

Local limit theorems on some non unimodular groups

Emile Le Page and Marc Peigné

Abstract. Let G_d be the semi-direct product of \mathbb{R}^{*+} and \mathbb{R}^d , $d \geq 1$ and let us consider the product group $G_{d,N} = G_d \times \mathbb{R}^N$, $N \geq 1$. For a large class of probability measures μ on $G_{d,N}$, one proves that there exists $\rho(\mu) \in]0, 1]$ such that the sequence of finite measures

$$\left\{ \frac{n^{(N+3)/2}}{\rho(\mu)^n} \mu^{*n} \right\}_{n \geq 1}$$

converges weakly to a non-degenerate measure.

Résumé. Soit G_d le produit semi-direct de \mathbb{R}^{*+} et de \mathbb{R}^d et $G_{d,N}$ le groupe produit $G_d \times \mathbb{R}^N$, $N \geq 0$. Pour une large classe de mesures de probabilité sur $G_{d,N}$ nous montrons qu'il existe $\rho(\mu) \in]0, 1]$ tel que la suite de mesures finies

$$\left\{ \frac{n^{(N+3)/2}}{\rho(\mu)^n} \mu^{*n} \right\}_{n \geq 1}$$

converge vaguement vers une mesure non nulle.

1. Introduction.

Fix two integers $d \geq 1$, $N \geq 0$ and choose a norm $\|\cdot\|$ on \mathbb{R}^d and \mathbb{R}^N (when $N \geq 1$). Let $G_{d,N}$ be the connected group $\mathbb{R}^{*+} \times \mathbb{R}^d \times \mathbb{R}^N$

with the composition law

$$\begin{aligned} \text{for all } g = (a, u, b), \text{ for all } g' = (a', u', b') \in G, \\ g \cdot g' = (a a', a u' + u, b + b'). \end{aligned}$$

We will note $g = (a(g), u(g), b(g))$ (or $g = (a, u, b)$ when there is no ambiguity). The group $(G_{d,N}, \cdot)$ is a non unimodular solvable group with exponential growth and the right Haar measure m_D on $G_{d,N}$ is

$$m_D(da du db_1 \cdots db_N) = \frac{da du db_1 \cdots db_N}{a}.$$

Note that $G_{d,0}$ is the semi-direct product of \mathbb{R}^{*+} and \mathbb{R}^d ; in particular $G_{1,0}$ is the affine group of the real line.

We consider a probability measure μ on G ; we denote by μ^{*n} its n^{th} power of convolution. Under quite general assumptions on μ we show that there exists $\rho(\mu) \in]0, 1[$ such that the sequence

$$\left\{ \frac{n^{(N+3)/2}}{\rho(\mu)^n} \mu^{*n} \right\}_{n \geq 0}$$

converges weakly to a non-degenerate measure. This problem has already been tackled by Ph. Bougerol in [3] where were established local limit theorems on some solvable groups with exponential growth; in particular, for a class R of probability measures μ on the affine group of the real line (that is $d = 1$ and $N = 0$) he showed that the sequence

$$\left\{ \frac{n^{3/2}}{\rho(\mu)^n} \mu^{*n} \right\}_{n \geq 0}$$

converges weakly to a non-degenerate measure. In [7] we extend this result to a quite large class of probability measures; the new ingredient in our proof was the fact that there exists closed connections between this problem and the theory of the fluctuations of a random walk on the real line. In the present paper, we extend this result to the case $N \geq 1$; we first obtain uniform upperbounds in the Local limit theorem for a random walk on \mathbb{R}^d and, secondly, we use a generalisation of the Wiener-Hopf's factorisation due to Ch. Sunyach [9].

This study is also related with the work by N. T. Varopoulos [10], [11] where upperbounds and lowerbounds for the asymptotic behaviour

of the convolution powers μ^{*n} of a large class of probability measures are given.

From now on, we will suppose that $N \geq 1$ and we set $G = G_{d,N}$. We introduce the following conditions on μ :

Hypothesis G1. *There exists $\alpha > 0$ such that*

$$\int_G (e^{\alpha|\log a|} + \|u\|^\alpha + \|b\|^2) \mu(da du db) < +\infty.$$

Hypothesis G2. $\int_G \text{Log } a \mu(da du db) = 0$ and $\int_G b \mu(da du db) = 0$.

Hypothesis G3. *The support of μ is included in $\mathbb{R}^{*+} \times (\mathbb{R}^+)^d \times \mathbb{R}^N$, the image of μ by the mapping $(a, u, b) \mapsto (\text{Log } a, b)$ is aperiodic in \mathbb{R}^{N+1} (see Definition 2.1) and there exists $\beta > 0$ such that*

$$\int_G \|u\|^{-\beta} \mu(da du db) < +\infty.$$

Hypothesis G'3. *The measure μ is absolutely continuous with respect to the Haar measure m_D on G and its density ϕ_μ satisfies*

$$\int_{]0,1] \times \mathbb{R}^N} \sqrt[q]{\int_{\mathbb{R}} \phi_\mu^q(a, u, b) du} \frac{da db}{a^\gamma} < +\infty.$$

for some γ and q in $]1, +\infty[$.

We have the

Theorem 1.1. *Let μ be a probability measure on G satisfying hypotheses G1, G2 and G3 (or G'3). Then, the sequence of finite measures $\{n^{(N+3)/2} \mu^{*n}\}_{n \geq 0}$ converges weakly to a non-degenerate Radon measure on G .*

Note that the asymptotic behavior of the sequence $\{\mu^{*n}\}_{n \geq 1}$ does not depend on d .

When μ is not centered, that is

$$\int_G \text{Log } a \mu(da du db) \neq 0$$

or

$$\int_G b \mu(da du db) \neq 0,$$

we bring back the study to the centered case as in [7]. We introduce the following conditions on μ :

Hypothesis G*1. *There exists $\alpha > 0$ such that*

$$\int_G (a^t + \|u\|^\alpha + \exp(t \|b\|)) \mu(da du db) < +\infty$$

for any $t \in \mathbb{R}$.

