

On the Holonomy or Algebraicity of Generating Functions Counting Lattice Walks in the Quarter-Plane

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Abstract. In two recent works [1, 2], it has been shown that the counting generating functions (CGF) for the 23 walks with small steps confined in a quarter-plane and associated with a finite group of birational transformations are holonomic, and even algebraic in 4 cases — in particular for the so-called Gessel’s walk. It turns out that the type of functional equations satisfied by these CGF appeared in a probabilistic context almost 40 years ago. Then a method of resolution was proposed in [4], involving at once algebraic techniques and reduction to boundary value problems. Recently this method has been developed in a combinatorics framework in [11], where a thorough study of the explicit expressions for the CGF is proposed. The aim of this paper is to derive the nature of the bivariate CGF by a direct use of some general theorems given in [4].

KEYWORDS: generating function, piecewise homogeneous lattice walk, quarter-plane, universal covering, Weierstrass elliptic functions, automorphism

AMS SUBJECT CLASSIFICATION: Primary 05A15; Secondary 30F10, 30D05

1. Introduction

The enumeration of planar lattice walks is a classical topic in combinatorics. For a given set \mathcal{S} of allowed jumps (or steps), it is a matter of counting the number of paths starting from some point and ending at some arbitrary point in a given time, and possibly restricted to some regions of the plane. A first basic

and natural question arises: how many such paths exist? A second question concerns the nature of the associated counting generating functions (CGF): are they rational, algebraic, holonomic (or D-finite, i.e. solution of a linear differential equation with polynomial coefficients)?

For instance, if no restriction is made on the paths, it is well-known that the CGF are rational and easy to obtain explicitly. As an other example, if the walks are supposed to remain in a half-plane, then the CGF can also be computed and turn out to be algebraic in this case, see e.g. [3].

Next it is quite natural to consider walks evolving in a domain formed by the intersection of two half-planes, for instance the positive quarter-plane \mathbb{Z}_+^2 . In this case, the problems become more intricate and multifarious results appeared: indeed, some walks admit of an algebraic generating function, see e.g. [7] and [8] for the walk with step set $\mathcal{S} = \{(-1, 0), (1, 1), (0, -1)\}$ and starting from $(0, 0)$, whereas some others admit a CGF which is even not holonomic, see e.g. [3] for the walk with $\mathcal{S} = \{(-1, 2), (2, -1)\}$ and starting from the point $(1, 1)$.

In this framework, Bousquet-Mélou and Mishna have initiated in [2] a systematic study of the nature of the walks confined to \mathbb{Z}_+^2 , starting from the origin and having steps of size 1, which means that \mathcal{S} is included in the set $\{(i, j) : |i|, |j| \leq 1\} \setminus \{(0, 0)\}$. Examples of such walks are shown in Figures 1 and 2 below.

A priori, there are 2^8 such models. In fact, after eliminating trivial cases and models equivalent to walks confined to a half-plane, and noting also that some models are obtained from others by symmetry, it is shown in [2] that one is left with 79 inherently different problems to analyze.

A common starting point to study these 79 walks relies on the following analytic approach. Let $q(i, j, k)$ denote the number of paths in \mathbb{Z}_+^2 starting from $(0, 0)$ and ending at (i, j) at time k . Then the corresponding CGF

$$F(x, y, z) = \sum_{i, j, k \geq 0} f(i, j, k) x^i y^j z^k \tag{1.1}$$

satisfies the functional equation obtained in [2]

$$K(x, y)F(x, y, z) = c(x)F(x, 0, z) + \tilde{c}(y)F(0, y, z) + c_0(x, y), \tag{1.2}$$

where

$$\begin{cases} K(x, y) = xy \left[\sum_{(i, j) \in \mathcal{S}} x^i y^j - 1/z \right], \\ c(x) = \sum_{(i, -1) \in \mathcal{S}} x^{i+1}, \\ \tilde{c}(y) = \sum_{(-1, j) \in \mathcal{S}} y^{j+1}, \\ c_0(x, y) = -\delta F(0, 0, z) - xy/z, \end{cases}$$

with $\delta = 1$ if $(-1, -1) \in \mathcal{S}$ [i.e. a south-west jump exists], $\delta = 0$ otherwise.

