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Some aspects of fluctuations of random walks on $\mathbb{R}$ and applications to random walks on $\mathbb{R}^+$ with non-elastic reflection at 0

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Abstract. In this article we refine well-known results concerning the fluctuations of one-dimensional random walks. More precisely, if $(S_n)_{n\geq 0}$ is a random walk starting from 0 and $r \geq 0$, we obtain the precise asymptotic behavior as $n \to \infty$ of $\mathbb{P}[\tau^{>r} = n, S_n \in K]$ and $\mathbb{P}[\tau^{>r} > n, S_n \in K]$, where $\tau^{>r}$ is the first time that the random walk reaches the set $]r, \infty[$, and K is a compact set. Our assumptions on the jumps of the random walks are optimal. Our results give an answer to a question of Lalley (1995), and are applied to obtain the asymptotic behavior of the return probabilities for random walks on $\mathbb{R}^+$ with non-elastic reflection at 0.

1. Introduction

General context. An essential aspect of fluctuation theory of discrete time random walks is the study of the two-dimensional renewal process formed by the successive maxima (or minima) of the random walk $(S_n)_{n\geq 0}$ and the corresponding times; this process is called the ascending (or descending) ladder process. It has been studied by many people, with major contributions by Baxter (1962/1963), Spitzer (1964), and others who introduced Wiener-Hopf techniques and established several fundamental identities that relate the distributions of the ascending and descending ladder processes to the law of the random walk.

Let $(S_n)_{n\geq 0}$ be a random walk defined on a probability space $(\Omega, \mathcal{T}, \mathbb{P})$ and starting from 0; in other words, $S_0 = 0$ and $S_n = Y_1 + \cdots + Y_n$ for $n \geq 1$, where

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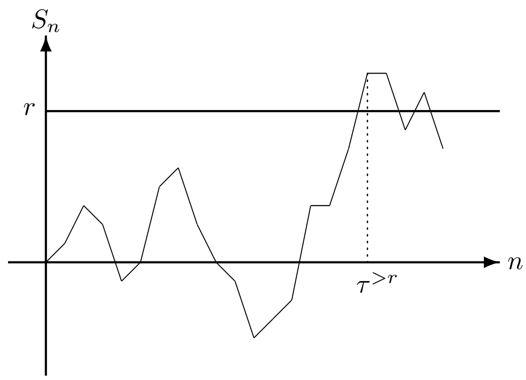

FIGURE 1.1. Definition of $\tau^{>r}$

$(Y_i)_{i\geq 1}$ is a sequence of independent and identically distributed (i.i.d.) random variables. The strict ascending ladder process $(T_n^{*+}, H_n)_{n\geq 0}$ is defined as follows:

$$T_0^{*+} = 0, \quad T_{n+1}^{*+} = \inf\{k > T_n^{*+} : S_k > S_{T_n^{*+}}\}, \quad \forall n \geq 0, \tag{1.1}$$

and

$$H_n = S_{T_n^{*+}}, \quad \forall n \geq 0.$$

There exists a large literature on this process, which typically focuses on so-called local limit theorems, and in particular on the behavior of the probabilities $\mathbb{P}[T_1^{*+} > n]$ and $\mathbb{P}[T_1^{*+} > n, H_1 \in K]$, where $K \subset \mathbb{R}$ is some compact set. Roughly speaking, when the variables $(Y_i)_{i\geq 1}$ admit moments of order 2 and are centered, one has the asymptotic behavior, as $n \to \infty$,

$$\mathbb{P}[T_1^{*+} > n] = \frac{a}{\sqrt{n}}(1 + o(1)), \qquad \mathbb{P}[T_1^{*+} > n, H_1 \in K] = \frac{b}{n^{3/2}}(1 + o(1)),$$

for some constants $a, b > 0$ to be specified (see for instance Le Page and Peigné (1997) and references therein).

These estimations are of great interest in several domains: one may cite for example branching processes in random environment (see for instance Geiger and Kersting (2000); Guivarc'h et al. (2003); Kozlov (1976)) and random walks on non-unimodular groups (see Le Page and Peigné (1997, 1999)); they also play a crucial role in several other less linear contexts, as in the study of return probabilities for random walks with reflecting zone on a half-line Lalley (1995).

In 1995, Lalley introduced for $r > 0$ the waiting time

$$\tau^{>r} = \inf\{n > 0 : S_n > r\},$$

see Figure 1.1, and first looked at the behavior, as $n \to \infty$, of the probability $\mathbb{P}[\tau^{>r} = n, S_n \in K]$, where K is a compact set. Under some strong conditions (namely, if the variables $(Y_i)_{i\geq 1}$ are lattice, bounded from above and centered), Lalley proved that

$$\mathbb{P}[\tau^{>r} = n, S_n \in K] = \frac{c}{n^{3/2}}(1 + o(1)), \quad n \to \infty, \tag{1.2}$$

for some non-explicit constant $c > 0$, and wrote that "[he] do[es] not know the minimal moment conditions necessary for [such an] estimate" (see Equation (3.18) and below in Lalley (1995, page 590)). His method is based on the Wiener-Hopf factorization and on a classical theorem of Darboux which, in this case, relates the asymptotic behavior of certain probabilities to the regularity of the underlying generating function in a neighborhood of its radius of convergence. In Lalley (1995), the fact that the jumps $(Y_i)_{i\geq 1}$ are bounded from above is crucial since it allows the author to verify that the generating function of the jumps $(Y_i)_{i\geq 1}$ is meromorphic in a neighborhood of its disc of convergence, with a non-essential pole at 0.

Aim and methods of this article. In this article we obtain the asymptotic behavior of the probability in (1.2), with besides an explicit formula for the constant c, under quite general hypotheses (Theorem 3.1). This in particular answers to Lalley's question. We will also obtain (Theorem 3.4) the asymptotic behavior of

$$\mathbb{P}[\tau^{>r} > n, S_n \in K], \quad n \to \infty. \tag{1.3}$$

To prove Theorems 3.1 and 3.4, we shall adopt another strategy as that in Lalley (1995), inspired by the works of Iglehart (1974), Le Page and Peigné (1997) (Sections 2 and 3). We will also propose an application of our main results to random walks on $\mathbb{R}^+$ with non-elastic reflection at 0 (Section 4). Finally, we shall emphasize the connections of our results with the ones of Denisov and Wachtel (2011), where quite a new approach is developed in any dimension, to find local limit theorems for random walks in cones (Section 5).

2. First results

2.1. *Notations.* We consider here a sequence $(Y_i)_{i\geq 1}$ of i.i.d. $\mathbb{R}$-valued random variables with law μ, defined on a probability space $(\Omega, \mathcal{T}, \mathbb{P})$. For any $n \geq 1$, we set $\mathcal{T}_n = \sigma(Y_1, \ldots, Y_n)$. Let $(S_n)_{n\geq 0}$ be the corresponding random walk on $\mathbb{R}$ starting from 0, i.e., $S_0 = 0$ and for $n \geq 1$, $S_n = Y_1 + \cdots + Y_n$. In order to study the fluctuations of $(S_n)_{n\geq 0}$, we introduce for $r \in \mathbb{R}$ the random variables $\tau^{\geq r}$, $\tau^{>r}$, $\tau^{\leq r}$ and $\tau^{<r}$, defined by

$$\begin{aligned}
\tau^{\geq r} &:= \inf\{n \geq 1 : S_n \geq r\},\\
\tau^{>r} &:= \inf\{n \geq 1 : S_n > r\},\\
\tau^{\leq r} &:= \inf\{n \geq 1 : S_n \leq r\},\\
\tau^{<r} &:= \inf\{n \geq 1 : S_n < r\}.
\end{aligned}$$

Throughout we shall use the convention $\inf\{\emptyset\} = \infty$. The latter variables are stopping times with respect to the canonical filtration $(\mathcal{T}_n)_{n\geq 1}$. When $r = 0$, in order to use standard notations, we shall rename $\tau^{\geq 0}$, $\tau^{>0}$, $\tau^{\leq 0}$ and $\tau^{<0}$ in τ^+, τ^{*+}, τ^- and τ^{*-}, respectively. As[1] $\mathbb{R}^- = \mathbb{R} \setminus \mathbb{R}^{*+}$ (resp. $\mathbb{R}^+ = \mathbb{R} \setminus \mathbb{R}^{*-}$), there will be some duality connections between τ^- and τ^{*+} (resp. τ^+ and τ^{*-}).