Hypothesis G*2. *One has*

$$\int_G \text{Log } a \mu(da du db) \neq 0$$

with $\mu\{g \in G : a(g) < 1\} > 0$ and $\mu\{g \in G : a(g) > 1\} > 0$.

When μ satisfies these two conditions, there exists a unique $(s_0, t_0) \in \mathbb{R} \times \mathbb{R}^N$ such that

$$\int_G a^{s_0} e^{\langle t_0, b \rangle} \mu(da du db) = \inf_{(s,t) \in \mathbb{R} \times \mathbb{R}^N} \int_G a^s e^{\langle t, b \rangle} \mu(da du db).$$

Furthermore,

$$\rho(\mu) = \int_G a^{s_0} e^{\langle t_0, b \rangle} \mu(da du db)$$

belongs to $]0, 1]$. Note that the probability measure

$$\mu_0(dg) = \frac{1}{\rho(\mu)} a(g)^{s_0} e^{\langle t_0, b(g) \rangle} \mu(dg)$$

satisfies hypotheses G1 and G2. The following condition is the equivalent of Hypothesis G'3 in the non centered case:

Hypothesis G*3. *The measure μ is absolutely continuous with respect to the Haar measure m_D on G and its density ϕ_μ satisfies*

$$\int_{]0,1] \times \mathbb{R}^N} \sqrt[q]{\int_{\mathbb{R}} \phi_\mu^q(a, u, b) du} \frac{da db}{a^\gamma} < +\infty$$

for some $q \in]1, +\infty[$ and $\gamma \in]1 - s_0, +\infty[$.

Theorem 1.2. *Let μ be a probability measure on G satisfying conditions G^*1 , G^*2 and $G3$ (or G^*3) and let*

$$\rho(\mu) = \inf_{(s,t) \in \mathbb{R} \times \mathbb{R}^N} \int_G a^s e^{\langle t,b \rangle} \mu(da du db).$$

Then, the sequence of finite measures

$$\left\{ \frac{n^{(N+3)/2}}{\rho(\mu)^n} \mu^{*n} \right\}_{n \geq 1}$$

weakly converges to a non-degenerate Radon measure on G .

The demonstration of Theorem 2.1 is closely related to the study of the fluctuations of a random walk $(X_1^n, Y_1^n)_{n \geq 0}$ on \mathbb{R}^{N+1} . In Section 2, we first state the classical local limit theorem on \mathbb{R}^{N+1} but we add in its statement uniform upperbounds relatively to the starting point of the random walk $(X_1^n, Y_1^n)_{n \geq 0}$. This result is thus very useful to obtain a precise equivalent in Theorem 2.5 of the joint law of the random walk $(X_1^n, Y_1^n)_{n \geq 0}$ with its first entrance time T_+ in the half space $\mathbb{R}^+ \times \mathbb{R}^N$; a local limit theorem for the process

$$(X_1^n, \max\{0, X_1^1, \dots, X_1^n\}, Y_1^n)_{n \geq 0}$$

is thus obtained (Theorem 2.6). In Section 3 we give the main steps of the proof of Theorem 1.1.

2. Fluctuations of a random walk on \mathbb{R}^{N+1} .

Fix an integer $N \geq 1$ and let $(X_1, Y_1), (X_2, Y_2), \dots$ be independent $\mathbb{R} \times \mathbb{R}^N$ -valued random variables with distribution p defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(X_1^n, Y_1^n)_{n \geq 0}$ be the associated random walk on $\mathbb{R} \times \mathbb{R}^N$ starting from $(0, 0)$ and defined by $X_1^0 = 0, Y_1^0 = 0$ and $X_1^n = X_1 + \dots + X_n, Y_1^n = Y_1 + \dots + Y_n$ for $n \geq 1$; the distribution of the couple (X_1^n, Y_1^n) is the n^{th} power of convolution p^{*n} of the measure p . Denote by \mathcal{F}_n the σ -algebra generated by $(X_1, Y_1), \dots, (X_n, Y_n), n \geq 1$.

Let us first recall the

Definition 2.1. Let p be a probability measure on \mathbb{R}^k , $k \geq 1$. The measure p is aperiodic on \mathbb{R}^k if there is no closed and proper subgroup H of \mathbb{R}^k and no $\alpha \in \mathbb{R}^k$ such that $p(\alpha + H) = 1$.

Denote by \hat{p} the characteristic function of p defined by $\hat{p}(u, v) = \mathbb{E}[e^{iuX_1 + ivY_1}]$ for any $(u, v) \in \mathbb{R} \times \mathbb{R}^N$. Recall that the probability measure p is aperiodic if and only if $|\hat{p}(u, v)| < 1$ for $(u, v) \neq (0, 0)$.

For any $\mathcal{A} \subset \mathbb{R} \times \mathbb{R}^N$ let $\{T_{\mathcal{A}}^{(k)}\}_{k \geq 0}$ be the successive times of visit of the random walk $(X_1^n, Y_1^n)_{n \geq 1}$ to the set \mathcal{A} ; one has $T_{\mathcal{A}}^{(0)} = 0$, $T_{\mathcal{A}}^{(1)} = \inf \{n \geq 1 : (X_1^n, Y_1^n) \in \mathcal{A}\}$ and $T_{\mathcal{A}}^{(k+1)} = \inf \{n \geq T_{\mathcal{A}}^{(k)} + 1 : (X_1^n, Y_1^n) \in \mathcal{A}\}$. Note that the $T_{\mathcal{A}}^{(k)}$ are stopping times with respect to the filtration $\{\mathcal{F}_n\}_{n \geq 1}$. We will associate to (p, \mathcal{A}) the transition kernel $P_{\mathcal{A}}$ defined by

$$P_{\mathcal{A}}((x, y), \mathcal{B}) = \int_{\mathbb{R} \times \mathbb{R}^N} \mathbf{1}_{\mathcal{A}^c \cap \mathcal{B}}(x + x', y + y') p(dx' dy'),$$

for any Borel set \mathcal{B} in $\mathbb{R} \times \mathbb{R}^N$; note that for any $k \geq 1$ one has $P_{\mathcal{A}}^k((0, 0), \mathcal{B}) = \mathbb{E}[[T_{\mathcal{A}} > k]; (X_1^k, Y_1^k) \in \mathcal{B}]$. In order to simplify the notations we will set $T_- = T_{\mathbb{R}^- \times \mathbb{R}^N}$, $P_- = P_{\mathbb{R}^- \times \mathbb{R}^N}$ and $T_-^{(k)} = T_{\mathbb{R}^- \times \mathbb{R}^N}^{(k)}$; similar notations will hold, with obvious modifications, when $\mathcal{A} = \mathbb{R}^{*-} \times \mathbb{R}^N$, $\mathbb{R}^+ \times \mathbb{R}^N$ and $\mathbb{R}^{*+} \times \mathbb{R}^N$.