For $z = 1/|\mathcal{S}|$, (1.2) plainly belongs to the generic class of functional equations (arising in the probabilistic context of random walks) studied and solved in the book [4], see Section A of our Appendix. For general values of z , the analysis of (1.2) for the 79 above-mentioned walks has been carried out in [11], where the integrand of the integral representations is studied in detail, via a complete characterization of ad hoc conformal gluing functions.

It turns out that one of the basic tools to solve equation (1.2) is to consider the group exhaustively studied in [4], and originally proposed in [10] where it was called the *group of the random walk*. Starting with this approach, the authors of [2] consider the group of birational transformations leaving invariant the generating function $\sum_{(i,j) \in \mathcal{S}} x^i y^j$, which is precisely the group $W = \langle \xi, \eta \rangle$ generated by

$$\xi(x, y) = \left(x, \frac{1}{y} \frac{\sum_{(i,-1) \in \mathcal{S}} x^i}{\sum_{(i,+1) \in \mathcal{S}} x^i} \right), \quad \eta(x, y) = \left(\frac{1}{x} \frac{\sum_{(-1,j) \in \mathcal{S}} y^j}{\sum_{(+1,j) \in \mathcal{S}} y^j}, y \right).$$

Clearly $\xi \circ \xi = \eta \circ \eta = \text{id}$, and W is a dihedral group of even order larger than 4. In [2] this order is calculated for each of the above-mentioned 79 cases: 56 walks admit an infinite group, while the group of the 23 remaining ones is finite.

It is also proved in [2] that among these 23 walks, W has order 4 for 16 walks (the ones with a step set having a vertical symmetry), W has order 6 for 5 walks (the 2 at the left in Figure 1 and the 3 ones at the left in Figure 2), and W has order 8 for the 2 walks on the right in Figures 1 and 2. Moreover, for these 23 walks, the answers to both main questions (explicit expression *and* nature of the CGF (1.1)) are known. In particular, the following results exist.

Theorem 1.1 ([2]). *For the 16 walks with a group of order 4 and for the 3 walks in Figure 1, the formal trivariate series (1.1) is holonomic non-algebraic. For the 3 walks on the left in Figure 2, the trivariate series (1.1) is algebraic.*

The proof of this theorem relies on skillful algebraic manipulations together with the calculation of adequate *orbit* and *half-orbit* sums.

Theorem 1.2 ([1]). *For the so-called Gessel’s walk on the right in Figure 2, the formal trivariate series (1.1) is algebraic.*

The proof given in [1] involves a powerful computer algebra system (Magma), allowing to carry out dense calculations.

The main goal of our paper will be to present another proof of Theorems 1.1 and 1.2 for the bivariate generating function $(x, y) \mapsto F(x, y, z)$, by means of the general and powerful approach proposed in the book [4]. In particular, we are going to show how chapter 4 of this book (which deals with walks having a finite

group) yields rather directly the nature of the bivariate generating functions (1.1) for the 23 walks associated with a finite group.

The rest of the paper has a simple organization. Indeed, Section 2 recalls the results of [4], used in Section 3, where we find the nature of the counting generating functions coming in (1.2).

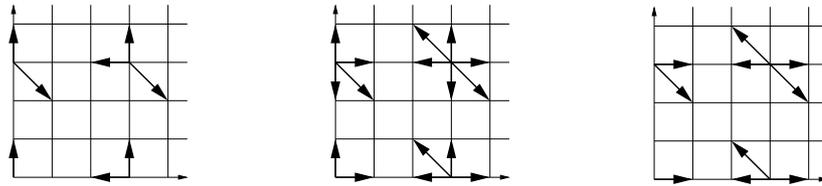


Figure 1. On the left, 2 walks with a group of order 6. On the right, 1 walk with a group of order 8.

