On the growth of nonuniform lattices in pinched negatively curved manifolds

Françoise Dal’bo 1, Marc Peigné & Jean-Claude Picaud 2, Andrea Sambusetti 3

1. Introduction

We study the relation between the exponential growth rate of volume in a pinched negatively curved manifold and the critical exponent of its lattices. These objects have a long and interesting story and are closely related to the geometry and the dynamical properties of the geodesic flow of the manifold (see e.g. [4], [9],[20] and references therein).

Throughout this paper, $X$ will denote a complete and simply connected Riemannian manifold of dimension $N \geq 2$ and we will assume that $X$ has pinched negative curvature, that is its sectional curvature $K_X$ is bounded between two negative constants $-b^2 \leq -a^2 < 0$. A Kleinian group of $X$ is a torsion free and discrete subgroup $\Gamma$ of $Is(X)$; then, $\Gamma$ operates freely and properly discontinuously on $X$ and the quotient manifold $M := X/\Gamma$ has a fundamental group which can be identified with $\Gamma$. The group $\Gamma$ is called a lattice when the volume of $M$ is finite; the lattice is said to be uniform if $M$ is compact.

Recall that the exponential growth rate of $X$, also known as the volume entropy of $X$, is defined as

$$\omega(X) = \limsup_{R \to +\infty} \frac{1}{R} \ln v_X(x, R)$$

where $v_X(x, R)$ is the volume of the open ball $B_X(x, R)$ of $X$, centered at the point $x$ and with radius $R$. By the triangular inequality, this quantity does not depend on the base point $x$; furthermore, under our pinching assumption, Bishop-Gunther’s comparison theorem (see [14]) implies

$$(N - 1)a \leq \omega(X) \leq (N - 1)b.$$  

The invariant $\omega(X)$ has been intensively studied when $Is(X)$ admits a uniform lattice $\Gamma$. It turns out that, in this case, $\omega(X)$ is a true limit and equals the topological entropy of the geodesic flow of the compact manifold $M$


1Françoise Dal’bo, IRMAR, Université de Rennes-I, Campus de Beaulieu, 35042 Rennes Cedex. mail : francoise.dalbo@univ-rennes1.fr
2Marc Peigné & Jean-Claude Picaud LMPT, UMR 6083, Faculté des Sciences etTechniques, Parc de Grandmont, 37200 Tours. mail : peigne@univ-tours.fr, jean-claude.picaud@univ-tours.fr
3Andrea Sambusetti Istituto di Matematica G. Castelnuovo Universita’ ”La Sapienza” di Roma P.le Aldo Moro 5 - 00185 Roma. mail : sambuset@mat.uniroma1.it
Furthermore, with a suitable normalization on the volume of \( M \), it is a complete invariant of locally symmetric metrics on \( M \) (see [4]).

The second object of our interest in this paper is the Poincaré series \( P_\Gamma(s, x) \) of a Kleinian group \( \Gamma \), defined by

\[
P_\Gamma(s, x) = \sum_{\gamma \in \Gamma} e^{-s d(x, \gamma x)},
\]

for \( x \in X \) and \( s \in \mathbb{R} \). Its abscissa of convergence, called the critical exponent of \( \Gamma \), is equal to

\[
\delta(\Gamma) = \limsup_{R \to \infty} \frac{1}{R} \ln v_\Gamma(x, R),
\]

where \( v_\Gamma(x, R) \) is the cardinality of the "ball" \( B_\Gamma(x, R) := \{ \gamma \in \Gamma / d(x, \gamma x) \leq R \} \); again, by the triangular inequality, \( \delta(\Gamma) \) does not depend on \( x \).

A way to understand the dynamic significance of the volume entropy \( \omega(X) \) and its relation with \( \delta(\Gamma) \) is to consider the Laplace transform of the \( \Gamma \)-invariant volume form \( dv_X \) on \( X \), namely

\[
I_X(s) = \int_{0}^{+\infty} e^{-sr}v_X(x, r)dr.
\]

The abscissa of convergence of \( I_X(s) \) coincides with \( \omega(X) \).

By a Fubini type argument, we also have \( I_X(s) = \frac{1}{s} \int_X e^{-sd(x, y)} dv_X(y) \). If \( D \) is a Borel fundamental domain for the action of \( \Gamma \) on \( X \), we get, by invariance of \( dv_X \):

\[
sI_X(s) = \sum_{\gamma \in \Gamma} \int_D e^{-sd(x, y)} dv_X(y) = \sum_{\gamma \in \Gamma} \int_D e^{-sd(\gamma^{-1}x, y)} dv_X(y)
\]

which, in turns, yields:

\[
(2) \quad P_\Gamma(s, x) \int_D e^{-sd(x, y)} dv_X(y) \leq sI_X(s) \leq P_\Gamma(s, x) \int_D e^{sd(x, y)} dv_X(y)
\]

From the left-hand side of (2) it immediately follows that we always have

\[
(3) \quad \delta(\Gamma) \leq \omega(X).
\]

Moreover, from the right-hand side of (2), we have \( \delta(\Gamma) = \omega(X) \) when \( \Gamma \) is a uniform lattice.

In this paper we shall investigate the case where \( X \) admits a non-uniform lattice \( \Gamma \). Let us emphasize that, under this assumption, if \( X \) also admits a uniform lattice \( \Gamma_0 \) then \( X \) is a symmetric space of non compact type (and rank 1). Actually, as the curvature does not vanish, the manifold \( X \) is not a Riemannian product; then (by [11], Corollary 9.2.2), either \( X \) is symmetric or the isometry group of \( X \) is discrete. But, in this last case, \( \Gamma_0 \) would have finite index in \( Is(X) \) (see [11] 1.9.34) and, if \( \varphi \) is a parabolic isometry of \( X \), then \( \varphi^n \) would belong to \( \Gamma_0 \) for some \( n \geq 1 \), which contradicts the fact that a uniform lattice contains only axial elements.

Somewhat surprisingly, the equality \( \delta(\Gamma) = \omega(X) \) may fail for a non uniform lattice \( \Gamma \); actually, in the last section of this paper, we shall prove...
Theorem 1.1. There exists a complete and simply connected Riemannian surface $X$ with pinched negative curvature which admits a non uniform lattice $\Gamma$ such that

$$\delta(\Gamma) < \omega(X).$$

Our construction extends to any dimension. To explain it, recall that to each cuspidal end of the quotient manifold $X/\Gamma$ corresponds a maximal parabolic subgroup $P \subset \Gamma$, which has a lower critical exponent:

$$\delta^-(P) = \liminf_{R \to \infty} \frac{1}{R} \ln v_P(x, R).$$

In strictly negative curvature, this exponent is nonzero, despite the fact that $P$ is virtually nilpotent (see [6]). The key point is that, in the variable curvature setting, $\delta^-(P)$ may be distinct from $\delta(P)$, as was suggested a long time ago by B. Bowditch; in contrast, it is well known that the critical exponent of any non elementary Kleinian group always is a true limit [19]. We shall show in Section 5 that the inequality $\omega(X) > \delta(\Gamma)$ may appear as soon as $\delta^-(P) < \delta(P)/2$.

On the other hand, our example induces us to introduce a notion of pinching for non uniform lattices which ensures that $\omega(X) = \delta(\Gamma)$. Namely, we say that $\Gamma$ is parabolically $1/2$-pinched if for any maximal parabolic subgroup $P \subset \Gamma$, we have

$$\frac{\delta(P)}{\delta^-(P)} \leq 2$$

We will prove

Theorem 1.2. Let $X$ be a complete, simply connected Riemannian manifold with pinched negative curvature. Then for any lattice $\Gamma \subset \text{Is}(X)$ which is parabolically $1/2$-pinched, we have $\delta(\Gamma) = \omega(X)$.

Moreover, we notice that, under the assumptions of this theorem, the invariant $\omega(X)$ is a true limit; this follows from Corollary 4.5, combined with the fact that $\delta(\Gamma)$ is a limit.

We shall see that Theorem 1.2 covers the case of lattices in any $1/4$-pinched negatively curved manifold (i.e. $\frac{k}{4\pi^2} \leq 4$). As far as we know, even in the classical case of Riemannian negatively curved symmetric spaces of rank one (which are $1/4$-pinched, cp. [15]), there does not exist an elementary proof of this result. Nevertheless, for those spaces, the equality $\omega(X) = \delta(\Gamma)$ can be easily deduced from a general and deep result of A. Eskin and C. McMullen in [13] on lattices of affine symmetric spaces, obtained by algebraic methods. In contrast, the context of variable negative curvature forces us to use only elementary geometric arguments.

The equality $\omega(X) = \delta(\Gamma)$ actually holds under a milder geometric assumption than $1/4$-pinched curvature. Namely, we will say that a manifold $M = X/\Gamma$ has asymptotically $1/4$-pinched curvature when, for any $\epsilon > 0$, there exists a compact set $C_\epsilon \subset M$, such that the metric is $(\frac{1}{4+\epsilon})$-pinched on $M \setminus C_\epsilon$. A direct consequence of Theorem 1.2 is
Corollary 1.3. Let $X$ be a complete, simply connected Riemannian manifold with pinched negative curvature and let $\Gamma$ be a lattice of $X$. If $M := X/\Gamma$ has asymptotically $1/4$-pinched curvature, then $\delta(\Gamma) = \omega(X)$.

