Discrete harmonic functions & random walks

Lecture #3

Analytic and probabilistic tools for lattice path enumeration

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77th Séminaire Lotharingien de Combinatoire
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Introduction & motivations

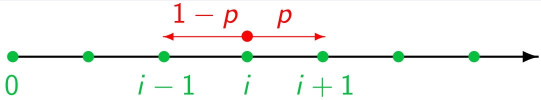
Applications in probability theory

Applications in combinatorics

Discrete harmonic functions in the quadrant

Introductory example & definition

Absorption probabilities for the SRW on $\mathbb{N}$



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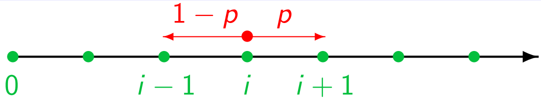

Probability $a(i) := \mathbb{P}_i[\exists n \geqslant 0 : \text{SRW } S(n) = 0]$ satisfies

- ▷ $a(0) = 1 \rightsquigarrow$ *initial condition*
- ▷ $a(i) = p \cdot a(i+1) + (1-p) \cdot a(i-1) \rightsquigarrow$ *recurrence*

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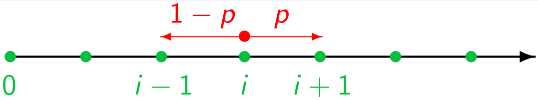

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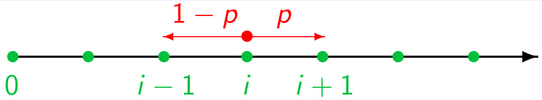

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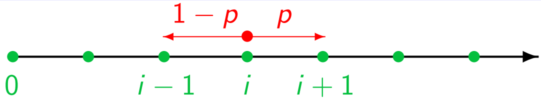

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- ▷ Multivariate linear recurrences with constant coefficients

✎ Bousquet-Mélou & Petkovšek '00

{History of/Questions on} preharmonic functions (1/2)

Classical (continuous) harmonic functions in $\mathbb{R}^d$

$$\Delta[f](x) = \sum_{i=1}^{d} \frac{\partial^2 f(x)}{\partial x_i^2} = 0$$

▷ Possibility of adding *weights* $\rightsquigarrow$ *elliptic operators*

▷ *Harmonic functions satisfy various properties:* maximum principle/mean value property/Harnack inequalities/Liouville's theorem/relations with analytic functions/etc.

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Do preharmonic functions satisfy similar properties?

▷ Dirichlet problem ✎ Phillips & Wiener '23; Bouligand '25

▷ Harnack inequalities ✎ Lawler & Polaski '92; Varopoulos '99

▷ Maximum principle, Liouville's theorem & related topics ✎ Heilbronn '48

▷ Cauchy-Riemann equations ✎ Duffin '55; Kiselman '05–'08

{History of/Questions on} preharmonic functions (2/2)

Further properties

- ▷ Rate of growth — ✎ Murdoch '63–'65; Ignatiuk-Robert '10
- ▷ Picard's theorem (sign of harmonic functions) & factorization — ✎ Murdoch '63–'65
- ▷ Absolute monotonicity — ✎ Lippner & Mangoubi '15

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▷ Relations between discrete & continuous harmonic functions ✎ Lusternik '26; Ferrand '44; Kesten '91; Varopoulos '09

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- ▷ Conformal invariance of lattice models ✎ Duminil-Copin & Smirnov '12

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- ▷ Conformal mappings ✎ Ferrand '44; Isaacs '52
- ▷ Discrete harmonic polynomials & discrete exponential functions ✎ Terracini '45–'46; Heilbronn '48; Isaacs '52; Duffin '55; Duffin & Peterson '68

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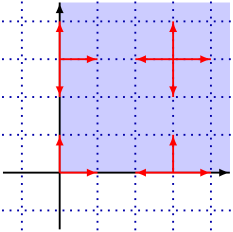
Potential theory

- ▷ Martin boundary ✎ Woess '92; Kurkova & Malyshev '98; Ignatiuk-Robert & Loree '10; Mustapha '15