Troughout this paragraph, for any $k \geq 1$, we denote by λ_k the Lebesgue measure on \mathbb{R}^k . Furthermore, for any $\delta > 0$, $\mathcal{H}_{\delta}(\mathbb{R}^k)$ is the space of \mathbb{C} -valued functions φ on \mathbb{R}^k such that

$$\sup_{x \in \mathbb{R}^k} (1 + \|x\|^{\delta})^k |\varphi(x)| < +\infty.$$

2.1. Preliminaries.

The local limit theorem gives the asymptotic behaviour of the sequence $\{p^{*n}(\varphi)\}_{n \geq 1}$ for any continuous function φ with compact support on \mathbb{R}^{N+1} ; we state it here and we precise some uniform upperbound for the sequence $\{p^{*n}(\varphi)\}_{n \geq 1}$ when φ belongs to $\mathcal{H}_{\delta}(\mathbb{R}^{N+1})$ with $\delta > 4$.

Theorem 2.2. Assume that:

i) the common distribution p of the variables (X_n, Y_n) , $n \geq 1$, is aperiodic on \mathbb{R}^{N+1} ,

ii) $\mathbb{E}[\|X_1\|^2 + \|Y_1\|^2] < +\infty$ and $\mathbb{E}[X_1] = 0, \mathbb{E}[Y_1] = 0$.

Then:

i) for any continuous function φ with compact support on \mathbb{R}^{N+1} one has

$$\begin{aligned} \lim_{n \rightarrow +\infty} n^{(N+1)/2} \mathbb{E}[\varphi(X_1^n, Y_1^n)] \\ = \frac{1}{(2\pi)^{(N+1)/2} \sqrt{|C|}} \int_{\mathbb{R}^{N+1}} \varphi(x, y) \lambda_1(dx) \lambda_N(dy), \end{aligned}$$

where $|C|$ denotes the determinant of the positive definite quadratic form

$$C(u, v) = \mathbb{E}[(u X_1 + \langle v, Y_1 \rangle)^2].$$

ii) For any function φ in $\mathcal{H}_\delta(\mathbb{R}^{N+1})$ with $\delta > 4$, the sequence $\{n^{(N+1)/2} \mathbb{E}[\varphi(x + X_1^n, y + Y_1^n)]\}_{n \geq 1}$ is bounded uniformly in $(x, y) \in \mathbb{R} \times \mathbb{R}^N$.

PROOF. The first assumption is the classical local limit theorem. To obtain the second claim, fix a non negative function ϕ whose Fourier transform has a compact support $K(\hat{\phi})$. Recall that

$$\hat{p}(u, v) = 1 - \frac{1}{2} C(u, v) (1 + \varepsilon(u, v))$$

with $\lim_{(u,v) \rightarrow (0,0)} \varepsilon(u, v) = 0$; so there exists $\delta > 0$ such that for $|u| + \|v\| < \delta$ one has

$$|\hat{p}(u, v)| \leq 1 - \frac{1}{4} C(u, v) \leq e^{-C(u,v)/4}.$$

On the other hand, by the aperiodicity of p there exists $\rho = \rho(p, K(\hat{\phi}))$ such that $|\hat{p}(u, v)| \leq \rho$ as soon as (u, v) belongs to $K(\hat{\phi})$ and $|u| + \|v\| \geq \delta$. It follows that

$$\begin{aligned} (2\pi n)^{(N+1)/2} \mathbb{E}[\phi(X_1^n, Y_1^n)] \\ \leq n^{(N+1)/2} \int_{|u| + \|v\| < \delta} |\hat{\phi}(u, v)| |\hat{p}(u, v)|^n \lambda_1(du) \lambda_N(dv) \\ + n^{(N+1)/2} \int_{|u| + \|v\| \geq \delta} |\hat{\phi}(u, v)| |\hat{p}(u, v)|^n \lambda_1(du) \lambda_N(dv) \end{aligned}$$

$$\begin{aligned}
&\leq n^{(N+1)/2} \int_{|u|+\|v\|<\delta n^{(N+1)/2}} \left| \hat{\phi}\left(\frac{u}{\sqrt{n}}, \frac{v}{\sqrt{n}}\right) \right| e^{-(n/4)C(u/\sqrt{n}, v/\sqrt{n})} \\
&\quad \cdot \lambda_1(du) \lambda_N(dv) \\
&\quad + n^{(N+1)/2} \rho^n \|\hat{\phi}\|_1 \\
&\leq \|\hat{\phi}\|_\infty \int_{\mathbb{R} \times \mathbb{R}^N} e^{-C(u,v)/4} \lambda_1(du) \lambda_N(dv) + n^{(N+1)/2} \rho^n \|\hat{\phi}\|_1 .
\end{aligned}$$

Now set $\phi_{x,y}(x', y') = \phi(x+x', y+y')$ for any $(x, y) \in \mathbb{R} \times \mathbb{R}^N$ and note that $\hat{\phi}_{x,y}(u, v) = e^{iux+i(v,y)} \hat{\phi}(u, v)$; the functions $\hat{\phi}_{x,y}$ and $\hat{\phi}$ thus have the same compact support and satisfies the equalities $\|\hat{\phi}_{x,y}\|_1 = \|\hat{\phi}\|_1$ and $\|\hat{\phi}_{x,y}\|_\infty = \|\hat{\phi}\|_\infty$. For any $(x, y) \in \mathbb{R} \times \mathbb{R}^N$ one thus has

$$\begin{aligned}
&|(2\pi n)^{(N+1)/2} \mathbb{E}[\phi_{x,y}(X_1^n, Y_1^n)]| \\
&\leq \|\hat{\phi}\|_\infty \int_{\mathbb{R} \times \mathbb{R}^N} e^{-C(u,v)/4} \lambda_1(du) \lambda_N(dv) + n^{(N+1)/2} \rho^n \|\hat{\phi}\|_1 .
\end{aligned}$$

The assertion ii) thus holds for any function ϕ whose Fourier transform has a compact support. To achieve the proof of ii) it suffices to show that for any function φ in $\mathcal{H}_\delta(\mathbb{R}^{N+1})$ with $\delta > 4$ there exists a function ϕ whose Fourier transform has a compact support and $|\varphi| \leq \phi$. It is an immediate consequence of the following result; we thank here J. P. Conze for helpful discussions about this fact.