We remark that the pinching constant $1/4$ is optimal because, for every $\epsilon > 0$, the example we construct in Theorem 1.1 can be chosen so that the curvature is $1/4+\epsilon$-pinched.

The paper is organized as follows. Section 2 deals with elementary geometrical estimates inside horoballs. In Section 3, we relate the volume growth of balls inside a horoball $H$ with the critical exponent of ample parabolic subgroups preserving $H$. In section 4, we first give an elementary proof of the equality $\omega(X) = \delta(\Gamma)$ for $1/4$-pinched manifolds; this is of interest since the main idea about the behavior of a ball intersecting a horoball appears clearly in the proof. The proofs of Theorem 1.2 and Corollary 1.3 will follow. Section 5 is devoted to the construction of the example of Theorem 1.1; this relies on pretty technical results about convex functions, postponed to the Appendix.

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We fix here once and for all some notation about asymptotic behavior of functions:

Notations. We shall write $f \leq g$ (or simply $f \preceq g$) when $f(R) \leq cg(R)$ for some constant $c > 0$ and $R$ large enough. The notation $f \succeq g$ (or simply $f \geq g$) means $f \geq g \geq f$.

Analogously, we shall write $f \preceq g$ (or simply $f \sim g$) when $|f(R) - g(R)| \leq c$ for some constant $c > 0$ and $R$ large enough.

The upper and lower exponential growth rates of a function $f$ are denoted by $\omega^+(f)$ (or simpler $\omega(f)$) and $\omega^-(f)$ respectively; namely we have

$$\omega^-(f) := \liminf_{R \to +\infty} \frac{\ln f(R)}{R} \quad \text{and} \quad \omega^+(f) = \omega(f) := \limsup_{R \to +\infty} \frac{\ln f(R)}{R}.$$ 

Finally, if $f$ and $g$ are two real functions, we denote by $f \star g$ the discrete convolution of $f$ with $g$, defined by $f \star g(R) = \sum_{n=0}^{[R]} f(n)g(R-n)$ for any $R \geq 0$.

2. Radial flow and geometry of horoballs

As the curvature is bounded from above by $-\alpha^2 < 0$, we have the following classical inequality:

Lemma 2.1. Let $T$ be a geodesic triangle with different vertices $x, y, z \in X$ and angle at $y$ greater than $\alpha > 0$. Then there is a constant $D = D(\alpha, a)$ such that

$$d(x, z) \geq d(x, y) + d(y, z) - D.$$ 

Proof. See [8].
Let $X(\infty)$ be the boundary at infinity of $X$. Fix a point $\xi$ in $X(\infty)$ and consider its associated radial semi-flow, $(\psi_{\xi,t})_{t \geq 0}$ defined as follows: for any $x \in X$, the point $\psi_{\xi,t}(x)$ lies on the geodesic ray $[x, \xi]$ at distance $t$ from $x$. For any horosphere $\partial H$ centered at $\xi$, we set $\partial H(t) = \psi_{\xi,t}(\partial H)$, and we let $d_t$ be the distance induced by $d$ on the horosphere $\partial H(t)$. For any points $x, y \in \partial H(t)$, we have (see [16])

$$
\frac{2}{a} \sinh \left( \frac{a}{2} d(x, y) \right) \leq d_t(x, y) \leq \frac{2}{b} \sinh \left( \frac{b}{2} d(x, y) \right).
$$

By [16], the differential of the map $\psi_{\xi,t} : \partial H \to \partial H(t)$ satisfies, for any vector $v \in T(\partial H)$ and any $t \geq 0$

$$
e^{-bt} ||v|| \leq ||d\psi_{\xi,t}(v)|| \leq e^{-at} ||v||.
$$

This readily implies the estimates

$$
e^{-b(N-1)t} \leq |Jac(\psi_{\xi,t})| \leq e^{-a(N-1)t}.
$$

In particular, if $\mu_t$ is the Riemannian measure induced on $\partial H(t)$ by the metric on $X$, we have, for any Borel set $A \subset \partial H$

$$
e^{-b(N-1)t} \mu_0(A) \leq \mu_t(\psi_{\xi,t}(A)) = \int_A |Jac(\psi_{\xi,t})|(x) d\mu_0(x) \leq e^{-a(N-1)t} \mu_0(A).
$$

If the points $x, y$ belong to the horosphere $\partial H$, we set

$$t_{x,y} = \inf \{ t \geq 0/ d_t(\psi_{\xi,t}(x), \psi_{\xi,t}(y)) \leq 1 \}.
$$

The next lemma, which precises Lemma 4 in [9], will be of major importance in the following.

**Lemma 2.2.** There exists a constant $c = c(a, b) > 0$, only depending on the bounds on the curvature, such that, for any horosphere $\partial H$ and any $x, y \in \partial H$, the arc $\gamma_{x,y}$ which is the ordered union of the three geodesic segments $[x, \psi_{\xi,t_{x,y}}(x)]$, $[\psi_{\xi,t_{x,y}}(x), \psi_{\xi,t_{x,y}}(y)]$ and $[\psi_{\xi,t_{x,y}}(y), y]$ is a $(1, c)$-quasigeodesic. Furthermore, for any $s, t \geq 0$, we have

$$d(\psi_{\xi,s}(x), \psi_{\xi,s}(x)) \lesssim \varphi(s, t)
$$

where $\varphi$ is the function defined on $\mathbb{R}_+ \times \mathbb{R}_+$ by

$$
\varphi(s, t) = \begin{cases} 
2t_{x,y} - s - t & \text{when } s, t \leq t_{x,y} \\
|s - t| & \text{otherwise.}
\end{cases}
$$

In particular, we have $d(x, y) \lesssim 2t_{x,y}$.

**Proof.** If $d_0(x, y) \leq 1$, the arc $\gamma_{x,y}$ is the geodesic segment $[x, y]$ and the lemma is obvious in this case. We now assume $d_0(x, y) > 1$. Let $x = \psi_{\xi,t_{x,y}}(x)$ and $y = \psi_{\xi,t_{x,y}}(y)$. From the right hand side of (5), the distance $d(x, y)$ is bounded from below by $b' := \frac{2}{b} \sinh^{-1} \frac{b}{2}$.

Let us now fix a point $\xi'$ on the boundary at infinity of the space $\mathbb{H}^N_a$ of constant curvature $-a^2$, and two points $x', y'$ on the same horosphere centered at $\xi'$, and at distance $b'$ each from the other on this space; comparing the triangles $x y \xi$ and $x' y' \xi'$ we deduce that $x y \xi \leq x' y' \xi' \leq \frac{\pi}{2} - \theta$, for some constant $\theta > 0$ depending only on $a$ and $b$. Since $x y \xi \geq \pi/2$, we have $x y \xi \leq \pi/2$ and so $x y \xi \geq \theta$. Applying Lemma 2.1 successively to the triangles $x x y$.
By Proposition 2.3, we deduce \( v \) (with \( \alpha \geq \pi/2 \)) and \( x \) y y (with \( \alpha \geq \theta \)) we obtain \( d(x, y) \sim d(x, x) + d(y, y) \). The second point follows from the first one, computing the distance between \( \psi_{x,y}(x) \) and \( \psi_{x,y}(y) \) along \( \gamma_{x,y} \). □

Applying this lemma, we obtain the

**Proposition 2.3.** There exists a constant \( c = c(a, b) > 0 \) such that for any point \( \xi \in X(\infty) \), any horoball \( \mathcal{H} \) centered at \( \xi \) and any \( x \in \partial \mathcal{H} \) and \( R > 0 \) we have

\[
B_X(\psi_{\xi,R/2}(x), R/2) \subset B_X(x, R) \cap \mathcal{H} \subset B_X(\psi_{\xi,R/2}(x), R/2 + c).
\]

**Proof.** We need only to prove the second inclusion, the first one being obvious. For \( z \in B_X(x, R) \cap \mathcal{H} \), denote by \( y \) the projection of \( z \) on \( \partial \mathcal{H} \) and by \( z_0 \) the intersection of the horosphere centered at \( \xi \) and containing \( z \) with the geodesic ray \( [x, \xi] \).

Assume first \( t_{x,y} \leq \max\{R/2, d(y, z)\} \); setting \( s = R/2 \) and \( t = d(y, z) \) in the previous lemma, we get \( d(\psi_{\xi,R/2}(x), z) \sim |s - t| = d(\psi_{\xi,R/2}(x), z_0) \leq R/2 \) (the last inequality following from the fact that \( d(x, z_0) \leq d(x, z) \leq R \)).

