1/4} Q_n \xrightarrow[n\to\infty]{(d)} \mathbf{S}. \tag{1}$$

Here, if $\mathbb{X} = (X, d)$ is a metric space and $a > 0$, we let $a\mathbb{X} = (X, ad)$ be the rescaled space, and we understand a map $\mathbf{m}$ as the metric space $(V(\mathbf{m}), d_{\mathbf{m}})$.

Le Gall [16] also gave a general method to prove such a limit theorem in a broader context, that applies in particular to uniform p-angulations (maps whose faces are of degree p) for any $p \in \{3, 4, 6, 8, 10 \ldots\}$. When this method applies, the scaling factor $n^{-1/4}$ and the limiting metric space $\mathbf{S}$ are the same, only the scaling constant $(9/8)^{1/4}$ may differ. One says that the Brownian map possesses a property of universality, and one actually expects this property to hold for many more "reasonable" classes of maps, see for instance [6, 2, 1].

A robust and widely used bijective encoding in obtaining such results is the Cori–Vauquelin–Schaeffer bijection [12, 23] and its generalization by Bouttier–Di Francesco–Guitter [8], see for instance [18, 20]. However, this bijection becomes technically hard to manipulate when dealing with non-bipartite maps (with the notable exception of triangulations).

In the work [7], we continue this line of research with another fundamental class of maps, namely uniform random plane maps with a prescribed number of edges. The key to our study is to use a combination of the Cori–Vauquelin–Schaeffer bijection, together with a recent bijection due to Ambjørn and Budd [3], that allows to couple directly a uniform (pointed) map with n edges and a uniform quadrangulation with n faces, while preserving distances asymptotically. This allows to transfer known results from uniform quadrangulations to uniform maps, in a way that is comparatively easier than a method based on the Bouttier–Di Francesco–Guitter bijection.

We let $\mathcal{M}_n$ be the set of rooted plane maps with n edges, and $\mathcal{M}_n^\bullet$ be the set of rooted and *pointed* plane maps with n edges, i.e., of pairs $(\mathbf{m}, v_*)$ where $\mathbf{m} \in \mathcal{M}_n$ and v_* is a distinguished element of $V(\mathbf{m})$.

Similarly, we let $\mathcal{Q}_n$ (resp. $\mathcal{Q}_n^\bullet$) be the set of rooted (resp. rooted and pointed) quadrangulations with n faces.

The Ambjørn–Budd (AB) bijection provides a natural coupling between a uniform random element (Q_n, v_*) of $\mathcal{Q}_n^\bullet$, and a uniform random element $(M_n^\bullet, v_*)$ of $\mathcal{M}_n^\bullet$. Using this coupling, it was observed already [3, 9] that the "two-point functions" that govern the limit distribution of the distances between two uniformly chosen points in $M_n^\bullet$ and Q_n coincide. In this work we show that much more is true.

Theorem. Let (Q_n, v_*) and $(M_n^\bullet, v_*)$ be uniform random elements of $\mathcal{Q}_n^\bullet$ and $\mathcal{M}_n^\bullet$ respectively, that are in correspondence *via* the Ambjørn–Budd bijection. Then we have the following joint convergence in distribution for the Gromov–Hausdorff topology

$$\left(\left(\frac{9}{8n}\right)^{1/4} M_n^\bullet, \left(\frac{9}{8n}\right)^{1/4} Q_n\right) \xrightarrow[n\to\infty]{(d)} (\mathbf{S}, \mathbf{S}),$$

where $\mathbf{S}$ is the Brownian map.

A very striking aspect of this is that the scaling constant $(9/8)^{1/4}$ is the same for $M_n^\bullet$ and for Q_n. This implies in particular that

$$n^{-1/4} \mathrm{d}_{\mathrm{GH}}(M_n^\bullet, Q_n) \xrightarrow[n\to\infty]{P} 0$$

where d_{GH} is the Gromov–Hausdorff distance beween two compact metric spaces, which, to paraphrase the title of [17], says that "the AB bijection is asymptotically an isometry." Although obtaining this scaling constant is theoretically possible using the methods of [20], the computation would be rather involved.

At the cost of an extra "de-pointing lemma," this implies our main result:

Corollary. Let M_n be a uniformly distributed random variable in $\mathcal{M}_n$. The following convergence in distribution holds for the Gromov–Hausdorff topology

$$\left(\frac{9}{8n}\right)^{1/4} M_n \xrightarrow[n\to\infty]{(d)} \mathbf{S}$$

where $\mathbf{S}$ is the Brownian map.

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Diagonals and D-finite functions

Marni Mishna
(joint work with Stephen Melczer)

1. Introduction

D-finite functions satisfy linear differential equations with polynomial coefficients. As innocuous as that may sound, they have been a rich treasure trove for number theorists, computer algebraists, and combinatorialists for over thirty years. Despite deep investigations from several different angles, they still hold many mysteries. For example, Gilles Christol conjectured over 20 years ago that all D-finite globally bounded functions are diagonals of rational functions [4]. This would imply that combinatorial classes with D-finite generating functions are very structured, but yet we have no firm grasp on the anatomy of such classes.

This abstract considers the connections between diagonals of rational functions; D-finite functions; and Asymptotic enumeration, particularly in the context of lattice path enumeration. There are many connections between these notions, exemplified by new explicit asymptotic enumeration results for lattice paths restricted to the first orthant, in d-dimensions.

2. Diagonals, D-finite functions, and asymptotic enumeration

The *diagonal* of the multivariate series

$$f(x_1, \ldots, x_k) = \sum_{i_0, i_1, \ldots, i_k} a(i_1, \ldots, i_k) x_1^{i_1} \ldots x_k^{i_k}$$

is defined as the series

$$\Delta f(x_1, \ldots, x_k) = \sum_{n \geq 0} a(n, \ldots, n) x_1^n.$$

If the series $g(x)$ is algebraic, satisfying $P(g(x), x) = 0$ for some bivariate polynomial $P(x, y)$, then Furstenberg [5] showed that it can be expressed as a diagonal of bivariate rational function, for example:

$$g(x) = \Delta \frac{x^2 P_x(x, xy)}{P(x, xy)}.$$

Diagonals of D-finite multivariate functions are also D-finite [7, 4], and there exist effective methods to translate between the system of differential equations satisfied by $f(x_1, \ldots, x_k)$ and the differential equation satisfied by Δf, although they are generally limited to a small number of variables owing to the underlying Gröbner bases computations.

Christol's conjecture considers the reverse direction, and begs the question is there a way to translate from the defining differential equation, to a diagonal formulation? There is a clear benefit to a diagonal data structure over the differential equation, thanks to recent developments in mulitvariate asymptotics, exemplified by the recent book of Pemantle and Wilson [9]. This burgeoning field provides a path to access the *complete* asymptotic developments for $a(n, \ldots, n)$ given $f(x_1, \ldots, x_k)$.

3. Lattice path models

Recent results in lattice path enumeration illustrate the harmony of these three subjects very well. A *lattice path model* is defined by a set of vectors $\mathcal{S} \subseteq \mathbb{Z}^d$ and a convex region $\mathcal{R} \subseteq \mathbb{Z}^d$ containing the origin. The combinatorial class is comprised of all finite sequences of elements of $\mathcal{S}$, or walks $w = s_1, s_2, \ldots, s_n$ such that every partial sum remains in the region. Our particular interest is the family of models using so-called *small steps* $\mathcal{S} \subseteq \{0, 1, -1\}^d$, which remain in the first orthant $\mathcal{R} \subseteq \mathbb{N}^d$.

These two restrictions seem strict, but the remaining models are still qutie insightful and structured. For example, when $d = 2$, there are only 79 non-isomorphic models, and they are well studied as a family. The models with D-finite generating functions succumb, with one exception, to an algorithmic strategy, known as the *orbit sum method*. It is a variant of the kernel method, but it was tailored in this case by Bouquet-Mélou and Mishna [3].

This method expresses the generating functions as diagonals of rational functions. The remaining model is algebraic, and hence by Furstenburg's result, it is also a diagonal of a rational function.

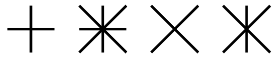
Observation. All D-finite small step lattice path models in restricted to $\mathbb{N}^2$ have generating functions which are globally bounded and which can be expressed as diagonals of rational functions.

As lattice path models can encode many other combinatorial classes, this observation may be more general than it initially appears.

4. Asymptotic enumeration of highly symmetric models in d dimensions

Next consider an explicit example of how to chain together the orbit sum method, and multivariable analytic combinatorics to deduce explicit asymptotic enumeration results.

A model $\mathcal{S}$ is *highly symmetric* if the following is true: $\mathbf{s} = (s_1, \ldots, s_d) \in \mathcal{S}$ implies that the vector where s_i is replaced by $-s_i$ in $\mathbf{s}$ is also in $\mathcal{S}$. The model

is symmetric across every axis. There are five non-isomorphic models in $d = 2$:


Set $\mathbf{x} = x_1, x_2, \ldots, x_d$ and $\overline{x} = \frac{1}{x}$. Define $Q(\mathbf{x}, t)$ to be the generating function counting the number of walks of length n with marked endpoint and $S(\mathbf{x}) = \sum_{s \in \mathcal{S}} x_1^{s_1} x_2^{s_2} \ldots x_d^{s_d}$. Finally, define

$$\begin{aligned} R(\mathbf{x}, t) &= \frac{(x_1 - \overline{x}_1) \cdots (x_d - \overline{x}_d)}{(x_1 \cdots x_d)(1 - tS(\mathbf{x}))} \\ G(\mathbf{x}, t)/H(\mathbf{x}, t) &= \frac{R\left(\overline{x}_1, \ldots, \overline{x}_d, x_1 \cdots x_d \cdot t\right)}{(1 - x_1) \cdots (1 - x_d)} \quad \text{with } G, H \text{ polynomials.} \end{aligned}$$

Then, applying the orbit sum method we can show that

$$Q(\mathbf{x}, t) = \Delta G(\mathbf{x}, t)/H(\mathbf{x}, t). \tag{1}$$

Consequently, all highly symmetric models have D-finite generating functions and we can apply the techniques of [9] to Eq. (1) and determine explicit asymptotic results.

Theorem (Melczer, Mishna [8]). Let $\mathcal{S} \subseteq \{-1, 0, 1\}^d \setminus \mathbf{0}$ be a highly symmetric d-dimensional lattice path model in the first orthant. Then the number of steps of length n, denoted s_n, satisfies:

$$s_n \sim \left[\left(s^{(1)} \cdots s^{(d)} \right)^{-1/2} \pi^{-d/2} |\mathcal{S}|^{d/2} \right] \cdot n^{-d/2} \cdot |\mathcal{S}|^n,$$

where $s^{(k)}$ denotes the number of steps in $\mathcal{S}$ which have k^{th} coordinate 1.

For example, the model with steps only along the axis, $\mathcal{S} = \{e_1, -e_1, \ldots, e_d, -e_d\}$ is asymptotically counted by $s_n \sim \left(\frac{2d}{\pi}\right)^{d/2} \cdot n^{-d/2} \cdot (2d)^n$. Similarly, the model with all possible steps $\mathcal{S} = \{0, 1, -1\}^d \setminus \mathbf{0}$ is asymptotically counted by $s_n \sim \left(\frac{(3^d-1)^{d/2}}{3^{d(d-1)/2} \cdot \pi^{d/2}}\right) \cdot n^{-d/2} \cdot (3^d - 1)^n$.

5. Future considerations

Our future considerations include: finding ways to better exploit the symmetry in this case; asymptotic analysis of models with (slightly) less symmetry; trying to develop a combinatorial understanding of diagonals to prove (or disprove) Christol's conjecture; consideration of models that are no longer restricted to small steps.

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Two variants of a face of the Birkhoff polytope

Alejandro H. Morales
(joint work with Karola Mészáros, Jessica Striker)

1. The Chan Robbins Yuen polytope

The Birkhoff polytope $\mathcal{B}_n$ is the polytope of all $n \times n$ matrices (b_{ij}) with nonnegative real entries that are doubly-stochastic, i.e. the sum of the rows and columns equals one. The vertices of this polytope are the $n \times n$ permutation matrices. This polytope is important in probability and optimization and has been studied since the 1940s. However, some basic questions like computing explicitly its volume for all n [2, 6] and finding explicitly its f-vector [3] remain open.

One face of $\mathcal{B}_n$ called the *Chan-Robbins-Yuen polytope* [4] or $\mathcal{CRY}_n$ is obtained by setting the entries b_{ij} of the doubly-stochastic matrix to zero if $i - j \geq 2$ (see Figure 1 (a),(b)). This polytope is an example of a *flow polytope* of the complete graph on $n + 1$ vertices. Postnikov-Stanley [10] and Baldoni-Vergne [1] showed that the (normalized) volume of this polytope is given by an evaluation of the *Kostant partition function* at a certain vector. This evaluation is the captivating product $Cat_1 Cat_2 \ldots Cat_{n-2}$ of Catalan numbers. This was proved analytically by Zeilberger in [13] using an identity closely related to Selberg's integral. No combinatorial proof is known. To shed some light on this volume formula we consider two variants of this polytope.