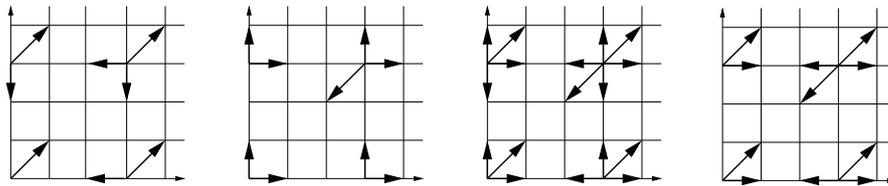


Figure 2. On the left, 3 walks with a group of order 6. On the right, 1 walk with a group of order 8.

2. Some results of [4] with miscellaneous extensions

In this section we use the notation of Appendix A, where the generic functional equation (A.1) studied in [4] is presented. It clearly contains (1.1) as a particular case, since the coefficients q, \tilde{q} and q_0 occurring in (A.1) may depend on the *two* variables x, y . Moreover, although this is not crucial, the unknown function $\pi(x)$ [resp. $\tilde{\pi}(y)$] is not necessarily equal to $\pi(x, 0)$ [resp. $\pi(0, y)$].

Chapter 5 of [4] explains how to solve (A.1) by reduction to boundary value problems of Riemann–Hilbert–Carleman type, and explicit expressions for the unknown functions are obtained.

Independently, *when the group of the walk is finite*, Chapter 4 of [4] provides another approach allowing to characterize completely the solutions of (A.1), and also to give *necessary and sufficient conditions* for these solutions *to be rational or algebraic*. We shall proceed along these lines in the sequel.

2.1. The group and the genus

Let $\mathbb{C}(x, y)$ be the field of rational functions in (x, y) over \mathbb{C} . Since Q is assumed to be irreducible in the general case, the quotient field $\mathbb{C}(x, y)$ denoted by $\mathbb{C}_Q(x, y)$ is also a field.

Definition 2.1. The *group of the random walk* is the Galois group $\mathcal{H} = \langle \xi, \eta \rangle$ of automorphisms of $\mathbb{C}_Q(x, y)$ generated by ξ and η given by

$$\xi(x, y) = \left(x, \frac{1 \sum_i p_{i,-1} x^i}{y \sum_i p_{i,+1} x^i} \right), \quad \eta(x, y) = \left(\frac{1 \sum_j p_{-1,j} y^j}{x \sum_j p_{+1,j} y^j}, y \right).$$

Let

$$\delta = \eta\xi.$$

Then \mathcal{H} has a normal cyclic subgroup $\mathcal{H}_0 = \{\delta^i, i \in \mathbb{Z}\}$, which is finite or infinite, and $\mathcal{H}/\mathcal{H}_0$ is a group of order 2. When the group \mathcal{H} is finite of order $2n$, we have $\delta^n = \text{id}$. We shall write $f_\alpha = \alpha(f)$ for any automorphism $\alpha \in \mathcal{H}$ and any function $f \in \mathbb{C}_Q(x, y)$.

In [2], the group W is introduced only in terms of two birational transformations. The difference between the two approaches is not only of a formal character. In fact W is defined in all of \mathbb{C}^2 , whereas \mathcal{H} in [4] would act only on an algebraic curve of the type

$$\left\{ (x, y) \in \mathbb{C}^2 : R(x, y) = xy \left[\sum_{i,j} r_{i,j} x^i y^j - 1/z \right] = 0 \right\}.$$

An immediate question arises: do the notions of order and finiteness coincide in the respective approaches? In general, the answer is clearly no. Indeed (adapting [4], Lemma 4.1.1) for the group to be of order 4 it is necessary and sufficient to have

$$\begin{vmatrix} r_{1,1} & r_{1,0} & r_{1,-1} \\ r_{0,1} & r_{0,0} - 1/z & r_{0,-1} \\ r_{-1,1} & r_{-1,0} & r_{-1,-1} \end{vmatrix} = 0.$$

and this condition depends on z , while W is independent of z . On the other hand, if W is finite so is \mathcal{H} , and conversely if \mathcal{H} is infinite so is W . In addition, with an obvious notation,

$$\text{Order}(\mathcal{H}) \leq \text{Order}(W). \tag{2.1}$$

In the sequel, we shall encounter groups of order not larger than 8.