Warning: lattice walk enum. vs. preharmonic functions

Multivariate recurrence relations in both cases



- ▷ $q(n; i, j) = \#_{\mathbb{N}^2}\{(0,0) \xrightarrow{n} (i,j)\}$
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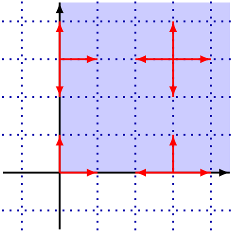

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Main differences & difficulties

- A unique solution vs. an unknown ($\leqslant \infty$) number of solutions
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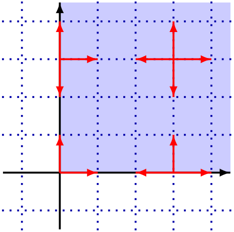

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- Preharmonic functions $\approx$ homogenized enumeration problem:

$$K(x,y)Q(x,y) = K(x,0)Q(x,0) + K(0,y)Q(0,y) - K(0,0)Q(0,0) - xy$$
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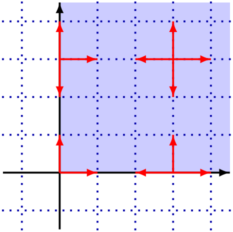

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- Preharmonic functions $\rightsquigarrow$ counting numbers asymptotics

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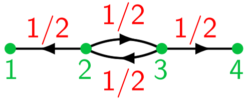
Markov chains: example



1 & 4: absorbing states

Absorption probabilities and statistical mechanics

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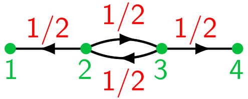

1 & 4: absorbing states

$f_i = \mathbb{P}_i[\text{hit } 4]$ satisfies $\begin{cases} f_1 = 0 \\ f_4 = 1 \\ f_2 = \frac{1}{2} f_1 + \frac{1}{2} f_3 \\ f_3 = \frac{1}{2} f_2 + \frac{1}{2} f_4 \end{cases}$

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Absorption probabilities and statistical mechanics

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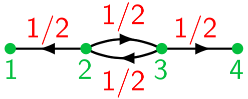

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Markov chains: general theorem

The hitting probabilities are characterized as being the *minimal non-negative solutions* to a system of *linear recurrences*.

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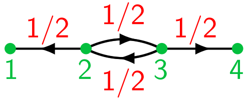

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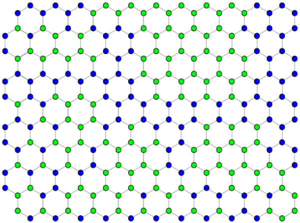
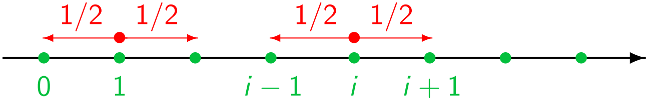


Discrete analyticity and convergence of the Fermionic observable

✎ Smirnov '10

Doob transform (1/2)

Example: construct a 1D process conditioned to stay in $\mathbb{N}$



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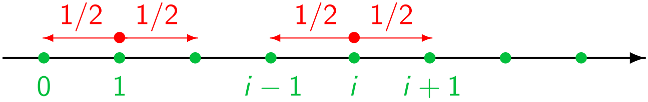

- Function $f(i) = i$ is positive harmonic and $f(0) = 0$
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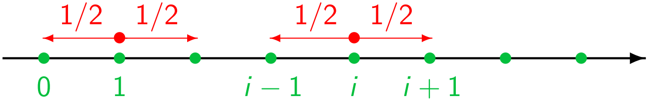

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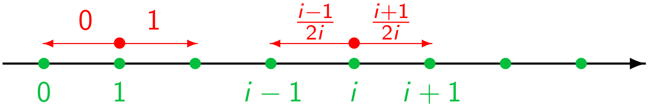

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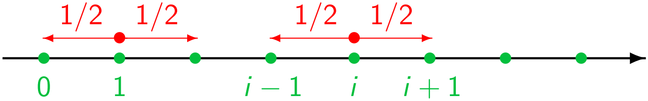

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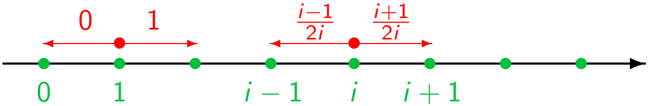

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Construction can be generalized

- *Random processes* conditioned never to leave *cones* of $\mathbb{Z}^d$
- Quantum random walks, eigenvalues of random matrices, non-colliding random walks, etc.