2. First variant: a face of the alternating sign matrix polytope

A generalization of permutation matrices of importance in statistical mechanics are *alternating sign matrices* or *ASM*s [9]. These are $n \times n$ matrices with entries $\{0, 1, -1\}$ such that the the sum of the rows and columns equals one, and the nonzero entries in each row and column alternate in sign. Just as the Birkhoff polytope $\mathcal{B}_n$ is the convex hull of the $n \times n$ permutation matrices, one can define a polytope $\mathcal{ASM}_n$ as the convex hull of the $n \times n$ alternating sing matrices [5, 12]. It turns out that the alternating sign matrices are exactly the vertices of this polytope. And its volume for all n or its f-vector is also unknown.

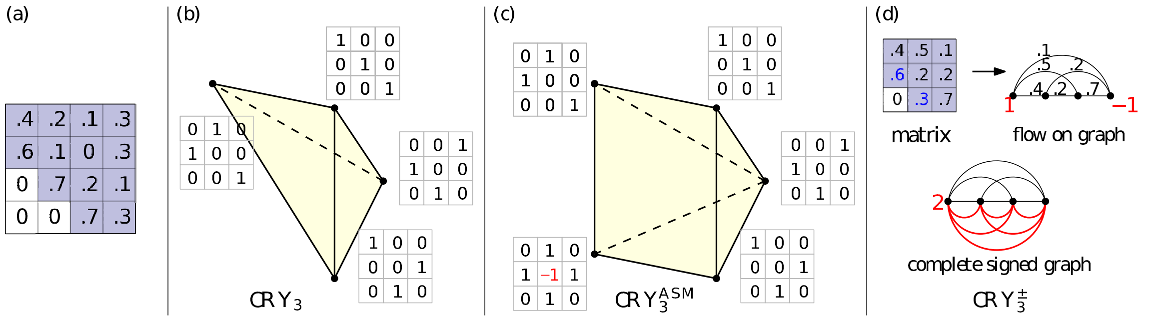

FIGURE 1. (a) A doubly-stochastic matrix in $\mathcal{CRY}_4$, (b) the polytope $\mathcal{CRY}_3$, (c) the polytope $\mathcal{CRY}_3^{ASM}$, (d) map from a matrix in $\mathcal{CRY}_3$ to a flow on K_4; graph $K_4^{\pm}$ associated to $\mathcal{CRY}_3^{\pm}$.

The first variant of $\mathcal{CRY}_n$ we consider in [8] is the corresponding face in $\mathcal{ASM}_n$ of all matrices (a_{ij}) in $\mathcal{ASM}_n$ where $a_{ij} = 0$ if $i-j \geq 2$. We call this new polytope $\mathcal{CRY}_n^{ASM}$ (see Figure 1 (c)). Surprisingly, this polytope is much simpler than the original $\mathcal{CRY}_n$. In fact, we show that $\mathcal{CRY}_n^{ASM}$ is equivalent to an *order polytope* [11] of the type A_{n-1} root poset. Therefore, its vertices are given by reverse order ideals in the poset (Cat_n many) and its volume is the number of linear extensions of this poset, i.e. the number $f_{(n-1,n-2,\ldots,2,1)}$ of standard Young tableaux of shape $(n-1, n-2, \ldots, 2, 1)$.

Corollary (Mészáros-M-Striker [8]).

$$\mathrm{vol}(\mathcal{CRY}_{\mathrm{n}}^{\mathrm{ASM}}) = \mathrm{f}_{(\mathrm{n}-1,\mathrm{n}-2,\ldots,2,1)} = \frac{\binom{\mathrm{n}}{2}!}{1^{\mathrm{n}-1}3^{\mathrm{n}-2}\cdots(2\mathrm{n}-3)^1}.$$

The polytopes $\mathcal{CRY}_n$ and $\mathcal{CRY}_n^{ASM}$ have the same dimension $\binom{n}{2}$ and the vertices of the former are a 2^{n-1}-subset of the vertices of the latter. It would be interesting to see how these two polytopes fit together.

3. Second variant: a flow polytope of a signed complete graph

Given a collection X of m vectors and a vector $\mathbf{b}$ all in $\mathbb{Z}^n$, let $k_X(\mathbf{b})$ be the number of ways of writing $\mathbf{b}$ as an $\mathbb{N}$-linear combination of vectors in X. The function $k_X(\mathbf{b})$ is called a *vector partition function*.

An important case is when X is the set of positive roots of a root system where the function $k_X(\mathbf{b})$ is called a *Kostant partition function*. These are very useful in representation theory for calculations of weight multiplicities and tensor product multiplicities. For type A_{n-1} we have $X = \{e_i - e_j \mid 1 \leq i < j \leq n\}$ (where e_i is the ith standard vector), and for other types like type D_n we have $X = \{e_i \pm e_j \mid 1 \leq i < j \leq n\}$.

If G is a directed acyclic graph on n vertices, let X_G be the set of vectors $e_i - e_j$ for each edge (i,j) in G with $i < j$. Note that X_G is also a subset of the positive roots of A_{n-1} mentioned above. In this case, the generating series of

$k_G(\mathbf{b}) := k_{X_G}(\mathbf{b})$ is

$$\sum_{\mathbf{b}\in\mathbb{Z}^{\mathbf{n}}} k_G(\mathbf{b})\mathbf{x}^{\mathbf{b}} = \prod_{\text{edges } (i,j) \text{ in } G} (1 - x_i x_j^{-1})^{-1}. \tag{1}$$

The *flow polytope* $\mathcal{F}_G(\mathbf{b}) := \mathcal{F}_{X_G}(\mathbf{b})$ consists of nonnegative real flows on the directed edges of G such that the *outflow* or *leak* on vertex i is b_i. It follows that the number of lattice points of $\mathcal{F}_G(\mathbf{b})$ are given by $k_{X_G}(\mathbf{b})$.

When G is the complete graph K_{n+1} the flow polytope $\mathcal{F}_{K_{n+1}}(e_1 - e_{n+1})$ is equivalent to $\mathcal{CRY}_n$ (see Figure 1 (c)). Postnikov-Stanley [10] and Baldoni-Vergne [1] showed that the volume of $\mathcal{F}_G(e_1 - e_{n+1})$ is given by the value of the partition function $k_{K_{n+1}}(\mathbf{b}')$ where $\mathbf{b}'$ only depends on G. Zeilberger [13] used this along with (1) and a constant term identity to prove the volume formula for $\mathcal{CRY}_n$.

One natural question to ask is whether there is an analogue of this result for other root systems. For this setting, we work with *signed* graphs $G^{\pm}$ that have negative edges $(i,j,-)$ corresponding to the roots $e_i - e_j$ $(i < j)$ and positive edges $(i,j,+)$ corresponding to the roots $e_i + e_j$ $(i < j)$. For this graph, the flow polytope $\mathcal{F}_{G^{\pm}}(\mathbf{b})$ is defined accordingly. In [7] we show that when $\mathbf{b} = 2e_1$, then the volume of $\mathcal{F}_{G^{\pm}}(2e_1)$ is given by a weighted partition function:

Theorem (Mészáros-M [7]). Given a signed graph $G^{\pm}$ with n vertices then the volume of the flow polytope $\mathcal{F}_{G^{\pm}}(2e_1)$ is

$$\mathrm{vol}(\mathcal{F}_{\mathrm{G}^{\pm}}(2\mathrm{e}_1)) = \mathrm{k}^{\mathrm{dyn}}_{\mathrm{G}^{\pm}}(0, \mathrm{d}_2 - 1, \ldots, \mathrm{d}_{\mathrm{n}-1} - 1, \mathrm{d}_\mathrm{n} - 1), \tag{2}$$

where d_i is the number of incoming negative edges to vertex i, and $k^{\mathrm{dyn}}_{G^{\pm}}$ has the following generating series:

$$\sum_{\mathbf{b}\in\mathbb{Z}^n} k^{\mathrm{dyn}}_{G^{\pm}}(\mathbf{b})\mathbf{x}^{\mathbf{b}} = \prod_{\text{edge } (i,j,-) \text{ in } G^{\pm}} (1 - x_i x_j^{-1})^{-1} \prod_{\text{edge } (i,j,+) \text{ in } G^{\pm}} (1 - x_i - x_j)^{-1}. \tag{3}$$

In [7] we look at the flow polytope $\mathcal{F}_{K_n^{\pm}}(2e_1)$ where $K_n^{\pm}$ is the complete signed graph with all edges $(i,j,\pm)$. We call this the *signed Chan-Robbins-Yuen polytope* or $\mathcal{CRY}_n^{\pm}$ (see Figure 1 (c)). By (2) its volume is given by $k^{dyn}_{K_n^{\pm}}(0,0,1,2,\ldots,n-2)$. Data suggest that this volume is as interesting as that of the $\mathcal{CRY}_n$ polytope:

Conjecture (Mészáros-M [7]). $\mathrm{vol}(\mathcal{CRY}_{\mathrm{n}}^{\pm}) = 2^{(\mathrm{n}-1)^2}\mathrm{Cat}_1\mathrm{Cat}_2\ldots\mathrm{Cat}_{\mathrm{n}-1}$.

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The shape of random combinatorial objects

Igor Pak
(joint work with T. Dokos and S. Miner)

In this talk, we study asymptotics of various classes of permutations, including **123**- and **132**-avoiding permutations, Baxter permutations and alternating permutations. The permutations are represented as $0-1$ matrices. We take their average and scale the function which then converges to a limiting surface we call the *limit shape*. We present explicit formulas in some cases and conjectures in other cases.

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Scaling Limits for Random Graphs from Subcritical Classes

Konstantinos Panagiotou
(joint work with Benedikt Stufler, Kerstin Weller)

Given a connected graph G with vertex set $V(G)$ and edge set $E(G)$ we can associate naturally to it a metric space $(V(G), d_G)$, where $d_G : V(G) \to \mathbb{N}_0$ denotes the shortest path distance in G. If G is a random graph, then this metric space is itself a random variable, and the aim of this research is to study its asymptotic properties when the size of G becomes large.

The classic example in this context is the case where $G = T_n$ and T_n is a uniform random tree with n vertices. Aldous [1], see also [4], showed that there is a constant $c > 0$, such that as $n \to \infty$

$$\left(V(T_n), \frac{d_{T_n}}{c\sqrt{n}}\right) \overset{(d)}{\to} \mathcal{T}, \tag{1}$$

where $\mathcal{T}$ is a continuous metric space, the so-called Brownian continuum random tree, and the convergence in distribution is in the sense of the Gromov-Hausdorff topology. This result can be extended to many families of simply generated trees (or equivalently, conditioned Galton-Watson trees), where up to the rescaling constant the same limiting behavior can be established [1, 2]. The convergence in (1) may be used to study many properties of large random trees, as for example the limiting distribution of the height or the distribution of the distances.

By now there are several beautiful results addressing the convergence of the metric space of many random objects, in particular of many families of random maps [4, 3]. However, except for trees, there are only very few results about families of random graphs. In this work we consider scaling limits as in (1) and extend the validity of this result to a large class of random graphs.

We consider so-called *block-stable* classes that can be defined as follows. Suppose that we are given a class $\mathcal{B}$ of 2-connected graphs, that may also include the graph consisting of a single edge. Then we let $\mathcal{C} = \mathcal{C}(\mathcal{B})$ be the class of all connected graphs whose blocks, i.e., maximal subgraphs that contain no cut-vertex, are in $\mathcal{B}$. For example, if $\mathcal{B}$ is the class of all 2-connected planar graphs (and the single edge), then $\mathcal{C}$ is the class of all connected planar graphs; if $\mathcal{B}$ is the class that contains only the graph that consists of a single edge, then we recover the class of trees.

In previous works [5] we showed that a global statistic of a random graph C_n with n vertices from a block-stable class $\mathcal{C}$ is determined by the critical quantity $t_B = \rho_B\, B''(\rho_B)$, where $B(z)$ is the exponential generating function enumerating the graphs in $\mathcal{B}$, and ρ_B is its singularity. In particular, if $t_B > 1$, then C_n is *subcritical* in the sense that the largest block in it has typically only $O(\log n)$ vertices; on the other hand, if $t_B < 1$ then C_n typically contains a giant block with $(1-t+o(1))n$ many vertices. Examples of subcritical classes are outerplanar and series-parallel graphs, while the class of planar graphs is not subcritical.