We also introduce, as in (1.1), the sequence $(T_n^{*+})_{n\geq 0}$ of successive ascending ladder epochs of the walk $(S_n)_{n\geq 0}$. One has $T_1^{*+} = \tau^{*+}$. Further, setting $\tau_{n+1}^{*+} := T_{n+1}^{*+} - T_n^{*+}$ for any $n \geq 0$, one may write $T_n^{*+} = \tau_1^{*+} + \cdots + \tau_n^{*+}$, where $(\tau_n^{*+})_{n\geq 1}$ is a sequence of i.i.d. random variables with the same law as τ^{*+}.[2]

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[1]Here and throughout, we note $\mathbb{R}^+ = [0, \infty[$, $\mathbb{R}^{*+} =]0, \infty[$, $\mathbb{R}^- =]-\infty, 0]$ and $\mathbb{R}^{*-} =]-\infty, 0[$.

[2]Similarly, we may also consider the sequences $(T_n^+)_{n\geq 0}$, $(T_n^-)_{n\geq 0}$ and $(T_n^{*-})_{n\geq 0}$ defined respectively by $T_0^+ = T_0^- = T_0^{*-} = 0$ and $T_{n+1}^+ = \inf\{k > T_n^+ : S_k \geq S_{T_n^+}\}$, $T_{n+1}^- = \inf\{k > T_n^- : S_k \leq S_{T_n^-}\}$ and $T_{n+1}^{*-} = \inf\{k > T_n^{*-} : S_k < S_{T_n^{*-}}\}$, for $n \geq 0$.

2.2. *Hypotheses.* Throughout this manuscript, we shall assume that the law μ satisfies one of the following moment conditions **M**:

M(k)**:** $\mathbb{E}[|Y_1|^k] < \infty$;
M$(\exp)$**:** $\mathbb{E}[\exp(\gamma Y_1)] < \infty$, *for all* $\gamma \in \mathbb{R}$;
M$(\exp^-)$**:** $\mathbb{E}[\exp(\gamma Y_1)] < \infty$, *for all* $\gamma \in \mathbb{R}^-$.

We shall also often suppose

C: $\mathbb{E}[Y_1] = 0$.

Under **M**(1) and **C**, the variables τ^+, τ^{*+}, τ^- and τ^{*-} are $\mathbb{P}$-a.s. finite, see Feller (1971),[3] and we denote by μ^+ (resp. $\mu^{*+}, \mu^-, \mu^{*-}$) the law of the variable S_{τ^+} (resp. $S_{\tau^{*+}}, S_{\tau^-}$ and $S_{\tau^{*-}}$).

We will also consider the two following couples of hypotheses **AA**:

AA$(\mathbb{Z})$**:** *the measure μ is adapted on $\mathbb{Z}$ (i.e., the group generated by the support S_μ of μ is equal to $\mathbb{Z}$) and aperiodic (i.e., the group generated by $S_\mu - S_\mu$ is equal to $\mathbb{Z}$);*

AA$(\mathbb{R})$**:** *the measure μ is adapted on $\mathbb{R}$ (i.e., the closed group generated by the support S_μ of μ is equal to $\mathbb{R}$) and aperiodic (i.e., the closed group generated by $S_\mu - S_\mu$ is equal to $\mathbb{R}$).*

2.3. *Classical results.* Let us now recall the result below, which concerns the probability (1.3) for $r = 0$.

Theorem 2.1 (Iglehart (1974); Le Page and Peigné (1997))**.** *Assume that the hypotheses* **AA**, **C** *and* **M**(2) *hold. Then for any continuous function ϕ with compact support on $\mathbb{R}$, one has*[4]

$$\lim_{n\to\infty} n^{3/2}\mathbb{E}[\tau^{*+} > n; \phi(S_n)] = a^-(\phi) := \int_{\mathbb{R}^-} \phi(t) a^-(\mathrm{d}t)$$
$$:= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}^-} \phi(t) \lambda^- * U^-(\mathrm{d}t),$$

where

- $\sigma^2 := \mathbb{E}[Y_1^2]$;
- λ^- *is the counting measure on $\mathbb{Z}^-$ when* **AA**$(\mathbb{Z})$ *holds (resp. the Lebesgue measure on $\mathbb{R}^-$ when* **AA**$(\mathbb{R})$ *holds);*[5]
- U^- *is the σ-finite potential* $U^- := \sum_{n\geq 0} (\mu^-)^{*n}$.

Since some arguments will be quite useful and used in the sequel, we give below a sketch of the proof of Theorem 2.1, following Iglehart (1974); Le Page and Peigné (1997). By a standard argument in measure theory (see Theorem 2 in Chapter XIII on Laplace transforms in the book Feller (1971)), it is sufficient to prove the above convergence for all functions ϕ of the form $\phi(t) = \exp(\alpha t)$, $\alpha > 0$ (indeed, notice

[3] Notice that this property also holds for symmetric laws μ without any moment assumption.

[4] Below and throughout, for any bounded random variable $Z : \Omega \to \mathbb{R}$ and any event $A \in \mathcal{T}$, one sets $\mathbb{E}[A; Z] := \mathbb{E}[Z \mathbb{1}_A]$.

[5] For an upcoming use, we also introduce
- the counting measures λ^{*-}, λ^+ and λ^{*+} on $\mathbb{Z}^{*-}$, $\mathbb{Z}^+$ and $\mathbb{Z}^{*+}$, respectively;
- the Lebesgue measures λ^{*-}, λ^+ and λ^{*+} on $\mathbb{R}^{*-}$, $\mathbb{R}^+$ and $\mathbb{R}^{*+}$, respectively.

Notice that $\lambda^{*-} = \lambda^-$ and $\lambda^{*+} = \lambda^+$ when **AA**$(\mathbb{R})$ holds, but we keep the two notations in order to unify the statements under the two types of hypotheses **AA**.

that the support of the limit measure $a^-(\mathrm{d}t)$ is included in $\mathbb{R}^-$). We shall use the same remark when proving Theorems 2.6 and 3.1.

Sketch of the proof of Theorem 2.1 in the case $\mathbf{AA}(\mathbb{Z})$*:* We shall use the following identity, which is a consequence of the Wiener-Hopf factorization (see Spitzer (1964, P5 in page 181)):

$$\phi_\alpha(s) := \sum_{n\geq 0} s^n \mathbb{E}[\tau^{*+} > n; e^{\alpha S_n}] = \exp B_\alpha(s), \quad \forall s \in [0,1[, \quad \forall \alpha > 0, \tag{2.1}$$

where

$$B_\alpha(s) := \sum_{n\geq 1} \frac{s^n}{n} \mathbb{E}[S_n \leq 0; e^{\alpha S_n}].$$

Further, by the classical local limit theorem on $\mathbb{Z}$ (this is here that we use $\mathbf{M(2)}$, see for instance Spitzer (1964, P10 in page 79)), one gets

$$\mathbb{E}[S_n \leq 0; e^{\alpha S_n}] = \frac{1}{\sigma\sqrt{2\pi n}} \frac{1}{1-e^{-\alpha}}(1+o(1)), \quad n \to \infty.$$

Accordingly, the sequence $(n^{3/2}\mathbb{E}[\tau^{*+} > n; e^{\alpha S_n}])_{n\geq 1}$ is bounded, thanks to Lemma 2.2 below (taken from Iglehart (1974, Lemma 2.1)), applied with $b_n := \mathbb{E}[S_n \leq 0; e^{\alpha S_n}]/n$ and $d_n := \mathbb{E}[\tau^{*+} > n; e^{\alpha S_n}]$.