Assume now \( t_{x,y} \geq \max\{R/2, d(y, z)\} \); applying twice the previous lemma, we get in this case

\[
\begin{cases}
  d(x, z) \sim 2t_{x,y} - d(z, y) & \text{(setting} \ s = 0 \ \text{and} \ t = d(y, z)) \\
  d(\psi_{\xi,R/2}(x), z) \sim 2t_{x,y} - d(z, y) - R/2 & \text{(setting} \ s = R/2 \ \text{and} \ t = d(y, z)).
\end{cases}
\]

Since \( z \in B_X(x, R) \) there, thus exists \( c > 0 \) such that \( d(\psi_{\xi,R/2}(x), z) \leq R/2 + c \). □

In the next section, we will consider discrete parabolic subgroups of \( Is(X) \); any such group fixes one point \( \xi \in X(\infty) \) and preserves any horoball \( \mathcal{H} \) centered at \( \xi \). We shall investigate the relation between the critical exponent of \( \mathcal{P} \) and the volume growth of \( X \). Here we shall limit ourselves to remark:

**Corollary 2.4.** If \( X \) is homogeneous, then for any discrete parabolic subgroup \( \mathcal{P} \) of \( Is(X) \), we have

\[
\delta(\mathcal{P}) \leq \omega(X)/2.
\]

This fact is well known when \( X \) is a rank one symmetric space; Proposition 2.3 allows to understand the geometrical reason of this inequality. Actually, let \( \mathcal{H} \) be an horoball preserved by \( \mathcal{P} \) and let \( x \in \partial \mathcal{H} \). As \( \mathcal{P} \) is discrete, we have \( d := \frac{1}{2} \inf_{p \in \mathcal{P}} d(x, px) > 0 \), then

\[
\bigcup_{p \in \mathcal{P}, d(x, px) \leq R} B_X(px, d) \times [0, 1] \subset B_X(x, R + d + 1) \cap \mathcal{H}.
\]

By Proposition 2.3, we deduce \( \nu_p(x, R) \leq \sup_{y \in \mathcal{H}} \nu_y \left( \frac{R + d + 1}{2} \right) + c \). As \( X \) is homogeneous, for any \( \epsilon > 0 \), we have \( \nu_y(\alpha, r) \leq e^{(\omega(X) + \epsilon)r} \) uniformly in \( y \). The Corollary follows. □
3. Growth of ample parabolic subgroups

Let be $\mathcal{P}$ a parabolic subgroup of $Is(X)$ fixing $\xi \in X(\infty)$. We shall say that $\mathcal{P}$ is ample if it acts cocompactly on every horoball $\partial \mathcal{H}$ centered at $\xi$. This holds in particular when $\mathcal{P}$ is a maximal parabolic subgroup of a non uniform lattice of $Is(X)$.

We then fix a (relatively compact) Borel fundamental domain $\mathcal{C} \subset \partial \mathcal{H}$ for the action of $\mathcal{P}$ on $\partial \mathcal{H}$. For any $t \geq 0$, the set $\mathcal{C}_t := \psi_{\xi,t}(\mathcal{C})$ is a fundamental domain for the action of $\mathcal{P}$ on $\partial \mathcal{H}(t)$; in the same way, the set $\mathcal{E} := \cup_{t \geq 0} \mathcal{C}_t$, which is canonically homeomorphic to $\mathcal{C} \times \mathbb{R}^+$, is a fundamental domain for the action of $\mathcal{P}$ on the horoball $\mathcal{H}$.

We now associate to any ample parabolic group $\mathcal{P}$ a function $A_\mathcal{P}$ which will play a crucial role in this paper:

**Definition 3.1.** The horospherical area of $\mathcal{P}$ is the function $A_\mathcal{P}(x,t)$ defined by

$$\forall x \in \partial \mathcal{H}, \forall t \geq 0 \quad A_\mathcal{P}(x,t) := \mu_t(\psi_{\xi,t}(\mathcal{C})).$$

The function $t \mapsto A_\mathcal{P}(x,t)$ is decreasing and does not depend on the choice of the fundamental domain $\mathcal{C}$; furthermore, by inequalities (8), for any $R$ and $R_0 > 0$, we have

$$e^{-(N-1)bR_0} \leq \frac{A_\mathcal{P}(x,R+R_0)}{A_\mathcal{P}(x,R)} \leq e^{-(N-1)aR_0}.$$

(9)

The following proposition stresses the relation between the function $A_\mathcal{P}$ and the orbital counting function $v_\mathcal{P}(x,R)$ of $\mathcal{P}$.

**Proposition 3.2.** There exists a constant $c = c(a,b,diam(\mathcal{C})) > 0$ such that for any $x \in X$

$$v_\mathcal{P}(x,R) \leq \frac{1}{A_\mathcal{P}(x,R)}.$$  

In particular, we have

(10)  \[ \delta(\mathcal{P}) = \omega\left(\frac{1}{A_\mathcal{P}(x,R)}\right) \quad \text{and} \quad \delta^-(\mathcal{P}) = \omega^-\left(\frac{1}{A_\mathcal{P}(x,R)}\right). \]

**Proof.** We recall that $d_t$ denotes the horospherical distance on the horosphere $\partial \mathcal{H}(t)$. We let $c$ be the constant of Lemma 2.2 such that $d(x,y) \lesssim 2 t_{x,y}$ for $x,y$ on $\partial \mathcal{H}$. If $d(x,y) = R$, as $t_{x,y} \gtrsim R$, we deduce

$$d_{\mathcal{B}_{\mathbb{H}}}(x(\frac{R+c}{2}), y(\frac{R+c}{2})) \leq 1 \quad \text{and} \quad d_{\mathcal{B}_{\mathbb{H}}}(x(\frac{R-c}{2}), y(\frac{R-c}{2})) \geq 1.$$  

This implies that $\psi_{\mathcal{B}_{\mathbb{H}}}(B_X(x,R) \cap \partial \mathcal{H}) \subset B_1$ and $\psi_{\mathcal{B}_{\mathbb{H}}}(B_X(x,R) \cap \partial \mathcal{H}) \subset B_2$ with

$$B_1 := B_{\partial \mathcal{H}(\frac{R+c}{2})}(x(\frac{R+c}{2},1)) \quad \text{and} \quad B_2 := B_{\partial \mathcal{H}(\frac{R-c}{2})}(x(\frac{R-c}{2},1)).$$  

Gauss equation implies that the sectional curvature of all horospheres for the induced metric is in between $a^2 - b^2$ and $2b(b-a)$ (see [7], section 1.4, example (iii)). Therefore, there exist positive constants $v^- = v^-(a,b,x)$ and $v^+ = v^+(a,b,x)$ such that $v^- \leq vol(B_i) \leq v^+$ for the induced volume form on the horospheres and $i = 1, 2.$
Now, there are at most $v_p(x, R)$ distinct fundamental domains $p(C)$ included in $B_X(x, R) \cap \partial \mathcal{H}$ and since the radial semi-flow $(\psi_{\xi,t})_{t \geq 0}$ is equivariant with respect to the action of $\mathcal{P}$ on the horospheres $\partial \mathcal{H}(t)$, there are also at most $v_p(x, R)$ distinct fundamental domains $p(C(\frac{R+\Delta}{2}))$ included in $\psi_{R+\Delta}(B_X(x, R) \cap \partial \mathcal{H})$. Therefore, we have $v_p(x, R) \leq \frac{v^+}{A_p(x, \frac{R+\Delta}{2})}$ and by (9), this leads to

$$v_p(x, R) \preceq \frac{1}{A_p(x, \frac{R}{2})}.$$ 

On the other hand, we can cover the set $B_X(x, R) \cap \partial \mathcal{H}$ with $v_p(x, R + d)$ distinct fundamental domains $p(C)$; by the equivariance of $(\psi_{\xi,t})$, we deduce again that $\psi_{R+\Delta}(B_X(x, R) \cap \mathcal{H})$ can be covered by $v_p(x, R + d)$ fundamental domains as well. Therefore, using (9) again

$$v_p(x, R) \succeq \frac{v^-}{A_p(x, \frac{R-c-d}{2})} \succeq \frac{1}{A_p(x, \frac{R}{2})}.$$ 

We now estimate the volume of a ball of radius $R$, inside the horoball $\mathcal{H}$. We have

**Proposition 3.3.** There exists a constant $c = c(a, b, \text{diam}(C)) > 0$ such that

$$\text{vol}(B_X(x, R) \cap \mathcal{H}) \preceq \int_0^R \frac{A_p(x, t)}{A_p(x, \frac{t+R}{2})} dt.$$ 

To get this result, we need the following refinement of Proposition 2.3.

**Lemma 3.4.** There exists a constant $\Delta = \Delta(a, b, \text{diam}(C))$ such that

$$p(C) \times [(2t_p-R+\Delta)^+, (R-\Delta)^+] \subset (p(\mathcal{E}) \cap B_X(x, R)) \subset p(C) \times [(2t_p-R-\Delta)^+, R].$$

**Proof.** Let $\Delta = c + \text{diam}(C)$, where $c$ is the constant of Lemma 2.2. We first prove the right hand side inclusion. Let $z = (z_0, t) \in p(C) \times \mathbb{R}^+$ and assume that this point belongs to $B_X(x, R)$. Clearly $t \leq R$ as $t = B_\xi(x, z) \leq d(x, z) \leq R$. If $t_p \leq \frac{R+\Delta}{2}$ there is nothing to prove; on the other hand, if $t_p > \frac{R+\Delta}{2}$, then $2t_p - t \preceq d(x, z) < R$ hence $t \in [(2t_p - R + \Delta)^+, R]$. Let us now consider the case where $z \in p(C) \times [(2t_p - R + \Delta)^+, (R-\Delta)^+]$. We may assume that $R \geq \Delta$ and $t_p \leq R - \Delta$, otherwise there is nothing to prove. If $t \geq t_p$ we have $d(x, z) \preceq t \leq R - \Delta$, otherwise we have $d(x, z) \preceq 2t_p - t \leq 2t_p - (2t_p - R + \Delta)^+$; therefore, in both cases $z \in B_X(x, R)$. 