Our main result implies that (1) is true if T_n is replaced by a uniform random graph C_n from a subcritical class with n vertices, and where $c = c(\mathcal{B}) > 0$. Given the dichotomy described in the previous paragraph, we may expect that C_n looks "tree-like" in the sense that its global structure is dominated by some underlying tree and that the macroscopic effect of its blocks vanishes as $n \to \infty$. We show that this is indeed the case. In particular, we describe the random generation of a graph from $\mathcal{C}$ with n vertices by a two-step procedure. First, we draw a simply generated tree S_n with n vertices and an appropriate offspring distribution. Then, in a second independent step we decorate randomly its vertices with blocks from $\mathcal{B}$. This allows us us to relate the distances in S_n to the distances of the corresponding vertices in C_n, and the convergence is established. As a side result, we obtain a combinatorial interpretation of the scaling factor c in (1): it corresponds to the average distance of two randomly selected vertices in a graph B from $\mathcal{B}$, where the size of B is itself a random and distributed as in a Boltzmann model.

Our methods cannot be applied to block-stable classes that are not subcritical, since there the a main contribution to the pairwise distances comes from the giant block. It it an open problem to investigate the behavior in this case, and in particular the class of planar graphs remains as the most important open problem.

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How many random permutations, conjugated by an adversary, generate S_n?

Robin Pemantle
(joint work with Yuval Peres and Igor Riven)

Let M be a random set of integers greater than 1, containing each integer n independently with probability $1/n$. Let $S(M)$ be the sumset of M, that is, the set of all sums of subsets of M. How many independent copies of S must one intersect in order to obtain a finite set?

This problem is the limiting form of a problem arising in computational Galois theory: How many permutations must one sample uniformly from the symmetric group S_n before there is at least an epsilon probability that these permutations generate S_n even when an adversary replaces each one with a conjugate (another permutation of the same cycle type)?

Dixon showed in 1992 that $C(\log n)^{1/2}$ sufficed, and conjectured that $O(1)$ was good enough, which was proved shortly thereafter by Luczak and Pyber. Their constant was roughly 2^{100} has not been improved until now, though is conjectured that it can be improved to 5 or 4. We show in fact that 4 permutations suffice (equivalently, four copies of S have finite intersection).

Combinatorics and algorithms for classes of pattern-avoiding permutations

Adeline Pierrot
(joint work with F. Bassino, M. Bouvel, C. Pivoteau, D. Rossin)

The notion of pattern in a permutation is a nice and simple notion that raises lots of fascinating and challenging questions in combinatorics and algorithmics.

In this talk, we present two examples of interactions between combinatorics and algorithms, in the case of pattern-avoiding permutations.

The first result provide automatic methods for enumeration and random generation. More precisely, we present an algorithm which derives a combinatorial specification for a permutation class given by its basis of excluded patterns. The second part deals with stack-sorting : we present a polynomial algorithm deciding whether a permutation given as input is sortable trough two stacks in series.

Both results are algorithms obtained thanks to a combinatorial study of permutations using recusive decompositions. Details can be found in [Pie13].

1. Structure in permutation classes

A permutation of size n is a word of n letters $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ on the alphabet $[1..n]$ containing each letter from 1 to n exactly once. A permutation $\pi = \pi_1\pi_2\ldots\pi_k$ is a *pattern* of a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ (denoted $\pi \preceq \sigma$) if and only if $k \leq n$ and there exist integers $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ such that $\sigma_{i_1}\ldots\sigma_{i_k}$ is order-isomorphic to π, *i.e.* such that $\sigma_{i_\ell} < \sigma_{i_m}$ whenever $\pi_\ell < \pi_m$. For example the permutation $\sigma = 316452$ contains $\pi = 2431$ as a pattern, whose occurrences are 3642 and 3652. But σ avoids the pattern 2413 as none of its subsequences of length 4 is order-isomorphic to 2413.

The pattern containment relation $\preceq$ is a partial order on permutations, and a *permutation class* $\mathcal{C}$ is a downset under this order: for any $\sigma \in \mathcal{C}$, if $\pi \preceq \sigma$, then we also have $\pi \in \mathcal{C}$. For every set B, $Av(B)$ denotes the set of permutations avoiding any pattern of B. Every class $\mathcal{C}$ can be written as $\mathcal{C} = Av(B)$ for a unique antichain B (*i.e.,* set of pairwise incomparable elements) called the *basis* of $\mathcal{C}$.

Initiated by [Knu73] almost forty years ago, the study of permutation classes has since received a lot of attention, mostly with respect to enumerative questions. Most articles are focused on a given class $\mathcal{C} = Av(B)$. Recently, some results describing general properties of permutation classes have been obtained. Our work falls into this new line of research.

We present an algorithm which derives a combinatorial specification for a permutation class $\mathcal{C} = Av(B)$ given by its basis B of excluded patterns. The specification is obtained if and only if the class contains a finite number of simple permutations, this condition being tested algorithmically. This algorithm rely on the *substitution decomposition* of permutations, and the *simple permutations* are the indecomposable ones. This works takes its root in the theorem of Albert and Atkinson [AA05] stating that every permutation class containing a finite number of simple permutations has a finite basis and an algebraic generating function.

By *combinatorial specification* of a class (see [FS09]), we mean an unambiguous system of combinatorial equations describing recursively the permutations of $\mathcal{C}$.

By routine algorithms, the specification yields a system of equations satisfied by the generating function of $\mathcal{C}$, this system being always positive and algebraic, and a Boltzmann uniform random sampler of permutations in $\mathcal{C}$, using the methods of [FS09] and [DFLS04] respectively.

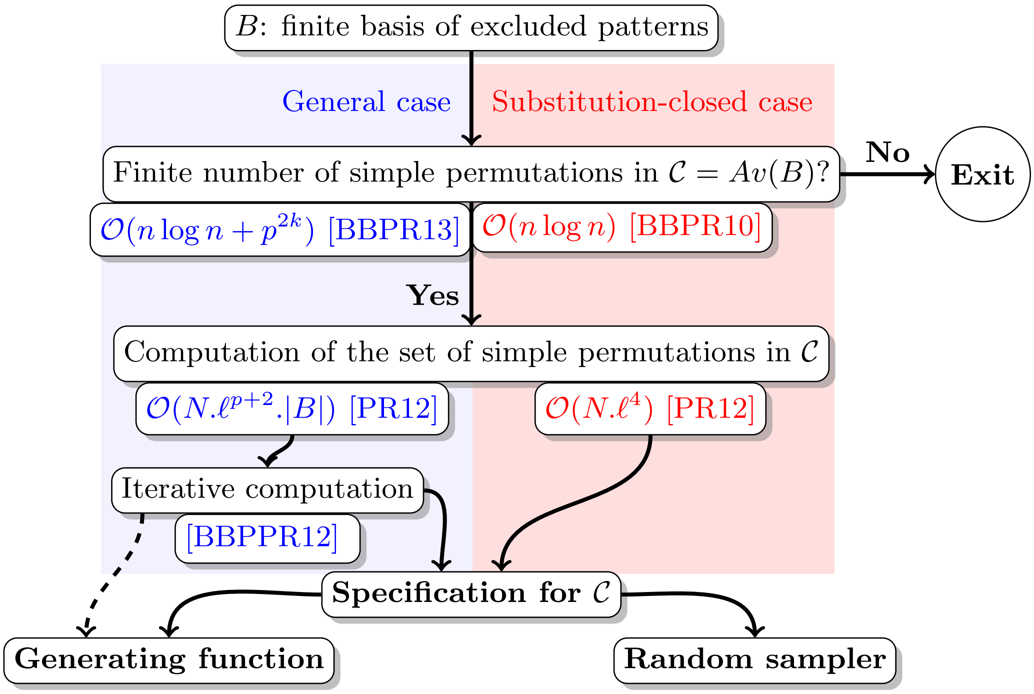


FIGURE 1. Automatic process from the basis of a permutation class to generating function and Boltzmann sampler.

2. STACK SORTING

Stack sorting has been studied first by Knuth in the sixties [Knu68]. Characterizing the stack-sortable permutations is the historical problem which led to define permutation patterns. Stack-sorting was then generalized by Tarjan, who introduced sorting networks [Tar72] allowing to sort more permutations, and many variations of this problem have been studied afterwards (see [B0́3] for a summary).

Here we study the decision problem "Is a given permutation σ sortable by two stacks connected in series?". The existence of a polynomial algorithm answering this question is a problem that stayed open for a long time, and the problem was even conjectured NP-complete in 2002.

Given two stacks H and V in series (see Figure 2) and a permutation σ, we want to sort the elements of σ using the stacks. We take σ as input: the elements σ_i are read one by one, from σ_1 to σ_n. We have three different operations:
ρ: Take the next element of σ in the input and push it on top of the first stack H.
λ: Pop the topmost element of stack H and push it on top of the second stack V.
μ: Pop the topmost element of stack V and write it to the output.

If there is a sequence of operations ρ, λ, μ leading to the identity $1 \dots n$ as output, the permutation σ is said 2-stack sortable. For example, 2431 is sortable using the following process:

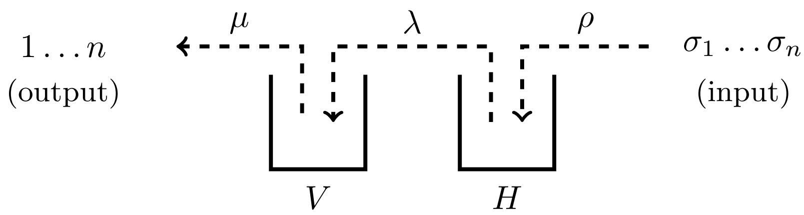
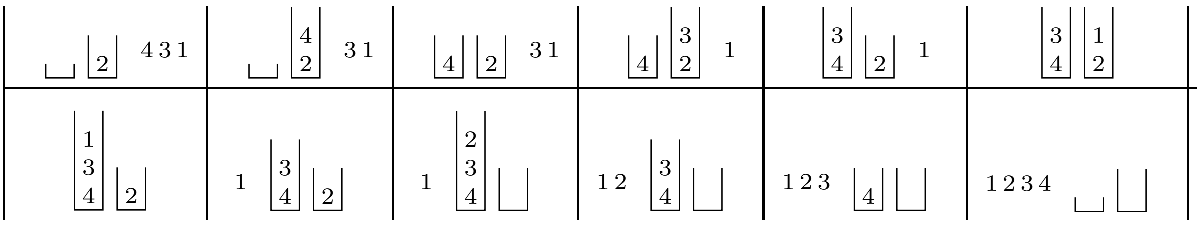


FIGURE 2. Sorting with two stacks in series.



The difficulty of this problem, whose statement is however very simple, lies in the fact that both stacks are considered at once, which gives a great liberty on which operation to apply at each step, and yields an exponential naive algorithm.

We give a polynomial decision algorithm by introducing a new notion, the pushall sorting, which is a restriction of the general stack sorting. We first solve the decision problem in the particular case of the pushall sorting, by encoding the sorting procedures through a bicoloring of the diagrams of the permutations. Then we solve the general case by showing that a sorting procedure in the general case corresponds to several steps of pushall sorting.

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Random planar graphs with given minimum degree

Lander Ramos
(joint work with Marc Noy)

In this work we enumerate planar graphs subject to a condition on the minimum degree δ, and to analyze the corresponding planar random graphs. Asking for $\delta \geq 1$ is not very interesting, since a random planar graph contains in expectation a constant number of isolated vertices. The condition $\delta \geq 2$ is directly related to the concept of the core of a graph. Given a connected graph $\mathcal{G}$, its *core* (also called 2-core in the literature) is the maximum subgraph $\mathcal{C}$ with minimum degree at least two. The core $\mathcal{C}$ is obtained from $\mathcal{G}$ by repeatedly removing vertices of degree one. Conversely, $\mathcal{G}$ is obtained by attaching rooted trees at the vertices of $\mathcal{C}$. The *kernel* of $\mathcal{G}$ is obtained by replacing each maximal path of vertices of degree two in the core $\mathcal{C}$ with a single edge. The kernel has minimum degree at least three, and $\mathcal{C}$ can be recovered from $\mathcal{K}$ by replacing edges with paths. Notice that $\mathcal{G}$ is planar if and only $\mathcal{C}$ is planar, if and only if $\mathcal{K}$ is planar.

It is convenient to introduce the following definitions: a *2-graph* is a connected graph with minimum degree at least two, and a *3-graph* is a connected graph with minimum degree at least three. In order to enumerate planar 2- and 3-graphs, we use generating functions. From now on all graphs are labelled and generating functions are of the exponential type. Let c_n, h_n and k_n be, respectively, the number of planar connected graphs, 2-graphs and 3-graphs with n vertices, and let

$$C(x) = \sum c_n \frac{x^n}{n!}, \qquad H(x) = \sum h_n \frac{x^n}{n!}, \qquad K(x) = \sum k_n \frac{x^n}{n!}$$

be the associated generating functions. Also, let $t_n = n^{n-1}$ be the number of (labelled) rooted trees with n vertices and let $T(x) = \sum t_n x^n/n!$. The decomposition of a connected graph into its core and the attached trees implies the following equation

$$C(x) = H(T(x)) + U(x), \tag{1}$$

where $U(x) = T(x) - T(x)^2/2$ is the generating functions of unrooted trees. Since $T(x) = xe^{T(x)}$, we can invert the above relation and obtain

$$H(x) = C(xe^{-x}) - x + \frac{x^2}{2}.$$

The equation defining $K(x)$ is more involved and requires the bivariate generating function

$$C(x, y) = \sum c_{n,k}\, y^k \frac{x^n}{n!},$$

where $c_{n,k}$ is the number of connected planar graphs with n vertices and k edges. We can express $K(x)$ in terms of $C(x, y)$ as

$$K(x) = C(A(x), B(x)) + E(x), \tag{2}$$

where $A(x), B(x), E(x)$ are explicit elementary functions.

