The compensation approach for walks with small steps in the quarter plane

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November 27, 2012

Abstract

This paper is the first application of the compensation approach (a well-established theory in probability theory) to counting problems. We discuss how this method can be applied to a general class of walks in the quarter plane $\mathbb{Z}_+^2$ with a step set that is a subset of $\{(-1,1),(-1,0),(-1,-1),(0,-1),(1,-1)\}$ in the interior of $\mathbb{Z}_+^2$. We derive an explicit expression for the generating function which turns out to be nonholonomic, and which can be used to obtain exact and asymptotic expressions for the counting numbers.

Keywords: lattice walks in the quarter plane, compensation approach, holonomic functions

AMS 2000 *Subject Classification*: 05A15, 05A16

1 Introduction

In the field of enumerative combinatorics, counting walks on a lattice is among the most classical topics. While counting problems have been largely resolved for unrestricted walks on $\mathbb{Z}^2$, walks that are confined to the quarter plane $\mathbb{Z}_+^2$ still pose considerable challenges. In recent years, much progress has been made, in particular for walks in the quarter plane with small steps, which means that the step set $\mathcal{S}$ is a subset of $\{(i,j) : |i|,|j| \leqslant 1\} \setminus \{(0,0)\}$. Bousquet-Mélou and Mishna [7] constructed a thorough classification of these models. By definition, there are 2^8 such models, but after eliminating trivial cases and exploiting equivalences, it is shown in [7] that there are 79 inherently different problems that need to be studied. Let $q_{i,j,k}$ denote the number of paths in $\mathbb{Z}_+^2$ having length k, starting from $(0,0)$ and ending in (i,j), and define the generating function (GF) as

$$Q(x,y;z) = \sum_{i,j,k=0}^{\infty} q_{i,j,k} x^i y^j z^k. \tag{1}$$

There are then two key challenges:

(i) Finding an explicit expression for $Q(x,y;z)$.

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(ii) Determining the nature of $Q(x,y;z)$: is it holonomic (the vector space over $\mathbb{C}(x,y,z)$—the field of rational functions in the three variables x,y,z—spanned by the set of all derivatives of $Q(x,y;z)$ is finite dimensional, see [11, Appendix B.4])? And in that event, is it algebraic, or even rational?

The common approach to address these challenges is to start from a functional equation for the GF, which for the walks with small steps takes the form (see [7, Section 4])

$$K(x,y;z)Q(x,y;z) = A(x)Q(x,0;z) + B(y)Q(0,y;z) - \delta Q(0,0;z) - xy/z, \tag{2}$$

where K, A and B are polynomials of degree two in x and/or y, and δ is a constant. For $z = 1/|\mathcal{S}|$, (2) belongs to the generic class of functional equations (arising in the probabilistic context of *random* walks) studied and solved in the book [8]. For general values of z, the analysis of (2) for the 79 above-mentioned models has been carried out in [13, 18], which settled Challenge (i).

In order to describe the results regarding Challenge (ii), it is worth to define the *group of the walk*, a notion introduced by Malyshev [15]. This is the group of birational transformations $W = \langle \xi, \eta \rangle$, with

$$\xi(x,y) = \left(x, \frac{1}{y} \frac{\sum_{(i,-1)\in\mathcal{S}} x^i}{\sum_{(i,+1)\in\mathcal{S}} x^i} \right), \quad \eta(x,y) = \left(\frac{1}{x} \frac{\sum_{(-1,j)\in\mathcal{S}} y^j}{\sum_{(+1,j)\in\mathcal{S}} y^j}, y \right), \tag{3}$$

which leaves invariant the function $\sum_{(i,j)\in\mathcal{S}} x^i y^j$. Clearly, $\xi \circ \xi = \eta \circ \eta = \mathrm{id}$, and W is a dihedral group of even order larger than or equal to four.

Challenge (ii) is now resolved for all 79 problems. It was first solved for the 23 models that have a finite group. The nature of the GF (as a function of the three variables x,y,z) was determined in [7] for 22 of these 23 models: 19 models turn out to have a GF that is holonomic but nonalgebraic, while 3 models have a GF that is algebraic. For the 23rd model, defined by $\mathcal{S} = \{(-1,0),(-1,-1),(1,0),(1,1)\}$ and known as *Gessel's walk*, it was proven in [5] that $Q(x,y;z)$ is algebraic. Alternative proofs for the nature of the GF for these 23 problems were given in [9]. For the remaining 56 models, which all have an infinite group, it was first shown that 5 of them (called singular) have a nonholonomic GF, see [16, 17]. Bousquet-Mélou and Mishna [7] have conjectured that the 51 other models also have a nonholonomic GF. Partial evidence was provided in [6, 18], and the proof was recently given in [14].

In this paper we consider walks on $\mathbb{Z}_+^2$ with small steps that do not fall into the class considered in [7]. The classification in [7] builds on the assumption that the steps on the boundaries are the same steps (if possible) as those in the interior. However, when the behavior on the boundaries is allowed to be different, we have many more models to consider. There are several motivations to permit arbitrary jumps on the boundary. First, this allows us to extend the existing results, and thus to have a better understanding of the combinatorics of the walks confined to a quarter plane. Second, some studies in probability theory (see, e.g., [12]) suggest that the behavior of the walk on the boundary has a deep and interesting influence on the quantities of interest. In this paper we first consider the walk in Figure 1, with steps taken from $\mathcal{S} = \{(-1,1),(-1,-1),(1,-1)\}$ in the interior, $\mathcal{S}_H = \{(-1,1),(-1,0),(1,0)\}$ on the horizontal boundary, $\mathcal{S}_V = \{(0,1),(0,-1),(1,-1)\}$ on the vertical boundary, and $\mathcal{S}_0 = \{(0,1),(1,0)\}$ in state $(0,0)$.

Notice that the step set $\mathcal{S} = \{(-1,1),(-1,-1),(1,-1)\}$ in the framework of [7] would render a trivial walk, since the walk could never depart from state $(0,0)$. However, by choosing the steps

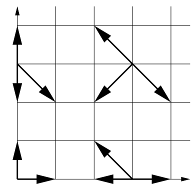

Figure 1: The jumps of the walk

on the boundaries as in $\mathcal{S}_H$, $\mathcal{S}_V$ and $\mathcal{S}_0$, it becomes possible to start walking from state $(0,0)$, and we have a rather intricate counting problem on our hands.

It turns out that our walk has an infinite group. To see this, observe that the interior step set in Figure 1 and the one represented in Figure 2 have isomorphic groups ([7, Lemma 2] says that two step sets differing by one of the eight symmetries of the square have isomorphic groups), and notice that the group associated with the step set in Figure 2 is infinite ([7, Section 3.1]).

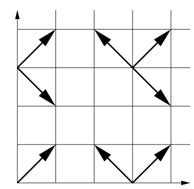

Figure 2: Walk considered in [17]

Since our walk has an infinite group, the approach in [7] cannot be applied. The methods developed in [17] also fail to work. Indeed, the main tool used there is an expression of $Q(x,0;z)$ and $Q(0,y;z)$ as series involving the iterates of the roots of the kernel. While these series are convergent when the transitions $(-1,0),(-1,-1),(0,-1)$ are absent, they become strongly divergent in our case, see [8, Chapter 6]. The approach via boundary value problems of [8] seems to apply, but this is more cumbersome than in [13, 18]. Indeed, contrary to the 79 models studied there, for which in (2), A and B depend on one variable, and δ is constant, these quantities are now polynomials in two variables, since the functional equation (2) for our walk becomes (for a proof, see Equations (12)–(17) in Section 2)

$$K(x,y;z)Q(x,y;z) = [1+x^2-x^2y-y]Q(x,0;z)+[1+y^2-xy^2-x]Q(0,y;z)+[x+y-1]Q(0,0;z)-xy/z \tag{4}$$

with the kernel

$$K(x,y;z) = 1 + x^2 + y^2 - xy/z. \tag{5}$$

We shall derive an explicit expression for $Q(x,y;z)$, which turns out to be a meromorphic function of x, y with infinitely many poles in x and in y. This implies that $Q(x,y;z)$ is nonholonomic as a function of x, and as a function of y. We note that for one variable, a function is holonomic if and only if it satisfies a linear differential equation with polynomial coefficients (see [11, Appendix B.4]), so that a holonomic function must have finitely many poles, since the latter are found among the zeros of the polynomial coefficients of the underlying differential equation. Accordingly, the trivariate function $Q(x,y;z)$ is nonholonomic as well, since the holonomy is stable by specialization of a variable (see [11]). As in [14, 18], this paper therefore illustrates the intimate relation between the infinite group case and nonholonomy.