Lemma 2.2 (Iglehart (1974))**.** *Let* $\sum_{n\geq 0} d_n s^n = \exp \sum_{n\geq 0} b_n s^n$*. If the sequence* $(n^{3/2} b_n)_{n\geq 1}$ *is bounded, the same holds for* $(n^{3/2} d_n)_{n\geq 1}$*.*

Differentiating the two members of (2.1) with respect to s, one gets

$$\phi'_\alpha(s) = \sum_{n\geq 1} n s^{n-1} \mathbb{E}[\tau^{*+} > n; e^{\alpha S_n}] = \phi_\alpha(s) \sum_{n\geq 1} s^{n-1} \mathbb{E}[S_n \leq 0; e^{\alpha S_n}].$$

We then make use of Lemma 2.3 (see Iglehart (1974, Lemma 2.2) for the original statement), applied with $c_n := \mathbb{E}[S_n \leq 0; e^{\alpha S_n}] = n b_n$, $d_n := \mathbb{E}[\tau^{*+} > n; e^{\alpha S_n}]$ and $a_n := n\mathbb{E}[\tau^{*+} > n; e^{\alpha S_n}]$.

Lemma 2.3 (Iglehart (1974))**.** *Let* $(c_n)_{n\geq 0}$ *and* $(d_n)_{n\geq 0}$ *be sequences of non-negative real numbers such that*

(1) $\lim_{n\to\infty} \sqrt{n} c_n = c > 0$*;*
(2) $\sum_{n\geq 0} d_n = D < \infty$*;*
(3) $(n d_n)_{n\geq 0}$ *is bounded.*

If $a_n = \sum_{0\leq k\leq n-1} d_k c_{n-k}$*, then* $\lim_{n\to\infty} \sqrt{n} a_n = cD$*.*

This way, one reaches the conclusion that

$$\lim_{n\to\infty} n^{3/2} \mathbb{E}[\tau^{*+} > n; e^{\alpha S_n}] = \frac{1}{\sigma\sqrt{2\pi}} \frac{1}{1-e^{-\alpha}} \sum_{n\geq 0} \mathbb{E}[\tau^{*+} > n; e^{\alpha S_n}].$$

To conclude, it remains to express differently the limit. First, the factor $1/(1-e^{-\alpha})$ is equal to $\int_{\mathbb{R}} e^{\alpha t} \lambda^-(\mathrm{d}t)$. Further, since the vectors $(Y_1, \ldots, Y_n)$ and $(Y_n, \ldots, Y_1)$

have the same law, one gets

$$\begin{aligned}\sum_{n\geq 0}\mathbb{E}[\tau^{*+}>n;e^{\alpha S_n}] &= \sum_{n\geq 0}\mathbb{E}[S_1\leq 0,S_2\leq 0,\dots,S_n\leq 0;e^{\alpha S_n}]\\ &= \sum_{n\geq 0}\mathbb{E}[S_n\leq S_{n-1},S_n\leq S_{n-2},\dots,S_n\leq 0;e^{\alpha S_n}]\\ &= \sum_{n\geq 0}\mathbb{E}[\exists \ell\geq 0: T_\ell^- = n;e^{\alpha S_n}]\\ &= \sum_{\ell\geq 0}\mathbb{E}[e^{\alpha S_{T_\ell^-}}] = U^-(x\mapsto e^{\alpha x}),\end{aligned}$$

i.e., $\sum_{n\geq 0}\mathbb{E}[\tau^{*+}>n;S_n\in \mathrm{d}x] = U^-(\mathrm{d}x)$, so that

$$\frac{1}{\sigma\sqrt{2\pi}}\frac{1}{1-e^{-\alpha}}\sum_{n\geq 0}\mathbb{E}[\tau^{*+}>n;e^{\alpha S_n}] = \frac{1}{\sigma\sqrt{2\pi}}\int_{\mathbb{R}^-} e^{\alpha t}\lambda^- * U^-(\mathrm{d}t).$$

The proof is complete. □

Remark 2.4. For similar reasons as in the proof of Theorem 2.1, one has

$$\begin{aligned}\textstyle\sum_{n\geq 0}\mathbb{E}[\tau^{+}>n;S_n\in \mathrm{d}x] = U^{*-}(\mathrm{d}x) := \sum_{n\geq 0}(\mu^{*-})^{*n}(\mathrm{d}x),\\ \textstyle\sum_{n\geq 0}\mathbb{E}[\tau^{*-}>n;S_n\in \mathrm{d}x] = U^{+}(\mathrm{d}x) := \sum_{n\geq 0}(\mu^{+})^{*n}(\mathrm{d}x),\\ \textstyle\sum_{n\geq 0}\mathbb{E}[\tau^{-}>n;S_n\in \mathrm{d}x] = U^{*+}(\mathrm{d}x) := \sum_{n\geq 0}(\mu^{*+})^{*n}(\mathrm{d}x),\end{aligned}$$

as well as the weak convergences, as $n\to\infty$,

$$\begin{aligned}n^{3/2}\mathbb{E}[\tau^{*+}>n;S_n\in \mathrm{d}x] &\longrightarrow a^-(\mathrm{d}x) := (1/\sigma\sqrt{2\pi})\lambda^- * U^-,\\ n^{3/2}\mathbb{E}[\tau^{+}>n;S_n\in \mathrm{d}x] &\longrightarrow a^{*-}(\mathrm{d}x) := (1/\sigma\sqrt{2\pi})\lambda^{*-} * U^{*-},\\ n^{3/2}\mathbb{E}[\tau^{*-}>n;S_n\in \mathrm{d}x] &\longrightarrow a^+(\mathrm{d}x) := (1/\sigma\sqrt{2\pi})\lambda^+ * U^+,\\ n^{3/2}\mathbb{E}[\tau^{-}>n;S_n\in \mathrm{d}x] &\longrightarrow a^{*+}(\mathrm{d}x) := (1/\sigma\sqrt{2\pi})\lambda^{*+} * U^{*+}.\end{aligned}$$

We conclude this part by finding the asymptotic behavior of $\mathbb{P}[\tau^{*+}>n]$. Using the well-known expansion

$$\sqrt{1-s} = \exp\left(\frac{1}{2}\ln(1-s)\right) = \exp\left(-\frac{1}{2}\sum_{n\geq 1}\frac{s^n}{n}\right)$$

and setting $\alpha = 0$ in (2.1), one gets that for s close to 1,

$$\sum_{n\geq 0}s^n\mathbb{P}[\tau^{*+}>n] = \exp\left(\sum_{n\geq 1}\frac{s^n}{n}\mathbb{P}[S_n\leq 0]\right) = \frac{\exp\kappa}{\sqrt{1-s}}(1+o(1)),$$

where

$$\kappa = \sum_{n\geq 1}\frac{1}{n}\left(\mathbb{P}[S_n\leq 0]-\frac{1}{2}\right). \tag{2.2}$$

Notice that the series in (2.2) is absolutely convergent, see Rosén (1962, Theorem 3).[6] By a standard Tauberian theorem, since the sequence $(\mathbb{P}[\tau^{*+}>n])_{n\geq 0}$ is

[6] There also exists the following expression for κ: $e^\kappa = (\sqrt{2}/\sigma)\mathbb{E}[S_{\tau^{*+}}]$, see Spitzer (1964, P5 in Section 18).

decreasing, one obtains (see Le Page and Peigné (1997))

$$\mathbb{P}[\tau^{*+} > n] = \frac{\exp \kappa}{\sqrt{\pi n}}(1 + o(1)), \quad n \to \infty. \tag{2.3}$$

Note that the monotonicity of the sequence $(\mathbb{P}[\tau^{*+} > n])_{n\geq 0}$ is crucial to replace the Cesàro means convergence by the usual convergence.

2.4. *Extensions.* Equation (2.3) shows that the asymptotic behavior of $\mathbb{P}[\tau^{*+} > n]$ is in $1/\sqrt{n}$ as $n \to \infty$. As for the probability $\mathbb{P}[\tau^{*+} = n]$, we have the following result, which is proved in Alili and Doney (1999); Èppel (1979).