From the expression of $C(x)$ as the solution of a system of functional-differential equations [1], it was shown that

$$c_n \sim \kappa n^{-7/2}\gamma^n n!,$$

where $\kappa \approx 0.4104 \cdot 10^{-5}$ and $\gamma \approx 27.2269$ are computable constants. In addition, analyzing the bivariate generating function $C(x, y)$ it is possible to obtain results on the number of edges and other basic parameters in random planar graphs. Our main goal is to extend these results to planar 2-graphs and 3-graphs.

Using Equations (1) and (2) we obtain precise asymptotic estimates for the number of planar 2- and 3-graphs:

$$\begin{aligned} h_n \sim \quad & \kappa_2 n^{-7/2}\gamma_2^n n!, \qquad && \gamma_2 \approx 26.2076, \quad \kappa_2 \approx 0.3724 \cdot 10^{-5}, \\ k_n \sim \quad & \kappa_3 n^{-7/2}\gamma_3^n n!, \qquad && \gamma_2 \approx 21.3102, \quad \kappa_3 \approx 0.3107 \cdot 10^{-5}. \end{aligned}$$

As is natural to expect, h_n and k_n are exponentially smaller than c_n. Also, the number of 2-connected planar graphs is known to be asymptotically $\kappa_c n^{-7/2} 26.1841^n n!$ (see [2]), smaller than the number of 2-graphs. This is consistent, since a 2-connected has minimum degree at least two.

By enriching Equations (1) and (2) taking into account the number of edges, we prove that the number of edges in random planar 2-graphs and 3-graphs are both asymptotically normal with linear expectation and variance. The expected number of edges in connected planar graphs was shown to be [1] asymptotically μn, where $\mu \approx 2.2133$. We show that the corresponding constants for planar 2-graphs and 3-graphs are

$$\mu_2 \approx 2.2614, \qquad \mu_3 \approx 2.4065.$$

This conforms to our intuition that increasing the minimum degree also increases the expected number of edges.

We also analyze the size X_n of the core in a random connected planar graph, and the size Y_n of the kernel in a random planar 2-graph. We show that both variables are asymptotically normal with linear expectation and variance and that

$$\begin{aligned} \mathbf{E}\, X_n \sim \quad & \lambda_2 n, \qquad && \lambda_2 \approx 0.9618, \\ \mathbf{E}\, Y_n \sim \quad & \lambda_3 n, \qquad && \lambda_3 \approx 0.8259. \end{aligned}$$

We remark that the value of λ_2 has been recently found by McDiarmid [3] using alternative methods. Also, we remark that the expected size of the largest block (2-connected component) in random connected planar graphs is asymptotically $0.9598n$ [4]. Again this is consistent since the largest block is contained in the core.

The picture is completed by analyzing the size of the trees attached to the core. We show that the number of trees with k vertices attached to the core is asymptotically normal with linear expectation and variance. The expected value is asymptotically

$$C\frac{k^{k-1}}{k!}\rho^k n,$$

where $C > 0$ is a constant and $\rho \approx 0.03673$ is the radius of convergence of $C(x)$. For k large, the previous quantity grows like

$$\frac{C}{\sqrt{2\pi}} \cdot k^{-3/2}(\rho e)^k n.$$

This quantity is negligible when $k \gg \log(n)/(\log(1/\rho e))$. Using the method of moments, we show that the size L_n of the largest tree attached to the core is in fact asymptotically

$$\frac{\log(n)}{\log(1/\rho e)}.$$

Moreover, we show that $L_n/\log n$ converges in law to a Gumbel distribution. This result provides new structural information on the structure of random planar graphs.

Our last result concerns the distribution of the vertex degrees in random planar 2-graphs and 3-graphs. We show that for each fixed $k \geq 2$ the probability that a random vertex has degree k in a random planar 2-graph tends to a positive constant $d_H(k)$, and for each fixed $k \geq 3$ the probability that a random vertex has degree k in a random planar 3-graph tends to a positive constant $d_K(k)$. Moreover $\sum_{k\geq 2} p_H(k) = \sum_{k\geq 3} p_K(k) = 1$, and the probability generating functions

$$p_H(w) = \sum_{k\geq 2} p_H(k) w^k, \qquad p_K(w) = \sum_{k\geq 3} p_K(k) w^k$$

are computable in terms of the probability generating function $p_C(w)$ of connected planar graphs, which was fully determined in [5].

It is natural to ask why we stop at minimum degree three. The reason is that there seems to be no combinatorial decomposition allowing to deal with planar graphs of minimum degree four or five (a planar graph has always a vertex of degree at most five). It is already an open problem to enumerate 4-regular planar graphs. In contrast, the enumeration of cubic planar graphs was completely solved in [6].

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On the exit time from a cone for random walks with drift and applications to the enumeration of walks in cones

Kilian Raschel
(joint work with Rodolphe Garbit)

This abstract is based on the paper "On the exit time from a cone for random walks with drift", see `http://arxiv.org/abs/1306.6761`.

In this article we consider d-dimensional random walks such that the law of the increments has all exponential moments. For a large class of cones, we compute the exponential decay of the probability for such random walks to stay in the cone up to time n, as n goes to infinity. We show that the latter equals the global minimum, on the dual cone, of the Laplace transform of the random walk increments.

Our results find applications in the counting of walks in orthants, a classical domain in enumerative combinatorics. Given a finite set $\mathfrak{S}$ of allowed steps, a now classical problem is to study $\mathfrak{S}$-walks in the orthant Q, that is walks confined to Q, starting at a fixed point x (often the origin) and using steps in $\mathfrak{S}$ only. Denote by $f_{\mathfrak{S}}(x, y; n)$ the number of such walks that end at y and use exactly n steps. Many properties of the counting numbers $f_{\mathfrak{S}}(x, y; n)$ have been recently analyzed (the seminal work in this area is [1]). First, exact properties of them were derived, via the study of their generating function (exact expression and algebraic nature). Such properties are now well established for the case of small steps walks in the quarter-plane, meaning that the step set $\mathfrak{S}$ is included in $\{0, \pm 1\}^2$. More qualitative properties of the $f_{\mathfrak{S}}(x, y; n)$ were also investigated, such as the asymptotic behavior, as $n \to \infty$, of the number of excursions $f_{\mathfrak{S}}(x, y; n)$ for fixed y, or that of the total number of walks,

$$f_{\mathfrak{S}}(x; n) = \sum_{y \in Q} f_{\mathfrak{S}}(x, y; n). \tag{1}$$

Concerning the excursions, several small steps cases have been treated by Bousquet-Mélou and Mishna [1] and by Fayolle and Raschel [3]. Later on, Denisov and Wachtel [2] obtained the very precise asymptotics of the excursions, for a quite large class of step sets and cones. As for the total number of walks (1), only very particular cases are solved, see again [1, 3]. In a most recent work [4], Johnson, Mishna and Yeats obtained an upper bound for the exponential growth constant, namely,

$$\limsup_{n \to \infty} f_{\mathfrak{S}}(x; n)^{1/n},$$

and proved by comparison with results of [3] that these bounds are tight for all small steps models in the quarter-plane. In the present article, we find the exponential growth constant of the total number of walks (1) in any dimension for any model such that:

(1) The step set $\mathfrak{S}$ is not included in a linear hyperplane;
(2) The step set $\mathfrak{S}$ is not included in a half-space u^-, with $u \in Q \setminus \{0\}$;

(3) The step set allows a path staying in Q from the origin to some point in the interior of Q.

In the sequel we shall say that a step set $\mathfrak{S}$ is proper if it satisfies the properties (1), (2) and (3). Note in particular that the well-known 79 models of walks in the quarter-plane studied in [1, 3] (including the so-called 5 singular walks) satisfy the hypotheses above.

Our theorem in combinatorics can be stated as follows. Let $\mathfrak{S}$ be any proper step set. The Laplace transform of $\mathfrak{S}$,

$$L_{\mathfrak{S}}(x) = \sum_{s \in \mathfrak{S}} e^{\langle x, s \rangle},$$

reaches a global minimum on Q at a unique point x_0, and for any starting point $x \in Q_\delta$,

$$\lim_{n \to \infty} f_{\mathfrak{S}}(x; n)^{1/n} = L_{\mathfrak{S}}(x_0).$$

As a consequence, we obtain the following result, which was conjectured in [4]: Let $\mathfrak{S} \subset \mathbb{Z}^d$ be a proper step set, and let $K_{\mathfrak{S}}$ be the growth constant for the total number of walks (1). Let $\mathcal{P}$ be the set of hyperplanes through the origin in $\mathbb{R}^d$ which do not meet the interior of the first orthant. Given $p \in \mathcal{P}$, let $K_{\mathfrak{S}}(p)$ be the growth constant of the walks on $\mathfrak{S}$ which are restricted to the side of p which includes the first orthant. Then $K_{\mathfrak{S}} = \min_{p \in \mathcal{P}} K_{\mathfrak{S}}(p)$.

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Multiple Binomial Sums

Bruno Salvy

(joint work with Alin Bostan and Pierre Lairez)

The computation of definite sums in computer algebra is classically performed by the method of *creative telescoping* initiated by Zeilberger [10, 11, 12]. In order to compute the sum of a sequence $(u(n,k))$, this method finds an identity of the form

$$a_0(n)u(n+p,k) + \cdots + a_p(n)u(n,k) = v(n,k+1) - v(n,k).$$

Provided that it is possible to sum both sides over k and that the sequence $(v(n,k))$ vanishes at the endpoints of the domain of summation, the left-hand side — called a *telescoper* — gives a recurrence for the sum. The right-hand side is then called the *certificate* of the identity. In the case of multiple sums, this idea leads to searching for a telescoping identity of the form

$$\begin{aligned}(1)\quad a_0(n)u(n+p,k_1,\dots,k_m)+\cdots+a_p(n)u(n,k_1,\dots,k_m) &= \\ v_1(n,k_1+1,k_2,\dots,k_m)-v_1(n,k_1,\dots,k_m)+\cdots& \\ +v_m(n,k_1,\dots,k_m+1)-v_m(n,k_1,\dots,k_m).&\end{aligned}$$

Again, under favorable circumstances the sums of the sequences on the right-hand side telescope, leaving a recurrence for the sum on the left-hand side.

This high-level presentation hides practical difficulties. The first one is that it is important to check that the sequences on both sides of the identities above are defined over the whole range of summation [1, 2]. The second one is a consequence of the first one: computing the certificate is not merely a useful by-product of the algorithm, but indeed a necessary part of the computation. Unfortunately, the size of the certificate may be much larger than that of the final recurrence and thus costly in terms of computational complexity.

The same difficulties occur in the differential case, when computing (multiple) integrals. However, we have showed that integration of multivariate rational functions over cycles can be achieved efficiently without computing the corresponding certificate [4]. In that case, the algorithm computes a linear differential equation for the parameterized integral. By passing to generating functions, a large number of multiple sums can be cast into problems of rational integration. The algorithmic consequences of this observation form the object of the present work.

More precisely, the algorithm we present handles all multiple sums of the form

$$\begin{aligned}(2)\quad S_n := \sum_{k_1,\dots,k_m}\binom{a_{1,1}k_1+\cdots+a_{1,m}k_m+c_1n+e_1}{b_{1,1}k_1+\cdots+b_{1,m}k_m+d_1n+f_1}\times\cdots& \\ \times\binom{a_{r,1}k_1+\cdots+a_{r,m}k_m+c_rn+e_r}{b_{r,1}k_1+\cdots+b_{r,m}k_m+d_rn+f_r}g_{k_1,\dots,k_m,n},&\end{aligned}$$

where the coefficients $a_{i,j}, b_{i,j}, c_i, d_i, e_i, f_i$ are integers, the sum is over all nonnegative integer values that make these binomial coefficients nonzero plus possible linear inequalities over the indices, and g is the sequence of coefficients of an algebraic series. This method also applies to multinomial coefficients, since they can be rewritten as products of binomial coefficients. The sequence g could also be taken as the sequence of coefficients of an arbitrary diagonal of a rational power series. The literature on which sequences can and cannot be encoded into the diagonal of a rational power series is large. We refer the reader to [3, 5] for more information on this subject. In particular, it is easy to see that diagonals are closed under sum, product and differentiation.