✎ Dyson '62; Biane '90–'92; Eichelsbacher & König '08

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- ▷ Important question: do we have $\mathbb{P}[\,\cdot\,|\{\tau = \infty\}] = \mathbb{P}^f$?

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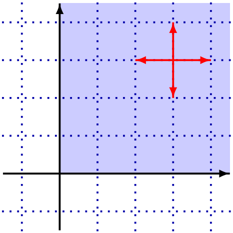
Example (1/3) in the quadrant: the simple walk



- Uniform weights $\frac{1}{4}$
- $f(i,j) = i \cdot j$
- *Unique preharmonic function* (up to multiplicative factors)
- Product form ✎ Picardello & Woess '92

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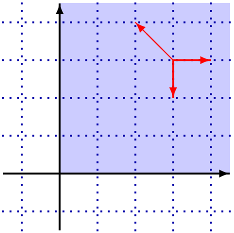
Example (2/3) in the quadrant: the Tandem walk



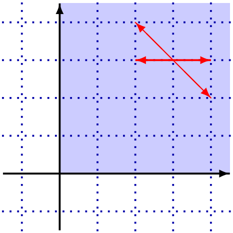
- Uniform weights $\frac{1}{3}$
- $f(i,j) = i \cdot j \cdot (i+j)$
- *Unique preharmonic function* (up to multiplicative factors) ✎ Biane '92

Doob transform (2/2)

A second way for the conditioning (with the first exit time)

- RW $\{S(n)\}_{n\geqslant 0}$; *first exit time* $\tau = \inf\{n \geqslant 0 : S(n) = 0\}$
- On the event $\{\tau = \infty\}$, the RW stays in $\{1, 2, 3, \ldots\}$
- Replace $\mathbb{P}[\,\cdot\,]$ by $\mathbb{P}[\,\cdot \mid \{\tau = \infty\}]$ to obtain a *conditioned RW*
- Important question: do we have $\mathbb{P}[\,\cdot \mid \{\tau = \infty\}] = \mathbb{P}^f$?
  ✎ Denisov & Wachtel '15 (general cones with zero drift); Courtiel, Melczer, Mishna & R. '16 (Gouyou-Beauchamps model with drift)
- Difficulty: exit time τ

Example (3/3) in the quadrant: the GB walk



- Uniform weights $\frac{1}{4}$
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Martin boundary theory

Rough description

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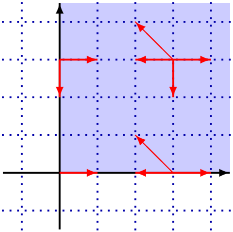
- General RW in cones: open problem (conjecture: uniqueness $\Longleftrightarrow$ drift $= 0$)

Asymptotics of some numbers of walks

Asymptotic statements



▷ *Total number of walks* starting at (k, ℓ):

$$q(n; k, \ell; \mathbb{N}^2) = \#_{\mathbb{N}^2}\{(k, \ell) \xrightarrow{n} \mathbb{N}^2\}$$
$$\sim f_1(k, \ell) \cdot \rho_1^n \cdot n^{\alpha_1}$$

✎ Not proved yet!

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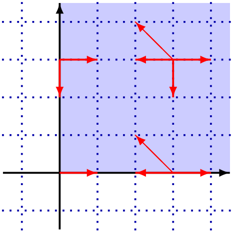

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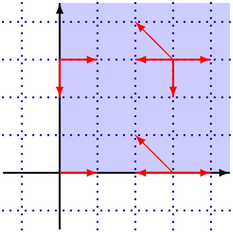

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- f_1 is ρ_1-harmonic & f_2 is ρ_2-harmonic

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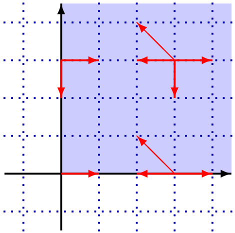