**Proof of Proposition 3.3.** We simply write $A(R) = A_p(x, R)$. Recall that

$$B_X(x, R) \cap \mathcal{H} = \bigsqcup_{p \in \mathcal{P}} B_X(x, R) \cap p(\mathcal{E}).$$
By Lemma 3.4, we have \( B_X(x, R) \cap p(\mathcal{E}) \subset p(\mathcal{C}) \times [(2t_p - R - \Delta)^+, R] \). Then, we find
\[
\sum_{p \in \mathcal{P}} \text{vol} \left( B_X(x, R) \cap p(\mathcal{E}) \right) = \sum_{t_p \leq R + \frac{\Delta}{2}} \int_{(2t_p - R - \Delta)^+}^{R} A(t) \, dt
\]
\[
= \sum_{t_p \leq R + \frac{\Delta}{2}} \int_{0}^{R} A(t) 1_{[(2t_p - R - \Delta)^+, +\infty]}(t) \, dt
\]
Now, as \( d(x, px) \sim 2t_p \leq c \leq \Delta \), for every fixed \( t \in [0, R] \) we have
\[
\sharp \{ p \in \mathcal{P} / t_p \leq R + \frac{\Delta}{2} \text{ and } 2t_p - R - \Delta \leq t \} \leq v_p \left( x, \frac{t + R + \Delta}{2} + \Delta \right)
\]
\[
\leq \frac{v^+}{A(t + R + \frac{3\Delta}{2})}
\]
\[
\leq \frac{1}{A(R + \frac{\Delta}{2})},
\]
where we have successively used Proposition 3.2 and (9). This yields
\[
\text{vol} (B_X(x, R) \cap \mathcal{H}) \leq \int_{0}^{R} \frac{A(t)}{A(R + \frac{\Delta}{2})} \, dt.
\]
We now prove the converse inequality. Again, by Proposition 3.4, we deduce
\[
B_X(x, R) \cap p(\mathcal{E}) \supset p(\mathcal{C}) \times [(2t_p - R + \Delta)^+, R - \Delta].
\]
We only consider those \( p \)'s such that \( \frac{R - \Delta}{2} \leq t_p \leq R - \Delta \); summing over these \( p \)'s, we find
\[
\sum_{\frac{R - \Delta}{2} \leq t_p \leq R - \Delta} \text{vol} \left( B_X(x, R) \cap p(\mathcal{E}) \right) = \sum_{\frac{R - \Delta}{2} \leq t_p \leq R - \Delta} \int_{2t_p - \Delta}^{R - \Delta} A(t) \, dt
\]
\[
\geq \sum_{\frac{R - \Delta}{2} \leq t_p \leq R - \Delta} \int_{R_0}^{R - \Delta} A(t) 1_{[2t_p - R + \Delta, R - \Delta]}(t) \, dt
\]
for any \( R_0 \geq 0 \). Now, for every fixed \( t \in [R_0, R - \Delta] \), we have
\[
\sharp \{ p \in \mathcal{P} / \frac{R - \Delta}{2} \leq t_p \leq R - \Delta \text{ and } 2t_p - R + \Delta \leq t \} \geq v_p \left( x, t + R - 2\Delta \right) - v_p \left( x, R \right)
\]
\[
\geq \frac{v^-}{A(R + \frac{\Delta}{2})} - \frac{v^+}{A(R)}
\]
\[
\geq \frac{1}{A(R + \frac{\Delta}{2})} \left( v^- \frac{A(R + \frac{\Delta}{2})}{A(R + \frac{3\Delta}{2})} - v^+ \frac{A(R)}{A(R)} \right)
\]
\[
\geq \frac{1}{A(R + \frac{\Delta}{2})} \left( v^- e^{-b(N-1)\Delta} - v^+ e^{-a(N-1)R_0/2} \right)
\]
by Proposition 3.2 and (9). Therefore, if \( R_0 \) is large enough, we find
\[
\text{vol} \left( B_X(x, R) \cap \mathcal{H} \right) \geq \int_{R_0}^{R - \Delta} \frac{A(t)}{A(R + \frac{\Delta}{2})} \, dt.
\]
We can replace this last integral by \( \int_0^R \frac{A(t)}{A(t/2)} \, dt \) since, for \( R \) large enough

\[
\int_{R_0}^{R-\Delta} \frac{A(t)}{A(t/2)} \, dt \geq \int_{R_0}^{2R_0} \frac{A(t)}{A(t/2)} \, dt \approx \frac{1}{A(R/2)} \int_{R_0}^{R_0} A(t) \, dt.
\]

As a direct consequence of Propositions 3.2 and 3.3, we obtain

**Corollary 3.5.** For any \( \epsilon > 0 \) and \( x \in \partial H \), we have

i) if \( \delta(\mathcal{P}) \geq 2\delta^- (\mathcal{P}) \) then

\[
e^{(\delta^- (\mathcal{P})-\epsilon)R} \lesssim \text{vol} \left( B_x (x, R) \cap \mathcal{H} \right) \lesssim e^{2 \left( \delta(\mathcal{P}) - \delta^- (\mathcal{P}) + \epsilon \right) R}
\]

ii) if \( \delta(\mathcal{P}) < 2\delta^- (\mathcal{P}) \) then

\[
e^{(\delta^- (\mathcal{P})-\epsilon)R} \lesssim \text{vol} \left( B_x (x, R) \cap \mathcal{H} \right) \lesssim e^{2 \left( \delta(\mathcal{P}) + \epsilon \right) R}.
\]

### 4. Growth of nonuniform lattices

We suppose now that the manifold \( X \) admits a nonuniform lattice \( \Gamma \). Let us recall some well known geometrical properties of \( \Gamma \) proved in the general context of geometrically finite groups in ([5]). Since the volume of \( M = X/\Gamma \) is finite, the limit set of \( \Gamma \) equals \( X(\infty) \) and is the disjoint union of its radial subset and of finitely many orbits \( \Gamma \xi_1, \ldots, \Gamma \xi_l \) of points, called bounded parabolic fixed points. By definition, a point \( \xi_i \) corresponds to a end of the manifold \( M \) and is fixed by a parabolic subgroup of \( \Gamma \). Denote \( \mathcal{P}_i \) the maximal parabolic subgroup fixing the point \( \xi_i \). This group preserves any horoball \( \mathcal{H} \) centered at \( \xi_i \) and acts cocompactly on the horosphere \( \partial \mathcal{H} \). By Margulis’ lemma (see [20]), there exist closed horoballs \( \mathcal{H}_{\xi_1}, \ldots, \mathcal{H}_{\xi_l} \) centered respectively at \( \xi_1, \ldots, \xi_l \), such that all the horoballs \( \gamma \mathcal{H}_{\xi_i} \), for \( 1 \leq i \leq l \) and \( \gamma \in \Gamma \), are disjoint or coincide.

We fix an origin \( o \in X \) and a convex Borel fundamental domain \( \mathcal{D} \) in \( X \) for the action of \( \Gamma \), containing the geodesic rays \( [o, \xi_1], \ldots, [o, \xi_l] \). For each \( 1 \leq i \leq l \), we set \( \mathcal{E}_i = \mathcal{D} \cap \mathcal{H}_{\xi_i} \) and \( \mathcal{C}_i = \mathcal{D} \cap \partial \mathcal{H}_{\xi_i} \). Those both sets are fundamental domains for the action of the group \( \mathcal{P}_i \) respectively on \( \mathcal{H}_{\xi_i} \) and \( \partial \mathcal{H}_{\xi_i} \). Moreover, the set \( \mathcal{C}_0 = \mathcal{D} \setminus \left( \bigcup_{i=1}^{l} \mathcal{E}_i \right) \), and hence each \( \mathcal{C}_i \), is relatively compact. We may assume that \( o \) belongs to the interior of \( \mathcal{C}_0 \).

The quotient manifold \( M \) is therefore decomposed into the disjoint union of a relatively compact set \( C_0 \) and finitely many ends of finite volume \( E_i = \mathcal{H}_{\xi_i}/\mathcal{P}_i \), which are the projections on \( M \) of the domains \( \mathcal{C}_0 \) and \( \mathcal{E}_i \) respectively.

We first precise some bounds on the critical exponent \( \delta(\Gamma) \) in terms of bounds on the curvature of \( X \).

**Lemma 4.1.** We have \( (N-1)a \leq \delta(\Gamma) \leq (N-1)b \).

In particular, when \( X \) is the real hyperbolic space \( \mathbb{H}^N \) of constant curvature \( -a^2 \), we have \( \delta(\Gamma) = (N-1)a \) and hence \( \delta(\Gamma) = \omega(\mathbb{H}^N) \).