A classical example in this area is one of the intermediate steps in Apéry's proof of the irrationality of $\zeta(3)$, which relies on a linear recurrence for the sum

$$A(n)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2.$$

This is exactly of the form above. Our algorithm will first rewrite this as the sequence of coefficients of the integral of a rational function in 5 variables around the origin, next this is further simplified automatically as

$$\oint \frac{dt_1 \wedge dt_2 \wedge dt_3}{t_1t_2t_3(1-t_1t_2-t_1t_2t_3)-(1+t_1)(1+t_2)(1+t_3)z}.$$

This triple integral is the generating series of the sequence $(A(n))$. Our algorithm from last year [4] computes a linear differential equation it satisfies, which is then translated into the desired (classical) recurrence:

$$(n+1)^3 A(n) - (2n+3)\left(17n^2+51n+39\right) A(n+1) + (n+2)^3 A(n+2) = 0.$$

Another example is provided by the sum $C(n)$

$$\sum_{r,s} (-1)^{n+r+s} \binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n},$$

for which telescoper and certificate can be computed [8, p. 33], but not so easily [9, Section 5.7.6]. It is however, readily encoded into

$$\frac{(1+t_3)^2}{t_1t_2t_3(1+t_3(1+t_1))(1+t_3(1+t_2))+z(1+t_1)(1+t_2)(1+t_3)^4},$$

from where we get a linear differential equation of order 3, whence the recurrence

$$(4n+3)(4n+4)(4n+5)C(n)+2(2n+3)(3n^2+9n+7)C(n+1) = (n+2)^3C(n+2).$$

Even for simple (as opposed to multiple) sums, our approach may compare favorably to the direct use of creative telescoping. For instance, the telescoper for

$$D(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{dk}{n}$$

computed by Zeilberger's algorithm has order $d-1$ [7]. Our algorithm readily expresses the generating series as

$$\sum_{n=0}^{\infty} D(n)t^n = \frac{1}{2\pi i}\oint \frac{dy}{y+t(1+y)^d-t}. \tag{3}$$

From there, since only the pole $y=0$ tends to 0 when t tends to 0, the generating series is obtained as the corresponding residue $1/(1+dt)$ and the first-order recurrence $D(n+1) = -dD(n)$ follows, showing that $D(n) = (-d)^n$.

In summary, our approach to binomial summation consists in three steps. First, we use the classical method of generating series (developed, e.g., by Egorychev [6]) to encode binomial sums of the form (2) into integrals of rational functions. We then use differential creative telescoping methods to compute linear differential equations satisfied by the generating series, and convert them back to recurrences. This offers an alternative to discrete creative telescoping for binomial sums. For a fair comparison, it should be stressed that our method deals only with binomial sums, while discrete creative telescoping handles other sums that cannot be put in the form of Eq. (2).

To get bounds on the computational complexity of our method, and on the size of the output recurrence, we import results on Picard-Fuchs differential equations satisfied by integrals of rational functions [4]. This lets us prove a complexity estimate for multiple binomial summation, under the mild assumption that $g = 1$ and the sum is over all $\mathbb{N}^m$ in Eq. (2). The recurrence has size (order, degree) that is bounded in terms of the sum of (absolute values of) the parameters $a_{i,j}, b_{i,j}, c_i, d_i, e_i, f_i$, raised to an exponent that grows linearly with the number r of binomial coefficients and the number m of nested sums. The complexity bound is of the same nature.

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Two enumerative tidbits

Richard Stanley

First tidbit. The first tidbit (with C. Bessenrodt) is a generalization of a classical result of Carlitz, Roselle, and Scoville [1]. Let λ be a partition of some integer $n \geq 1$. We identitfy λ with its Young diagram. Let λ^* be the augmented diagram (shape) obtained by adding a border strip around the southeast boundary of λ.

Place the number 1 in the squares of λ^*/λ. Then place in the square t of λ the integer n_t such that the largest square subdiagram of λ^* with upper left-hand corner t, regarded as a matrix, has determinant one. It is easy to see that the integers n_t exist and are unique. The figure below shows this filling for $\lambda = (3, 2)$.

9	5	2	1
3	2	1	1
1	1	1	

Let $\lambda(t)$ be the largest subdiagram of λ with t as the upper left-hand corner. Let u_t be the number of partitions μ whose diagram is contained in $\lambda(t)$, i.e., $\mu \leq \lambda(t)$ in Young's lattice. The Carlitz-Roselle-Scoville theorem asserts that $n_t = u_t$.

We extend this result in two ways. We give a multivariate refinement of u_t, and we compute not just the determinant, but rather the Smith normal form (SNF), which a priori need not exist. More specifically, for each square (i,j) of λ (using matrix coordinates, so $(1,1)$ is the upper left-hand corner), associate an indeterminate x_{ij}. Define a refinement $u_\lambda(x)$ of u_λ by

$$u_\lambda(x) = \sum_{\mu \subseteq \lambda} \prod_{(i,j) \in \lambda/\mu} x_{ij}.$$

For $t \in \lambda$ let

$$A_t = \prod_{(i,j) \in \lambda(t)} x_{ij}.$$

Theorem. *Let $t = (i,j)$. Then M_t has Smith normal form*

$$\mathrm{diag}(A_{ij}, A_{i-1,j-1}, \ldots, 1).$$

The second tidbit (with Fu Liu = 刘拂). The second tidbit arose from a problem submitted by Ron Graham to the *New York Times Numberplay Blog* of March 25, 2013. Namely, if S is any 8-element subset of $\mathbb{Z}$, can you two-color S such that there is no three-term monochromatic arithmetic progression? This question was finally answered affirmatively by Noam Elkies. His proof involved the following concept. Let $1 \leq i < j < k \leq n$ and $1 \leq a < b < c \leq n$. Define $\{i,j,k\}$ and $\{a,b,c\}$ to be *compatible* if there exist integers $x_1 < x_2 < \cdots < x_n$ such that x_i, x_j, x_k is an arithmetic progression and x_a, x_b, x_c is an arithmetic progression. Let $\binom{[n]}{3}$ denote the set of all 3-element subsets of $\{1,2,\ldots,n\}$. Let M_n denote the collection of all subsets S of $\binom{[n]}{3}$ such that any two elements of S are compatible. Elkies needed to look at the elements of M_8. He was led to the following conjectures.

Conjecture 1. $\#M_n = 2^{\binom{n-1}{2}}$.

Conjecture 2. The number of elements in M_n of maximum size is

$$g(n) = \begin{cases} 2^{m(m-1)}, & n = 2m+1 \\ 2^{(m-1)(m-2)}(2^m - 1), & n = 2m. \end{cases}$$

NOTE: The maximum size (number of elements) $\sigma(n)$ of an element of M_n is given by

$$\sigma(n) = \begin{cases} m^2, & n = 2m+1 \\ m(m-1), & n = 2m. \end{cases}$$

We give proofs of both conjectures. The proof of Conjecture 1 is based on an observation of Jim Propp that $\#M_n$ is equal to the number of antichains (or equivalently, order ideals) in a certain poset P_n, and hence the number of elements of the distributive lattice $L_n = J(P_n)$ of order ideals of P_n. By examining the join-irreducibles of L_n we show that it is isomorphic to the set of all semistandard Young tableaux of shape $(n-2, n-3, \ldots, 1)$ with largest part at most $n-1$. Standard results from the theory of symmetric functions imply that the number of such tableaux is $2^{\binom{n-1}{2}}$, as desired.

For the second conjecture, we use a theorem of Dilworth that the elements of a finite distributive lattice $J(P)$ corresponding to maximum size antichains of P forms a sublattice, necessarily distributive. Thus again we need to find the number of elements in a distributive lattice D_n . We do this by analyzing the join-irreducibles of D_n and showing that they are closely related to the join-irreducibles of $M_{\lfloor (n+2)/2 \rfloor}$.

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Floors and Ceilings of the k-Catalan arrangement

Marko Thiel

For a crystallographic root system Φ with ambient space V, we define the k-Catalan arrangement as the hyperplane arrangement given by all the hyperplanes $H_\alpha^r = \{x \in V \mid \langle x, \alpha \rangle = r\}$ for $\alpha \in \Phi$ and $-k \leq r \leq k$. Let us choose a simple system S for Φ with associated positive system Φ^+. Then the number of regions of the k-Catalan arrangement that are contained in the dominant chamber $C = \{x \in V \mid \langle x, \alpha \rangle > 0 \text{ for all } \alpha \in S\}$ is the k-th Fuß-Catalan number $\mathrm{Cat}^{(k)}(\Phi)$ of Φ [3]. This number also occurs in other contexts: it also counts the k-divisible noncrossing partitions of Φ as well as the facets of the k-generalised cluster complex of Φ [1]. Why this is so is still somewhat mysterious, as every known proof of this fact appeals to the classification of irreducible crystallographic root systems.

For a dominant region R of the k-Catalan arrangement, we call the supporting hyperplanes of facets of R its *walls*. Those walls of R that do not contain the origin and have R and the origin at the same side we call the *ceilings* of R. The other walls of R we call its *floors*. We call a hyperplane of the form H_α^r for some $\alpha \in \Phi^+$ *r-coloured*. The number of dominant regions of the k-Catalan arrangement that have exactly i k-coloured floors is the Fuß-Narayana number $\mathrm{Nar}^{(k)}(\Phi, i)$. We find

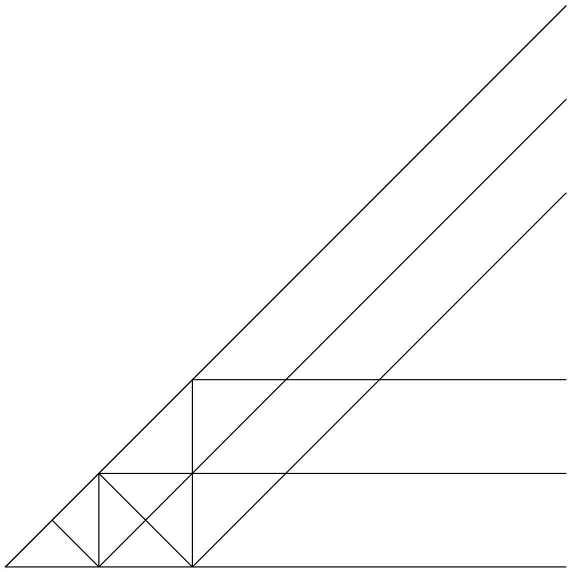

FIGURE 1. The dominant chamber of the 2-Catalan arrangement of type B_2.

the same numbers as the rank numbers of the poset of k-divisible noncrossing partitions of Φ and also as entries of the h-vector of the k-generalised cluster complex of Φ [1, Definition 3.5.4] [4] [5, Theorem 3.2] [7, Theorem 1]. Similarly, counting bounded dominant regions by their number of k-coloured ceilings gives the entries of the h-vector of the positive part of the k-generalised cluster complex of Φ [2, Conjecture 1.2] [7, Corollary 5].

These coincidences suggest that it is interesting to study the refined enumeration of dominant regions of the k-Catalan arrangement by floors and ceilings. We find a very close relationship, given by the following theorem.

Theorem ([6, Theorem 1]). For any set $M = \{H_{\alpha_1}^{i_1}, \dots, H_{\alpha_m}^{i_m}\}$ of hyperplanes in the k-Catalan arrangement, with $i_j > 0$ for all $j \in [m]$, there is a bijection between the dominant regions R such that all hyperplanes in M are floors of R and the dominant regions R' such that all hyperplanes in M are ceilings of R'.

From this we can deduce a number of enumerative corollaries, such as the following.

Corollary ([1, Conjecture 5.1.24], [6, Corollary 3]). For any $i \in \mathbb{Z}_{>0}$ and $j \in \mathbb{Z}_{\geq 0}$, the number of dominant regions R that have exactly j i-coloured floors equals the number of dominant regions R' that have exactly j i-coloured ceilings.

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Tamari lattice, m-Tamari, (a, b)-Tamari and beyond

Xavier Viennot
(joint work with Louis-Francois Preville-Ratelle)

In recent years, much work has been done on the associahedron and its underlying lattice called the Tamari lattice. The vertices of this lattice are binary trees or equivalently Dyck paths (or ballot paths), enumerated by the Catalan numbers C_n. The order relation is generated by the covering relation defined with the so-called rotation on binary trees, or equivalently a certain elementary transformation on ballot paths. The number of intervals in the Tamari lattice is given by $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$ (Chapoton, [4]), and this number also counts the rooted triangulations in the plane.

Motivated by the higher diagonal coinvariant spaces of the symmetric group, Bergeron [2] introduced the m-Tamari lattice for every integer m by mimicking the construction defining the covering relation on ballot paths, corresponding to $m = 1$. The elements of this lattice are m-ballot paths, that is paths located above the diagonal with slope $1/m$. Much work has been done in algebra and in combinatorics around these diagonal coinvariant spaces, which are representations of the symmetric group indexed by a number of sets of variables. The topic was initiated 20 years ago by Garsia and Haiman, in relation with Macdonald polynomials, and many deep conjectures are still open even in the bivariate case (i.e. with two sets of variables). Bergeron conjectured that the number of intervals in the m-Tamari lattice is the dimension of the alternants of the trivariate higher diagonal coinvariant spaces. He also conjectured that the number of such intervals is given by $\frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1}$, extending the formula of Chapoton. This last conjecture was proved by Bousquet-Melou, Fusy and Preville-Ratelle [3].