Proposition 2.5. *Assume that the hypotheses* **AA**, **C** *and* **M**(2) *hold. Then the sequence* $(n^{3/2}\mathbb{P}[\tau^{*+} = n])_{n\geq 0}$ *converges to some positive constant.*

We now refine Proposition 2.5, by adding in the probability the information of the position of the walk at time τ^{*+}. Using the same approach as for Theorem 2.1, we may obtain the following theorem, which we did not find in the literature:

Theorem 2.6. *Assume that the hypotheses* **AA**, **C** *and* **M**(2) *hold. Then for any continuous function* ϕ *with compact support on* $\mathbb{R}$, *one has*

$$\lim_{n\to\infty} n^{3/2}\mathbb{E}[\tau^{*+} = n; \phi(S_n)] = b^{*+}(\phi) := \int_{\mathbb{R}^+} \phi(t) b^{*+}(\mathrm{d}t)$$
$$:= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}^+} \phi(t)\lambda^{*+} * \mu^{*+}(\mathrm{d}t),$$

where λ^{*+} *is the counting measure on* $\mathbb{Z}^{*+}$ *when* **AA**$(\mathbb{Z})$ *holds (resp. the Lebesgue measure on* $\mathbb{R}^{*+}$ *when* **AA**$(\mathbb{R})$ *holds).*

Sketch of the proof of Theorem 2.6 in the case **AA**$(\mathbb{Z})$*:* We shall use the following identity, which as (2.1) is a consequence of the Wiener-Hopf factorization:

$$\psi_\alpha(s) := \sum_{n\geq 0} s^n \mathbb{E}[\tau^{*+} = n; e^{-\alpha S_n}] = 1 - \exp -\widetilde{B}_\alpha(s), \quad \forall s \in [0, 1[, \quad \forall \alpha > 0, \tag{2.4}$$

where

$$\widetilde{B}_\alpha(s) := \sum_{n\geq 1} \frac{s^n}{n} \mathbb{E}[S_n > 0; e^{-\alpha S_n}].$$

Setting $d_n := \mathbb{E}[\tau^{*+} = n; e^{-\alpha S_n}]$, the same argument as in the proof of Theorem 2.1 (via Lemma 2.2) implies that the sequence $(n^{3/2}d_n)_{n\geq 1}$ is bounded (we notice that in Lemma 2.2, the sequences $(b_n)_{n\geq 0}$ and $(d_n)_{n\geq 0}$ are not necessarily non-negative, so it can be applied in the present situation).

Differentiating the two members of (2.4) with respect to s then yields

$$\psi'_\alpha(s) = \sum_{n\geq 1} n s^{n-1} \mathbb{E}[\tau^{*+} = n; e^{-\alpha S_n}] = (1 - \psi_\alpha(s)) \sum_{n\geq 1} s^{n-1} \mathbb{E}[S_n > 0; e^{-\alpha S_n}],$$

and Theorem 2.6 is thus a consequence of Lemma 2.3, applied with $c_n := \mathbb{E}[S_n > 0; e^{-\alpha S_n}]$, $d_n := \mathbb{1}_{\{n=0\}} - \mathbb{E}[\tau^{*+} = n; e^{-\alpha S_n}]$ and $a_n := n\mathbb{E}[\tau^{*+} = n; e^{-\alpha S_n}]$. □

According to the previous proof, we also have, as $n \to \infty$, the weak convergences below:

$$\begin{aligned} n^{3/2}\mathbb{E}[\tau^{*+} = n; S_n \in \mathrm{d}x] &\longrightarrow b^{*+}(\mathrm{d}x) := (1/\sigma\sqrt{2\pi})\lambda^{*+} * \mu^{*+}, \\ n^{3/2}\mathbb{E}[\tau^{+} = n; S_n \in \mathrm{d}x] &\longrightarrow b^{+}(\mathrm{d}x) := (1/\sigma\sqrt{2\pi})\lambda^{+} * \mu^{+}, \\ n^{3/2}\mathbb{E}[\tau^{*-} = n; S_n \in \mathrm{d}x] &\longrightarrow b^{*-}(\mathrm{d}x) := (1/\sigma\sqrt{2\pi})\lambda^{*-} * \mu^{*-}, \\ n^{3/2}\mathbb{E}[\tau^{-} = n; S_n \in \mathrm{d}x] &\longrightarrow b^{-}(\mathrm{d}x) := (1/\sigma\sqrt{2\pi})\lambda^{-} * \mu^{-}. \end{aligned}$$

3. Main results

In this section we are first interested in the expectation $\mathbb{E}[\tau^{>r} = n; \phi(S_n)]$, for any fixed value of $r > 0$. In Theorem 3.1 we find its asymptotic behavior as $n \to \infty$, for any continuous function ϕ with compact support on $\mathbb{R}$. Then in Proposition 3.3 we take ϕ identically equal to 1, and we prove that the sequence $(n\mathbb{P}[\tau^{>r} = n])_{n\geq 0}$ is bounded. We then consider the expectation $\mathbb{E}[\tau^{>r} > n; \phi(S_n)]$. We first derive its asymptotic behavior as $n \to \infty$, in Theorem 3.4. Finally, in Proposition 3.5 we obtain the asymptotics of the probability $\mathbb{P}[\tau^{>r} > n]$ for large values of n. The theorems stated in Section 3 concern the hitting time $\tau^{>r}$; similar statements (obtained exactly along the same lines) exist for the hitting times $\tau^{\geq r}$, $\tau^{<r}$ and $\tau^{\leq r}$.

Theorem 3.1. *Assume that the hypotheses* **AA**, **C** *and* **M(2)** *hold. Then for any continuous function ϕ with compact support on $]r, \infty[$, one has*

$$\lim_{n\to\infty} n^{3/2}\mathbb{E}[\tau^{>r} = n; \phi(S_n)] = \iint_{\Delta_r} \phi(x+y) U^{*+}(\mathrm{d}x) b^{*+}(\mathrm{d}y) + \iint_{\Delta_r} \phi(x+y) a^{*+}(\mathrm{d}x)\mu^{*+}(\mathrm{d}y),$$

where $\Delta_r := \{(x, y) \in \mathbb{R}^{*+} \times \mathbb{R}^{*+} : 0 \leq x \leq r, x + y > r\}$.

Proof: Since ϕ has compact support in $]r, \infty[$, one has

$$\begin{aligned} \mathbb{E}[\tau^{>r} = n; \phi(S_n)] &= \sum_{0\leq k\leq n} \mathbb{E}[\exists \ell \geq 0, T_\ell^{*+} = k, S_k \leq r, n-k = \tau_{\ell+1}^{*+}, S_n > r; \phi(S_n)] \\ &= \sum_{0\leq k\leq n} \iint_{\Delta_r} \phi(x+y)\mathbb{P}[\exists \ell \geq 0, T_\ell^{*+} = k, S_k \in \mathrm{d}x]\times \\ &\qquad\qquad \times \mathbb{P}[\tau^{*+} = n-k, S_{n-k} \in \mathrm{d}y] \\ &= \sum_{0\leq k\leq n} I_{n,k}(r, \phi), \end{aligned}$$

where we have set

$$I_{n,k}(r, \phi) := \iint_{\Delta_r} \phi(x+y)\mathbb{P}[\tau^{-} > k, S_k \in \mathrm{d}x]\mathbb{P}[\tau^{*+} = n-k, S_{n-k} \in \mathrm{d}y]. \quad (3.1)$$

In Equation (3.1) above, we have used the equality $\mathbb{P}[\exists \ell \geq 0, T_\ell^{*+} = k, S_k \in \mathrm{d}x] = \mathbb{P}[\tau^{-} > k, S_k \in \mathrm{d}x]$. It follows by the same arguments as in the proof of Theorem 2.1 (below Lemma 2.3). To pursue the proof, we shall use the following elementary result (see Le Page and Peigné (1997, Lemma II.8) for the original statement and its proof):