Lemma 2.3. *Set*

$$h_\varepsilon(x) = \frac{1}{1 + |x|^{4+\varepsilon}} ,$$

for any $x \in \mathbb{R}$. If $\varepsilon > 0$ there exists a function \overline{h}_ε greater than h_ε and whose Fourier transform has a compact support in \mathbb{R} .

PROOF. Set

$$\overline{h}_\varepsilon(x) = C \left(\frac{\sin^2 x}{x^2} + \frac{\sin^2 \alpha x}{x^2} \right)$$

for some α and C in \mathbb{R}^{*+} which will depend on ε . Assume $\alpha \notin \mathbb{Q}$, the function \overline{h}_ε is strictly positive on \mathbb{R} ; it thus suffices to show that there exists $\alpha \notin \mathbb{Q}$ such that

$$\lim_{x \rightarrow +\infty} x^{2+\varepsilon} (\sin^2 x + \sin^2(\alpha x)) = +\infty .$$

If such a real did not exist, then for any $\alpha \notin \mathbb{Q}$ there should exist a sequence $\{x_n\}_{n \geq 1}$ which tends to $+\infty$ and a constant $C_\varepsilon > 0$ such that for all $n \geq 1$,

$$\sin^2 x_n + \sin^2(\alpha x_n) \leq \frac{C}{x_n^{2+\varepsilon}} .$$

So there should exist two strictly increasing sequences of integers $\{k_n\}_{n \geq 1}$ and $\{l_n\}_{n \geq 1}$ such that

$$|x_n - k_n \pi| \leq \frac{C'}{x_n^{1+\varepsilon/2}} , \quad |\alpha x_n - l_n \pi| \leq \frac{C'}{x_n^{1+\varepsilon/2}}$$

which implies

$$\left| \alpha - \frac{l_n}{k_n} \right| \leq \frac{C''}{k_n^{2+\varepsilon/2}}$$

for some positive constants C' and C'' . This leads to a contradiction because for almost all $\alpha \in \mathbb{R}$ (with respect with the Lebesgue measure), this last inequality has at most a finite number of solutions in \mathbb{N}^2 [2]. The lemma is proved.

2.2. A local limit theorem for a killed random walk on a half space.

In [7], we proved the following

Theorem 2.4. *Let the hypotheses of Theorem 2.2 hold. Then for any continuous function with compact support φ on \mathbb{R}^- we have*

$$\lim_{n \rightarrow +\infty} n^{3/2} \mathbb{E}[[T_+ > n]; \varphi(X_1^n)] = \frac{1}{\sigma(X_1) \sqrt{2\pi}} \int_{-\infty}^0 \varphi(x) \lambda_1^- * U^{*-}(dx) ,$$

where λ_1^- denotes the restriction of the Lebesgue measure on \mathbb{R}^- and U^{*-} is the σ -finite measure on \mathbb{R}^- defined by

$$U^{*-}(\mathcal{B}) = \sum_{k=1}^{+\infty} \mathbb{E}[\mathbf{1}_{\mathcal{B}}(X_1^{T_*^{(k)}})]$$

for any Borel set \mathcal{B} . In the same way, one has

$$\lim_{n \rightarrow +\infty} n^{3/2} \mathbb{E}[[T_{*+} > n]; \varphi(X_1^n)] = \frac{1}{\sigma(X_1) \sqrt{2\pi}} \int_{-\infty}^0 \varphi(x) \lambda_1^- * U^-(dx) ,$$

where U^- is the σ -finite measure on \mathbb{R}^- defined by

$$U^-(\mathcal{B}) = \sum_{k=1}^{+\infty} \mathbb{E}[\mathbf{1}_{\mathcal{B}}(X_1^{T_1^{(k)}})]$$

for any Borel set \mathcal{B} .

Recall that the random walks $\{X_1^{T_1^{(k)}}\}_{k \geq 1}$ and $\{X_1^{T_{*+}^{(k)}}\}_{k \geq 1}$ are transient on \mathbb{R}^- ; it follows that the series $\sum_{k=0}^{+\infty} \mathbb{E}[[T_+ > k]; \varphi(x + X_1^k)]$ and $\sum_{k=0}^{+\infty} \mathbb{E}[[T_{*+} > k]; \varphi(x + X_1^k)]$ do converge. Furthermore one has

$$\sum_{k=0}^{+\infty} \mathbb{E}[[T_+ > k]; \varphi(x + X_1^k)] = \int_{-\infty}^0 \varphi(x) U^{*-}(dx)$$

and

$$\sum_{k=0}^{+\infty} \mathbb{E}[[T_{*+} > k]; \varphi(x + X_1^k)] = \int_{-\infty}^0 \varphi(x) U^-(dx).$$

Let us now state the following

Theorem 2.5. *Let the hypotheses of Theorem 2.2 hold. Then:*

i) *For any continuous function φ with compact support on $\mathbb{R}^- \times \mathbb{R}^N$ one has*

$$\begin{aligned} & \lim_{n \rightarrow +\infty} n^{(N+3)/2} \mathbb{E}[[T_+ > n]; \varphi(X_1^n, Y_1^n)] \\ &= \frac{1}{(2\pi)^{(N+1)/2} \sqrt{|C|}} \int_{\mathbb{R}^- \times \mathbb{R}^N} \varphi(x, y) \lambda_1^- * U^{*-}(dx) \lambda_N(dy). \end{aligned}$$

ii) *For any continuous function f with compact support on \mathbb{R} and any g in $\mathcal{H}_\delta(\mathbb{R}^N)$ with $\delta > 4$, the sequence*

$$\{n^{(N+3)/2} \mathbb{E}[[T_+ > n]; f(X_1^n) g(y + Y_1^n)]\}_{n \geq 1}$$

is bounded, uniformly in $y \in \mathbb{R}^N$.