Some authors [6, 1] extended such combinatorics to paths above a line of rational slope a/b (where a and b are relatively prime). This is the so-called "rational Catalan combinatorics" [1]. The problem to find a generalization of the Tamari lattice for (a, b) is open and the question is posed by Elizalde in his research report for this workshop.

In the talk of this workshop, we propose a solution to this problem, and a much more general extension of the notion of Tamari lattice [10], containing m-Tamari and (a, b)-Tamari. For any path v, made of elementary north and east steps on the square grid, we define a poset $\mathrm{Tam}(v)$. The elements are the paths u with the same endpoints as v and weakly above v. Its covering relations are defined by certain transformations on the paths u that depend on v. If the path v is the path just above the line $ax = by$, we get the (a, b)-Tamari lattice, and recover the particular cases of the ordinary Tamari and m-Tamari lattices.

Then we prove the following three propositions. 1) $\mathrm{Tam}(v)$ is a lattice. 2) The lattice $\mathrm{Tam}(\overleftarrow{v})$, where $\overleftarrow{v}$ is the mirror image of v, exchanging east and north steps, is isomorphic to the dual of $\mathrm{Tam}(v)$. Thus (b, a)-Tamari is dual to (a, b)-Tamari. This property was already briefly mentioned in the thesis of Preville-Ratelle [8], and it is proved here in full generality. 3) We define a partition of the ordinary Tamari lattice (on binary trees with n vertices) into 2^{n-1} intervals $\mathrm{I}(v)$, where each interval is isomorphic to the lattice $\mathrm{Tam}(v)$. Thus all the lattices m-Tamari, (a, b)-Tamari and extensions, are simply contained in the ordinary Tamari lattice. Note that this embedding of the m-Tamari lattice in the ordinary Tamari lattice is not the same as some previous embedding already given in the literature.

The key idea is in defining a bijection between binary trees and pair of paths (u, v), where u is a path above the path v, and where the path v is called the canopy of the binary tree. This notion of canopy was introduced by Loday and Ronco (without giving a name) [7], in some algebraic considerations about the 3 Hopf algebra associated to the trilogy: hypercube, associahedron, permutahedron, with respective dimensions $2^{n-1}, C_n$ and $n!$. The bijection between binary trees and pair of paths (u, v) was introduced in a different form by Delest and Viennot [5]. We describe here a new version of the bijection which involves a "push-gliding" algorithm, and fits to our purpose.

By studying the behavior of the canopy via the rotation on binary trees, we first prove that the set $\mathrm{I}(v)$ of binary trees having a given canopy v is an interval of the (ordinary) Tamari lattice. Then we prove that this interval $\mathrm{I}(v)$ is isomorphic to the poset $\mathrm{Tam}(v)$, using various equivalent definitions of the canopy, and some combinatorics of binary trees and of the "push-gliding" bijection. The three propositions above follow immediately. The duality between the lattices $\mathrm{Tam}(v)$ and $\mathrm{Tam}(\overleftarrow{v})$ follows from the simple fact that the mirror image of a binary tree exchanges the "right" rotation and the "left" rotation defining the covering relation in each of theses lattices.

We mention that in a forthcoming paper [9], Preville-Ratelle has proved that the total number of intervals in the lattices $\mathrm{Tam}(v)$, for all the paths v of length n, is given by $\frac{2(3n+3)!}{(n+2)!(2n+3)!}$, which is the same as the number of rooted non-separable planar maps with $n + 2$ edges. This gives an answer to some questions posed by the audience after our talk. Also in his talk at this workshop, Armstrong gave a construction of a simplicial complex for any pair of relatively prime integers (a, b), called the rational associahedra $\mathrm{Ass}(a, b)$ (see [1]). It will be interesting to compare the constructions $\mathrm{Ass}(a, b)$ and (a, b)-Tamari.

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Connectivity and related properties for graph classes – overview of my thesis

Kerstin Weller

(joint work with Mireille Bousquet-Mélou and Colin McDiarmid)

There has been much recent interest in random graphs sampled uniformly from the set of (labelled) graphs on n vertices in a suitably structured class $\mathcal{A}$. An important and well-studied example of such a suitable structure is *bridge-addability*, introduced in [4] in the course of studying random planar graphs. A class $\mathcal{A}$ is bridge-addable when the following holds: if we take any graph G in $\mathcal{A}$ and any pair u, v of vertices that are in different components in G, then the graph $G^{'}$ obtained by adding the edge uv to G is also in $\mathcal{A}$. It was shown that for a random graph sampled from a bridge-addable class, the probability that it is connected is always bounded away from 0, and the number of components is bounded above by a Poisson law.

What happens if 'bridge-addable' is replaced by something weaker? In my thesis, this question is explored in several different directions and in my talk I chose to present two different aspects of my research.

First part – minor-closed classes of graphs
Together with Mireille Bousquet-Mélou, I investigated minor-closed, labelled classes of graphs. The excluded minors are always assumed to be connected, which is equivalent to the class $\mathcal{A}$ being *decomposable* - a graph is in $\mathcal{A}$ if and only if every component of the graph is in $\mathcal{A}$. When $\mathcal{A}$ is minor-closed, decomposable and bridge-addable various properties are known [5], generalizing results for planar

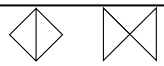
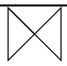
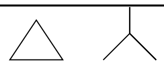
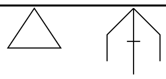
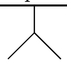
Excluded minors	$C(\rho)$	Sing. of $C(z)$	$\lim p_n$	number N_n of comp.	root comp. S_n	largest comp. L_n
2-connected	$< \infty$	?	$\geq 1/\sqrt{e}$ < 1	$O(1)$ Poisson	$n - R_n$ $R_n \to$ disc.	$n - F_n$ $F_n \to$ disc.
at least a spoon, but no tree	$< \infty$	$(1 - ze)^{3/2}$	> 0 $\leq 1/\sqrt{e}$	id.	id.	id.
	∞	log $(+\sqrt{})$	0	$\log n$ gaussian	n $\frac{1}{4}(1-x)^{-3/4}$	$\mathrm{PD}^{(1)}(1/4)$
	∞	$1/\sqrt{}$	0	$n^{1/3}$ gaussian	$n^{2/3}$ $2\sqrt{x/\pi}e^{-x}$	?
 (path forests)	∞	simple pole	0	$\sqrt{n}$ gaussian	$\sqrt{n}$ xe^{-x}	$\sqrt{n}\log n$ Gumbel
 (forests of caterpillars)	∞	id.	0	id.	id.	?
 (max. deg. 2)	∞	id. (+ log)	0	id.	id.	?
all conn. graphs of size $k+1$	∞	entire (polynomial)	0	n/k gaussian	k Dirac	k Dirac

Table 1. Summary of the results: for each quantity N_n, S_n and L_n, we give an equivalent of the expected value (up to a multiplicative constant, except in the last line where constants are exact) and a description (name or density) of the limit law. The examples are ordered according to the speed of divergence of $C(z)$ near its radius ρ. Spoons are graphs made of a triangle with a path attached to one vertex of the triangle. As we get lower in the table, the graphs have more components, of a smaller size. The symbol $\mathrm{PD}^{(1)}(1/4)$ stands for the first component of a Poisson-Dirichlet distribution of parameter $1/4$.

graphs. A minor-closed class is decomposable and bridge-addable if and only if all excluded minors are 2-connected. In our paper [2], we present a series of examples where the excluded minors are not 2-connected, analysed using generating functions as well as techniques from graph theory. This is a step towards a classification of connectivity behaviour for minor-closed classes of graphs. In contrast to the bridge-addable case, different types of behaviours are observed and the results are summarised in Table 1. Furthermore, we investigate a parameter that has not received any attention in this context yet: the size of the root component. It follows non-gaussian limit laws (beta and gamma), and clearly deserves a systematic investigation (work in progress).

Second part – relatively bridge-addable classes of graphs

Together with my supervisor Colin McDiarmid I generalize the result for bridge-addable classes of graphs in [4] to *relatively bridge-addable classes of graphs* where some but not necessarily all of the possible bridges are allowed to be introduced. We start with a bridge-addable class $\mathcal{A}$ and a host graph H, and consider the set of subgraphs of H in $\mathcal{A}$. This set of subgraphs is then relatively bridge-addable with respect to H and the notion was first introduced in [6]. Our connectivity bound then involves the edge-expansion properties of the host graph H. We also give a bound on the expected number of vertices not in the largest component. Furthermore, we investigate whether these bounds are tight, and in particular give detailed results about random forests in balanced complete multipartite graphs [7].

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Enumeration of graphs with a heavy-tailed degree sequence

Nick Wormald

(joint work with Pu Gao)

For a positive integer n, let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be a non-negative integer vector. How many simple graphs are there with degree sequence $\mathbf{d}$? We denote this number by $g(\mathbf{d})$. No simple exact formula is known for $g(\mathbf{d})$, but some illuminating formulae have been obtained for the asymptotic behaviour of $g(\mathbf{d})$ as $n \to \infty$, provided certain restrictions are satisfied by the degree sequence $\mathbf{d}$.

We asymptotically enumerate graphs with a given degree sequence $\mathbf{d} = (d_1, \ldots, d_n)$ satisfying restrictions designed to permit heavy-tailed sequences in the sparse case (i.e. where the average degree is rather small). For stating such restrictions, we often use the notations $M_k = \sum_{i=1}^{n} [d_i]_k$ for any integer $k \geq 1$, where $[x]_k = x(x-1)\cdots(x-k+1)$ for any nonnegative integer k. Note that M_1 is simply twice the number of edges in the graphs. We also define $\Delta = \max_i d_i$.

Our general result requires upper bounds on M_k for a few small integers $k \geq 1$. As special cases, we asymptotically enumerate graphs with

- degree sequences satisfying $M_2 = o(M_1^{9/8})$;
- degree sequences following a power law with parameter $\gamma > 5/2$;

- degree sequences following a certain 'long tailed' power laws;
- bi-valued degree sequences satisfying certain conditions.

A previous result on enumeration of sparse graphs by McKay and the speaker [3] applies to a wide range of degree sequences but requires $\Delta = o(M_1^{1/3})$, where Δ is the maximum degree. This immediately requires $\Delta = o(n^{1/2})$. The new result applies in some cases when Δ is only barely $o(M_1^{3/5})$, or $o(n^{2/3})$. Case (i) above extends a result of Janson [1, 2] which requires $M_2 = O(M_1) = O(n)$ (and hence $\Delta = O({M_1}^{1/2}) = O(n^{1/2})$).

Random graphs with given degree sequence $\mathbf{d}$ can be generated by the pairing model. This is a probability space consisting of n distinct bins v_i (representing the n vertices), $1 \le i \le n$, each containing d_i points, and all points are uniformly at random paired (i.e. the points are partitioned uniformly at random subject to each part containing exactly two points). We call each element in this probability space a *pairing*, and two paired points (points contained in the same part) is called a *pair*. Let Φ denote the set of all pairings. Then $|\Phi|$ equals the number of matchings on M_1 points, and

$$|\Phi| = \frac{M_1!}{2^{M_1/2}(M_1/2)!} = \sqrt{2}(M_1/e)^{M_1/2}\big(1 + O(M_1^{-1})\big).$$

For each pairing $P \in \Phi$, consider the multigraph generated by P by representing bins as vertices and pairs as edges. This has degree sequence $\mathbf{d}$. It is easy to see that every simple graph with degree sequence $\mathbf{d}$ corresponds to exactly $\prod_{i=1}^n d_i!$ distinct pairings in Φ. Hence, letting $\mathcal{S}(n, \mathbf{d})$ denote the probability that the random multigraph generated by the pairing model is simple, we have

$$g(\mathbf{d}) = \frac{|\Phi|}{\prod_{i=1}^n d_i!} \mathcal{S}(n, \mathbf{d}),$$

and we are left with estimating $\mathcal{S}(n, \mathbf{d})$. The results mentioned above are obtained in this way. In both [3] and [1], $\mathcal{S}(n, \mathbf{d})$ is estimated by using what are called switchings, and in [2] it is done by the method of moments, which will not be effective when the probability that the graph is simple tends quickly to 0. Our main result gives an estimate of $\mathcal{S}(n, \mathbf{d})$ under certain new conditions. Since these conditions are quite complicated, we just give some special cases here.

Theorem 1. Let $\mathbf{d}$ have minimum component at least 1 and satisfy $M_2 = o(M_1^{9/8})$. Then with $\lambda_{i,j} = d_i d_j / M_1$ and $|\Phi|$ given above,

$$\mathcal{S}(n, \mathbf{d}) = \big(1 + O(\sqrt{\xi})\big) \exp\left(-\frac{M_1}{2} + \frac{M_2}{2M_1} - \frac{M_3}{3M_1^2} + \frac{3}{4} + \sum_{i<j} \Big(\log(1 + \lambda_{i,j}) \Big) \right),$$

where $\xi = M_2^4 / M_1^{9/2} + M_2^{3/2}/M_1^2 + 1/M_1$ and necessarily $\xi = o(1)$.