The technique we are using is the so-called *compensation approach*. This technique has been developed in a series of papers [1, 3, 4] in the probabilistic context of *random* walks; see [2] for an overview. It does not aim directly at obtaining a solution for the GF, but rather tries to find a solution for its coefficients

$$q_{i,j}(z) = \sum_{k=0}^{\infty} q_{i,j,k} z^k. \tag{6}$$

These coefficients satisfy certain recursion relations, which differ depending on whether the state (i,j) lies on the boundary or not. The idea is then to express $q_{i,j}(z)$ as a linear combination of products $\alpha^i \beta^j$, for pairs (α, β) such that

$$K(1/\alpha, 1/\beta; z) = 0. \tag{7}$$

By choosing only pairs (α,β) for which (7) is satisfied, the recursion relations for $q_{i,j}(z)$ in the interior of the quarter plane are satisfied by any linear combination of products $\alpha^i\beta^j$, by virtue of the linearity of the recursion relations. The products have to be chosen such that the recursion relations on the boundaries are satisfied as well. As it turns out, this can be done by alternatingly compensating for the errors on the two boundaries, which eventually leads to an infinite series of products.

This paper is organized as follows. In Section 2 we obtain an explicit expression for the generating function $Q(x,y;z)$ by applying the compensation approach. In Section 3 we derive an asymptotic expression for the coefficients $q_{i,j,k}$ for large values of k, using the technique of singularity analysis. Because this paper is the first application of the compensation approach to counting problems, we also shortly discuss in Section 4 for which classes of walks this compensation approach might work.

2 The compensation approach

We start from the following recursion relations:

$$q_{i,j,k+1} = q_{i-1,j+1,k} + q_{i+1,j-1,k} + q_{i+1,j+1,k}, \qquad i,j \geqslant 1, k \geqslant 0, \tag{8}$$

$$q_{i,0,k+1} = q_{i-1,1,k} + q_{i+1,1,k} + q_{i-1,0,k} + q_{i+1,0,k}, \qquad i \geqslant 1, k \geqslant 0, \tag{9}$$

$$q_{0,j,k+1} = q_{1,j-1,k} + q_{1,j+1,k} + q_{0,j-1,k} + q_{0,j+1,k}, \qquad j \geqslant 1, k \geqslant 0, \tag{10}$$

$$q_{0,0,k+1} = q_{0,1,k} + q_{1,1,k} + q_{1,0,k}, \qquad k \geqslant 0. \tag{11}$$

Since $q_{i,j,0} = 0$ if $i+j>0$ and $q_{0,0,0} = 1$, these relations uniquely determine all the counting numbers $q_{i,j,k}$. Multiplying the relations (8)–(11) by z^k and summing w.r.t. $k \geqslant 0$ leads to (with the generating functions $q_{i,j} = q_{i,j}(z)$ defined in (6))

$$q_{i,j}/z = q_{i-1,j+1} + q_{i+1,j-1} + q_{i+1,j+1}, \qquad i,j \geqslant 1, \tag{12}$$

$$q_{i,0}/z = q_{i-1,1} + q_{i+1,1} + q_{i-1,0} + q_{i+1,0}, \qquad i \geqslant 2, \tag{13}$$

$$q_{0,j}/z = q_{1,j-1} + q_{1,j+1} + q_{0,j-1} + q_{0,j+1}, \qquad j \geqslant 2, \tag{14}$$

and

$$q_{1,0}/z = q_{0,1} + q_{2,1} + q_{0,0} + q_{2,0}, \tag{15}$$

$$q_{0,1}/z = q_{1,0} + q_{1,2} + q_{0,0} + q_{0,2}, \tag{16}$$

$$q_{0,0}/z = 1/z + q_{0,1} + q_{1,1} + q_{1,0}. \tag{17}$$

For reasons that will become clear below, we want to discuss separately the equations involving $q_{0,0}$, and we therefore do not merge (15) (resp. (16)) and (13) (resp. (14).

Lemma 1. *Equations (12)–(17) have a unique solution in the form of formal power series.*

Proof. Substituting the power series $q_{i,j}$ into Equations (12)–(17), and equating coefficients of z^k, yields Equations (8)–(11). The latter obviously have a unique solution for the coefficients $q_{i,j,k}$, because these counting numbers can be determined recursively using $q_{0,0,0} = 1$. □

In order to find the unique solution for $q_{i,j}$, we shall employ the compensation approach, which consists of three steps:

- Characterize all products $\alpha^i\beta^j$ for which the inner equations (12) are satisfied, and construct linear combinations of these products, which in addition to being formal solutions to (12), also satisfy (13) and (14).
- Prove that these solutions are formal power series.
- Determine the complete unique solution $q_{i,j}$ by taking into account the boundary conditions (15)–(17).

2.1 Linear combinations of products

Substituting the product $\alpha^i\beta^j$ into the inner equations (12), and dividing by common powers, yields

$$\alpha\beta/z = \beta^2 + \alpha^2 + \alpha^2\beta^2. \tag{18}$$

Incidentally, note that (18) is nothing else but (7) (see (5)). Hence, a product $\alpha^i\beta^j$ is a solution of (12) if and only if (18) is satisfied, and any linear combination of such products will satisfy (12). Figure 3 depicts the curve (18) in $\mathbb{R}_+^2$ for $z = 1/4$.

Now we construct a linear combination of the products introduced above, which will give a formal solution to the equations (12)–(14). The first term of this combination, say $\alpha_0^i\beta_0^j$, we require to satisfy both (12) and (13). In other words, the pair (α_0, β_0) has to satisfy (18) as well as

$$\alpha\beta/z = \beta^2 + \alpha^2\beta^2 + \beta + \alpha^2\beta. \tag{19}$$

The motivation to start with a term satisfying both (18) and (19) will be explained at the end of this subsection, see Remark 5.

Lemma 2. *There exists a unique pair (α_0, β_0), namely*

$$\alpha_0 = \frac{1 - \sqrt{1 - 8z^2}}{4z}, \quad \beta_0 = \frac{\alpha_0^2}{1 + \alpha_0^2}, \tag{20}$$

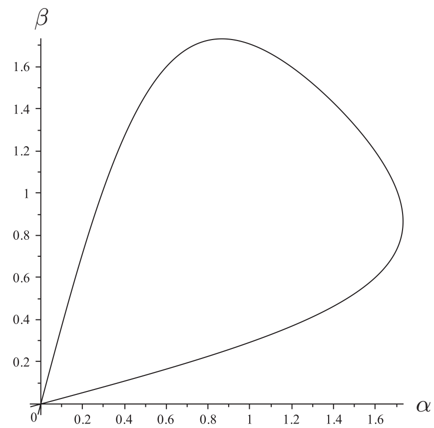

Figure 3: The curve (18) in $\mathbb{R}_+^2$ for $z = 1/4$.

of formal power series in z such that the product $\alpha_0^i \beta_0^j$ satisfies both (12) and (13).

Proof. Equation (19) yields

$$\alpha/z = \beta + \alpha^2\beta + 1 + \alpha^2 = (1+\beta)(1+\alpha^2). \tag{21}$$

Subtracting (19) from (18) gives $\beta(1+\alpha^2) = \alpha^2$, so that $(1+\beta)(1+\alpha^2) = 1 + 2\alpha^2$, and together with (21) this gives $\alpha/z = 1 + 2\alpha^2$. The last equation has the two solutions $(1 \pm \sqrt{1-8z^2})/(4z)$, and only the solution with the $-$ sign is a formal power series in z. □

The function $\alpha_0^i \beta_0^j$ thus satisfies (12) and (13), but it fails to satisfy (14). Indeed, if $\alpha^i \beta^j$ is a solution to (14), then (α, β) should satisfy

$$\alpha\beta/z = \alpha^2 + \alpha^2\beta^2 + \alpha + \alpha\beta^2, \tag{22}$$

and (α_0, β_0) is certainly not a solution to (22).