Besides, in [4], the algebraic curve is associated with a Riemann surface which is of *genus* $g = 0$ (the sphere) or $g = 1$ (the torus). Accordingly, the *universal covering* of this surface is

- the Riemann sphere \mathbb{P}^1 if $g = 0$;

- the finite complex plane \mathbb{C}_ω if $g = 1$.

All 23 random walks we consider in this paper correspond indeed to $g = 1$, and an efficient uniformization is briefly described in Appendix B. Moreover, they have a finite group, say of order $2n$ (this will be made precise in Section 3).

2.2. Algebraicity and holonomy

For any $h \in \mathbb{C}_Q(x, y)$, let the *norm* $N(h)$ be defined as

$$N(h) \stackrel{\text{def}}{=} \prod_{i=0}^{n-1} h_{\delta^i}. \tag{2.2}$$

Written on $Q(x, y) = 0$, equation (A.1) yields the system

$$\begin{cases} \pi = \pi_\xi, \\ \pi_\delta - f\pi = \psi, \end{cases} \tag{2.3}$$

where

$$f = \frac{q\tilde{q}_\eta}{\tilde{q}q_\eta}, \quad \psi = \frac{q_0\tilde{q}_\eta}{q_\eta\tilde{q}} - \frac{(q_0)_\eta}{q_\eta}.$$

It turns out that the analysis of (A.1) given in [4] deeply differs, according to $N(f) = 1$ or $N(f) \neq 1$. In Section 3, as for the above-mentioned walks, it will be shown that we are in the case $N(f) = 1$ and we need the following important theorem stated in a terse form.

Theorem 2.1. *If $N(f) = 1$, then the general solution of the fundamental system (2.3) has the form $\pi = w_1 + w_2 + w/r$, where*

- the functions w_1 and r are rational;
- the function w_2 is given by

$$w_2 = \frac{\tilde{\Phi}}{n} \sum_{k=0}^{n-1} \left(\frac{\psi_{\delta^k}}{\prod_{i=1}^k f_{\delta^i}} \right); \tag{2.4}$$

- on the universal covering \mathbb{C}_ω , we have

$$\tilde{\Phi} \left(\omega + \frac{\omega_2}{2} \right) \stackrel{\text{def}}{=} \frac{\omega_1}{2i\pi} \zeta(\omega; \omega_1, \omega_3) - \frac{\omega}{i\pi} \zeta \left(\frac{\omega_1}{2}; \omega_1, \omega_3 \right), \tag{2.5}$$

where $\zeta(\omega; \omega_1, \omega_3) \stackrel{\text{def}}{=} \zeta_{1,3}(\omega)$ stands for the classical Weierstrass ζ -function (see [9]) with quasi periods ω_1, ω_3 ;

- the function w is algebraic and satisfies the automorphy conditions $w = w_\xi = w_\delta$.

The quantities $\omega_1, \omega_2, \omega_3$ have explicit forms (see [4], Lemmas 3.3.2 and 3.3.3) and the automorphism δ writes simply as the translation $\delta(\omega) = \omega + \omega_3$, for any $\omega \in \mathbb{C}_\omega$. In addition, when the group is finite of order $2n$, there exists an integer k relatively prime with n satisfying the relation $n\omega_3 = k\omega_2$, which in turn allows to prove (see [4], Lemma 4.3.5) that $\tilde{\Phi}(\omega)$ defined in (2.5) is *not* algebraic in $x(\omega)$. We are now in a position to derive the following corollary.