In the next example, we consider degree sequences $\mathbf{d}$ that follow a so-called power law with parameter $\gamma > 1$, i.e. the number n_i of vertices of degree i is approximately $ci^{-\gamma}n$ for some constant $c > 0$. We relax these conditions a little to say that $\mathbf{d}$ is a *power-law bounded sequence* with parameter γ if $n_i = O(i^{-\gamma} n)$ for all $i \geq 1$.

Theorem 2. Assume that $\mathbf{d}$ is a power-law bounded sequence with parameter $\gamma > 5/2$. Then putting $M_i^* = M_i + M_1$ for $i = 2$ and 3,

$$\mathcal{S}(n, \mathbf{d}) = \exp\left(-\frac{M_2}{2M_1} - \frac{M_2^2}{4M_1^2} + \frac{M_3^2}{6M_1^3} + O(n^{5/\gamma-2})\right).$$

Note that in the case of a 'strict' power law, where $\mathbf{d}$ is a sequence with $n_i = \Theta(i^{-\gamma} n)$ for $i \leq \Delta = \Theta(n^{1/\gamma})$, with $5/2 < \gamma < 3$ constant, the whole exponential factor in this theorem is $\exp\big(-\Theta(n^{6\gamma-2})\big)$.

The main advantage of our results over existing ones is for the case when the degree sequence is far from that of a regular graph. Two special cases of our main result (to be presented later), exemplify this. One general example is for degree sequences with only two distinct degrees, which we call *bi-valued.*

Theorem 3. Let $3 \leq \delta \leq \Delta$ be integers depending on n, and assume that $d_i \in \{\delta, \Delta\}$ for $1 \leq i \leq n$. Let ℓ denote the number of vertices with degree Δ. If

(a) $\Delta = O(\sqrt{\delta n + \Delta \ell})$ and $\xi := (\Delta^7 \ell^3 + \Delta^3 \delta^4 n^2 \ell + \delta^7 n^3)/(\delta^4 n^4 + \Delta^4 \ell^4) = o(1)$, or

(b) $\Delta = \Omega(\sqrt{\delta n})$ and $\xi := \dfrac{\Delta^5 \ell^3}{\delta^3 n^3} + \dfrac{\Delta^5 \ell^2}{\delta^2 n^3} + \dfrac{\delta^3}{n} + \dfrac{\Delta^3 \ell}{n^2} = o(1)$,

then

$$\mathcal{S}(n, \mathbf{d}) = \exp\left(-\frac{M_1}{2} + \frac{M_2}{2M_1} + \frac{3}{4} + \sum_{i<j} \log(1 + d_i d_j / M_1) + O\big(\sqrt{\xi}\big)\right)$$

where M_i is simply $[\Delta]_i \ell + [\delta]_i (n - \ell)$.

Note that the summation in the exponent in this theorem is easy to express in terms of δ etc. as there are only three possible values of $d_i d_j$. Also, part (b) applies to some instances of bi-valued sequences where the minimum degree is around $n^{1/3-\epsilon}$ and simultaneously there are up to n^ϵ vertices with maximum degree as large as $o(n^{2/3-\epsilon})$. These are much higher degrees than can be reached by any previously published results on enumeration of sparse graphs with given degree sequence.

To prove our main result, we estimate $\mathcal{S}(n, \mathbf{d})$ using a switching method that 'eliminates' loops and multiple edges from random pairings. We extend the definition of the switchings used in [3] in a natural way to handle loops and multiple edges of arbitrarily high multiplicities. On the other hand, our method of analysis of the switchings is very different.

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Problems presented at the problem session

Edited by Svante Linusson

Einar Steingrimsson: The number of 1324-avoiding permutations and a related conjecture

An occurrence of the pattern 1324 in a permutation $a_1a_2\dots a_n$ of the integers $\{1,2,\dots,n\}$ is a subsequence $a_ia_ja_ka_\ell$, with $i<j<k<\ell$ and $a_i<a_k<a_j<a_\ell$. A permutation *avoids* 1324 if it has no occurrence of 1324. For example, 2638 is an occurrence of 1324 in 52763148, whereas 426513 avoids 1324.

Let $\mathrm{A}_n(1324)$ be the number of permutations of length n that avoid 1324. The problem is to find a closed formula for $\mathrm{A}_n(1324)$ or else the *Stanley-Wilf limit* for 1324, namely,

$$\mathrm{SW}(1324)=\lim_{n\to\infty}\sqrt[n]{\mathrm{A}_n(1324)}.$$

The currently best known published bounds (see [3, Section 3] for attributions) are

$$9.47<\mathrm{SW}(1324)<13.93,$$

although a new upper bound of 13.73718 has recently been found by Bóna [1], and a new lower bound of 9.81 by David Bevan (yet unpublished). It seems likely that the actual number lies between 11 and 12 and, after the presentation of this problem, Tony Guttmann came up with the estimate $\mathrm{SW}(1324)\approx 11.598\pm 0.003$, based on extrapolation of known values for $n\le 36$.

A conjecture [2, Conjecture 13] whose confirmation would lead to an upper bound of 13.02 is this, where an inversion is a pair (i,j) with $i<j$ and $a_i>a_j$:

> Let $\mathrm{A}_n^k(1324)$ be the number of permutations in $\mathrm{A}_n(1324)$ with exactly k inversions. Then, for each k, $\mathrm{A}_n^k(1324)$ is weakly increasing as a function of n.

In fact, it is also conjectured in [2, Conjecture 20] that this weakly increasing property holds for *all* patterns except the strictly increasing ones.

References

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Drew Armstrong: The expected length of the longest k-alternating run of a permutation

For each permutation $w \in \mathfrak{S}_n$ we let $\mathsf{as}_k(w)$ be the length of the longest "k-alternating subsequence" in w. That is, the length of the longest sequence

$$w_{i_1} > w_{i_2} < w_{i_3} > w_{i_4} < \cdots$$

where $|w_{i_j} - w_{i_{j+1}}| \geq k$ for all j. Let $\mathsf{as}_k(n)$ denote the average length of the longest k-alternating subsequence:

$$\mathsf{as}_k(n) := \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \mathsf{as}_k(w).$$

Conjecture: For $1 \leq k \leq n-1$ we have

$$\mathsf{as}_k(n) = \frac{4(n-k)+1}{6}.$$

Remark: The case $k = 1$ was proved by Stanley (`http://arxiv.org/abs/math/0511419v1`).

Bonus. Explain why $\mathsf{as}_k(n)$ depends only on the difference $n - k$.

Miklos Bona: The probability that a random vertex of a rooted plane tree is a leaf

Let T_n be the set of all rooted plane trees on vertex set $[n]$ in which the label of each child is less than that of its parent. It is straightforward to prove by generating functions or induction that $|T_n| = (2n-3)!!$, and that the total number of leaves in all elements of T_n is $(2n+1)!!/3$. Therefore, a randomly selected vertex of a random tree in T_n has asymptotically $2/3$ chance of being a leaf. Is there a simple combinatorial proof of this fact? (No generating functions, no induction.) In other words, prove that if p_n is the probability that a random vertex of a random element of T_n is a leaf, then p_n converges to $2/3$ as n goes to infinity.

Richard Stanley has come up with a simple solution *if we assume* that $L = \lim_\infty p_n$ exists. It would still be preferable to find a simple combinatorial proof that does not need this asumption.

Brendan McKay: Enumeration of sparse symmetric matrices without 1

Let J, J^* be subsets of the non-negative integers, and let $\boldsymbol{d} = \boldsymbol{d}(n) = (d_1, \ldots, d_n)$ be a vector of non-negative integers. Let $M(n, J, J^*)$ be the number of symmetric matrices whose diagonal entries are drawn from J^* and off-diagonal entries from J, whose row sums are $d_1, \ldots, d_n$. As usual in graph theory, entries on the diagonal are counted twice. We are interested in the asymptotic value of $M(n, J, J^*)$ in the sparse case, where the row sums do not grow very quickly with n. The most general result completes the case where $0 \in J^*$ and $0, 1 \in J$ (C. S. Greenhill and B.

D. McKay, Asymptotic enumeration of sparse nonnegative integer matrices with specified row and column sums, *Adv. Appl. Math.*, **41** (2008) 459–481). The requirements that $0 \in J^*$ and $0 \in J$ are easy disposed of; for example if 0 is not permitted on the diagonal we can just subtract 1 from each diagonal entry and 2 from each d_i. However, disposing of the requirement that $1 \in J$ is not so easy. The simplest non-trivial case is $J^* = \{0\}$ and $J = \{0, 2, 3\}$.

Colin McDiarmid: Connectivity for an unlabelled bridge-addable graph class

Call a class $\mathcal{A}$ of graphs *bridge-addable* if, whenever a graph G in $\mathcal{A}$ has vertices u and v in distinct components, then the graph $G + uv$ (obtained by adding the edge uv) is also in $\mathcal{A}$.

Let $\tilde{\mathcal{A}}_n$ denote the set of graphs in $\mathcal{A}$ on n unlabelled vertices. Let us use the notation $\tilde{R}_n \in_u \tilde{\mathcal{A}}$ to indicate that the random unlabelled graph $\tilde{R}_n$ is sampled uniformly from $\tilde{\mathcal{A}}_n$. We make two conjectures about the probability of being connected.

Conjecture 1 There is a $\delta > 0$ such that, if $\tilde{\mathcal{A}}$ is bridge-addable and $\tilde{R}_n \in_u \tilde{\mathcal{A}}$, then

$$\mathbb{P}(\tilde{R}_n \text{ is connected}) \geq \delta \quad \text{for each } n. \tag{1}$$

A first step would be to show that there is such a δ which may depend on $\mathcal{A}$ - even that would be interesting. For forests (which are bridge-addable) we have $\mathbb{P}(\tilde{R}_n \text{ is connected}) \to \tau \approx e^{-0.5226} \approx 0.5930$ as $n \to \infty$, see for example the first line of table 3 in [1].

The second conjecture is more speculative. Call the class $\mathcal{A}$ of graphs *decomposable* when a graph is in the class if and only if each component is.

Conjecture 2 If $\tilde{\mathcal{A}}$ is decomposable and bridge-addable and $\tilde{R}_n \in_u \tilde{\mathcal{A}}$ then

$$\liminf_n \mathbb{P}(\tilde{R}_n \text{ is connected}) \geq \tau. \tag{2}$$

The *fragment size* $\text{frag}(G)$ of a graph G is the number of vertices less the maximum number of vertices in a component. The final conjecture is also a little speculative.

Conjecture 3 There is a constant c such that, if $\mathcal{A}$ is decomposable and bridge-addable and $\tilde{R}_n \in_u \tilde{\mathcal{A}}$, then $\mathbb{E}[\text{frag}(\tilde{R}_n)] \leq c$.

For labelled graphs the corresponding statements are known, see e.g. [2].

References

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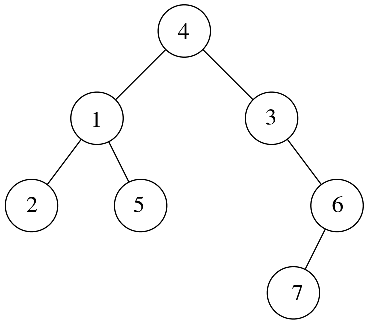
Ira Gessel: Enumeration of labeled binary trees and the enumeration of regions of subarrangements of the Catalan arrangement

We count labeled binary trees according to the number of left and right ascents and descents. In the following figure,



vertex 1 is a left descent because it is a left child and is smaller than its parent; vertex 6 is a right ascent because it is a right child and larger than its parent. Let

$$B_n(u_1, u_2, v_1, v_2) = \sum_T u_1^{\mathrm{LA}(T)} u_2^{\mathrm{LD}(T)} v_1^{\mathrm{RA}(T)} v_2^{\mathrm{RD}(T)}$$

where the sum is over all $n!\, C_n = \frac{n!}{n+1}\binom{2n}{n}$ labeled binary trees on $[n]$, $\mathrm{LA}(T)$ is the number of left ascents of T, and so on.

The first few values of $B_n = B_n(u_1, u_2, v_1, v_2)$ are $B_1 = 1$, $B_2 = u_1+v_1+u_2+v_2$, and $B_3 = u_1^2 + v_1^2 + u_2^2 + v_2^2 + 4(u_1 + v_2)(u_2 + v_1) + 5(u_1v_2 + u_2v_1)$.

It is surprising that evaluations of $B_n(u_1, u_2, v_1, v_2)$ count regions of certain hyperplane arrangements (all subarrangements of the braid arrangement):

$B_n(1, 1, 1, 1) = n!\, C_n$ is the number of regions of the Catalan arrangement, $x_i - x_j = 0, \pm 1$, for $1 \le i < j$.