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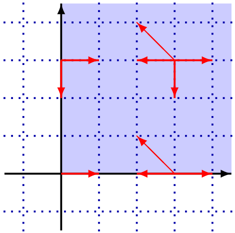

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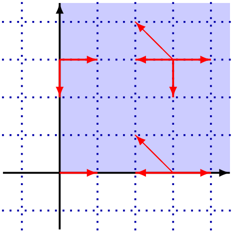

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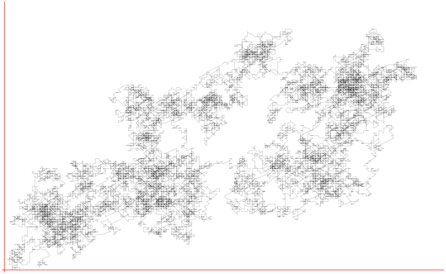
- Drift zero: unique harmonic function $\Longrightarrow f_1$, f_2 and f_2'

Random generation

Aim: generate efficiently a long walk (e.g., confined to a region)



A walk of length 18000

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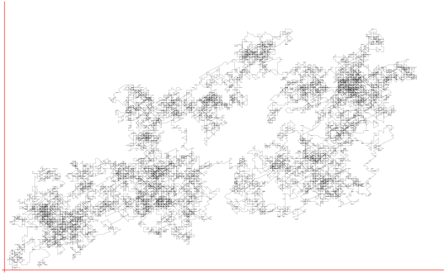

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Different methods

- *Recursive method* (step-by-step construction)
- *Bijections* (if existing) For Kreweras see ✎ Bernardi '07
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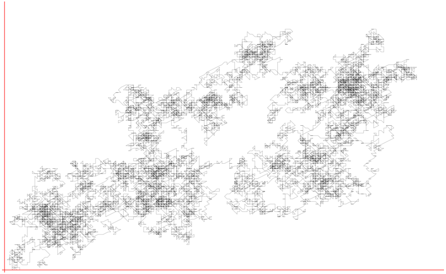

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- ▷ Preharmonic functions and *Doob transform* ✎ Fusy '16
  (Difficulty: after Doob transform, non-uniform walks)

Potential theoretic tools

Counting numbers are caloric functions

- Asymptotics of numbers of quadrant walks (also with inhomogeneities) ✎ D'Arco, Lacivita & Mustapha '16
- Asymptotics in three quarter of plane ✎ Mustapha '16

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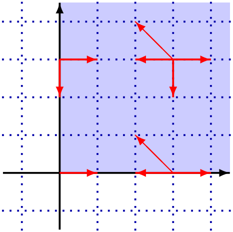
Non-zero drift case: Cramér's transform & ongoing work

- ▷ Works if drift with $\leqslant 0$ coordinates
- ▷ Ongoing work in the remaining cases ✎ Garbit, Mustapha & R.

Functional equation & Tutte's invariants

A functional equation reminiscent of the enumeration



- $F(x,y)=\sum_{i,j\geqslant 1} f(i,j)x^{i-1}y^{j-1}$
- $K'(x,y)=xy\{\sum_{-1\leqslant k,\ell\leqslant 1} p(k,\ell)x^{-k}y^{-\ell}-1\}$
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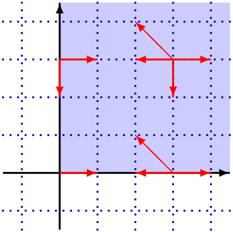

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Definition of Tutte's invariants

- ▷ Introduced to count q-colored triangulations & planar maps

  ✎ Tutte '73; Bernardi & Bousquet-Mélou '11
- ▷ Define X_0 & X_1 by $K'(X_0,y) = K'(X_1,y) = 0$
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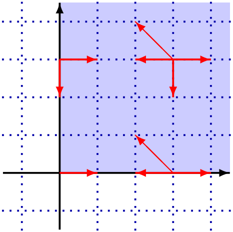

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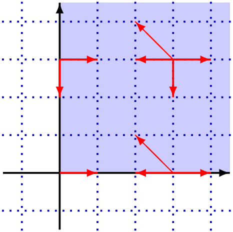

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Does this characterize the sections?