Corollary 2.1. *Assume the group is finite of order $2n$ and $N(f) = 1$. Then the solution $\pi(x)$ of (A.1) is holonomic. Moreover it is algebraic if and only if on $Q(x, y) = 0$,*

$$\sum_{k=0}^{n-1} \frac{\psi_{\delta^k}}{\prod_{i=1}^k f_{\delta^i}} = 0. \tag{2.6}$$

In fact, the proof is an immediate consequence of the next lemma.

Lemma 2.1. *In the uniformization given in Appendix B, let us write*

$$\wp(\omega) \stackrel{\text{def}}{=} g(x(\omega)),$$

where g is a fractional linear transform. Then the function $w_2(\wp^{-1}(g(x)))$ defined by (2.4) is holonomic in x , where \wp^{-1} denotes the elliptic integral inverse function of \wp .

Proof. Since w_2 is the product of $\tilde{\Phi}$ by an algebraic function, it suffices to prove that $\tilde{\Phi}(\wp^{-1}(g))$ is holonomic. It is known that the class of holonomic functions is closed under indefinite integration (see e.g. [6]), thus it is enough to prove that $[\tilde{\Phi}(\wp^{-1}(g))]'$ is holonomic. In fact, we are going to show that $[\tilde{\Phi}(\wp^{-1}(g))]'$ is algebraic — we recall that any algebraic function is holonomic. Indeed, use on the one hand that the derivative of $-\zeta_{1,3}$ is $\wp_{1,3}$, the Weierstrass elliptic function with periods ω_1, ω_3 , and on the other hand that

$$[\wp^{-1}]'(u) = 1/[4u^3 - g_2u - g_3]^{1/2},$$

g_2 and g_3 being the so-called invariants of \wp — for these two properties see e.g. [9]. In this way we get

$$-2i\pi[\tilde{\Phi}(\wp^{-1}(g))]' = \frac{g'}{[4g^3 - g_2g - g_3]^{1/2}} [\omega_1\wp_{1,3}(\wp^{-1}(g) - \omega_2/2) + 2\zeta_{1,3}(\omega_1/2)].$$

The function g being rational, $g'/[4g^3 - g_2g - g_3]^{1/2}$ is algebraic and in order to conclude it is enough to prove that $\wp_{1,3}(\wp^{-1}(g) - \omega_2/2)$ is algebraic. Since $n\omega_3 = k\omega_2$, it is shown in [4], Lemma 4.3.3 that both \wp and $\wp_{1,3}$ are rational functions of the Weierstrass elliptic function with periods $\omega_1, k\omega_2$. Then it is immediate that $\wp_{1,3}$ is also algebraic function of \wp . Moreover, by the well-known addition theorem for the Weierstrass elliptic functions (see e.g. [9]), \wp is

a rational function of $\wp(\cdot + \omega_2/2)$, so that $\wp_{1,3}$ is also an algebraic function of $\wp(\cdot + \omega_2/2)$. In particular $\wp_{1,3}(\wp^{-1}(g) - \omega_2/2)$ is algebraic and Lemma 2.1 is proved. \square

Of course similar results can be written for $\tilde{\pi}$. In particular, it is easy to check that if $N(f) = 1$ then also $\tilde{N}(f) = 1$, where $\tilde{N}(f) = 1$ is defined in terms of $\tilde{\delta} \stackrel{\text{def}}{=} \xi\eta$. Thus we are in a position to state the following general result.