Lemma 3.2. *Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences of non-negative real numbers such that $\lim_{n\to\infty} n^{3/2}a_n = a \in \mathbb{R}^{*+}$ and $\lim_{n\to\infty} n^{3/2}b_n = b \in \mathbb{R}^{*+}$. Then:*

- *there exists $C > 0$ such that, for any $n \geq 1$ and any $0 < i < n-j < n$,*

$$n^{3/2} \sum_{i+1\leq k\leq n-j} a_k b_{n-k} \leq C\left(\frac{1}{\sqrt{i}} + \frac{1}{\sqrt{j}}\right);$$

- *setting $A := \sum_{n\geq 0} a_n$ and $B := \sum_{n\geq 0} b_n$, one has*

$$\lim_{n\to\infty} n^{3/2} \sum_{k=0}^{n} a_k b_{n-k} = aB + bA.$$

Since ϕ is non-negative with compact support in $]r,\infty[$, there exists a constant $c_\phi > 0$ such that $\phi(t) \leq c_\phi e^{-t}$, for all $t \geq 0$. This yields that for any $0 < i < n-j < n$,

$$\sum_{i+1\leq k\leq n-j} I_{n,k}(r,\phi) \leq c_\phi \sum_{i+1\leq k\leq n-j} a_k b_{n-k},$$

with $a_k := \mathbb{E}[\tau^- > k; e^{-S_k}]$ and $b_k := \mathbb{E}[\tau^{*+} = k; e^{-S_k}]$. With Lemma 3.2 we deduce that there exists some constant $C > 0$ such that

$$\sum_{i+1\leq k\leq n-j} I_{n,k}(r,\phi) \leq C\left(\frac{1}{\sqrt{i}} + \frac{1}{\sqrt{j}}\right).$$

On the other hand, for any fixed $k \geq 1$ and $x \in [0,r]$, one has by Theorem 2.6

$$\lim_{n\to\infty} n^{3/2} \int_{\{y\geq 0\}} \phi(x+y)\mathbb{P}[\tau^{*+} = n-k, S_{n-k} \in \mathrm{d}y] = \int_{\{y\geq 0\}} \phi(x+y) b^{*+}(\mathrm{d}y).$$

Further, for any $k \geq 1$, the function

$$x \mapsto n^{3/2} \int_{\{y\leq 0\}} \phi(x+y)\mathbb{P}[\tau^{*+} = n-k, S_{n-k} \in \mathrm{d}y]$$

is dominated on $[0,r]$ by $x \mapsto c_\phi(\sup_{n\geq 1} n^{3/2}\mathbb{E}[\tau^{*+} = n-k; e^{-S_{n-k}}])e^{-x}$, which is bounded (by Theorem 2.6) and so integrable with respect to the measure $\mathbb{P}[\tau^- > k, S_k \in \mathrm{d}x]$. The dominated convergence theorem thus yields

$$\lim_{n\to\infty} n^{3/2} \sum_{0\leq k\leq i} I_{n,k}(r,\phi) = \sum_{0\leq k\leq i} \iint_{\Delta_r} \phi(x+y)\mathbb{P}[\tau^- > k, S_k \in \mathrm{d}x] b^{*+}(\mathrm{d}y).$$

The same argument leads to

$$\lim_{n\to\infty} n^{3/2} \sum_{n-j\leq k\leq n} I_{n,k}(r,\phi) = \sum_{0\leq k\leq j} \iint_{\Delta_r} \phi(x+y) a^{*+}(\mathrm{d}x)\mathbb{P}[\tau^{*+} = k, S_k \in \mathrm{d}y].$$

Letting $i, j \to \infty$ and using the equalities

$$\sum_{k\geq 0} \mathbb{E}[\tau^- > k; S_k \in \mathrm{d}x] = U^{*+}(\mathrm{d}x), \qquad \sum_{k\geq 0} \mathbb{E}[\tau^{*+} = k; S_k \in \mathrm{d}y] = \mu^{*+}(\mathrm{d}y),$$

one concludes. □

Proposition 3.3. *Assume that the hypotheses **AA**, **C** and **M(2)** hold. Then for any $r \in \mathbb{R}^+$, the sequence $(n\mathbb{P}[\tau^{>r} = n])_{n\geq 0}$ is bounded.*

Proof: By the proof of Theorem 3.1, one may decompose the probability $\mathbb{P}[\tau^{>r} = n]$ as $\sum_{0\leq k\leq n} I_{n,k}(r,1)$, with $I_{n,k}$ defined in (3.1). One easily obtains that

$$I_{n,k}(r,1) \leq \mathbb{P}[\tau^- > k, S_k \in [0,r]]\mathbb{P}[\tau^{*+} = n-k],$$

Δ_r being defined as in Theorem 3.1. One concludes by applying Remark 2.4 (we obtain the estimate $1/k^{3/2}$ for the first probability above), Proposition 3.3 (we deduce the estimate $1/(n-k)^{3/2}$ for the second probability) and Lemma 3.2. □

We now pass to the second part of Section 3, which is concerned with the expectation $\mathbb{E}[\tau^{>r} > n; \phi(S_n)]$.

Theorem 3.4. *Assume that the hypotheses* **AA**, **C** *and* **M**(2) *hold. Then for any continuous function* ϕ *with compact support on* $\mathbb{R}$, *one has*

$$\lim_{n\to\infty} n^{3/2}\mathbb{E}[\tau^{>r} > n; \phi(S_n)] = \iint_{D_r} \phi(x+y)U^{*+}(\mathrm{d}x)a^-(\mathrm{d}y) + \iint_{D_r} \phi(x+y)a^{*+}(\mathrm{d}x)U^-(\mathrm{d}y),$$

where $D_r := \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq r, y \leq 0\} = [0,r] \times \mathbb{R}^-$.

We do not write the proof of Theorem 3.4 in full details, for the three following reasons. First, it is similar to that of Theorem 3.1. We just emphasize the unique but crucial difference in the decomposition of the expectation $\mathbb{E}[\tau^{>r} > n; \phi(S_n)]$, namely:

$$\mathbb{E}[\tau^{>r} > n; \phi(S_n)] = \sum_{0\leq k\leq n} \iint_{D_r} \phi(x+y)\mathbb{P}[\tau^- > k, S_k \in \mathrm{d}x]\mathbb{P}[\tau^{*+} > n-k, S_{n-k} \in \mathrm{d}y]. \quad (3.2)$$

The second reason is that Theorem 3.4 is equivalent to Le Page and Peigné (1997, Theorem II.7). Indeed, the event $[\tau^{>r} > n]$ can be written as $[M_n \leq r]$, where $M_n = \max(0, S_1, \ldots, S_n)$. Likewise, Proposition 3.5 below on the asymptotics of $\mathbb{P}[\tau^{>r} > n]$ can be found in Le Page and Peigné (1997). Finally, Theorem 3.4 is also proved in the recent article Doney (2012), see in particular Proposition 11.

Proposition 3.5. *Assume that the hypotheses* **AA**, **C** *and* **M**(2) *hold. One has*

$$\mathbb{P}[\tau^{>r} > n] = \frac{\exp\kappa}{\sqrt{\pi n}} U^{*+}([0,r])(1+o(1)), \quad n \to \infty. \quad (3.3)$$

Proof: By (3.2), the probability $\mathbb{P}[\tau^{>r} > n]$ may be decomposed as $\sum_{0\leq k\leq n} J_{n,k}(r)$, with

$$\begin{aligned} J_{n,k}(r) &= \iint_{D_r} \mathbb{P}[\tau^- > k, S_k \in \mathrm{d}x]\mathbb{P}[\tau^{*+} > n-k, S_{n-k} \in \mathrm{d}y] \\ &= \mathbb{P}[\tau^- > k, S_k \in [0,r]]\mathbb{P}[\tau^{*+} > n-k], \end{aligned}$$

where the domain D_r is defined in Theorem 3.4. One concludes, using the following three facts. Firstly, by Remark 2.4, one has $n^{3/2}\mathbb{P}[\tau^- > n, S_n \in [0,r]] \to a^{*+}([0,r])$ as $n \to \infty$. Secondly, by Equation (2.3), one has $\sqrt{n}\mathbb{P}[\tau^{*+} > n] \to e^\kappa/\sqrt{\pi}$ as $n \to \infty$. Thirdly, one has $\sum_{n\geq 0} \mathbb{P}[\tau^- > n, S_n \in [0,r]] = U^{*+}([0,r])$, also thanks to Remark 2.4. □