In the same way, one has

$$\begin{aligned} & \lim_{n \rightarrow +\infty} n^{(N+3)/2} \mathbb{E}[[T_{*+} > n]; \varphi(X_1^n, Y_1^n)] \\ &= \frac{1}{(2\pi)^{(N+1)/2} \sqrt{|C|}} \int_{\mathbb{R}^- \times \mathbb{R}^N} \varphi(x, y) \lambda_1^- * U^-(dx) \lambda_N(dy) \end{aligned}$$

and the sequence

$$\{n^{(N+3)/2} \mathbb{E} [[T_+ > n]; f(X_1^n) g(y + Y_1^n)]\}_{n \geq 1}$$

is bounded, uniformly in $y \in \mathbb{R}^N$.

PROOF. We prove this theorem by induction over N . Theorem 2.2 deals with the case $N = 0$; we will suppose that this result hold for some $N \geq 0$ and we consider a sequence $(X_n, Y_n, Z_n)_{n \geq 1}$ of independent identically distributed random variables on $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$. By a classical argument in probability theory, it suffices to show the above convergence hold for $\varphi(x, y, z) = e^{ax} \mathbf{1}_{\mathbb{R}^-}(x) \phi(y) \psi(z)$ where $a \in \mathbb{R}^{*+}$ and ϕ, ψ are \mathbb{C} -valued functions whose Fourier transform are continuous with compact supports. By the inverse Fourier transform one has

$$\begin{aligned} I_n &= \mathbb{E} [[T_+ > n]; e^{aX_1^n} \phi(Y_1^n) \psi(Z_1^n)] \\ &= \frac{1}{(2\pi)^{(N+1)/2}} \int_{\mathbb{R}^N \times \mathbb{R}} \hat{\phi}(v) \hat{\psi}(w) \alpha_n(a, v, w) \lambda_N(dv) \lambda_1(dw) \end{aligned}$$

with $\alpha_n(a, v, w) = \mathbb{E} [[T_+ > n]; e^{aX_1^n + i\langle v, Y_1^n \rangle + iwZ_1^n}]$.

The Spitzer's factorisation for random walks on \mathbb{R} gives for all $a > 0$, for all $s \in [0, 1[$

$$\sum_{n=0}^{+\infty} s^n \mathbb{E} [[T_+ > n]; e^{aX_1^n}] = \exp \left(\sum_{n=1}^{+\infty} \frac{s^n}{n} \mathbb{E} [[X_1^n < 0]; e^{aX_1^n}] \right).$$

Using the fact that $\mathbb{R}^+ \times \mathbb{R}^{N+1}$ and $\mathbb{R}^{*-} \times \mathbb{R}^{N+1}$ are semi-groups in \mathbb{R}^{N+2} , Ch. Sunyach extended this factorisation to the multidimensional case ([9, Corollary 3, p. 553 and Theorem 5, p. 556]); for any $a > 0$, $v \in \mathbb{R}^N$, $w \in \mathbb{R}$ and $s \in [0, 1[$ one thus has

$$\begin{aligned} \sum_{n=0}^{+\infty} s^n \mathbb{E} [[T_+ > n]; e^{aX_1^n + i\langle v, Y_1^n \rangle + iwZ_1^n}] \\ = \exp \left(\sum_{n=1}^{+\infty} \frac{s^n}{n} \mathbb{E} [[X_1^n < 0]; e^{aX_1^n + i\langle v, Y_1^n \rangle + iwZ_1^n}] \right) \end{aligned}$$

that is

$$(n + 1) \alpha_{n+1}(a, v, w) = \sum_{k=0}^n \beta_{n+1-k}(a, v, w) \alpha_k(a, v, w)$$

with $\beta_n(a, v, w) = \mathbb{E}[[X_1^n < 0]; e^{aX_1^n + i\langle v, Y_1^n \rangle + iwZ_1^n}]$. Finally

$$I_n = \frac{1}{n+1} \sum_{k=0}^n I_{n,k}$$

with

$$I_{n,k} = \frac{1}{(2\pi)^{(N+1)/2}} \int_{\mathbb{R}^N \times \mathbb{R}} \beta_{n+1-k}(a, v, w) \alpha_k(a, v, w) \\ \cdot \hat{\phi}(v) \hat{\psi}(w) \lambda_N(dv) \lambda_1(dw).$$

Set

$$I = \frac{1}{(2\pi)^{(N+2)/2} \sqrt{|C|}} \int_{\mathbb{R}^N \times \mathbb{R}} \sum_{k=0}^{+\infty} \mathbb{E}[[T_+ > k]; \frac{e^{aX_1^k}}{a}] \\ \cdot \phi(y) \psi(z) \lambda_N(dy) \lambda_1(dz),$$

since

$$I = \lambda_1^- * U^{*-} (e^{a\cdot}) \lambda_N(\phi) \lambda_1(\psi),$$

it suffices to show that $\{n^{(N+4)/2} I_n\}_{n \geq 1}$ converges to I , that is

- 1) for all $k > 0$, $\lim_{n \rightarrow +\infty} n^{(N+2)/2} I_{n,k} = I_{*k}$,
- 2) $\sum_{k=0}^{+\infty} |I_{*k}| < +\infty$ and $\sum_{k=0}^{+\infty} I_{*k} = I$,
- 3) $\limsup_{l \rightarrow +\infty} \limsup_{n \rightarrow +\infty} n^{(N+2)/2} \sum_{k=l}^n |I_{n,k}| = 0$.