Theorem 2.2. *Assume the group is finite of order $2n$ and $N(f) = 1$. Then the bivariate series $\pi(x, y)$ solution of (A.1) is holonomic. Furthermore $\pi(x, y)$ is algebraic if and only if (2.6) and $(\widetilde{2.6})$ hold on $Q(x, y) = 0$, $(\widetilde{2.6})$ denoting the condition for $\tilde{\pi}(y)$ to be algebraic, obtained by symmetry from (2.6).*

3. Nature of the counting generating functions

We return now to equation (1.2) and make use of the machinery of Section 2 with

$$f = \frac{c\tilde{c}_\eta}{\tilde{c}c_\eta}, \quad \psi = \frac{c_0\tilde{c}_\eta}{c_\eta\tilde{c}} - \frac{(c_0)_\eta}{c_\eta}. \tag{3.1}$$

Beforehand it is important to note that the variable z plays here the role of a *parameter*. Moreover, the CGF (1.1) is well defined and analytic in a region containing the domain

$$\{(x, y, z) \in \mathbb{C}^3 : |x| \leq 1, |y| \leq 1, |z| < 1/|\mathcal{S}|\}.$$

The analysis in [4] aimed at finding the stationary measure of the random walk, in which case $z = 1$. But it is possible to check *mutatis mutandis* that most of the topological results remains valid at least for $\Re(z) \geq 0$ [see for instance [5] or [11] for time dependent problems].

Lemma 3.1. *For any finite n and f given by (3.1), we have $N(f) = 1$.*

Proof. Since c [resp. \tilde{c}] is a function solely of x [resp. y], we have $c = c_\xi$ and $\tilde{c} = \tilde{c}_\eta$. Hence

$$f = \frac{c}{c_\eta} = \frac{c_\delta}{c_{\delta\eta}} = \frac{c_\delta}{c},$$

which yields immediately $N(f) = 1$ from the definition (2.2). \square

In order to obtain Theorems 1.1 and 1.2 for the bivariate function $(x, y) \mapsto F(x, y, z)$, it is thus enough, by Theorem 2.2 and Lemma 3.1, to prove the following final proposition.

Proposition 3.1. *For the 4 walks in Figure 2, (2.6) and $(\widetilde{2.6})$ hold in \mathbb{C}^2 . As for the 19 other walks (3 in Figure 1 and the 16 ones with a vertically symmetric step set \mathcal{S}), (2.6) or $(\widetilde{2.6})$ do not hold on $\{(x, y) \in \mathbb{C}^2 : K(x, y) = xy[\sum_{(i,j) \in \mathcal{S}} x^i y^j - 1/z] = 0\}$, for any z .*

Proof. The proof proceeds in three stages.

(i) The walks in Figure 2

Let us check (2.6) for the 4 walks in Figure 2. We begin with the popular Gessel’s walk, i.e. the rightest one in Figure 2. Here $c = 1$ in (1.2), so that by (3.1) $f = 1$ and $\psi = [(xy)_\eta - xy]/z$. Moreover, the order of the group W was shown in [2] to be equal to 8. According to (2.1), we know that $Order(\mathcal{H}) \leq 8$. On the other hand, by Lemmas 4.1.1 and 4.1.2 of [4], a simple algebra (details are omitted) can show that the order of \mathcal{H} cannot be equal to 4 neither to 6, which entails $Order(\mathcal{H}) = 8$, hence $n = 4$.

Then

$$\sum_{k=0}^{n-1} \frac{\psi_{\delta^k}}{\prod_{i=1}^k f_{\delta^i}} = \sum_{k=0}^3 \psi_{\delta^k} = \frac{1}{z} \sum_{k=0}^3 [(xy)_{\eta\delta^k} - (xy)_{\delta^k}] = -\frac{1}{z} \sum_{\rho \in \mathcal{H}} \text{sign}(\rho)(xy)_\rho. \tag{3.2}$$

The last quantity in (3.2) can be understood as $(-1/z)$ times the orbit sum of xy through the group \mathcal{H} , and it was remarked in [2] that this orbit sum equals zero for Gessel’s walk, which is equivalent to (2.6) in \mathbb{C}^2 .

For the second walk in Figure 2, $Order(W) = Order(\mathcal{H}) = 6$ ($n = 3$) and still $c = 1$, so that the above argument applies.