Now we start adding compensation terms. We consider $c_0\alpha_0^i\beta_0^j + d_1\alpha^i\beta^j$, where (α, β) satisfies (18) and is such that $c_0\alpha_0^i\beta_0^j + d_1\alpha^i\beta^j$ satisfies (14). The identity (14), which has to be true for any $j \geqslant 2$, forces us to take $\beta = \beta_0$. Then, thanks to (18), we see that α is the conjugate root of α_0 in (18). Denote this new root by α_1. Hence, α_0 and α_1 are the two roots of Equation (18), where β is replaced by β_0. In other words, α_0 and α_1 satisfy

$$(1+\beta_0^2)\alpha^2 - (\beta_0/z)\alpha + \beta_0^2 = 0. \tag{23}$$

Due to the root-coefficient relationships, we obtain

$$\frac{\beta_0/z}{1+\beta_0^2} = \alpha_0 + \alpha_1. \tag{24}$$

We now determine d_1 in terms of c_0. Note that $c_0\alpha_0^i\beta_0^j + d_1\alpha_1^i\beta_0^j$ is a solution to (14) if and only if

$$(\beta_0/z)(c_0 + d_1) = (\alpha_0 c_0 + \alpha_1 d_1)(1+\beta_0^2) + (c_0+d_1)(1+\beta_0^2). \tag{25}$$

The latter identity can be rewritten as

$$\frac{\beta_0/z}{1+\beta_0^2}(c_0+d_1) = (\alpha_0 c_0 + \alpha_1 d_1) + (c_0 + d_1), \tag{26}$$

and thus, using (24),

$$d_1 = -\frac{1-\alpha_1}{1-\alpha_0} c_0. \tag{27}$$

The series $c_0\alpha_0^i\beta_0^j + d_1\alpha_1^i\beta_0^j$ after one compensation step satisfies (12) and (14) for the interior and the vertical boundary. However, the compensation term $d_1\alpha_1^i\beta_0^j$ has generated a new error at the horizontal boundary. To compensate for this, we must add another compensation term, and so on. In this way, the compensation approach can be continued, which eventually leads to

$$x_{i,j} = \underbrace{\overbrace{c_0\alpha_0^i\beta_0^j}^{\text{H}} + \overbrace{d_1\alpha_1^i\beta_0^j}^{\text{H}}}_{\text{V}} + \underbrace{\overbrace{c_1\alpha_1^i\beta_1^j}^{\text{H}} + \overbrace{d_2\alpha_2^i\beta_1^j}^{\text{H}}}_{\text{V}} + c_2\alpha_2^i\beta_2^j + \cdots \tag{28}$$

where, by construction and (18), for all $k \geqslant 0$ we have

$$\beta_k = f(\alpha_k), \quad \alpha_{k+1} = f(\beta_k), \quad f(t) = \frac{1-\sqrt{1-4z^2(1+t^2)}}{2z(1+t^2)} t, \tag{29}$$

and the function $f(t)$ follows from (18). We shall prove in Subsection 2.2 the convergence of the series (28) as a formal power series in z. The construction is such that each term in (28) satisfies the equations in the interior of the state space, the sum of two terms with the same α satisfies the horizontal boundary conditions (H) and the sum of two terms with the same β satisfies the vertical boundary conditions (V). Hence $x_{i,j}$ satisfies all equations (12)–(14). See also Figure 4. Furthermore,

$$d_{k+1} = -\frac{1-\alpha_{k+1}}{1-\alpha_k} c_k, \quad c_{k+1} = -\frac{1-\beta_{k+1}}{1-\beta_k} d_{k+1}, \quad k \geqslant 0. \tag{30}$$

An easy calculation starting from (30) yields

$$d_{k+1} = -\frac{(1-\alpha_{k+1})(1-\beta_k)}{(1-\alpha_0)(1-\beta_0)} c_0, \quad c_{k+1} = \frac{(1-\beta_{k+1})(1-\alpha_{k+1})}{(1-\alpha_0)(1-\beta_0)} c_0, \quad k \geqslant 0, \tag{31}$$

so that choosing (arbitrarily) $c_0 = (1-\alpha_0)(1-\beta_0)$ finally gives

$$x_{i,j} = \sum_{k=0}^{\infty} (1-\beta_k)\beta_k^j [(1-\alpha_k)\alpha_k^i - (1-\alpha_{k+1})\alpha_{k+1}^i]. \tag{32}$$

By symmetry, we also obtain that $x_{j,i}$ is a formal solution to (12)–(14). This leaves to consider the solution for the states $(0,1)$, $(0,0)$ and $(1,0)$, i.e., the boundary conditions (15)–(17).

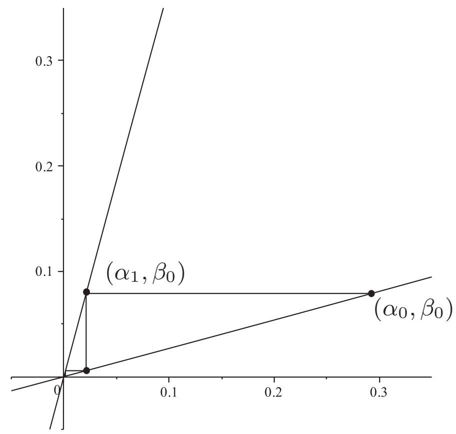

Figure 4: The curve (18) near 0 for $z = 1/4$, and the construction of the products: the α_k and β_k are given by the intersections of the curve (18) with the broken line.

2.2 Formal power series

Proposition 3. *The sequences $\{\alpha_k\}_{k\geqslant 0}$ and $\{\beta_k\}_{k\geqslant 0}$ appearing in (28)–(29) satisfy the following property: for all $k \geqslant 0$, $\alpha_k = z^{2k+1}\widehat{\alpha}_k(z)$ and $\beta_k = z^{2k+2}\widehat{\beta}_k(z)$, where $\widehat{\alpha}_k$ and $\widehat{\beta}_k$ are formal power series such that $\widehat{\alpha}_k(0) = \widehat{\beta}_k(0) = 1$.*

Proof. For f defined in (29), all $p \geqslant 0$, and all real numbers $s_1, s_2, \ldots$,

$$f(z^p[1 + s_1 z + s_2 z^2 + \cdots]) = z^{p+1}[1 + s_1 z + (1 + s_2)z^2 + \cdots]. \tag{33}$$

Because (20) yields $\alpha_0 = z + 2z^3 + \cdots$, the proof is completed via (29). □

The following result is an immediate consequence of Proposition 3.

Corollary 4. *For $i, j \geqslant 0$, $x_{i,j}$ and $x_{j,i}$ defined in (32) are formal power series in z.*

Remark 5. The approach outlined in Subsection 2.1 is initialized with a term satisfying both (18) and (19). Alternatively, we could start with an arbitrarily chosen term with power series α_0 and β_0 satisfying (18) only. This term would violate (19) as well as (22), and therefore generate two sequences of terms, one starting with compensation of (19) and the other with (22). It is readily seen that in one of the two sequences, some of the parameters α or β will not be power series. Indeed, the two sequences of terms would be generated as in (29), with instead of f the function

$$f_\pm(t) = \frac{1 \pm \sqrt{1 - 4z^2(1 + t^2)}}{2z(1 + t^2)} t. \tag{34}$$

In Subsection 2.1, we already described the construction of the sequence with the choice $f_- = f$. For $f = f_+$, notice that for all $p \geqslant 0$, and all real numbers $s_1, \ldots$, we have

$$f_+(z^p[1 + s_1 z + \cdots]) = z^{p-1}[1 + s_1 z + \cdots]. \tag{35}$$

Hence, the resulting solution constructed as in (29) would fail to be a formal power series, cf. Equation (32) and Subsection 2.3.