To prove the assertion 1), note that

$$I_{n,k} = \mathbb{E}[[T_+ > k] \cap [X_{k+1}^{n+1} > 0]; e^{aX_1^{n+1}} \phi(Y_1^{n+1}) \psi(Z_1^{n+1})],$$

by the local limit theorem on \mathbb{R}^{N+2} the assertion 1) follows with

$$I_{*k} = \frac{1}{2\pi^{(N+2)/2} \sqrt{|C|}} \frac{\mathbb{E}[[T_+ > k]; e^{aX_1^k}]}{a} \\ \cdot \int_{\mathbb{R}^N} \phi(y) \lambda_N(dy) \int_{\mathbb{R}} \psi(z) \lambda_1(dz).$$

The fact that the series $\sum_{k=0}^{+\infty} |I_{*k}|$ converges is a direct consequence of Theorem 2.4. To prove the assertion 3), note that

$$\begin{aligned}
 |I_{n,k}| &\leq \mathbb{E}[[T_+ > k] \cap [X_{k+1}^{n+1} < 0]; e^{aX_1^{n+1}} |\phi(Y_1^{n+1})| |\psi(Z_1^{n+1})|] \\
 &\leq \mathbb{E}\left[[T_+ > k]; e^{aX_1^k} \int_{\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}} e^{ax} |\phi(y + Y_1^k)| \right. \\
 &\quad \left. \cdot |\psi(z + Z_1^k)| p^{*(n+1-k)}(dx dy dz)\right] \\
 &\leq \frac{C(a, \phi, \psi)}{(n+1-k)^{(N+2)/2}} \mathbb{E}[[T_+ > k]; e^{aX_1^k}] \quad \text{by Theorem 2.2.ii)} \\
 &\leq \frac{C_1}{(n+1-k)^{(N+2)/2} k^{3/2}} \quad \text{by Theorem 2.4.}
 \end{aligned}$$

On the other hand

$$\begin{aligned}
 |I_{n,k}| &\leq \|\psi\|_\infty \int_{\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}} \mathbb{E}[[T_+ > k]; e^{aX_1^k} |\psi(y + Y_1^k)| \\
 &\quad \cdot e^{ax} p^{*(n+1-k)}(dx dy dz)] \\
 &\leq \frac{\|\psi\|_\infty C(a, \phi)}{k^{(N+3)/2}} \\
 &\quad \cdot \mathbb{E}[[X_{k+1}^{n+1} < 0]; e^{aX_{k+1}^{n+1}}] \quad \text{by hypothesis of induction} \\
 &\leq \frac{C_2}{k^{(N+3)/2} \sqrt{n+1-k}}.
 \end{aligned}$$

The assertion 3) follows since for any $\varepsilon > 0$ one has

$$\begin{aligned}
 n^{(N+2)/2} \sum_{k=l}^n |I_{n,k}| &\leq C_1 \sum_{k=l}^{[n(1-\varepsilon)]} \frac{n^{(N+2)/2}}{k^{3/2} (n+1-k)^{(N+2)/2}} \\
 &\quad + C_2 \sum_{[n(1-\varepsilon)+1]}^n \frac{n^{(N+2)/2}}{k^{(N+3)/2} \sqrt{n+1-k}} \\
 &\leq \frac{C_1}{\varepsilon^{(N+2)/2}} \sum_{k=l}^{[n(1-\varepsilon)]} \frac{1}{k^{3/2}} \\
 &\quad + \frac{C_2}{\sqrt{n} (1-\varepsilon)^{(N+3)/2}} \sum_{[n(1-\varepsilon)+1]}^n \frac{1}{\sqrt{n+1-k}}
 \end{aligned}$$

$$\leq C \left(\frac{1}{\varepsilon^{(N+2)/2} \sqrt{l}} + \frac{\sqrt{\varepsilon}}{(1-\varepsilon)^{(N+3)/2}} \right).$$

Since ε is arbitrarily small, the assertion 3) follows.

The proof of ii) is also made by induction over N . If $g \in \mathcal{H}_\delta(\mathbb{R}^{N+1})$ there exist $\phi \in \mathcal{H}_\delta(\mathbb{R}^N)$ and $\psi \in \mathcal{H}_\delta(\mathbb{R}^1)$ such that $|g| \leq \phi \otimes \psi$. We set

$$I_n(y, z) = \mathbb{E}[[T_+ > n]; e^{aX_1^n} \phi(y + Y_1^n) \psi(z + Z_1^n)]$$

and we have

$$I_n(y, z) = \frac{1}{n+1} \sum_{k=0}^n I_{n,k}(y, z)$$

with

$$I_{n,k}(y, z) = \mathbb{E}[[T_+ > k] \cap [X_{k+1}^{n+1} < 0]; e^{aX_1^{n+1}} \phi(y + Y_1^{n+1}) \psi(z + Z_1^{n+1})].$$

As above, one has

$$|I_{n,k}(y, z)| \leq \inf \left\{ \frac{C_1}{(n+1-k)^{(N+2)/2} k^{3/2}}, \frac{C_2}{k^{(N+3)/2} \sqrt{n+1-k}} \right\}$$

which proves that the sequence

$$\left\{ n^{(N+2)/2} \sum_{k=0}^n |I_{n,k}(y, z)| \right\}_{n \geq 1}$$

is uniformly bounded in y, z . This achieves the proof of ii).

The convergence of the sequence

$$\{n^{(N+3)/2} \mathbb{E}[[T_{*+} > n]; \varphi(X_1^n, Y_1^n)]\}_{n \geq 1}$$

is obtained with similar arguments.