$B_n(1, 0, 1, 0) = n!$ is the number of regions of the braid arrangement, $x_i - x_j = 0$, for $1 \le i < j \le n$.

$B_n(1, 1, 1, 0) = (n+1)^{n-1}$ is the number of regions of the Shi arrangement, $x_i - x_j = 0, 1$ for $1 \le i < j \le n$.

$B_n(0, 1, 1, 0)$ is the number of regions of the Linial arrangement, $x_i - x_j = 1$, for $1 \le i < j \le n$.

Is there a bijective explanation? Such a bijection should lead to additional results, as there are other interesting specializations (e.g., giving the Eulerian polynomials) and generalizations of the polynomials $B_n(u_1, u_2, v_1, v_2)$ (notably a symmetric function version, corresponding to trees with repeated labels) and many generalizations of the arrangements described above that share some of their properties. If there is no bijective explanation, one could at least hope for some sort of combinatorial explanation, e.g., perhaps involving Möbius inversion.

Sylvie Corteel: Combinatorics of Rogers-Ramanujan identites

The Rogers-Ramanujan identities were proved in the 1910s by Rogers and Ramanujan. They are

$$\sum_{n\geq 0} \frac{q^{n(n+i)}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}.$$

with $i = 0, 1$ and $(a,q)_n = \prod_{i=0}^{n-1}(1-aq^i)$.

A lot of generalizations were done in the XX^{th} century For example the Andrews-Gordon identities proven in the 60s are

$$\sum_{n_1,\ldots,n_k} \frac{q^{\sum_{j=1}^k n_j^2 + \sum_{j=i}^k n_j}}{(q)_{n_1-n_2}\cdots(q)_{n_{k-1}-n_k}(q)_{n_k}} = \frac{(q^{2k+3};q^{2k+3})_\infty}{(q;q)_\infty}\theta(q^i;q^{2k+3})_\infty.$$

with $\theta(a,q) = (a;q)_\infty (q/a;q)_\infty$. They all have combinatorial interpretations in terms of integer partitions.

Recently, Foda and Wheeler (private communication), gave an interpretation of the Andrews-Gordon identities' sum side thanks to cylindric plane partitions (introduced by Gessel and Krattenthaler (1990s) and then studied by Borodin (2000s)). A lot of combinatorial problems can be attacked thanks to this new approach and open some new hope for the combinatorics of the Rogers-Ramanujan identities.

Another breakthrough is due to Bartlett and Warnaar (2013) followed by Griffith, Ono and Warnaar (2014). They give some $A_{2n}^{(2)}$-analogues of the Rogers-Ramanujan identities. Namely

$$\sum_{\lambda;\lambda_1\leq m} P_{2\lambda}(1,q,\ldots;q^{2n-1}) = \frac{(q^\kappa;q^\kappa)_\infty}{(q;q)_\infty} \prod_{1\leq i<j\leq m} \theta(q^{j-i};q^\kappa\theta(q^{i+j-1};q^\kappa)_\infty.$$

with $\kappa = 2m+2n+1$.

The combinatorial interpretation of this identity (and the other identities of the paper) is a wide open problem.

Tony Guttmann: The asymptotic behaviour of the enumeration of Dyck paths weighted by height

Let $d_{n,h}$ be the number of Dyck paths of length $2n$ and height (maximal y-coordinate) h. Then

$$D(x,y) = \sum d_{n,h} x^{2n} y^h.$$

For $y < 1$, calculate the asymptotic behaviour of

$$[x^{2n}]D(x,y) = \sum_h d_{n,h} y^h,$$

which I expect to behave as

$$const.4^n \mu^{n^\sigma} n^g$$

where both *const* and μ are y-dependent.

Solution Robin Pemantle almost immediately produced an argument that $\sigma = \frac{1}{3}$, and a day later Brendan McKay produced the solution giving the explicit y-dependence of both *const* and μ.

Mireille Bousquet-Mélou: Corners in square lattice loops

We consider square lattice walks, with North, South, East and West steps, starting and ending at the origin. We call them *loops*. We focus on those that stay in the first (that is, nonnegative) quadrant $\{(i,j) : i \geq 0, j \geq 0\}$.

Let $Q_n(a)$ be the polynomial that counts quadrant loops of length $2n$ by the number of NW and ES factors. For instance, the loop NEENSESWWW contains one such factor (ES) and contributes a to the polynomial $Q_5(a)$.

A recent study of permutations that can be sorted by two parallel stacks [1], joint with Michael Albert (Dunedin, New-Zealand), has led us to a pair of intriguing conjectures dealing with $Q_n(a)$.

Conjecture The expansion of $Q_n(a)$ in powers of $(a+1)$ reads

$$Q_n(a) = \sum_{i=0}^{n-1} q_{i,n}(a+1)^i,$$

where $q_{i,n} \geq 0$. We say that $Q_n(a)$ is $(a+1)$-*positive.*

Evidence for this conjecture. True for $n \leq 200$, even if we refine by fixing the number of horizontal and vertical steps: the polynomials $Q_{i,j}(a)$ that count loops with $2i$ horizontal steps and $2j$ vertical steps seem to be $(a+1)$-positive. Proved for general loops (unconfined) and for loops confined to the upper half-plane. Also, we know the value when $a = -1$, and it is remarkably nice:

$$Q_{i,j}(-1) = \binom{i+j}{i} C_i C_j.$$

We have no combinatorial explanation of this identity, and its proof is not particularly easy.

Our second conjecture deals with the exponential growth of the (positive) numbers $Q_n(a)$, for $a \geq -1$, or equivalently, with the radius of the series $Q(a,u) = \sum_{n \geq 0} u^n Q_n(a)$. We conjecture it to be the same as for unconfined loops, for all $a \geq -1$. The exact (algebraic) value is given in [1], where the reader will also find details, proofs and (possibly helpful) comments.

References

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Richard Stanley: D-finiteness of certain series associated with group algebras

Let G be a group and $\mathbb{Z}G$ its integral group algebra. For $u \in \mathbb{Z}G$ let $f_u(n) = [1]u^n$, the coefficient of the identity element of G when u^n is expanded in terms of the basis G. Set $F_u(x) = \sum_{n\geq 1} f_u(n)x^n$. If $F = F_d$, the free group on d generators, then it is known that $F_u(x)$ is algebraic. This goes back to Chomsky and Schützenberger and seems first to have been explicitly stated by Haiman. If $G = \mathbb{Z}^d$ then it follows from standard facts about D-finite series that $F_u(x)$ is D-finite, though it need not be algebraic.

Maxim Kontsevich asked whether $F_u(x)$ is always D-finite when $G = \mathrm{GL}(d, \mathbb{Z})$. This remains open, though it is known that the question of whether $F_u(x) = 0$ is undecidable. More generally, we can ask for which groups G is $F_u(x)$ algebraic for all $u \in \mathbb{Z}G$, and for which groups is $F_u(x)$ D-finite for all $u \in \mathbb{Z}G$.

Nick Wormald: Reduction of degree in the coefficients of a generating function

Problem: Let $[y]_k$ denote $y(y-1)\cdots(y-k+1)$ and $A_t(y,z)$ the coefficient of x^t in

$$\log \sum_{k\geq 0} \frac{[y]_k[z]_k}{k!}x^k.$$

Clearly A_t has total degree at most $2t$. Show that A_t has total degree at most $t+1$ for $t \geq 1$.

Notes:

(1) A short solution was quickly found by Ira Gessel, Gilles Schaeffer and Richard Stanley, each independently.

(2) A similar question: show that for $t \geq 1$ the coefficient of x^t in

$$\log \sum_{k\geq 0} \frac{[y]_{2m}}{m!}x^m$$

has degree $t+1$. This was also solved by Ira Gessel, using the solution to the main problem.

(3) Ira Gessel has obtained the leading coefficients in both questions.

(4) The effect of 'reduction of degree' occurring in a similar context is explained in a paper of Valentin Féray, "Asymptotic behavior of some statistics in Ewens random permutations" (Electron. J. Probab. 18 (2013), no. 76, 32 pp).

Jim Haglund: The sweep map on rational Catalan paths

Let (m, n) be a pair of relatively prime positive integers. Let $\mathrm{Grid}(m, n)$ be the $n \times m$ grid of labelled squares whose upper-left-hand square is labelled with $(n-1)(m-1)-1$, and whose labels decrease by m as you go down columns and by n as you go across rows. For example, $\mathrm{Grid}(3, 7)$ is the array

(3)

11	4	-3
8	1	-6
5	-2	-9
2	-5	-12
-1	-8	-15
-4	-11	-18
-7	-14	-21

By a lattice path we mean a sequence of North N steps and East E steps, starting at the lower-left-hand corner and ending at the upper-right-hand corner. Let $\mathfrak{D}_{(m,n)}$ denote the set of lattice paths π for which none of the squares with negative labels are above π. We call the set of corners which are touched by π the "vertices" of π. Given a path π, let $S(\pi)$ be the set of labels of those squares whose upper-left-hand-corners are vertices of π. A given label in $S(\pi)$ is called an N label if the vertex associated to it is the start of an N step, otherwise it is called an E label. Here is an example:
$\pi = NNNNNNEENE, \quad S(\pi) = \{-10, -7, -4, -1, 2, 5, 8, 1, -6, -3\}$.

We now define the "sweep map" of [1], denoted ϕ, from $\mathfrak{D}_{(m,n)} \to \mathfrak{D}_{(m,n)}$ as follows: Given $\pi \in \mathfrak{D}_{(m,n)}$, order the elements of $S(\pi)$ in increasing order to create a vector of labels $E(\pi) = (e_1, e_2, \ldots, e_{m+n})$. Then create a path $\phi(\pi)$ by defining the ith step of $\phi(\pi)$ to be an N step (E step) if e_i is an N label (E label). For the example above, we have

(4) $$E(\pi) = (-10, -7, -6, -4, -3, -1, 1, 2, 5, 8) \quad \phi(\pi) = NNNNENENNE.$$

Problem: Prove that ϕ is a bijection from $\mathfrak{D}_{(m,n)} \to \mathfrak{D}_{(m,n)}$.

If $m = n+1$ the ϕ map reduces to the ζ map described on page 50 of [3], which is known to be a bijection. More generally, in [2] it is shown that ϕ is a bijection whenever $m = kn+1$ or $m = kn-1$ for some positive integer k.

References

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Jang Soo Kim: Fillings of the shifted staircase

Let $a_1 < a_2 < \cdots < a_n$ be a sequence of positive integers. We define $Y(a_1, a_2, \ldots, a_n)$ to be the set of fillings of the shifted staircase of size n with $1, 2, \ldots, \binom{n}{2}$ such that $a_1, a_2, \ldots, a_n$ are in the diagonal cells and the entries are increasing from left to right and from top to bottom. For instance, the following is an element in $Y(1, 3, 8, 10)$.

1	2	4	6
	3	5	7
		8	9
			10

We define $S(a_1, a_2, \ldots, a_n)$ to be the set of fillings of the shifted staircase of size n with $1, 2, \ldots, \binom{n}{2}$ such that $a_1, a_2, \ldots, a_n$ are in the diagonal cells and every non-diagonal entry is bigger than the diagonal entry in the same row and smaller than the diagonal entry in the same column. For instance, the following is an element in $S(1, 3, 8, 10)$.

1	2	7	4
	3	5	6
		8	9
			10

Kim and Oh showed that

$$|S(a_1, a_2, \ldots, a_n)| = |Y(a_1, a_2, \ldots, a_n)| 1!2! \ldots (n-1)!.$$

Find a bijective proof of the above equation. This problem was motivated from the Selberg integral formula.

Svante Linusson: Narayana numbers in an identity involving Semistandard Young Tableaux

Set $\mathrm{SSYT}_{r,k}(m)$ to be the number of semistandard Young tableaux on a shape of two columns of lengths $r \geq l$ with no number exceeding m. We are interested in the following quantity.

$Y^{\beta}_{r,l}(m) :=$ number of SSYTs on two columns $r \geq l$ with no entry exceeding m such that the number β appears somewhere in the second column.

In [1] the following identity was proven.

Theorem For $m \geq r \geq l \geq 1$ and $1 \leq \beta < m$ we have

$$Y^{\beta}_{r,l}(m) = \sum_{1 \leq f \leq e \leq \beta} N_{e-1,f-1} \cdot \mathrm{SSYT}_{r-f,l-f}(m-e),$$

where

$$N_{e,f} = \frac{1}{e}\binom{e}{f+1}\binom{e}{f}$$

are the Narayana numbers.

The proof is recursive and does not give any insight into why the Narayana numbers occurr here.

Problem: Find a proof of the formula that explains the presence of the Narayana numbers.

References

[1] Arvind Ayyer and Svante Linusson, Correlations in the Multispecies TASEP and a Conjecture by Lam, arXiv:1404.6679

Reporter: Marko Thiel