Remark 3.6. Theorem 3.1 (for which $r > 0$) formally implies Theorem 2.6 ($r = 0$). To see this, it is enough to check that for $r = 0$, the constant in the asymptotics of $\mathbb{E}[\tau^{>r} = n; \phi(S_n)]$ coincides with the one in the asymptotics of $\mathbb{E}[\tau^{*+} = n; \phi(S_n)]$. To that purpose, we first notice that for $r = 0$, the domain Δ_r degenerates in $\{0\} \times \mathbb{R}^{*+}$. Furthermore, $U^{*+}(0) = 1$ and $a^{*+}(0) = 0$. Accordingly,

$$\iint_{\Delta_r} \phi(x+y) U^{*+}(\mathrm{d}x) a^{-}(\mathrm{d}y) + \iint_{\Delta_r} \phi(x+y) a^{*+}(\mathrm{d}x) U^{-}(\mathrm{d}y) = \int_{\mathbb{R}^{*+}} \phi(y) b^{*+}(\mathrm{d}y).$$

In the right-hand side of the equation above, $\mathbb{R}^{*+}$ can be replaced by $\mathbb{R}^{+}$, as $b^{*+}(0) = 0$. We then obtain the right constant in Theorem 2.6. Likewise, we could see that Theorem 3.4 formally implies Theorem 2.1.

4. Applications to random walks on $\mathbb{R}^+$ with non-elastic reflection at 0

In this section we consider a sequence $(Y_i)_{i\geq 1}$ of i.i.d. random variables defined on a probability space $(\Omega, \mathcal{T}, \mathbb{P})$, and we define the random walk $(X_n)_{n\geq 0}$ on $\mathbb{R}^+$ with non-elastic reflection at 0 (or absorbed at 0) recursively, as follows:

$$X_{n+1} := \max(X_n + Y_{n+1}, 0), \quad \forall n \geq 0,$$

where X_0 is a given $\mathbb{R}^+$-valued random variable. The process $(X_n)_{n\geq 0}$ is a Markov chain on $\mathbb{R}^+$. We obviously have that for all $n \geq 0$, $X_{n+1} = f_{Y_{n+1}}(X_n)$, with

$$f_y(x) := \max(x + y, 0), \quad \forall x, y \in \mathbb{R}.$$

The chain $(X_n)_{n\geq 0}$ is thus a random dynamical system; we refer the reader to Peigné and Woess (2011a,b) for precise notions and for a complete description of recurrence properties of such Markov processes.

The profound difference between this chain and the classical random walk $(S_n)_{n\geq 0}$ on $\mathbb{Z}$ or $\mathbb{R}$ is due to the reflection at 0. We therefore introduce the successive absorption times $(\mathbf{a}_\ell)_{\ell\geq 0}$:

$$\begin{aligned} \mathbf{a_0} &:= 0, \\ \mathbf{a} = \mathbf{a}_1 &:= \inf\{n > 0 : X_0 + Y_1 + \cdots + Y_n < 0\}, \\ \mathbf{a}_\ell &:= \inf\{n > \mathbf{a}_{\ell-1} : Y_{\mathbf{a}_{\ell-1}+1} + \cdots + Y_{\mathbf{a}_{\ell-1}+n} < 0\}, \quad \forall \ell \geq 2. \end{aligned}$$

Let us assume the first moment condition $\mathbf{M(1)}$ (i.e., that $\mathbb{E}[|Y_1|] < \infty$). If in addition $\mathbb{E}[Y_1] > 0$, the absorption times are not $\mathbb{P}$-a.s. finite, and in this case, the chain is transient. Indeed, one has $X_n \geq X_0 + Y_1 + \cdots + Y_n$, with $Y_1 + \cdots + Y_n \to \infty$, $\mathbb{P}$-a.s. If $\mathbb{E}[Y_1] \leq 0$, all the $\mathbf{a}_\ell$, $\ell \geq 1$, are $\mathbb{P}$-a.s. finite, and the equality $X_{\mathbf{a}_\ell} \mathbb{1}_{\{\mathbf{a}_\ell < \infty\}} = 0$, $\mathbb{P}$-a.s., readily implies that $(X_n)_{n\geq 0}$ visits 0 infinitely often. On the event $[X_0 = 0]$, the first return time of $(X_n)_{n\geq 0}$ at the origin equals τ^-. In the subcase $\mathbb{E}[Y_1] = 0$, it has infinite expectation, and $(X_n)_{n\geq 0}$ is null recurrent. If $\mathbb{E}[Y_1] < 0$, one has $\mathbb{E}[\tau^-] < \infty$, and the chain $(X_n)_{n\geq 0}$ is positive recurrent. In particular, when $\mathbb{E}[Y_1] \geq 0$, for any $x \geq 0$ and any continuous function ϕ with compact support included in $\mathbb{R}^+$, one has

$$\lim_{n\to\infty} \mathbb{E}[\phi(X_n) | X_0 = x] = 0. \tag{4.1}$$

We shall here focus our attention on the speed of convergence in (4.1), by proving the following result:

Theorem 4.1. *Assume that the hypotheses* **AA**, **C** *and* **M(2)** *are satisfied. Then, for any* $x \geq 0$ *and any continuous function* ϕ *with compact support on* $\mathbb{R}^+$*, one has*

$$\lim_{n\to\infty} \sqrt{n}\mathbb{E}[\phi(X_n)|X_0 = x] = \frac{\widetilde{\kappa}}{\sqrt{\pi}} \int_{\mathbb{R}^+} \phi(t)U^+(\mathrm{d}t),$$

where[7]

$$\widetilde{\kappa} := \exp\left(\sum_{n\geq 1} \frac{\mathbb{P}[S_n < 0] - 1/2}{n}\right). \tag{4.2}$$

If $\mathbb{E}[Y_1] > 0$ *and if furthermore* **AA** *and* $\mathbf{M}(\exp^-)$ *hold,*[8] *there exists* $\rho = \rho(\mu) \in]0,1[$ *and a positive constant* $C(\phi)$ *(which can be computed explicitly) such that*

$$\lim_{n\to\infty} \frac{n^{3/2}}{\rho^n}\mathbb{E}[\phi(X_n)|X_0 = x] = C(\phi).$$

Proof: We first assume that $X_0 = 0$. On the event $[T_\ell^{*-} \leq n < T_{\ell+1}^{*-}]$, one has that $X_n = S_n - S_{T_\ell^{*-}}$. It readily follows that

$$\begin{aligned}
&\mathbb{E}[\phi(X_n)|X_0 = 0]\\
&= \sum_{\ell\geq 0} \mathbb{E}[\mathbf{a}_\ell \leq n < \mathbf{a}_{\ell+1}; \phi(X_n)|X_0 = 0]\\
&= \sum_{\ell\geq 0} \mathbb{E}[T_\ell^{*-} \leq n < T_{\ell+1}^{*-}; \phi(X_n)|X_0 = 0]\\
&= \sum_{\ell\geq 0} \mathbb{E}[T_\ell^{*-} \leq n < T_{\ell+1}^{*-}; \phi(S_n - S_{T_\ell^{*-}})]\\
&= \sum_{\ell\geq 0}\sum_{0\leq k\leq n} \mathbb{E}[T_\ell^{*-} = k, Y_{k+1} \geq 0, \ldots, Y_{k+1} + \cdots + Y_n \geq 0; \phi(Y_{k+1} + \cdots + Y_n)]\\
&= \sum_{0\leq k\leq n}\sum_{\ell\geq 0} \mathbb{P}[T_\ell^{*-} = k]\mathbb{E}[Y_{k+1} \geq 0, \ldots, Y_{k+1} + \cdots + Y_n \geq 0; \phi(Y_{k+1} + \cdots + Y_n)].
\end{aligned}$$