2.3 Determining the unique solution

We now determine the generating functions $q_{i,j}$. Define

$$\widehat{x}_{i,j} = x_{i,j} + x_{j,i}, \tag{36}$$

so that in particular

$$\widehat{x}_{0,0} = 2\sum_{k=0}^{\infty}(1-\beta_k)(\alpha_{k+1}-\alpha_k). \tag{37}$$

Proposition 6. *The expressions*

$$q_{0,0} = \frac{1+\widehat{x}_{0,0}}{1-2z+z\widehat{x}_{0,0}} \tag{38}$$

and

$$q_{i,j} = c\widehat{x}_{i,j}, \quad i+j>0, \tag{39}$$

with

$$c = \frac{1}{1-2z+z\widehat{x}_{0,0}}, \tag{40}$$

are the unique solutions to Equations (12)–(17) *in the form of formal power series.*

Proof. We shall look for a solution satisfying, for $i+j>0$,

$$q_{i,j} = cx_{i,j} + \widetilde{c}x_{j,i}. \tag{41}$$

By symmetry we must have $c = \widetilde{c}$. Such combinations have been shown to satisfy (12)–(14), which do not involve $q_{0,0}$. We still need to determine the solution for which also the boundary conditions (15) (or by symmetry (16)) and (17) are satisfied. This can be achieved by choosing c and $q_{0,0}$ appropriately.

By using (17) and (15) (or (17) and (16)), we obtain that

$$q_{0,0} = 1 + zc[\widehat{x}_{0,1} + \widehat{x}_{1,1} + \widehat{x}_{1,0}] \tag{42}$$

and

$$c = \frac{z}{\widehat{x}_{1,0} - z[\widehat{x}_{0,1} + \widehat{x}_{2,1} + \widehat{x}_{2,0}] - z^2[\widehat{x}_{1,0} + \widehat{x}_{0,1} + \widehat{x}_{1,1}]}. \tag{43}$$

In addition, tedious calculations starting from (32) give

$$\widehat{x}_{1,0}/z - [\widehat{x}_{0,1} + \widehat{x}_{2,1} + \widehat{x}_{2,0} + \widehat{x}_{0,0}] = 1, \quad (1/z-1)\widehat{x}_{0,0} + 2 - [\widehat{x}_{1,0} + \widehat{x}_{0,1} + \widehat{x}_{1,1}] = 0. \tag{44}$$

Thanks to the equations in (44), the denominator of (43) can be written as $z[1-2z+z\widehat{x}_{0,0}]$. The functions defined in (38) and (39) are the solutions to Equations (12)–(17) by construction. Finally, the uniqueness follows from Lemma 1. □

Proposition 7. *We have*

$$Q(x,y;z) = \sum_{i,j=0}^{\infty} q_{i,j} x^i y^j \tag{45}$$

$$= c + c\sum_{k=0}^{\infty}\left(\frac{1-\beta_k}{1-\beta_k y}\left[\frac{1-\alpha_k}{1-\alpha_k x} - \frac{1-\alpha_{k+1}}{1-\alpha_{k+1}x}\right] + \frac{1-\beta_k}{1-\beta_k x}\left[\frac{1-\alpha_k}{1-\alpha_k y} - \frac{1-\alpha_{k+1}}{1-\alpha_{k+1}y}\right]\right), \tag{46}$$

with $q_{i,j}$ *as in Proposition 6,* c *as in (40), and* α_k *and* β_k *as in (20) and (29), respectively. Furthermore,*

$$Q(0,0;z) = c(1+\widehat{x}_{0,0}), \quad Q(1,0;z) = Q(0,1;z) = c(1-\alpha_0(z)), \quad Q(1,1;z) = c. \tag{47}$$

Proof. This is immediate from (32), (37)–(40). □

2.4 Retrieving coefficients

Now that we have an explicit expression for $q_{i,j}$, we briefly present an efficient procedure for calculating its coefficients $q_{i,j,k}$.

To compute $q_{i,j}$ we need to calculate in principle an infinite series, or to solve the recurrence relations. However, we shall prove that if we are only interested in a finite number of coefficients $q_{i,j,k}$, then it is enough to take into account a finite number of α_k and β_k. For $k \geqslant 0$, denote by $k \vee 1$ the maximum of k and 1. Define

$$N_p^{i,j} = 1 + \left\lfloor \tfrac{1}{4}\max\{p - (i\vee 1 + 2(j\vee 1)), p - (2(i\vee 1) + j \vee 1)\}\right\rfloor. \tag{48}$$

Proposition 8. *For any* $i,j \geqslant 0$*, the first* p *coefficients of* $q_{i,j}$ *only require the series expansions of order* p *of* $\alpha_0, \beta_0, \ldots \alpha_{N_p^{i,j}}, \beta_{N_p^{i,j}}$*.*

Proof. In order to obtain the series expansion of $q_{0,0}$ of order p, it is enough to know the series expansion of $\widehat{x}_{0,0}$ of order p, see (38). From (32) we obtain

$$\widehat{x}_{0,0} = 2\sum_{k=0}^{\infty}(1-\beta_k)(\alpha_{k+1}-\alpha_k) = -2\alpha_0 + 2\sum_{k=0}^{\infty}\beta_k(\alpha_k - \alpha_{k+1}). \tag{49}$$

With Proposition 3, $\beta_k\alpha_k = O(z^{4k+3})$ and $\beta_k\alpha_{k+1} = O(z^{4k+5})$, so that in order to obtain the series expansion of $\widehat{x}_{0,0}$ of order p, it is enough to consider in (49) the values of k such that $4k+3 \leqslant p$. In other words, we have to deal with $\alpha_0, \beta_0, \ldots, \alpha_k, \beta_k, \alpha_{k+1}$ for $4k+3 \leqslant p$. Since $N_p^{0,0} = 1 + \lfloor (p-3)/4 \rfloor$, see (48), Proposition 8 is shown for $i=j=0$. The proof for other values of i and j is similar and therefore omitted. □

As an application of Proposition 8, let us find the numbers $q_{0,0,k}$ for $k \in \{0,\ldots,10\}$. Taking $i = j = 0$, we have $N_{10}^{0,0} = 2$; we then calculate the series expansions of order 10 of $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2$. After using (38) and (49), we obtain the numbers:

$$1, 0, 2, 2, 10, 16, 64, 126, 454, 1004, 3404. \tag{50}$$

3 Asymptotic analysis

In this section we derive an asymptotic expression for the coefficients $q_{i,j,k}$ for large values of k, using the technique of *singularity analysis* (see Flajolet and Sedgewick [11] for an elaborate exposition). This requires the investigation of the function $q_{i,j} = q_{i,j}(z)$ near its dominant singularity (closest to the origin) in the z-plane. First, notice that

$$\sum_{i,j=0}^{\infty} q_{i,j,k} \leqslant 3^k, \tag{51}$$

because there are at most 3^k paths of length k. This clearly implies that $q_{i,j,k} \leqslant 3^k$, so that $q_{i,j}$ is analytic at least for $|z| < 1/3$, and the singularities z of $q_{i,j}$ must satisfy $|z| \geqslant 1/3$.

In fact, the singularities of $q_{i,j}$ will be found among the singularities of $\widehat{x}_{i,j}$ and the zeros of the denominator of c in (40), see (38) and (39) . In what follows, we first show in Subsection 3.1 that $\widehat{x}_{i,j}$ converges in the domain $|z| \leqslant 1/\sqrt{8}$. We then find in Subsection 3.2 the first singularity of $q_{i,j}$. Finally, we state and prove our asymptotic results in Subsection 3.3.