**Proof.** The inequality \( \delta(\Gamma) \leq (N-1)b \) follows from (3) and (1). It remains to prove the left hand side inequality of the Lemma.
If $\delta(\Gamma) = \omega(X)$, the inequality follows from (1). Assume now $\delta(\Gamma) < \omega(X)$ and consider $s \in ]\delta(\Gamma), \omega(X)[$. Inequality (2) implies
\[
\int_{\mathcal{D}} e^{sd(o,x)} dv_X(x) = +\infty
\]
which, by the decomposition $\mathcal{D} = \mathcal{C}_0 \cup \left( \bigcup_{i=1}^{l} \mathcal{E}_i \right)$, is equivalent to
\[
\max_{i \in \{1, \ldots, l\}} \int_{\mathcal{E}_i} e^{sd(o,x)} dv_X(x) = +\infty.
\]
Note now that for $x \in \mathcal{E}_i$, we have $B_{\xi_i}(o, x) \leq d(o, x) \leq B_{\xi_i}(o, x) + \text{diam}(C_i)$ where $B_{\xi_i}(\cdot, \cdot)$ denotes the Busemann function centered at $\xi_i$. Therefore the integrals $\int_{\mathcal{E}_i} e^{sd(o,x)} dv_X(x)$ and $\int_{\mathcal{E}_i} e^{sB_{\xi_i}(o,x)} dv_X(x)$ are of the same nature.

By (8), we have
\[
\int_{\mathcal{E}_i} e^{sB_{\xi_i}(o, x)} dv_X(x) = \int_{d(o, C_i)}^{+\infty} e^{st} \mu_t(\psi_{\xi_i,t}(C_i)) dt \leq \mu_0(C_i) \int_0^{+\infty} e^{s[N-(N-1)a]} dt
\]
and the last integral must be divergent for all $s \in ]\delta(\Gamma), \omega(X)[$, so $\delta(\Gamma) \geq (N-1)a$. \hfill \Box

Recall that $v_X(o, R)$ denotes the volume of the open ball $B_X(o, R)$ and that $v_T(o, R)$ represents the cardinality of the intersection of this ball with $\Gamma(o)$. The following estimate will be used to obtain a upper bound for $\delta(\Gamma)$.

**Proposition 4.2.** There exists a constant $\Delta = \Delta(a, b, \text{diam}(C_0)) > 0$ such that, for all $R > 0$, we have
\[
v_X(o, R - \Delta) \leq v_T(o, R) + \sum_{i=1}^{l} \sum_{n=0}^{[R]} v_T(o, n+1) \times \text{vol}\left( B_X(x_i, R-n+\Delta) \cap \mathcal{H}_{\xi_i} \right)
\]
where $x_i$ denotes the intersection of the geodesic ray $[o, \xi_i]$ with the horosphere $\partial \mathcal{H}_{\xi_i}$.

**Proof.** Set $d_0 = \text{diam}(C_0)$. We have
\[
B_X(o, R) = \left( B_X(o, R) \cap \Gamma.C_0 \right) \cup \left( \bigcup_{1 \leq i \leq l} \left( B_X(o, R) \cap \Gamma.\mathcal{H}_{\xi_i} \right) \right)
\]
whence
\[
B_X(o, R) \cap \Gamma.C_0 \subset \bigcup_{\gamma \in B_T(o, R+d_0)} \gamma(C_0)
\]
and
\[
\text{vol}\left( B_X(o, R) \cap \Gamma.C_0 \right) \leq v_T(R + d_0).
\]
Now, for each $i \in \{1, \ldots, l\}$ we define a map on $\Gamma$ as follows: for any $\gamma \in \Gamma$, let $x_{\gamma,i}$ be the intersection of the ray $[o, \gamma(\xi_i)]$ with the horosphere $\gamma(\partial \mathcal{H}_{\xi_i})$. Since $C_i$ is a fundamental domain for the action of $\mathcal{P}_i$ on $\partial \mathcal{H}_{\xi_i}$ there exist a finite number of elements $\tilde{\gamma}$ in $\gamma \mathcal{P}_i$ such that $x_{\gamma,i} \in \tilde{\gamma}(C_i)$. Choose one of those
elements and denote it by $\tilde{\gamma}_i$. Let $\tilde{\Gamma}_i$ be the set of all $\tilde{\gamma}_i$ for $\gamma$ in $\Gamma$. Since $d(x_{\gamma,i}, \tilde{\gamma}_i o) \leq d_0$, and since the angle at $x_{\gamma,i}$ between the geodesic segments $[x_{\gamma,i}, o]$ and $[x_{\gamma,i}, x]$ is greater than $\pi/2$, by lemma 2.1 there exists a constant $d_1 > 0$ such that for every $\gamma \in \Gamma$ and $x \in \gamma H_{\xi_i} \cap B_X(o, R)$, we have:

$$d(o, \tilde{\gamma}_i o) + d(\tilde{\gamma}_i o, x) - d_1 \leq d(o, x).$$

We have by (13)

$$B_X(o, R) \cap \Gamma H_{\xi_i} \subset \left( \bigcup_{0 \leq n \leq [R + d_0]} \bigcup_{\tau \in \tilde{\Gamma}_i} B_X(\gamma o, R - n + d_1) \cap \gamma H_{\xi_i} \right).$$

For each $i$ denote $x_i$ the intersection of the geodesic ray $[o, \xi_i]$ with the horosphere $\partial H_{\xi_i}$. One has

$$vol \left( B_X(\gamma o, R - n + d_1) \cap \gamma H_{\xi_i} \right) \leq vol \left( B_X(x_i, R - n + d_1 + d_0) \cap H_{\xi_i} \right),$$

while

$$\# \{ \gamma \in \tilde{\Gamma}_i / n \leq d(o, \gamma o) < n + 1 \} \leq \nu_\Gamma(o, n + 1),$$

so

$$v_X(o, R - d_0) \leq v_\Gamma(o, R) + \sum_{i=1}^{[R]} \sum_{n=0}^{[R]} v_\Gamma(o, n + 1) \times vol \left( B_X(x_i, R - n + d_1) \cap H_{\xi_i} \right).$$

The lemma follows with $\Delta \geq \max(d_0, d_1)$. \qed

Proposition 4.2 is crucial to establish Theorem 1.2; we first give an elementary proof of this result, in the case where $X$ is 1/4-pinched.

**4.1. Proof of Theorem 1.2 : the $\frac{1}{4}$–pinched curvature case.** We prove here that if $(X,g)$ is a complete, simply connected Riemannian manifold with 1/4-pinched negative curvature, then for any lattice $\Gamma \subset Is(X)$, we have $\delta(\Gamma) = \omega(X)$.

We use the notations of Proposition 4.2.

By (3), we need only to show that $\omega(X) \leq \delta(\Gamma)$. By Proposition 2.3, we know that for $r > 0$ the set $B_X(x_i, r) \cap H_{\xi_i}$ is included in the ball of radius $r/2 + c$ centered at the point $\psi_{\xi_i, r/2}(x_i)$. Then, (12) leads to the following inequality

$$v_X(o, R - \Delta) \leq v_\Gamma(o, R) + \sum_{n=0}^{[R]} v_\Gamma(o, n + 1) \times \sup_{x \in B_X(x, \frac{R - n + \Delta}{2}) \subset \Theta} vol \left( B_X(x, \frac{R - n + \Delta}{2}) \right).$$

From Bishop Gunther’s theorem and the fact that $b^2 \leq 4a^2$, we have

$$vol \left( B_X(x, r) \cap \Theta \right) \leq v_X(x, r) \leq e^{b(N-1)r} \leq e^{2a(N-1)r},$$

for any $x \in X$ and $r > 0$. We conclude that $\omega(X) \leq (N - 1)a \leq \delta(\Gamma)$ using Lemma 4.1. \qed
Remark - The above proof uses in a crucial way Lemma 4.1 and it still works if we relax the pinching assumption as follows:

For any \( \epsilon > 0 \), there exists a compact set \( C_\epsilon \subset M \) such that the curvature on \( M \setminus C_\epsilon \) belongs to \([- (4 + \epsilon) a^2, -a^2]\).

However, this condition is much stronger than the \((\frac{4}{1+\epsilon})\)-pinching assumption and the proof of Corollary 1.3 requires the more precise estimates of the volume of balls obtained in the previous section.

4.2. Proof of Theorem 1.2: the general case. We fix here a nonuniform lattice \( \Gamma \subset Is(X) \) and apply the results of Section 3 to each maximal parabolic subgroup \( P_i \) of \( \Gamma \). We first set the

\textbf{Definition 4.3.} Let \( M = X/\Gamma \) be a complete Riemannian manifold of finite volume with \(-b^2 \leq K_X \leq -a^2 < 0\) and with ends \( E_1, ..., E_l \). For \( 1 \leq i \leq l \), the cuspidal function \( F_i \) associated with \( E_i \) is defined by

\[ F_i(x, R) = \int_0^R A_i(x, t) \frac{A_i(x, t+R)}{A_i(x, t+\frac{R}{2})} dt \]

where \( A_i(x, t) \) is the horospherical area function associated with \( E_i \).