Using the fact that for any $k \geq 0$, the events $[T_\ell^{*-} = k]$, $\ell \geq 0$, are pairwise disjoint together with the fact that $\mathcal{L}(Y_1, \ldots, Y_n) = \mathcal{L}(Y_n, \ldots, Y_1)$, one gets

$$\begin{aligned}
\sum_{\ell\geq 0} \mathbb{P}[T_\ell^{*-} = k] &= \mathbb{P}[\exists \ell \geq 0, T_\ell^{*-} = k]\\
&= \mathbb{P}[S_k < 0, S_k < S_1, \ldots, S_k < S_{k-1}] = \mathbb{P}[\tau^+ > k],
\end{aligned}$$

which in turn implies that

$$\mathbb{E}[\phi(X_n)|X_0 = 0] = \sum_{0\leq k\leq n} \mathbb{P}[\tau^+ > k]\mathbb{E}[\tau^{*-} > n - k; \phi(S_{n-k})]. \tag{4.3}$$

The situation is more complicated when the starting point is $x \geq 0$. In that case, one has the decomposition

$$\mathbb{E}[\phi(X_n)|X_0 = x] = E_1(x,n) + E_2(x,n), \tag{4.4}$$

[7] We refer to Footnote 6 for another expression of $\widetilde{\kappa}$.

[8] In fact, it would be sufficient to assume that $\mathbb{E}[e^{\gamma Y_1}] < \infty$ for γ belonging to some interval $[a,0]$, if $[a,0]$ is such that the convex function $\gamma \mapsto \mathbb{E}[e^{\gamma Y_1}]$ reaches its minimum at a point $\gamma_0 \in]a,0[$.

with $E_1(x,n) := \mathbb{E}[\mathbf{a} > n; \phi(X_n)|X_0 = x]$ and $E_2(x,n) := \mathbb{E}[\mathbf{a} \leq n; \phi(X_n)|X_0 = x]$. From the definition of $\mathbf{a}$, one gets $E_1(x,n) = \mathbb{E}[\tau^{<-x} > n; \phi(x+S_n)]$. Similarly, by the Markov property and the fact that $X_{\mathbf{a}} = 0$, $\mathbb{P}$-a.s., one may write

$$E_2(x,n) = \sum_{0\leq \ell \leq n} \mathbb{P}[\tau^{<-x} = \ell]\mathbb{E}[\phi(X_{n-\ell})|X_0 = 0].$$

The centered case. We first assume that hypotheses **AA** and **M(2)** are satisfied and that the $(Y_i)_{i\geq 1}$ are centered (hypothesis **C**). In this case, by fluctuation theory of centered random walks, one gets $\mathbb{P}[\mathbf{a}_\ell < \infty] = 1$ for any $\ell \geq 0$ and any initial distribution $\mathcal{L}(X_0)$.

We first consider the case when $X_0 = 0$ and we use the identity (4.3). By Le Page and Peigné (1997, Theorem II.2) (see also how (2.3) is obtained), one gets

$$\lim_{n\to\infty} \sqrt{n}\mathbb{P}[\tau^+ > n] = \frac{\widetilde{\kappa}}{\sqrt{\pi}},$$

with $\widetilde{\kappa}$ defined in (4.2). On the other hand, by Remark 2.4 in Section 2 we know that

$$\lim_{n\to\infty} n^{3/2}\mathbb{E}[\tau^{*-} > n; \phi(S_n)] = a^+(\phi).$$

We conclude, setting $c_n := \mathbb{P}[\tau^+ > n$, $d_n := \mathbb{E}[\tau^{*-} > n; \phi(S_n)]$, thus $c := \widetilde{\kappa}/\sqrt{\pi}$ and $D := \sum_{n\geq 0}\mathbb{E}[\tau^{*-} > n; \phi(S_n)] = U^+(\phi)$, in Lemma 2.3.

In the general case (when $X_0 = x$), we use identity (4.4). By the results of Section 3 (Theorem 3.4 with $\tau^{<-x}$ instead of $\tau^{>r}$), one gets $E_1(x,n) = O(n^{-3/2})$.[9] On the other hand, by the Markov property, since $X_{\mathbf{a}} = 0$, $\mathbb{P}$-a.s., one has

$$\begin{aligned} E_2(x,n) &= \sum_{0\leq k\leq n} \mathbb{E}[\mathbf{a} = k; \phi(X_n)|X_0 = x] \\ &= \sum_{0\leq k\leq n} \mathbb{P}[\mathbf{a} = k|X_0 = x]\mathbb{E}[\phi(X_{n-k})|X_0 = 0] \\ &= \sum_{0\leq k\leq n} \mathbb{P}[\tau^{<-x} = k]\mathbb{E}[\phi(X_{n-k})|X_0 = 0]. \end{aligned}$$

Recall that $\lim_{n\to\infty}\sqrt{n}\mathbb{E}[\phi(X_n)|X_0 = 0] = (\widetilde{\kappa}/\sqrt{\pi})U^+(\phi)$; on the other hand, it follows from Proposition 3.3 (with $\tau^{<-x}$ instead of $\tau^{>r}$) that $(n\mathbb{P}[\tau^{<-x} = n])_{n\geq 0}$ is bounded. Furthermore, $\sum_{n\geq 1}\mathbb{P}[\tau^{<-x} = n] = \mathbb{P}[\tau^{<-x} < \infty] = 1$. One may thus apply Lemma 2.3, which yields

$$\lim_{n\to\infty}\sqrt{n}\mathbb{E}[\phi(X_{n-k})|X_0 = x] = \lim_{n\to\infty}\sqrt{n}E_2(x,n) = \frac{\widetilde{\kappa}}{\sqrt{\pi}}U^+(\phi).$$

The non-centered case. Hereafter, we assume that hypotheses $\mathbf{M}(1)$, $\mathbf{M}(\exp^-)$ and **AA** hold, and that in addition $\mathbb{E}[Y_1] > 0$. We use the standard relativisation procedure that we now recall: the function

$$\widehat{\mu}(\gamma) := \mathbb{E}[e^{\gamma Y_1}]$$

is well defined on $\mathbb{R}^-$, tends to ∞ as $\gamma \to -\infty$, and has derivative $\mathbb{E}[Y_1] > 0$ at 0. It thus achieves its minimum at a point $\gamma_0 < 0$, and we have $\rho := \widehat{\mu}(\gamma_0) \in]0,1[$. The measure

$$\widetilde{\mu}(\mathrm{d}x) := (1/\rho)e^{\gamma_0 x}\mu(\mathrm{d}x)$$

[9]Notice that in the preceding formula, $O(n^{-3/2})$ depends on x.

is a probability on $\mathbb{R}$. Furthermore, if $(\widetilde{Y}_i)_{i\geq 1}$ is a sequence of i.i.d. random variables with law $\widetilde{\mu}$ and $(\widetilde{S}_n)_{n\geq 1}$ is the corresponding random walk on $\mathbb{R}$ starting from 0, one gets

$$\mathbb{E}[\varphi(Y_1,\ldots,Y_n)] = \rho^n \mathbb{E}[\varphi(\widetilde{Y}_1,\ldots,\widetilde{Y}_n)e^{-\gamma_0 \widetilde{S}_n}]$$

for any $n \geq 1$ and any bounded test Borel function $\varphi : \mathbb{R}^n \to \mathbb{R}$. Denoting by $\widetilde{\tau}^+$ and $\widetilde{\tau}^{*-}$ the first entrance times of $(\widetilde{S}_n)_{n\geq 1}$ in $\mathbb{R}^+$ and $\mathbb{R}^{*-}$, respectively, we may thus write (4.3) as

$$\mathbb{E}[\phi(X_n)|X_0 = 0] = \rho^n \sum_{0\leq k\leq n} \mathbb{E}[\widetilde{\tau}^+ > k; e^{-\gamma_0 \widetilde{S}_k}]\mathbb{E}[\widetilde{\tau}^{*-} > n-k; \phi(\widetilde{S}_{n-k})e^{-\gamma_0 \widetilde{S}_{n-k}}],$$

and by Lemma 3.2 the sequence $((n^{3/2}/\rho^n)\mathbb{E}[\phi(X_n)|X_0 = x])_{n\geq 0}$ converges to some constant $C(\phi) > 0$.