3.1 Convergence of the solutions

Proposition 9. *The sequences $\{\alpha_k\}_{k\geqslant 0}$ and $\{\beta_k\}_{k\geqslant 0}$ appearing in (28)–(29) satisfy the following properties:*

(i) *For any $k \geqslant 0$, α_k and β_k are analytic in the disc with center at 0 and radius $1/\sqrt{8}$.*

(ii) *For all $|z| \leqslant 1/\sqrt{8}$, we have $1/\sqrt{2} \geqslant |\alpha_0| > |\beta_0| > |\alpha_1| > |\beta_1| > \cdots$.*

(iii) *For all $|z| \leqslant 1/\sqrt{8}$ and all $k \geqslant 0$, we have $|\alpha_k| \leqslant 1/\sqrt{2}^{2k+1}$ and $|\beta_k| \leqslant 1/\sqrt{2}^{2k+2}$.*

Proof. We first prove (ii) and (iii). To that purpose, it is enough, thanks to (29) and since $\alpha_0 = f(1)$, to prove that for all $|t| \leqslant 1$ and $|z| \leqslant 1/\sqrt{8}$,

$$|f(t)| \leqslant \frac{|t|}{\sqrt{2}}. \tag{52}$$

For this, we notice that as soon as $4|z|^2|1+t^2| \leqslant 1$ (in particular for $|t| \leqslant 1$ and $|z| \leqslant 1/\sqrt{8}$), we have

$$\frac{f(t)}{t} = \sum_{k=0}^{\infty} C_n z^{2n+1}(1+t^2)^n, \tag{53}$$

where $C_n = \binom{2n}{n}/(n+1)$ is the nth Catalan number, and thus

$$\left|\frac{f(t)}{t}\right| \leqslant \sum_{n=0}^{\infty} C_n |z|^{2n+1}(1+|t|^2)^n \leqslant \sum_{n=0}^{\infty} C_n |z|^{2n+1} 2^n = \frac{1-\sqrt{1-8|z|^2}}{4|z|}. \tag{54}$$

For $|z| \leqslant 1/\sqrt{8}$, the last quantity is less than $1/\sqrt{2}$, and (52) is proved.

We now show (i). To that aim we remark the following: let γ be a power series in z. If

(a) the radius of convergence of γ is larger than or equal to $1/\sqrt{8}$,

(b) $\max_{|z|\leqslant 1/\sqrt{8}} |\gamma| \leqslant 1$,

then with f as in (29), $f(\gamma)$ is a power series satisfying (a) and (b). (This lemma is an obvious consequence of the expression (29) of function f.)

The function α_0 defined in (20) clearly satisfies (a) and (b). Further, the functions $\alpha_0, \beta_0, \alpha_1, \beta_1, \ldots$ follow from the iterative scheme (29), so that with (ii) (or with (iii)), we obtain that (a) and (b) hold. We then conclude to (i). □

Corollary 10. *For $i, j \geqslant 0$, $x_{i,j}$ and $x_{j,i}$ defined in (32) are convergent for $|z| \leqslant 1/\sqrt{8}$. Further, for each $|z| \leqslant 1/\sqrt{8}$,*

$$\sum_{i,j=0}^{\infty} |x_{i,j}(z)| < \infty. \tag{55}$$

Proof. The first statement in Corollary 10 immediately follows from Proposition 9 and from the expression for $x_{i,j}$. Next, use (32) to get that for $i+j>0$,

$$|x_{i,j}| \leqslant \sum_{k=0}^{\infty} |1-\beta_k||\beta_k|^j \left[|1-\alpha_k||\alpha_k|^i + |1-\alpha_{k+1}||\alpha_{k+1}|^i\right]. \tag{56}$$

Using then that $|1-\beta_k|$, $|1-\alpha_k|$ and $|1-\alpha_{k+1}|$ are bounded, see Proposition 9(ii), and applying Proposition 9(iii) gives that there is some constant constant $C>0$ such that for all $i+j>0$,

$$|x_{i,j}| \leqslant C \sum_{k=0}^{\infty} 1/\sqrt{2}^{(2k+1)i+(2k+2)j}. \tag{57}$$

Likewise, we can prove that for some constant $C>0$,

$$|x_{0,0}| \leqslant C \sum_{k=0}^{\infty} 1/\sqrt{2}^{(2k+1)}. \tag{58}$$

The second claim in Corollary 10 immediately follows. □

We conclude this section with the following statement.

Proposition 11. *For all $0<|z|<1/\sqrt{8}$, the functions $\sum_{i,j=0}^{\infty} q_{i,j} x^i y^j$, $\sum_{i=0}^{\infty} q_{i,0} x^i$ and $\sum_{j=0}^{\infty} q_{0,j} y^j$ have infinitely many poles.*

Proof. This is a direct consequence of Propositions 3, 7 and 9, and of Corollary 10. □

It follows immediately from Proposition 11 that both $\sum_{i=0}^{\infty} q_{i,0} x^i$ and $\sum_{j=0}^{\infty} q_{0,j} y^j$ are nonholonomic. The holonomy being maintained after specialization to a variable (see [11]), we reach the conclusion that the trivariate generating function $Q(x,y;z)$ is nonholonomic as well.

3.2 Dominant singularity of $q_{i,j}$

A consequence of the results in Subsection 3.1 is that $q_{i,j}$ is analytic for $|z|<1/\sqrt{8}$, except possibly at points z such that the denominator of c in (40) equals 0. Denote this denominator by

$$h(z) = 1 - 2z + z\widehat{x}_{0,0}. \tag{59}$$

Lemma 12. *The series $h(z)$ has a unique root ρ in the interval $(1/3, 1/\sqrt{8})$, which is the dominant singularity of c.*

The proof of Lemma 12 is presented in Appendix A. Furthermore, in Corollary 22 we shall prove that $\rho \in [0.34499975, 0.34499976]$

Remark 13. It seems hard to find a closed-form expression for ρ or to decide whether ρ is algebraic or not. From this point of view, the situation is quite different from that for the 74 (nonsingular) models of walks discussed in Section 1. Indeed, for all these 74 models, the analogue of ρ (which eventually will be the first positive singularity of $Q(0,0;z)$, $Q(1,0;z)$, $Q(0,1;z)$ and $Q(1,1;z)$, see below) is always algebraic (of some degree between 1 and 7), see [10].

Remark 14. In fact, it will be proved in Subsection 3.3 that the series $q_{i,j}$, $Q(0,0;z)$, $Q(1,0;z)$, $Q(0,1;z)$ and $Q(1,1;z)$ are analytic within the domain

$$\{z \in \mathbb{C} : |z| < (1+\epsilon)\rho\} \setminus [\rho, (1+\epsilon)\rho), \tag{60}$$

for $\epsilon > 0$ small enough, with a unique pole at ρ.

3.3 Asymptotic results

Here are our main results on the asymptotic behavior of large counting numbers:

Proposition 15. *The asymptotics of the numbers of walks $q_{i,j,k}$ as $k \to \infty$ is*

$$q_{i,j,k} \sim C_{i,j}\rho^{-k}, \tag{61}$$

with ρ as defined in Lemma 12,

$$C_{0,0} = \frac{3\rho - 1}{-\rho^2 h'(\rho)}, \tag{62}$$

and

$$C_{i,j} = \frac{\widehat{x}_{i,j}(\rho)}{-\rho h'(\rho)}, \quad i + j \geqslant 1. \tag{63}$$

Proof. Consider first the case $i = j = 0$. Thanks to (38), (59) and Lemma 12, we obtain that $q_{0,0}$ has a pole at ρ, and is analytic within the domain (60) for any $\epsilon > 0$ small enough. Moreover, the pole of $q_{0,0}$ at ρ is of order one, as we show now. For this it is sufficient to prove that $1 + \widehat{x}_{0,0}(\rho) \neq 0$ and that $h'(\rho) \neq 0$, see again (38) and (59). In this respect, note that with (59) and Lemma 12 we have $h(\rho) = 1 - 2\rho + \rho\widehat{x}_{0,0}(\rho) = 0$, so that $1 + \widehat{x}_{0,0}(\rho) = (3\rho - 1)/\rho$, which is positive by Lemma 12. On the other hand, it is a consequence of Lemma 24 that $h'(\rho) \neq 0$. In particular, the behavior of $q_{0,0}$ near ρ is given by

$$q_{0,0} = \frac{1 + \widehat{x}_{0,0}(\rho)}{h'(\rho)(z - \rho)[1 + O(z - \rho)]} = \frac{3\rho - 1}{-\rho^2 h'(\rho)(1 - z/\rho)}[1 + O(z - \rho)]. \tag{64}$$

The analyticity of $q_{0,0}$ within the domain (60) and the behavior (64) of $q_{0,0}$ near ρ immediately give the asymptotics (61) for $i = j = 0$ (see, e.g., [11, Chapter 6]). The proof for other values of i and j is similar and therefore omitted. ☐

A result similar to Proposition 15 also holds for the total number of walks of length k (not necessarily ending in state $(0,0)$). Indeed, from formula (47), we obtain that

$$Q(1,0;z) = \frac{1-\alpha_0(z)}{h(z)}, \quad Q(1,1;z) = \frac{1}{h(z)}, \tag{65}$$

with $h(z)$ as in (59). Accordingly, the exponential growth rate of these generating functions is the same as that of $Q(0,0;z)$, namely ρ (defined in Lemma 12). An identical reasoning as in the proof of Proposition 15 then gives the following result.