For the walk on the left in Figure 2, we have

$$\xi = (x, 1/(xy)), \quad \eta = (1/(xy), y),$$

hence $\xi\eta = (1/(xy), x)$. Moreover $c = x$, thus by using (3.1) we get $f = x^2y$ and $\psi = [y - (xy)^2]/z$. Next one easily computes $f_\delta = 1/(xy^2)$, $f_{\delta^2} = y/x$, $\psi_\delta = [x - 1/y^2]/z$ and $\psi_{\delta^2} = [1/(xy) - 1/x^2]/z$. Finally an immediate calculation shows that (2.6) holds in \mathbb{C}^2 .

As for the last walk in Figure 2, we could check $n = 3$ and again (2.6) holds in \mathbb{C}^2 .

Concerning equation $(\widetilde{2.6})$, only Gessel’s walk needs some special care, since the 3 others give rise to symmetric conditions. In fact, from a direct calculation along the same lines as above, it is not difficult to see that $(\widetilde{2.6})$ holds in \mathbb{C}^2 .

(ii) The 16 walks with a vertical symmetry

We will show the equality

$$\sum_{k=0}^{n-1} \frac{\psi_{\delta^k}}{\prod_{i=1}^k f_{\delta^i}} = -\frac{x^2}{zc}(x - x_\eta)(y - y_\xi), \tag{3.3}$$

which clearly implies that (2.6) will not be satisfied on $\{(x, y) \in \mathbb{C}^2 : K(x, y) = 0\}$, for any z .

For these 16 walks we have $Order(W) = 4 = Order(\mathcal{H})$, that is $n = 2$. Moreover, the vertical symmetry of the jump set \mathcal{S} has two consequences : first,

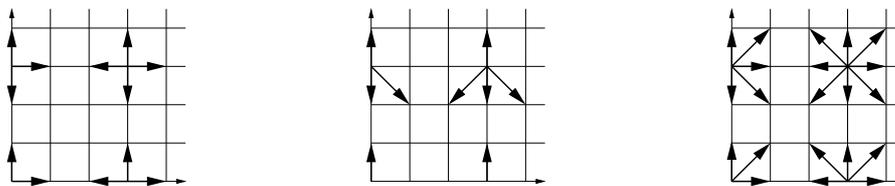


Figure 3. Three examples of walks having a group of order 4, with symmetrical *West* and *East* jumps.

necessarily $\eta = (1/x, y)$; second, the coefficient c in (1.2) must be equal to x , $1 + x^2$ or $1 + x + x^2$. In addition, for any of these three possible values of c , we have $f = c/c_\eta = x^2$. Hence finally

$$\sum_{k=0}^{n-1} \frac{\psi_{\delta^k}}{\prod_{i=1}^k f_{\delta^i}} = \psi + \frac{\psi_\delta}{f_\delta} = \frac{(xy)_\eta - xy}{zc_\eta} + \frac{(xy)_\xi - (xy)_\delta}{zc_\xi f_\delta} = \frac{y(x_\eta - x)}{zc_\eta} + \frac{y_\xi(x - x_\eta)}{zc(x^2)_\delta}.$$

Factorizing by x^2/c and using again $c/c_\eta = x^2$, we obtain (3.3).

(iii) The 3 walks in Figure 1

We are going to show that (2.6) does not hold on $K(x, y) = 0$, for any z .

By similar calculations as in (3.3), we obtain that for the 2 first walks in Figure 1 (which are such that $n = 3$),

$$\sum_{k=0}^{n-1} \frac{\psi_{\delta^k}}{\prod_{i=1}^k f_{\delta^i}} = \frac{(x - y^2)(1 - xy)(y - x^2)}{zy^3t},$$

with $t = y$ for the walk at the left and $t = x + y$ for the walk in the middle, so that, clearly, (2.6) does not hold on $\{(x, y) \in \mathbb{C}^2 : K(x, y) = 0\}$, for any z .