2.3. Behaviour of the process $((X_1^n, \max\{0, X_1^1, \dots, X_1^n\}, Y_1^n))_{n \geq 0}$.

For any $n \geq 0$ set $\mathcal{X}_1^n = \max\{0, X_1^1, \dots, X_1^n\}$ and let T_n be the random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ by $T_n = \inf\{0 \leq k \leq n : \mathcal{X}_1^n =$

X_1^k }; for any continuous function φ with compact support on \mathbb{R}^{N+1} we have

$$\begin{aligned}
 & \mathbb{E}[\varphi(\mathcal{X}_1^n, \mathcal{X}_1^n - X_1^n, Y_1^n)] \\
 &= \sum_{k=0}^n \mathbb{E}[[T_n = k]; \varphi(X_1^k, -X_{k+1}^n, Y_1^n)] \\
 &= \sum_{k=0}^n \mathbb{E}[[0 < X_1^k, X_1^1 < X_1^k, \dots, X_1^{k-1} < X_1^k, \\
 &\quad X_1^{k+1} \leq X_1^k, \dots, X_1^n \leq X_1^k]; \varphi(X_1^k, -X_{k+1}^n, Y_1^n)] \\
 &= \sum_{k=0}^n \mathbb{E}[[X_1^1 > 0, \dots, X_1^k > 0] \cap [X_{k+1}^{k+1} \leq 0, \dots, X_{k+1}^n \leq 0]; \\
 &\quad \varphi(X_1^k, -X_{k+1}^n, Y_1^n)].
 \end{aligned}$$

One obtains the following factorisation

$$\mathbb{E}[\varphi(\mathcal{X}_1^n, \mathcal{X}_1^n - X_1^n, Y_1^n)] = \sum_{k=0}^n J_{n,k}(\varphi)$$

with

$$\begin{aligned}
 & J_{n,k}(\varphi) \\
 &= \int_{\mathbb{R}^{N+1}} \varphi(x, -x', y + y') P_-^k((0, 0), dx dy) P_{*+}^{n-k}((0, 0), dx' dy').
 \end{aligned}$$

The behaviour of the process $(\mathcal{X}_1^n, \mathcal{X}_1^n - X_1^n, Y_1^n)$ is thus closely related to the one of the iterates of the transition kernels P_- and P_{*+} . Using this factorisation one proves the

Theorem 2.6. *Suppose that the hypotheses of Theorem 2.2 hold.*

Then, for any continuous function with compact support on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^N$ the sequence

$$\{n^{(N+3)/2} \mathbb{E}[\varphi(\mathcal{X}_1^n, \mathcal{X}_1^n - X_1^n, Y_1^n)]\}_{n \geq 1}$$

converges to

$$\begin{aligned} & \frac{1}{(2\pi)^{(N+1)/2} \sqrt{|C|}} \int_{\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^N} \varphi(s, -t, y) U^{*+}(ds) \lambda_1^- * U^-(dt) \lambda_N(dy) \\ & + \frac{1}{(2\pi)^{(N+1)/2} \sqrt{|C|}} \\ & \cdot \int_{\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^N} \varphi(s, -t, y) \lambda_+ * U^{*+}(ds) U^-(dt) \lambda_N(dy). \end{aligned}$$

Furthermore, for any continuous function f with compact support on $\mathbb{R}^+ \times \mathbb{R}^+$ and any g in $\mathcal{H}_\delta(\mathbb{R}^N)$, the sequence

$$\{n^{(N+3)/2} \mathbb{E}[f(\mathcal{X}_1^n, \mathcal{X}_1^n - X_1^n) g(y + Y_1^n)]\}_{n \geq 1}$$

is bounded, uniformly in $y \in \mathbb{R}^N$.

PROOF. We only proof the first assertion; the second one may obtained with obvious modifications as in Theorem 2.5. Set $\varphi(x, t, y) = \varphi_1(x) \varphi_2(t) \varphi_3(y)$ where φ_1, φ_2 and φ_3 are continuous with compact support. Fix $k \geq 0$; by Theorem 2.5, the sequence

$$\left\{ n^{(N+3)/2} \int_{\mathbb{R}^- \times \mathbb{R}^N} \varphi_2(x') \varphi_3(y + y') P_{*+}^{n-k}((0, 0), dx' dy') \right\}_{n \geq 1}$$

is bounded uniformly in $y \in \mathbb{R}^N$ and converges to

$$\frac{1}{(2\pi)^{(N+1)/2} \sqrt{|C|}} \int_{-\infty}^0 \varphi_2(-t) \lambda_1^- * U^-(dt) \lambda_N(\varphi_3).$$

By the dominated convergence theorem, one thus obtains, for any fixed $i \geq 1$

$$\begin{aligned} & \lim_{n \rightarrow +\infty} n^{(N+3)/2} \sum_{k=0}^i J_{n,k}(\varphi) \\ & = \frac{1}{(2\pi)^{(N+1)/2} \sqrt{|C|}} \sum_{k=0}^i \mathbb{E}[[T_- > k]; \varphi_1(X_1^k)] \\ & \cdot \int_{-\infty}^0 \varphi_2(-t) \lambda_1^- * U^-(dt) \lambda_N(\varphi_3). \end{aligned}$$

In the same way one has

$$\begin{aligned} \lim_{n \rightarrow +\infty} n^{(N+3)/2} \sum_{k=n-i+1}^n J_{n,k}(\varphi) \\ = \frac{1}{(2\pi)^{(N+1)/2} \sqrt{|C|}} \sum_{k=0}^i \mathbb{E}[[T_{*+} > k]; \varphi_2(-X_1^k)] \\ \cdot \lambda_1^+ * U^{*+}(\varphi_1) \lambda_N(\varphi_3). \end{aligned}$$

Note that the sums $\sum_{k=0}^i \mathbb{E}[[T_- > k]; \varphi_1(X_1^k)]$ and $\sum_{k=0}^i \mathbb{E}[[T_{*+} > k]; \varphi_2(X_1^k)]$ converges respectively to $U^{*+}(\varphi_1)$ and $\int_{-\infty}^0 \varphi_2(-t) U^-(dt)$. To obtain the theorem it suffices to check that

$$\limsup_{i \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \left| n^{(N+3)/2} \sum_{k=i+1}^{n-i} J_{n,k}(\varphi) \right| = 0,$$

one has

$$\begin{aligned} \left| n^{(N+3)/2} \sum_{k=i+1}^{[n/2]} J_{n,k}(\varphi) \right| \\ \leq n^{(N+3)/2} \sum_{k=i+1}^{[n/2]} \mathbb{E}[[T_- > k]; |\varphi_1(X_1^k)|] \\ \cdot \int_{\mathbb{R}^- \times \mathbb{R}^N} |\varphi_2(x')| |\varphi_3(y+y')| P_{*+}^{n-k}((0,0), dx' dy') \\ \leq C(\varphi_2, \varphi_3) \sum_{k=i+1}^{[n/2]} \mathbb{E}[[T_- > k]; |\varphi_1(X_1^k)|] \\ \cdot \frac{n^{(N+3)/2}}{(n-k)^{(N+3)/2}} \quad \text{by Theorem 2.5.ii)} \\ \leq C(\varphi) \sum_{k=i+1}^{+\infty} \frac{1}{k^{3/2}}. \end{aligned}$$

The same upperbound holds for the term

$$n^{(N+3)/2} \sum_{k=[n/2]+1}^{n-i} J_{n,k}(\varphi).$$

This achieves the proof.