Proposition 16. *The asymptotics of the total number of walks* $\sum_{i,j=0}^{\infty} q_{i,j,k}$ *as* $k \to \infty$ *is*

$$\sum_{i,j=0}^{\infty} q_{i,j,k} \sim \frac{1}{-\rho h'(\rho)} \rho^{-k}. \tag{66}$$

The asymptotics of the total number of walks ending on the horizontal axis (or, equivalently, on the vertical axis) $\sum_{i=0}^{\infty} q_{i,0,k}$ *as* $k \to \infty$ *is*

$$\sum_{i=0}^{\infty} q_{i,0,k} \sim \frac{1-\alpha_0(\rho)}{-\rho h'(\rho)} \rho^{-k}. \tag{67}$$

Remark 17. A priori, the fact that

$$Q(0,0;z), \quad Q(1,0;z), \quad Q(0,1;z), \quad Q(1,1;z) \tag{68}$$

have the same first positive singularity (namely, ρ) is far from obvious. This appears to be related to the fact that the two coordinates of the drift vector of our $\mathcal{S}$ are negative (they both equal -2). Here, for any step set $\mathcal{S}$, the drift vector is given by $(\sum_{-1\leqslant i\leqslant 1} \delta_{i,j}^{\mathcal{S}}, \sum_{-1\leqslant j\leqslant 1} \delta_{i,j}^{\mathcal{S}})$, where $\delta_{i,j}^{\mathcal{S}} = 1$ if $(i,j) \in \mathcal{S}$, and 0 otherwise. Indeed, the following phenomenon has been observed in [10] for the 79 models of walks mentioned in Section 1: if the two coordinates of the drift vector are negative, then the four functions (68) do have the same first positive singularity.

Let us illustrate Proposition 15 with the following table, obtained by approximating ρ by 0.34499975 and $C_{0,0}$ by 0.0531, see Corollary 22 and Lemma 24 in Appendix A.

Value of k	Exact value of $q_{0,0,k}$ (see Section 2.4)	Approximation of $q_{0,0,k}$ (see Proposition 15)	Ratio
10	$3.404 \cdot 10^{3}$	$2.222 \cdot 10^{3}$	0.653
20	$1.106 \cdot 10^{8}$	$9.305 \cdot 10^{7}$	0.840
30	$4.254 \cdot 10^{12}$	$3.895 \cdot 10^{12}$	0.915
40	$1.714 \cdot 10^{17}$	$1.630 \cdot 10^{17}$	0.951
50	$7.037 \cdot 10^{21}$	$6.825 \cdot 10^{21}$	0.969
60	$2.913 \cdot 10^{26}$	$2.857 \cdot 10^{26}$	0.980
70	$1.211 \cdot 10^{31}$	$1.196 \cdot 10^{31}$	0.987
80	$5.051 \cdot 10^{35}$	$5.007 \cdot 10^{35}$	0.991
90	$2.109 \cdot 10^{40}$	$2.096 \cdot 10^{40}$	0.993
100	$8.814 \cdot 10^{44}$	$8.774 \cdot 10^{44}$	0.995

4 Discussion

4.1 A wider range of applicability

The compensation approach for our counting problem leads to an exact expression for the generating function $Q(x, y; z)$. A detailed exposition of the compensation approach can be found in [1, 3, 4], in which it has been shown to work for two-dimensional random walks on the lattice of the first quadrant that obey the following conditions:

- Step size: Only transitions to neighboring states.
- Forbidden steps: No transitions from interior states to the North, North-East, and East.
- Homogeneity: The same transitions occur for all interior points, and similarly for all points on the horizontal boundary, and for all points on the vertical boundary.

Although the theory has been developed for the *bivariate* transform of stationary distributions of recurrent two-dimensional random walks, this paper shows that is can also be applied to obtain a series expression for the *trivariate* function $Q(x, y; z)$. The fact that $Q(x, y; z)$ has one additional variable does not seem to matter much. We therefore expect that the compensation approach will work for walks in the quarter plane that obey the three conditions mentioned above. This will be a topic for future research. Another topic is to see whether the above conditions can be relaxed, and in fact, in the next section we present an example of a walk, not satisfying these properties, yet amenable to the compensation approach.

4.2 Another example

We consider the walk with $\mathcal{S} = \{(-1, 0), (-1, -1), (0, -1), (1, -1), (1, 0)\}$ in the interior, $\mathcal{S}_H = \{(-1, 0), (1, 0)\}$ on the horizontal boundary, $\mathcal{S}_V = \{(0, 1), (0, -1), (1, -1), (1, 0)\}$ on the vertical boundary, and $\mathcal{S}_0 = \{(0, 1), (1, 0)\}$; see the left display in Figure 5. This walk has a rather special behavior, as it can only move upwards on the vertical boundary. By simple enumeration, we get the following recursion relations:

$$q_{i,j}/z = q_{i-1,j} + q_{i-1,j+1} + q_{i,j+1} + q_{i+1,j} + q_{i+1,j+1}, \qquad i, j > 0, \tag{69}$$

$$q_{i,0}/z = q_{i-1,0} + q_{i-1,1} + q_{i,1} + q_{i+1,0} + q_{i+1,1}, \qquad i > 0, \tag{70}$$

$$q_{0,j}/z = q_{0,j-1} + q_{0,j+1} + q_{1,j} + q_{1,j+1}, \qquad j > 0, \tag{71}$$

$$q_{0,0}/z = 1/z + q_{0,1} + q_{1,1} + q_{1,0}. \tag{72}$$

Notice that Equations (69) and (69) could be merged into a single equation, valid for $i > 0$ and $j \geq 0$, and that this simplification is, in part, responsible for the simplicity of the solution that follows.

Proposition 18. *The unique solution to Equations (69)–(72) is given by $q_{i,j} = c_0 \alpha_0^i \beta_0^j$, with β_0 defined as the unique formal power series in z and solving*

$$\beta/z = 1 + \beta^2 + \beta^2(1 + \beta)^2, \tag{73}$$

$\alpha_0 = \beta_0(1+\beta_0)$ and $c_0 = \beta_0/z$. Therefore, the GF is algebraic (and even rational in x, y) and given by

$$Q(x,y;z) = \sum_{i,j,k=0}^{\infty} q_{i,j,k} x^i y^j z^k = \frac{c_0}{(1-\alpha_0 x)(1-\beta_0 y)}. \tag{74}$$

Proof. Let us first construct all pairs (α, β) such that $c\alpha^i\beta^j$ solves (69), (70) and (71). We obtain that α and β must satisfy

$$\alpha\beta/z = \beta + \beta^2 + \alpha\beta^2 + \alpha^2\beta + \alpha^2\beta^2, \tag{75}$$

$$\alpha\beta/z = \alpha + \alpha\beta^2 + \alpha^2\beta + \alpha^2\beta^2. \tag{76}$$