By (9), the growth rates \( \omega^\pm(F_i(x, \cdot)) \) depend only on the ends \( E_i \) of \( M \) as for any points \( x, y \in X \) and any \( R_0 > 0 \) fixed, we have \( F_i(x, R) \sim F_i(y, R) \). Those functions are of major importance in order to estimate \( v_X(x, R) \); namely, we have the

\textbf{Proposition 4.4.} There exists \( \Delta = \Delta(a, b, \text{diam}(C_0)) > 0 \) such that

\begin{align*}
(i) & \quad v_X(\cdot, R + \Delta) \geq v_\Gamma(\cdot, R) + \sum_{i=1}^l F_i(\cdot, R) \\
(ii) & \quad v_X(\cdot, R + \Delta) \leq v_\Gamma(\cdot, R) + \sum_{i=1}^l v_\Gamma(\cdot, \cdot) * F_i(\cdot, \cdot)(R)
\end{align*}

which leads to the

\textbf{Corollary 4.5.} We have \( \omega^{\pm}(X) = \max\left(\delta(\Gamma), \omega^{\pm}(F_1), ..., \omega^{\pm}(F_l)\right) \).

\textbf{Proof of Proposition 4.4. Part (i).} We have

\[ B_X(o, R) \supset \bigcup_{\gamma \in B_1(o, R-d_0)} \gamma(C_0) \cup \bigcup_{i=1}^l (B_X(o, R) \cap H_i) \cdot \]

On the other hand \( B_X(o, R) \cap H_i \supset B_X(x_i, R-d_0) \cap H_i \), and by Proposition 3.3, we have

\[ v_X(o, R) \geq v_\Gamma(o, R-d_0) + \sum_{i=1}^l F_i(x_i, R) \]

with \( F_i(x_i, R) \sim F_i(o, R) \); the first inequality follows.

Part (ii) follows by plugging Proposition 3.3 in (12). \( \Box \)
Proof of Theorem 1.2 By Corollary 4.5, it is enough to show that \( \omega(\mathcal{F}_i) \leq \delta(\Gamma) \) for \( 1 \leq i \leq l \). By Proposition 3.2, we have, for any \( \epsilon > 0 \):

\[
\mathcal{A}_i(t) \leq e^{-(2\delta^{-}(\mathcal{P})-\epsilon)t} \quad \text{and} \quad \mathcal{A}_i\left(\frac{t + R}{2}\right) \geq e^{-(\delta(\mathcal{P})+\epsilon)(t+R)}.
\]

So, we obtain \( \mathcal{F}_i(t) \leq e^{(\delta(\mathcal{P})+\epsilon)R} \int_0^R e^{(\delta(\mathcal{P})-2(\delta^{-}(\mathcal{P})+2\epsilon))t} dt \leq e^{(\delta(\mathcal{P})+3\epsilon)R} \) as \( \delta(\mathcal{P}) - 2\delta^{-}(\mathcal{P}) \leq 0 \), therefore \( \omega(\mathcal{F}_i) \leq \delta(\mathcal{P}) \leq \delta(\Gamma) \). \( \Box \).

Proof of Corollary 1.3 Assume that \( M = X/\Gamma \) is asymptotically \( \frac{1}{4} \)-pinched. Then, for any fixed \( \epsilon > 0 \) we know that outside a compact subset \( C \) the curvature of \( M \) is between \( -\beta^2 \) and \( -\alpha^2 \), with \( \beta^2 \leq (4 + \epsilon)\alpha^2 \). Therefore we have

\[
e^{-\beta(N-1)t} \leq \mathcal{A}_i(t) \leq e^{-\alpha(N-1)t}
\]

hence, by Proposition 3.2, we deduce that

\[
\frac{\delta(\mathcal{P})}{\delta^{-}(\mathcal{P})} \leq \frac{\beta}{\alpha} \leq 2 + \epsilon
\]

for every maximal parabolic subgroup of \( \Gamma \). As \( \epsilon \) is arbitrary, we deduce that \( M \) is parabolically \( \frac{1}{4} \)-pinched, and we conclude by Theorem 1.2. \( \Box \).

Remark. We have seen that, under the assumptions of Theorem 1.2, we have \( \omega(\mathcal{F}_i) \leq \delta(\Gamma) \) for all \( 1 \leq i \leq l \); in particular, \( \omega(X) \) is a limit in this case.

5. An end with the leading role

We shall construct in this section a pinched, negatively curved surface \( S = X/\Gamma \) of finite volume such that \( \omega(X) > \delta(\Gamma) \). The surface we exhibit is homeomorphic to a 3-punctured sphere, and we shall deform a hyperbolic metric on one end \( E \) of \( S \).

Our construction rests on two main ideas:

i) we can deform the metric in the end \( E \) varying the sectional curvature from \( \alpha^2 \) to \( \beta^2 \) on different bands of \( E \), in order that the function \( \mathcal{A} \) associated to \( E \) satisfies \( \omega(\mathcal{F}) > \delta(\mathcal{P}) \).

ii) we set \( \epsilon := \omega(\mathcal{F}) - \delta(\mathcal{P}) \) and we show that the above deformation of the metric can be performed in such a way that \( \delta(\Gamma) < \delta(\mathcal{P}) + \epsilon \) also.

By Corollary 4.5 we conclude that \( \omega(X) > \delta(\Gamma) \).

Fix positive real numbers \( \alpha \) and \( \beta \) such that \( \beta > 2\alpha \). We can construct sequences of disjoint intervals \([p_n, q_n], [r_n, s_n]\) included in \([\Delta^{n-1}, \Delta^n] \) (for some \( \Delta > 1 \)), and a \( C^2 \) convex, decreasing function \( \mathcal{A}(t) \) on \([\Delta, +\infty]\) whose restrictions to \([p_n, q_n]\) and \([r_n, s_n]\) coincide respectively with \( e^{-\alpha t} \) and \( e^{-\beta t} \). More precisely, we can arrange the points \( p_n, q_n, r_n \) and \( s_n \) in order that \( q_n \geq p_n + 1 \) and \( t \in [p_n, q_n] \iff \frac{t + \Delta}{2} \in [r_n, s_n] \), and we can choose \( \mathcal{A} \) such that \( e^{-\beta t} \leq \mathcal{A}(t) \leq e^{-\beta t} \) and \( \frac{\mathcal{A}(t)}{\mathcal{A}(0)} \in [\alpha^2 - \eta, \beta^2 + \eta] \) for all \( t \in [\Delta, +\infty] \) and some \( \eta > 0 \). The existence of such intervals and of the function \( \mathcal{A} \) is rather technical and we postponed the details of proof to the Appendix (Section 6).
By construction, the function \( \mathcal{F}(R) := \int_0^R \frac{A(t)}{A(t+R)} \, dt \) satisfies:

\[
\omega(\mathcal{F}) \geq \limsup_{n \to +\infty} \frac{1}{\Delta_n} \ln \int_{p_n}^{q_n} \frac{A(t)}{A(t+\Delta_n)} \, dt > \beta/2.
\]

We can now construct the surface of Theorem 1.1. Start from a 3-punctured sphere \( S \) with a metric \( g_0 \) of finite volume and constant curvature \(-\alpha^2\). Let \( \Gamma = \pi_1(S) \) and let \( \mathcal{P} \) be the maximal parabolic subgroup associated with the end \( E \) of \( S \). Consider the horospherical parametrization \( \sigma : [0, 1] \times \mathbb{R}^+ \to \mathcal{E} \) of \( \mathcal{E} \); with respect to these coordinates, the hyperbolic metric writes \( g = e^{-2\alpha t} dx^2 + dt^2 \).

We now perturb \( g \) on \( E_n = \sigma([0, 1] \times [p_n, +\infty[) \) to obtain a new \( C^2 \)-metric \( g_n \) such that \( g_n = A^2(t)dx^2 + dt^2 \) on \( E_n \), for \( A \) defined above. We shall denote by \( d \) and \( d_n \) the distances on \( X \) associated respectively to \( g \) and \( g_n \) and we let \( \delta_n(\Gamma) \) be the critical exponents of \( \Gamma \) and \( \mathcal{P} \) relatively to the new metric \( g_n \).

Notice that \( K_X = -\frac{A''}{A} \) is pinched between \(-\beta^2 - \eta\) and \(-\alpha^2 + \eta\); furthermore \( A(R) \) is precisely the horospherical area (length) function of \( \mathcal{P} \), with respect to \( g_n \), so \( \delta_n(\mathcal{P}) = \beta/2 \) for all \( n \), by Proposition 3.2 (while \( \delta_n^{-}(\mathcal{P}) \leq \alpha/2 \)). Since we know that \( \omega(\mathcal{F}) = \beta/2 + \epsilon \) for some \( \epsilon > 0 \), it will be enough to show that:

**Proposition 5.1.** For \( n \) large enough, we have \( \delta_n(\Gamma) \in ]\delta_n(\mathcal{P}), \delta_n(\mathcal{P}) + \epsilon[ \).