Example: the SRW

A product-form generating function

$$f(i,j) = i \cdot j \Longrightarrow \boxed{F(x,y) = \sum_{i,j\geqslant 1} i \cdot j \cdot x^{i-1} y^{j-1} = \frac{1}{(1-x)^2(1-y)^2}}$$

Kernel: $K'(x,y) = xy\{\frac{x}{4} + \frac{1}{4x} + \frac{y}{4} + \frac{1}{4y} - 1\} = \frac{y(x-1)^2}{4} + \frac{x(y-1)^2}{4}$

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- $I(X_0) = I(X_1) \overset{X_0X_1=1}{\Longrightarrow} I(x) = I(\frac{1}{x}) \Longrightarrow I$ function of $x + \frac{1}{x}$
- $\boxed{K'(x,0)F(x,0) = \frac{x}{4}\frac{1}{(1-x)^2} = \frac{1}{4}\frac{1}{x+\frac{1}{x}-2}}$ is an invariant

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$$\begin{aligned} K'(x,y)F(x,y) &= K'(x,0)F(x,0) + K'(0,y)F(0,y) - K'(0,0)F(0,0) \\ &= \frac{x}{4} \times \frac{1}{(1-x)^2} + \frac{y}{4} \times \frac{1}{(1-y)^2} - 0 \times 1 \end{aligned}$$

Tutte's invariants

- $I(X_0) = I(X_1) \overset{X_0X_1=1}{\Longrightarrow} I(x) = I(\frac{1}{x}) \Longrightarrow I$ function of $x + \frac{1}{x}$
- $\boxed{K'(x,0)F(x,0) = \frac{x}{4}\frac{1}{(1-x)^2} = \frac{1}{4}\frac{1}{x+\frac{1}{x}-2}}$ is an invariant

Why *this* function of $x + \frac{1}{x}$?

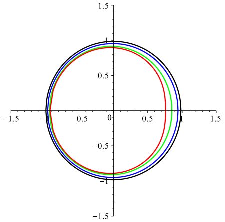
- Of order 1 in $x + \frac{1}{x} \rightsquigarrow$ *Minimality* (conformal mappings)
- $F(1,0) = \infty \rightsquigarrow$ *Liouville's theorem*

Tutte's invariants & conformal mappings

A general theorem

$K'(x,0)F(x,0) = w(x)$, *characterized by*

- Conformal mapping of a certain domain
- $w(x) = w(\overline{x})$
- $w(1) = \infty$
- Same for $K'(0,y)F(0,y)$



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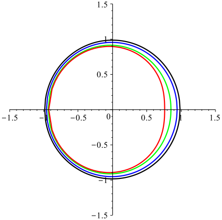
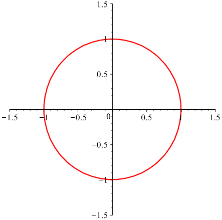


Going back to the SRW

$K'(x,0)F(x,0) = \frac{x}{4(1-x)^2}$, *characterized by*

- ▷ Conformal mapping of the unit disc
- ▷ $w(e^{i\theta}) = w(e^{-i\theta})$
- ▷ $w(1) = \infty$
- ▷ Same for $K'(0,y)F(0,y) = \frac{y}{4(1-y)^2}$



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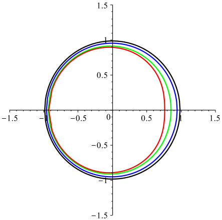

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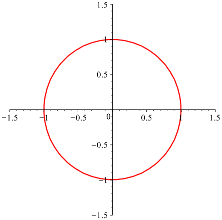
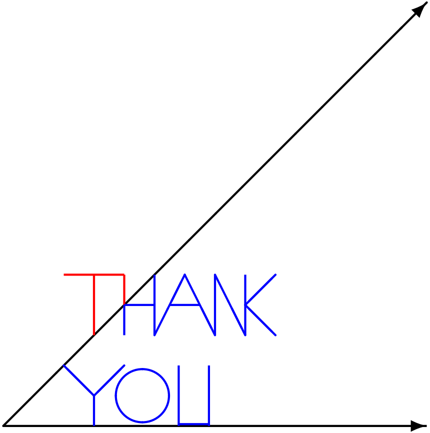


Question

How deep is this *connection conformal maps/harmonic functions*?



cnrs

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