(Note that the reason why we obtain two and not three equations is that Equation (70) and (69) can be merged into a single equation.) Further, all pairs (α, β) that satisfy both (75) and (76) are such that $\beta(1+\beta) = \alpha$. Substituting the latter into (75) gives (73). Denote by β_k, $k \in \{0, \dots, 3\}$, the four roots of (73). Among the β_k, only one, say β_0, is a unique formal power series in z, see [20, Proposition 6.1.8]. Letting

$$\alpha_0 = \beta_0(1+\beta_0), \tag{77}$$

we conclude that for any c_0, $c_0\alpha_0^i\beta_0^j$ is a solution to (69), (70) and (71). With the particular choice $c_0 = 1/[1 - z(\alpha_0 + \alpha_0\beta_0 + \beta_0)]$, $c_0\alpha_0^i\beta_0^j$ is also solution to (72). By uniqueness of the solution to the recursion relations (see Lemma 1), we conclude that $q_{i,j} = c_0\alpha_0^i\beta_0^j$. Finally, by (73) and (77), we can simplify c_0 into β_0/z. □

Figure 5: Two walks with different boundary behavior

We now modify the boundary behavior of this walk and we show that this has severe consequences for the GF. Consider again the walk with $\mathcal{S} = \{(-1,0), (-1,-1), (0,-1), (1,-1), (1,0)\}$ in the interior, but this time with $\mathcal{S}_V = \{(0,-1), (1,-1), (1,0)\}$ on the vertical boundary, $\mathcal{S}_0 = \{(1,0)\}$, and the following rather special step set on the horizontal boundary: if the walk is in state $(i,0)$ with $i > 0$, the possible steps are $\{(-1,0), (1,0)\}$ and a *big* step to $(0,i)$; see the right display in Figure 5. We should note that a *random* walk, with similar unusual boundary behavior, has already been considered in [19]. The equations for $q_{i,j}$ are then given by (69), (70), (72) and

$$q_{0,j}/z = q_{j,0} + q_{0,j+1} + q_{1,j} + q_{1,j+1}, \quad j > 0. \tag{78}$$

Proposition 19. *Let*

$$g(t) = \frac{1 - tz - \sqrt{(1-tz)^2 - 4z^2(1+t)^2}}{2z(1+t)}, \tag{79}$$

and define the sequences α_k, β_k and c_k by

$$\alpha_k = g(\beta_k), \quad \beta_{k+1} = \alpha_k, \quad c_{k+1} = c_k \frac{\alpha_{k+1}}{1+\beta_{k+1}}, \quad k \geqslant 0. \tag{80}$$

starting with $\beta_0 = 0$ and

$$1/c_0 = \sum_{k=0}^{\infty} \frac{\beta_2 \cdots \beta_{k+1}}{(1+\beta_1)\cdots(1+\beta_k)} [1 - z(\beta_k + \beta_{k+1} + \beta_k \beta_{k+1})] \tag{81}$$

The unique solution to Equations (69), (70), (72) and (78) is given by

$$q_{i,j} = \sum_{k=0}^{\infty} c_k \alpha_k^i \beta_k^j. \tag{82}$$

Therefore, the GF is given by

$$Q(x, y; z) = \sum_{k=0}^{\infty} \frac{c_k}{(1-\alpha_k x)(1-\beta_k y)}, \tag{83}$$

which has infinitely many poles in x and y, and is therefore nonholonomic as a trivariate function.

Proof. We start searching for a solution of the form (82) with $\alpha_k = g(\beta_k)$. Since Equation (69) for the interior and Equation (70) for the horizontal boundary are then fulfilled, we only have to consider Equation (78) for the vertical boundary, which reads

$$\begin{aligned} \sum_{k=0}^{\infty} c_k \alpha_k^j &= \sum_{k=0}^{\infty} c_k \beta_k^j (1/z - \beta_k - \alpha_k - \alpha_k \beta_k) \\ &= c_0 \beta_0^j (1/z - \beta_0 - \alpha_0 - \alpha_0 \beta_0) + \sum_{k=0}^{\infty} c_{k+1} \beta_{k+1}^j (1/z - \beta_{k+1} - \alpha_{k+1} - \alpha_{k+1}\beta_{k+1}). \end{aligned}$$

Then an obvious (formal) solution is given by (80).

We now determine c_0 so as to satisfy the remaining Equation (72). With (80), that we have proved above, we obtain

$$\frac{c_k}{c_0} = \frac{\beta_2 \cdots \beta_{k+1}}{(1+\beta_1)\cdots(1+\beta_k)}, \quad k \geqslant 0. \tag{84}$$

Further, from (72) we have

$$\frac{c_0}{z} \sum_{k=0}^{\infty} \frac{c_k}{c_0} = \frac{1}{z} + c_0 \sum_{k=0}^{\infty} \frac{c_k}{c_0} (\alpha_k + \beta_k + \alpha_k \beta_k). \tag{85}$$

Together with (84), we obtain (81). The α_k and β_k converge to the unique formal power series β_* solution to $g(\beta_*) = \beta_*$. Using (79), we obtain that

$$\beta_*^3 + 2\beta_*^2 + \beta_*(1 - 1/z) + 1 = 0. \tag{86}$$

The series β_* is such that $\beta_*(0) = 0$. It remains to show that $q_{i,j}$ defined in (82) is a (well-defined) formal power series. There is here an interesting phenomenon: for the general compensation approach of Section 2, $q_{i,j}$ was a formal power series because $\alpha_k = z^{2k+1}\widehat{\alpha}_k$ and $\beta_k = z^{2k+2}\widehat{\beta}_k$, where $\widehat{\alpha}_k$ and $\widehat{\beta}_k$ are formal power series (see Proposition 3). But here, we have $\alpha_k = z\widehat{\alpha}_k$ and $\beta_k = z\widehat{\beta}_k$, where $\widehat{\alpha}_k$ and $\widehat{\beta}_k$ are formal power series not vanishing at 0 (this follows from (79) and (80)). On the other hand, we have $c_k = z^k\widehat{c}_k$, where $\widehat{c}_k$ is a formal power series. Therefore, in the particular case of Proposition 19, $q_{i,j}$ is a formal power series thanks to the constants c_k (see (80) and (82)). The nonholonomy of the trivariate function $Q(x, y; z)$ follows by the same arguments as that given below Proposition 11. □

Acknowledgments. We are grateful to an anonymous referee for numerous valuable and constructive comments. The work of IA and JvL was supported by an ERC Starting Grant and a Free Competition grant from the Netherlands Organisation for Scientific Research. The work of KR was partially supported by CRC 701, Spectral Structures and Topological Methods in Mathematics at the University of Bielefeld.

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A Remaining proofs

Let us start by expressing h, defined in (59), as an alternating sum. The reason why we wish to formulate h differently is twofold. First, this will enable us to show that $\rho \in [0.34499975, 0.34499976]$, see Corollary 22—and in fact, in a similar way, we can approximate ρ up to any level of precision. Also, this is actually a key lemma for proving Lemma 12.

Lemma 20.

$$h(z) = 1 + 2z\left(- 1 - \alpha_0 + \sum_{k=0}^{\infty} (-1)^k T_k \right), \quad T_k = \begin{cases} \beta_{k/2}\alpha_{k/2} & \text{if } k \text{ is even,} \\ \beta_{(k-1)/2}\alpha_{(k+1)/2} & \text{if } k \text{ is odd.} \end{cases} \tag{87}$$

Proof. Equation (87) follows immediately from (49) and (59). □

As a preliminary result, we also need the following refinement of Proposition 9:

Lemma 21. *Assume that $z \in (0, 1/\sqrt{8})$. Then both sequences $\{\alpha_k\}_{k\geqslant 0}$ and $\{\beta_k\}_{k\geqslant 0}$ are positive and decreasing. Moreover, for all $k \geqslant 0$*

$$0 \leqslant \alpha_k \leqslant 1/\sqrt{2}^{2k+1}, \quad 0 \leqslant \beta_k \leqslant 1/\sqrt{2}^{2k+2}. \tag{88}$$

Proof. It is enough to prove that for all $|t| \leqslant 1$ and $z \in (0, 1/\sqrt{8})$,

$$0 \leqslant f(t) \leqslant \frac{t}{\sqrt{2}}. \tag{89}$$

The proof of this inequality is similar to that of (52) and therefore omitted. □

Corollary 22. *The series $h(z)$ defined in (59) or (87) has a radius of convergence at least $1/\sqrt{8}$. Its first positive zero ρ lies in $[0.34499975, 0.34499976]$.*

Proof. For $z \in (0, 1/\sqrt{8})$, the sequence $\{T_k\}_{k \geqslant 0}$ defined in Lemma 20 is positive and decreasing, see Lemma 21. In particular, denoting

$$\Lambda^p = \sum_{k=0}^{p} (-1)^k T_k \tag{90}$$

and using (87), for all $p \geqslant 0$ and all $z \in (0, 1/\sqrt{8})$ we have

$$1 + 2z(-1 - \alpha_0 + \Lambda^{2p+1}) < h(z) < 1 + 2z(-1 - \alpha_0 + \Lambda^{2p}). \tag{91}$$

Applying the last inequalities to $p = 4$ and noting that the right-hand side (resp. left-hand side) evaluated at 0.34499976 (resp. 0.34499975) is negative (resp. positive) concludes the proof. □

Before doing the proof of Lemma 12, let us recall Rouché's theorem (see, e.g., [11, Chapter 4]).