**Proof.** Let \( p \) be a generator of \( \mathcal{P} \) and choose another parabolic element \( q \in \Gamma \) such that \( \Gamma \) is the free non abelian group over \( p \) and \( q \). Fix \( N \geq 2 \); each element \( \gamma \in \Gamma \setminus \{id\} \) can be written in a unique way as

\[
\gamma = p^{l_1}q^{m_1} \cdots p^{l_k}q^{m_k},
\]

where \( l_i, m_i \in \mathbb{Z}^* \) except for \( l_1 \) and \( m_k \) which may be zero. Given this decomposition, we select those \( l_i \) such that \( |l_i| \geq N \), say \( l_{i_1}, \ldots, l_{i_r} \), and write

\[
\gamma = Q_1p^{l_{i_1}}Q_2 \cdots p^{l_{i_r}}Q_r
\]

where each \( Q_i \) is a subword of the expression (15), containing powers of \( p \) and powers of \( q \) not exceeding \( N \) in absolute value. Note that decomposition (16) is still unique. We denote by \( \mathcal{Q}_N \) the subset of elements \( \gamma \in \Gamma \) which write simply \( \gamma = Q_1 \) in (16).

Now let \( o \in X \) and \( D \) be the Dirichlet domain for the action of \( \Gamma \), centered at \( o \). Roughly speaking, the union of the geodesic segments

\[
[o, Q_1(o)], [Q_1(o), Q_1p^{l_{i_1}}(o)], \ldots, [Q_1 \cdots p^{l_{i_r}}(o), \gamma(o)]
\]

represents a quasigeodesic which stays close to \([o, \gamma(o)]\) and each of its subsegments corresponds to the excursion of the geodesic loop \( \gamma \) alternatively outside or inside the cusp \( E \). We now precise this argument.

As \( K_X \leq -\alpha^2 + \eta \), there exits a minimal angle \( \theta_0 > 0 \) such that for all \( x \in p^{\pm 1}(D) \) and all \( y \in q^{\pm 1}(D) \), we have \( x^\infty \cdot y^\infty \geq \theta_0 \). Then, when \( Q_1 \neq id \) in (16), by a ping-pong argument we deduce that \( \angle_o(Q_1^{-1}(o), p^{l_{i_1}}Q_2 \cdots Q_r(o)) \geq \theta_0 \), as \( l_i \geq N \geq 2 \). Therefore, by Lemma 2.1, there exists a constant \( d = d(\alpha, \theta_0) > 0 \) such that

\[
d_n(o, \gamma(o)) \geq d_n(o, Q_1(o)) + d_n(o, p^{l_{i_1}}Q_2 \cdots p^{l_{i_r-1}}Q_r(o) - d
\]
Repeating this argument yields
\[ d_n(o, \gamma(o)) \geq \sum_{i=0}^{r} d_n(o, Q_i(o)) + \sum_{j=1}^{r-1} d_n(o, p^{l_j}(o)) - 2rd. \]

Consequently
\[
\sum_{\gamma \in \Gamma} e^{-sd_n(o, \gamma(o))} \leq \sum_{\gamma \in Q_N} e^{-sd_n(o, \gamma(o))} + \sum_{r \geq 1} \left( e^{2sd} \sum_{|k| \geq N} e^{-sd_n(o, p^k(o))} \sum_{\gamma \in Q_N} e^{-sd_n(o, \gamma(o))} \right)^r
\]

If \( n \) is large enough with respect to \( N \), every element of \( Q_N \) correspond to a geodesic loop staying in the part of \( S \) where the curvature is constant equal to \(-\alpha^2\). For that choice of \( n \) and \( s = \frac{\beta + \epsilon}{2} \), we have
\[
\sum_{\gamma \in Q_N} e^{-sd_n(o, \gamma(o))} \leq \sum_{\gamma \in \Gamma} e^{-sd(o, \gamma(o))} := A.
\]

The latter series converges because the value of the critical exponent of any lattice in the space of constant curvature case \(-\alpha^2\) is \( \alpha \) and \( \alpha < s \).

Furthermore
\[
\sum_{|k| \geq N} e^{-sd(o, p^k(o))} \leq \sum_{m \geq d(o, p^N(o))} v_P(o, m) e^{-sm}
\]
\[
\leq \sum_{m \geq d(o, p^N(o))} \frac{e^{-sm}}{A(m/2)}
\]
\[
\leq \sum_{m \geq d(o, p^N(o))} e^{-(s-\frac{\beta}{2})m} = \sum_{m \geq d(o, p^N(o))} e^{-\epsilon m/2}
\]

so that \( \sum_{|k| \geq N} e^{-sd(o, p^k(o))} \to 0 \) when \( N \to +\infty \). Then, we can choose \( N \) and \( n \) such that
\[
\sum_{\gamma \in Q_N} e^{-sd(o, \gamma(o))} \leq A < +\infty \quad \text{and} \quad \left( e^{2sd} \sum_{|k| \geq N} e^{-sd(o, p^k(o))} A \right) < 1.
\]

For that choice, (17) implies that the Poincaré series associated with \( \Gamma \) converges at \( s \) and consequently : \( \delta(\Gamma) \leq s < \delta(\Gamma) + \epsilon \).

**Remark.** Notice that the curvature of \( S \) is not asymptotically \( \frac{1}{4} \)-pinched as \( \beta > 2\alpha \); but, letting \( \alpha \to \beta/2 \) and \( \eta \to 0 \), the metric can be choosen so that \( K_S \) is asymptotically \( \left( \frac{1}{4+\epsilon} \right) \)-pinched, for any \( \epsilon > 0 \).

\[
\square
\]

### 6. Appendix

Let \( t_0, t_1, t_2, t_3 \) be four real numbers satisfying \( t_0 < t_1 < t_2 < t_3 \). Denote by \( \varphi_1 \) a \( C^2 \) convex and decreasing function on \( [t_0, t_1] \) and \( \varphi_2 \) a \( C^2 \) convex
and decreasing function on \([t_2, t_3]\). A straightforward geometric argument on epigraphs of \(\varphi_1\) and \(\varphi_2\) shows that the following inequalities:

\[
\varphi_1'(t_1)(t_2 - t_1) \leq \varphi_2(t_2) - \varphi_1(t_1) \leq \varphi_2'(t_2)(t_2 - t_1)
\]

are necessary and sufficient for the existence of a \(C^2\) convex decreasing function \(\psi\) on \([t_0, t_3]\) such that \(\psi|_{[t_0, t_1]} \equiv \varphi_1\) and \(\psi|_{[t_2, t_3]} \equiv \varphi_2\).

**Lemma 6.1.** Let \(\alpha, \beta\) two positive reals such that \(\alpha < \beta\).

(I) Inequalities (18) are satisfied for \(\varphi_1(t) = e^{-\alpha t}\) and \(\varphi_2(t) = e^{-\beta t}\) when \(t_2 - t_1 > \frac{1}{\alpha}\).

(II) Inequalities (18) are satisfied for \(\varphi_1(t) = e^{-\beta t}\) and \(\varphi_2(t) = e^{-\alpha t}\) when \(t_2 > (\frac{\beta}{\alpha} + \epsilon)t_1\) for any \(\epsilon > 0\).

**Proof.** Case (I):

\[(a) \iff e^{-\beta t_2 + \alpha t_1} + \alpha(t_2 - t_1) > 1\]

and the second inequality is satisfied when \(t_2 - t_1 > \frac{1}{\alpha}\). Note that this condition is optimal if we want such an inequality to be satisfied for arbitrary large \(t_1\) because with \(u = t_2 - t_1\), this inequality becomes

\[e^{(\alpha - \beta)t_1 - \beta u} + \alpha u > 1\]

and this inequality can be satisfied for small \(u\) when \(t_1\) is too large.

With the previous notations,

\[(b) \iff e^{\beta u}e^{(\beta - \alpha)t_1} - \beta u - 1 > 0\]

and the latter inequality is always satisfied because \(e^{x} - x - 1 > 0\) for all \(x > 0\).

Case (II):

\[(a) \iff e^{-\alpha t_2 + \beta t_1} + \beta(t_2 - t_1) > 1\]

and this second inequality is satisfied when \(t_2 - t_1 > \frac{1}{\beta}\). The same remark as in the case (I).