Following the same way, for any $x \geq 0$ one can decompose as above $\mathbb{E}[\phi(X_n)|X_0 = x]$ as $E_1(x,n) + E_2(x,n)$, with

$$\begin{aligned} E_1(x,n) &= \rho^n \mathbb{E}[\widetilde{\tau}^{<-x} > n; \phi(\widetilde{S}_n)e^{-\gamma_0 \widetilde{S}_{n-k}}], \\ E_2(x,n) &= \sum_{0\leq k\leq n} \rho^k \mathbb{E}[\widetilde{\tau}^{<-x} = k; e^{-\gamma_0 \widetilde{S}_k}]\mathbb{E}[\phi(X_n)|X_0 = 0]. \end{aligned}$$

One concludes using Section 3 (Theorem 3.1 with $\tau^{<-x}$ instead of $\tau^{>r}$) for the behavior of the sequence $(\mathbb{E}[\widetilde{\tau}^{<-x} = n; e^{-\gamma_0 \widetilde{S}_n}])_{n\geq 0}$ and the previous estimation for the behavior of $(\mathbb{E}[\phi(X_n)|X_0 = 0])_{n\geq 0}$. □

5. Local limit theorems and links with results by Denisov and Wachtel

Hereafter, we shall assume that $\mathbf{AA}(\mathbb{Z})$ holds; in particular, the random walk $(S_n)_{n\geq 0}$ is $\mathbb{Z}$-valued. Taking $\phi(S_n) = \mathbb{1}_{\{S_n = i\}}$, Theorem 3.4 immediately leads to:

Corollary 5.1. *Assume that the hypotheses* $\mathbf{AA}(\mathbb{Z})$, $\mathbf{C}$ *and* $\mathbf{M}(2)$ *hold. Then for* $i \leq r$,

$$\mathbb{P}[\tau^{>r} > n, S_n = i] = \frac{Z(r,i)}{n^{3/2}}(1 + o(1)), \quad n \to \infty,$$

where we have set

$$Z(r,i) = \sum_{\max\{i,0\}\leq k\leq r} [a^-(i-k)U^{*+}(k) + U^-(i-k)a^{*+}(k)]. \tag{5.1}$$

It is worth noting that the definition of a^- implies that for $y \in \mathbb{Z}^{*+}$, $a^-(y) = 0$, and for $y \in \mathbb{Z}^-$,

$$a^-(y) = \frac{1}{\sigma\sqrt{2\pi}} \sum_{n\geq 0} \mathbb{E}[\tau^{*+} > n; S_n \in [y, 0]].$$

Likewise, for $y \in \mathbb{Z}^-$, $a^{*+}(y) = 0$, and for $y \in \mathbb{Z}^{*+}$,

$$a^{*+}(y) = \frac{1}{\sigma\sqrt{2\pi}} \sum_{n\geq 0} \mathbb{E}[\tau^- > n; S_n \in]0, y]].$$

Remark 5.2. Using these facts and similar remarks for the potentials U^{*+} and U^-, we obtain that the quantity (5.1) can also be written as a sum of two convolution terms:

$$Z(r,i) = \sum_{-\infty<k<r} [a^-(i-k)U^{*+}(k) + U^-(i-k)a^{*+}(k)] \tag{5.2}$$

$$= \sum_{-\infty<k<\infty} [a^-(i-k)U^{*+}(k)\mathbb{1}_{\{k\le r\}} + U^-(i-k)a^{*+}(k)\mathbb{1}_{\{k\le r\}}]. \tag{5.3}$$

In the remaining of this section we compare the local limit theorem of Corollary 5.1 with the one in Denisov and Wachtel (2011). All results taken from Denisov and Wachtel (2011) make the assumptions that the $(Y_i)_{i\ge 1}$ have moments of order $2+\epsilon$, with $\epsilon > 0$. To state the local limit theorem Denisov and Wachtel (2011, Theorem 7), we need to introduce the function (see Denisov and Wachtel (2011, Section 2.4) for more details)

$$V(x) := -\mathbb{E}[S_{\tau^{\le -x}}] = -\mathbb{E}[S_{\tau^{<-x+1}}]. \tag{5.4}$$

This function is positive on $\mathbb{R}^+$ and is harmonic for the random walk $(S_n)_{n\ge 0}$ killed when reaching $\mathbb{R}^-$; it means that for $x > 0$,

$$\mathbb{E}[V(x+Y_1); \tau^{<-x} > 1] = V(x).$$

Define V' as the harmonic function for the random walk with increments $(-Y_i)_{i\ge 1}$ with the same construction as (5.4). We have the following result:

Theorem 5.3 (Denisov and Wachtel (2011))**.** *Assume that the hypotheses* $\mathbf{AA}(\mathbb{Z})$, $\mathbf{C}$ *and* $\mathbf{M}(2+\epsilon)$ *hold. Then for* $i \le r$,

$$\mathbb{P}[\tau^{>r} > n, S_n = i] = \frac{1}{\sigma}\sqrt{\frac{2}{\pi}}\frac{V'((r+1)/\sigma)V((r+1-i)/\sigma)}{n^{3/2}}(1+o(1)), \quad n\to\infty.$$

Proof: Theorem 7 in Denisov and Wachtel (2011) states that if $(\widetilde{S}_n)_{n\ge 0}$ is a random walk on a lattice $h\mathbb{Z}$ starting from 0 and with increments $(\widetilde{Y}_i)_{i\ge 0}$ having a variance equal to 1, the following local limit theorem holds:

$$\mathbb{P}[x+\widetilde{S}_n = y, \tau^{\le -x} > n] = h\sqrt{\frac{2}{\pi}}\frac{V(x)V'(y)}{n^{3/2}}(1+o(1)), \quad n\to\infty.$$

Applying this result to the random walk $(\widetilde{S}_n)_{n\ge 0} := (-S_n/\sigma)_{n\ge 0}$, and letting $x := (r+1)/\sigma$ and $y := (r+1-i)/\sigma$, we obtain Theorem 5.3. ☐

By Corollary 5.1 and Theorem 5.3, we must have

$$Z(r,i) = \frac{1}{\sigma}\sqrt{\frac{2}{\pi}}V'((r+1)/\sigma)V((r+1-i)/\sigma). \tag{5.5}$$

However:

Question 1. It is an open problem to show by a direct computation that (5.5) holds.

To conclude Section 5, we prove (5.5) for the simple random walk, with probabilities of transition $\mathbb{P}[Y_i = -1] = \mathbb{P}[Y_i = 1] = p$ and $\mathbb{P}[Y_i = 0] = 1-2p$. In this case the harmonic functions have the simple form $V(x) = V'(x) = x$, and obviously $\sigma = \sqrt{2p}$. We deduce that the constant in Theorem 5.3 is

$$\frac{(r+1)(r+1-i)}{2p^{3/2}\sqrt{\pi}}. \tag{5.6}$$

To compute $Z(r,i)$, we start from the formulation (5.1), where we assume that $i \geq 0$ (the computation for $i < 0$ would be similar). We recall that for the simple random walk one has $U^{*+}(k) = \mathbb{1}_{\{k\geq 0\}}$ and $U^{-}(k) = \mathbb{1}_{\{k\leq 0\}}/p$. Then for $k \leq 0$, $a^{-}(k) = (|k|+1)/(p\sigma\sqrt{2\pi})$ and for $k \geq 0$, $a^{*+}(k) = k/(\sigma\sqrt{2\pi})$. We deduce that

$$Z(r,i) = \frac{1}{p\sigma\sqrt{2\pi}} \sum_{i\leq k\leq r} [(k-i+1)+k].$$

It is then an easy exercise to show that $Z(r,i)$ equals (5.6).

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