Theorem 23 (Rouché's theorem)**.** *Let the functions F and G be analytic in a simply connected domain of $\mathbb{C}$ containing in its interior the closed simple curve γ . Assume that F and G satisfy $|F(r) - G(r)| < |G(r)|$ for r on the curve γ . Then F and G have the same number of zeros inside the interior domain delimited by γ .*

Proof of Lemma 12. First, using the explicit expression of α_k and β_k, and employing calculus software, we obtain that the algebraic function $1 + 2z(-1 - \alpha_0 + \Lambda^5)$ has only one zero within the circle of radius $1/\sqrt{8}$, and that

$$\inf_{|z|=1/\sqrt{8}} |1 + 2z(-1 - \alpha_0 + \Lambda^5)| > 10^{-2}. \tag{92}$$

Note that the lower bound 10^{-2} in (92) is almost optimal; it is rather small because like ρ, the unique zero of $1 + 2z(-1 - \alpha_0 + \Lambda^5)$ within the disc of radius $1/\sqrt{8}$ is very close to $1/\sqrt{8}$.

By Rouché's theorem (see Theorem 23), applied to the circle with radius $1/\sqrt{8}$, it is now enough to prove that

$$\sup_{|z|=1/\sqrt{8}} \left| [1 + 2z(-1 - \alpha_0 + \Lambda^5)] - [1 + 2z(-1 - \alpha_0 + \Lambda^\infty)] \right| < 10^{-2}. \tag{93}$$

For this we write

$$|\Lambda^\infty - \Lambda^p| = \left| \sum_{k=p+1}^{\infty} (-1)^k T_k \right| \leqslant \sum_{k=p+1}^{\infty} |T_k| \leqslant \frac{1/\sqrt{2}^{2p+5}}{1 - (1/\sqrt{2})^2} < 1/\sqrt{2}^{2p+3}, \tag{94}$$

where the second upper bound is obtained from the inequality $|T_k| \leqslant 1/\sqrt{2}^{2k+3}$, see (87) and Lemma 21. By (94) we then obtain

$$|[1 + 2z(-1 - \alpha_0 + \Lambda^p)] - [1 + 2z(-1 - \alpha_0 + \Lambda^\infty)]| \leqslant 2|z| \frac{1}{2^{p+3/2}}. \tag{95}$$

If $|z| < 1/\sqrt{8}$, the last quantity is bounded from above by $1/2^{p+2}$. For $p = 5$, we obviously get $1/2^{p+2} < 10^{-2}$, and (93) is proven. □

Lemma 24.

$$C_{0,0} = 0.0531 \cdot [1 + O(10^{-3})]. \tag{96}$$

Proof. As in the proof of Proposition 15, we can write

$$C_{0,0} = \frac{3\rho - 1}{\rho[1 - \rho^2 \widehat{x}'_{0,0}(\rho)]}, \tag{97}$$

so that the main difficulty lies in approximating $\widehat{x}'_{0,0}(\rho)$. For this, we use (49) and (87) to write

$$\widehat{x}'_{0,0}(z) = -2\alpha'_0(z) + \sum_{k=0}^{\infty} (-1)^k T'_k(z). \tag{98}$$

While it is easy to control the series $\sum_{k=0}^{\infty} (-1)^k T_k(z)$, because $\alpha_k(z)$ and $\beta_k(z)$, and hence $T_k(z)$, decrease exponentially fast to 0 as $k \to \infty$, see Proposition 3 and Lemma 21, it is not obvious how we should deal with $\sum_{k=0}^{\infty} (-1)^k T'_k(z)$, where terms like $\alpha'_k(z)$ and $\beta'_k(z)$ appear. We next show that $\alpha'_k(z)$ and $\beta'_k(z)$ actually also decrease exponentially fast to 0 as $k \to \infty$, at least for $z \in [1/4, 0.35]$. Note that this assumption on z is not restrictive, since ρ belongs to the interval $[1/4, 0.35]$ by Corollary 22.

Consider the sequence $\{\gamma_k(z)\}_{k \geqslant 0}$ defined by $\gamma_0(z) = \alpha_0(z)$ and, for $k \geqslant 0$, by $\gamma_{k+1}(z) = f(\gamma_k(z))$, with $\alpha_0(z)$ and f as defined in (20) and (29), respectively. Note that (29) gives $\gamma_{2k}(z) = \alpha_k(z)$ and $\gamma_{2k+1}(z) = \beta_k(z)$. The sequence $\{\gamma'_k(z)\}_{k \geqslant 0}$ satisfies the recurrence relation

$$\gamma'_{k+1}(z) = \gamma'_k(z) \partial_t f(\gamma_k(z)) + \partial_z f(\gamma_k(z)). \tag{99}$$

Below we study the coefficients $\partial_t f(\gamma_k(z))$ and $\partial_z f(\gamma_k(z))$ of (99). By using expression (29) of f, we easily obtain that for all $z \in [0, 1/\sqrt{8}]$ and all $t \in [0, a_0(z)]$

$$|\partial_t f(\gamma_k(z))| \leqslant 4\sqrt{2}/9. \tag{100}$$

In addition, $\partial_z f(t) = f(t)/[z\sqrt{1 - 4z^2(1 + t^2)}]$, in such a way that $|\partial_z f(t)| \leqslant |2/z| \cdot |f(t)|$ for all $z \in [0, 1/\sqrt{8}]$ and all $t \in [0, a_0(z)]$. Using then Lemma 21, we obtain

$$|\partial_z f(\gamma_k(z))| \leqslant |2/z| \cdot |f(\gamma_k(z))| = |2/z| \cdot |\gamma_{k+1}(z)| \leqslant |2/z| \cdot 1/\sqrt{2}^{k+1} \leqslant 8/\sqrt{2}^{k+1}, \tag{101}$$

where the last inequality follows from the assumption $z \in [1/4, 0.35]$. From (99)–(101) we get

$$|\gamma'_{k+1}(z)| \leqslant \eta \cdot |\gamma'_{k+1}(z)| + \xi_k, \quad \eta = 4\sqrt{2}/9, \quad \xi_k = 8/\sqrt{2}^{k+1}. \tag{102}$$

We deduce that

$$|\gamma'_{k+1}(z)| \leqslant \eta^{k+1} \cdot |\gamma'_0(z)| + \sum_{p=0}^{k} \eta^p \xi_{k-p} \leqslant \eta^{k+1} \cdot |\gamma'_0(z)| + 72/\sqrt{2}^{k+1} \leqslant 100/\sqrt{2}^{k+1}, \tag{103}$$

where the last inequality follows from $\sup_{z\in[0.25,0.35]} |\gamma'_0(z)| \leqslant 13$ and $\eta \leqslant 1/\sqrt{2}$. With Lemma 21 and (103), we obtain

$$|T'_k(z)| \leqslant 200/\sqrt{2}^{k+2}, \tag{104}$$

and thanks to (98) we can obtain approximations of $\widehat{x}'_{0,0}(\rho)$ up to any level of precision. Lemma 24 follows. □