With the previous notations too,

\[(b) \iff e^{\alpha u}e^{(\alpha - \beta)t_1} - \alpha u - 1 > 0\]

with \(u = t_2 - t_1\). If we set \(t_2 = (1 + x)t_1 + f(t_1)\) and substitute in the last term, a necessary condition in order to realise (b) is \((x + 1) \geq \frac{\beta}{\alpha}\) and if we set \((x + 1) = \frac{\beta}{\alpha}\) and replace, we get \(e^{\alpha f(t_1)} - (\beta - \alpha)t_1 - f(t_1) - 1 > 0\). The conclusion follows.

\[\square\]

**Lemma 6.2.** Let \(t_0 < t_1 < t_2 < t_3\) and \(\eta > 0\). There exists \(A = A(\eta, \alpha, \beta) > 0\) and \(B = B(\alpha, \beta) > 0\) such that if \(t_2 > A.t_1\) and \(t_0 > B\),

(I) There exists a \(C^2\) convex and decreasing function \(\psi\) on \([t_0, t_3]\) satisfying:

\[
(C_1) \quad \begin{cases} 
\forall t \in [t_0, t_1], & \psi(t) = e^{-\alpha t} \\
\forall t \in [t_2, t_3], & \psi(t) = e^{-\beta t} \\
\forall t \in [t_0, t_3], & \alpha^2 - \eta \leq \frac{\psi'(t)}{\psi(t)} \leq \beta^2 - \eta \quad \text{and} \quad \psi(t) \geq e^{-\beta t}
\end{cases}
\]
(II) There exists a $C^2$ convex and decreasing function $\psi$ on $[t_0, t_3]$ such that we have

\[
\begin{align*}
(C_2) & \quad \forall t \in [t_0, t_1], \quad \psi(t) = e^{-\beta t} \\
& \quad \forall t \in [t_2, t_3], \quad \psi(t) = e^{-\alpha t} \\
& \quad \forall t \in [t_0, t_3], \quad \alpha^2 - \eta \leq \frac{\psi''(t)}{\psi(t)} \leq \beta^2 + \eta \quad \text{and} \quad \psi(t) \geq e^{-\beta t}
\end{align*}
\]

**Proof.** By the previous remark, if we choose $A > \frac{\beta}{\alpha}$ and $B > \frac{1}{\beta - \alpha}$, inequalities (18) are satisfied. In both cases, set

\[
\psi(t) = e^{-t\varphi(t)} \quad t \in [t_0, t_3]
\]

where $\varphi$ is constant on $[t_0, t_1]$ and $[t_2, t_3]$ (depending in an obvious way on case I or II). Consider a $C^2$ function $\phi : [0, 1] \to [\alpha, \beta]$; set $s = \lambda(t - t_1)$ where $\lambda = \frac{1}{t_2 - t_1}$ and put $\varphi(t) = \phi(s)$ for $t \in [t_1, t_2]$. A straightforward calculus gives, for $s \in [0, 1]$:

\[
\frac{\psi''(t)}{\psi(t)} = \frac{(s\phi(s)')' + \lambda t_1 \phi'(s)^2}{(s\phi(s))'} = \frac{(k'(s))^2 + \lambda t_1 (k'(s)^2 + \phi'(s) + \phi''(s)) - \phi'(s)^2 - \phi''(s))}{(k'(s))^2 + \theta(\lambda)}
\]

where $k(s) := s\phi(s)$ and $\theta$ is a function such that $\theta(\lambda) \to 0$ when $\lambda \to 0$.

Set $M_i = \sup_{s \in [0, 1]} |\phi^{(i)}(s)|$ for $i = 1, 2$ (which depend only on $(\alpha, \beta)$) and $C = \frac{1}{8(\beta + 1)(M_1 + M_2 + \beta)}$. The previous equalities imply

\[
(k'(s))^2 - \frac{\eta}{2} \leq \frac{\psi''(t)}{\psi(t)} \leq (k'(s))^2 + \frac{\eta}{2}
\]

when $\lambda t_1 < C\eta$ i.e. for $t_2 > (1 + \frac{1}{C\eta})t_1 := A.t_1$. We show in both cases that we can choose a $C^2$ function $\psi$ with values in $[\alpha, \beta]$ such that for all $s \in [0, 1]$:

\[
\alpha - \frac{\eta}{4} \leq k'(s) \leq \beta + \frac{\eta}{4}
\]

Case (I) : choose $\phi : [0, 1] \to [\alpha, \beta]$ non decreasing satisfying $\phi(0) = \alpha$, $\phi(1) = \beta$ and $\phi'(0) = \phi'(1) = \phi''(0) = \phi''(1) = 0$. Then, the function $\varphi$ can be extend on $[t_0, t_3]$ in a $C^2$ manner and on $[0, 1]$, we have $k'(s) = (s\phi(s))' = \phi'(s) + s\phi''(s)$ and $\phi(s) \leq \beta$ so that $\psi''/\psi \geq \alpha^2 - \eta$ and $\psi(t) \geq e^{-\beta t}$ are both satisfied on $[t_0, t_3]$. It implies in particular that the function $\psi$ constructed is convex on $[t_0, t_3]$. Note that in this case, the inequality $\lambda t_1 < C\eta$ must be satisfied, for, in the second expression of $\frac{\psi''}{\psi}$, the term $(t_1(\phi'(s))^2 - \phi''(s))$ is negative in the neighborhood of $s_0 = \inf\{s ; \phi'(s) = 0\}$.

It is left to show that $\phi$ or equivalently $k$ can be choosen so that $k'(s) = \phi'(s) + s\phi''(s) \leq \beta + \frac{\eta}{4}$. The boundary conditions for $\phi$ up to the first order translate to $k(0) = 0$, $k(1) = \beta$, $k'(0) = \alpha$ and $k'(1) = \beta$. For $\epsilon_1 \in [0, 1]$, consider the $C^0$-piecewise affine function $\tilde{k}$ defined on $[0, \epsilon_1]$ by $\tilde{k}(t) = \alpha t$, on $[1 - \epsilon_1, 1]$ by $\tilde{k}(t) = \beta t$ and affine on $[\epsilon_1, 1 - \epsilon_1]$. If we choose $\epsilon_1$ small enough (depending on $\eta$ and $\alpha$), we can smooth $\tilde{k}$ to obtain a $C^2$ function $k$ on $[0, 1]$ in such a way that the derivative $k'$ satisfies

\[
\begin{align*}
& \quad \left\{ \begin{array}{ll}
k'(s) = \alpha & s \in [0, \epsilon_1/2] \\
k'(s) \leq \beta + \eta/(4\beta) & s \in [\epsilon_1/2, 1 - \epsilon_1/2] \\
k'(s) = \beta & s \in [1 - \epsilon_1/2, 1]
\end{array} \right.
\]

\]
so that \((k'(s))^2 \leq \beta^2 - \eta/2\).

Case (II): this case is similar. We choose \(\phi : [0, 1] \to [\alpha, \beta]\) non increasing satisfying \(\phi(0) = \beta, \phi(1) = \alpha\) and \(\phi'(0) = \phi'(1) = \phi''(0) = \phi''(1) = 0\), or equivalently (up to the first order), we choose \(k(s) = s\phi(s)\) satisfying \(k(0) = 0, k(1) = \int_0^1 k'(s) ds = \alpha, k'(0) = \beta\) and \(k'(1) = \alpha\). The construction is symmetric to the previous one. In both cases, the desired inequalities: (20), (19) and \(e^{-\alpha t} \leq \psi(t) \leq e^{-\beta t}\) are satisfied.

Let us now construct the sequences of intervals \([p_n, q_n], [r_n, s_n]\) and the function \(A\) we used in Section 4. Let \(A > 1\) and \(B > 0\) given by Lemma 6.2.

We set

\[
\begin{align*}
p_n &= (1 - \lambda_0)\Delta_n^{n-1} + \lambda_0\Delta_n^n \quad \text{and} \quad r_n = \frac{p_n + \Delta_n^n}{2} \\
q_n &= (1 - \mu_0)\Delta_n^{n-1} + \mu_0\Delta_n^n \quad \text{and} \quad s_n = \frac{q_n + \Delta_n^n}{2}
\end{align*}
\]

for \(\Delta, \lambda_0\) and \(\mu_0\) to be defined.

Fix \((\lambda_0, \mu_0)\) in the (nonempty) set \((0, 1]\cap\{(\lambda, \mu) : 1 + \lambda - 2A\mu > 0 \land \mu > \lambda\}\). The polynomial function \(P(x) = 2\lambda_0 x^2 + ((2 - A) - 2\lambda_0 - A\mu_0)x - A(1 - \mu_0)\) tends to infinity as \(x \to +\infty\); thus, we can choose a positive real number \(\Delta\) such that both inequalities

\[
\begin{align*}
&\Delta > \frac{2A - 1 + \lambda_0 - 2A\mu_0}{1 + \lambda_0 - 2A\mu_0} \quad \text{(21)} \\
&P(q_0) > 0 \quad \text{(22)}
\end{align*}
\]

are satisfied.

Inequality (21) insures that \(r_n > Aq_n\) and inequality (22) insures that \(p_{n+1} > As_n\). By Lemma 6.2, there exists \(n_0 \in \mathbb{N}^*\) and a \(C^2\)-convex and decreasing function \(A\) on \([\Delta_n^{n_0-1}, +\infty[\) satisfying \(\frac{A''(t)}{A(t)} \geq \alpha^2 - \eta\) and \(A(t) \geq e^{-\beta t}\) for all \(t \in [\Delta_n^{n_0-1}, +\infty[\), and such that for \(n \geq n_0\), we have:

\[
\begin{align*}
&\forall t \in [p_n, q_n], \quad A(t) = e^{-\alpha t} \\
&\forall t \in [r_n, s_n], \quad A(t) = e^{-\beta t}
\end{align*}
\]

Note that by construction \(t \in [p_n, q_n] \iff \frac{t + R_n}{2} \in [r_n, s_n]\).


Bibliographie