At last, for the walk at the right in Figure 1, we have $n = 4$, and we get

$$\sum_{k=0}^{n-1} \frac{\psi_{\delta^k}}{\prod_{i=1}^k f_{\delta^i}} = \frac{(y - 1)(x^2 - 1)(x^2 - y)(x^2 - y^2)}{xy^4z},$$

which is obviously not identically to zero on $\{(x, y) \in \mathbb{C}^2 : K(x, y) = 0\}$, for any z . □

A. The basic functional equations (see [4], Chapters 2 and 5)

In a probabilistic framework, one considers a piecewise homogeneous random walk with sample paths in \mathbb{Z}_+^2 . In the interior of \mathbb{Z}_+^2 , the size of the jumps is 1. On the other hand, no assumption is made about the boundedness of

the upward jumps on the axes, neither at $(0, 0)$. In addition, the downward jumps on the x [resp. y] axis are bounded by L [resp. M], where L and M are arbitrary finite integers. Then the invariant measure $\{\pi_{i,j}, i, j \geq 0\}$ does satisfy the fundamental functional equation

$$Q(x, y)\pi(x, y) = q(x, y)\pi(x) + \tilde{q}(x, y)\tilde{\pi}(y) + q_0(x, y), \tag{A.1}$$

with

$$\left\{ \begin{array}{l} \pi(x, y) = \sum_{i,j \geq 1} \pi_{ij} x^{i-1} y^{j-1}, \\ \pi(x) = \sum_{i \geq 1} \pi_{i0} x^{i-1}, \quad \tilde{\pi}(y) = \sum_{j \geq 1} \pi_{0j} y^{j-1}, \\ Q(x, y) = xy \left[1 - \sum_{i,j \in \mathcal{S}} p_{ij} x^i y^j \right], \quad \sum_{i,j \in \mathcal{S}} p_{ij} = 1, \\ q(x, y) = x \left[\sum_{i \geq -1, j \geq 0} p'_{ij} x^i y^j - 1 \right], \\ \tilde{q}(x, y) = y \left[\sum_{i \geq 0, j \geq -1} p''_{ij} x^i y^j - 1 \right], \\ q_0(x, y) = \pi_{00} [P_{00}(x, y) - 1]. \end{array} \right.$$

In equation (A.1), the functions $\pi(x, y), \pi(x), \tilde{\pi}(y)$ are unknown and sought to be analytic in the region $\{(x, y) \in \mathbb{C}^2 : |x| < 1, |y| < 1\}$ and continuous on their respective boundaries, while the q, \tilde{q}, q_0 (up to a constant) are given probability generating functions supposed to have suitable analytic continuations (as a rule, they are polynomials when the jumps are bounded). The function $Q(x, y)$ is often referred to as the *kernel* of (A.1).

B. About the uniformization of $Q(x, y) = 0$ (see [4], Chapter 3)

When the associated Riemann surface is of genus 1, the algebraic curve $Q(x, y) = 0$ admits a uniformization given in terms of the Weierstrass \wp function with periods ω_1, ω_2 and its derivatives. Indeed, setting

$$\begin{aligned} Q(x, y) &= a(x)y^2 + b(x)y + c(x), \\ D(x) &= b^2(x) - 4a(x)c(x) = d_4x^4 + d_3x^3 + d_2x^2 + d_1x + d_0, \\ u &= 2a(x)y + b(x), \end{aligned}$$

the following formulae hold (see [4], Lemma 3.3.1).

1. If $d_4 \neq 0$ (4 finite branch points x_1, \dots, x_4), then $D'(x_4) > 0$ and

$$\begin{cases} x(\omega) = x_4 + \frac{D'(x_4)}{\wp(\omega) - (1/6)D''(x_4)}, \\ u(\omega) = \frac{D'(x_4)\wp'(\omega)}{2(\wp(\omega) - (1/6)D''(x_4))^2}. \end{cases}$$

2. If $d_4 = 0$ (3 finite branch points x_1, x_2, x_3 and $x_4 = \infty$), then

$$\begin{cases} x(\omega) = \frac{\wp(\omega) - d_2/3}{d_3}, \\ u(\omega) = -\frac{\wp'(\omega)}{2d_3}. \end{cases}$$

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