3. A local limit theorem for a particular class of solvable groups.

Recall that $G = G_{d,N} = \mathbb{R}^{*+} \times \mathbb{R}^d \times \mathbb{R}^N$ with the composition law

$$g \cdot g' = (a a', a u' + u, b + b'),$$

for all $g = (a, u, b)$, for all $g' = (a', u', b') \in G_{d,N}$.

The proof of Theorem 1.1 is closed to the one of the local limit theorem for the affine group of the real line given in [7]; we just give here the main steps of the demonstration.

Let us first introduce some helpful notations. Let $g_n = (a_n, u_n, b_n)$, $n = 1, 2, \dots$ be independent and identically distributed random variables with distribution μ . Denote by \mathcal{F}_n the σ -algebra generated by the variables g_1, g_2, \dots, g_n , $n \geq 1$. For any $n \geq 1$, set $G_1^n = g_1 \cdots g_n = (A_1^n, U_1^n, B_1^n)$; we have $A_1^n = a_1 \cdots a_n$, $U_1^n = \sum_{k=1}^n a_1 \cdots a_{k-1} u_k$ and $B_1^n = b_1 + \cdots + b_n$. More generally, if $1 \leq m \leq n$, set $A_m^n = a_m \cdots a_n$, $U_m^n = \sum_{k=m}^n a_m \cdots a_{k-1} u_k$, $B_m^n = b_m + \cdots + b_n$ and set $A_m^n = 1$, $U_m^n = 0$, $B_m^n = 0$ otherwise.

Let $\tilde{\mu}$ be the image of μ by the map

$$g = (a, u, b) \mapsto \tilde{g} = \left(\frac{1}{a}, \frac{u}{a}, b \right),$$

if $\tilde{g}_n = (\tilde{a}_n, \tilde{u}_n, \tilde{b}_n)$, $n = 1, 2, \dots$ are independent and identically distributed random variables with distribution $\tilde{\mu}$ on G , set $\tilde{G}_m^n = \tilde{g}_m \cdots \tilde{g}_n = (\tilde{A}_m^n, \tilde{U}_m^n, \tilde{B}_m^n)$.

In order to obtain the asymptotic behaviour of the power of convolution μ^{*n} we use the fact that the sequence $\{U_1^n\}_{n \geq 1}$ behaves like the maximum of the variables A_1^1, \dots, A_1^n . These idea was already used in [7]. Set $\mathcal{A} = \{g = (a, u, b) \in G : a > 1\}$ and consider the transition kernel $P_{\mathcal{A}}$ associated with (μ, \mathcal{A}) and defined by

$$P_{\mathcal{A}}(g, \mathcal{B}) = \int_G \mathbf{1}_{\mathcal{A}^c \cap \mathcal{B}}(gh) \mu(dh)$$

for any Borel set $\mathcal{B} \subset G$ and any $g \in G$. The probabilistic interpretation of $P_{\mathcal{A}}$ is the following one: if $T_{\mathcal{A}} = \inf \{n \geq 1 : G_1^n \in \mathcal{A}\}$ is the first entrance time in \mathcal{A} of the random walk $\{G_1^n\}_{n \geq 0}$ then

$$P_{\mathcal{A}}^n(e, \mathcal{B}) = \mathbb{P}[[T_{\mathcal{A}} > n] \cap [G_1^n \in \mathcal{B}]], \quad \text{for all } n \geq 1.$$

In the same way, set $\mathcal{A}' = \{g \in G : a(g) \geq 1\}$, let $\tilde{P}_{\mathcal{A}'}$ be the operator associated with $(\tilde{\mu}, \mathcal{A}')$ and denote by $\tilde{T}_{\mathcal{A}'}$ the first entrance time in \mathcal{A}' of the random walk $\{G_1^n\}_{n \geq 1}$; one has

$$\tilde{P}_{\mathcal{A}'}^n(e, \mathcal{B}) = \mathbb{P}[[\tilde{T}_{\mathcal{A}'} > n] \cap [\tilde{G}_1^n \in \mathcal{B}]], \quad \text{for all } n \geq 1.$$

As in Section 2.3, we introduce the first time at which the random walk $\{A_1^n\}_{n \geq 1}$ reaches its maximum on \mathbb{R}^{*+} ; for any continuous function φ with compact support on G , we thus obtain

$$\mathbb{E}[\varphi(G_1^n)] = \sum_{k=0}^n I_{n,k}(\varphi),$$

where

$$I_{n,k}(\varphi) = \int_{G \times G} \varphi\left(\frac{a'}{a}, \frac{u+u'}{a}, b+b'\right) \tilde{P}_{\mathcal{A}'}^k(e, da du db) P_{\mathcal{A}}^{n-k}(e, da' du' db').$$

We now give the main steps of the proof of Theorem 1.1 under hypothesis G1, G2 and G3.

First step. *Control of the central terms of the sum $\sum_{k=0}^n I_{n,k}(\varphi)$.*

We show here that

$$\limsup_{i \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \sum_{k=i}^{n-i} I_{n,k}(\varphi) = 0.$$

Without loss of generality, one may suppose that the support of φ is included in $\mathbb{R}^{*+} \times (\mathbb{R}^{*+})^d \times \mathbb{R}^N$; for any $\varepsilon > 0$ there exist a constant $C > 0$ and a positive function ϕ with compact support on \mathbb{R}^N such that

$$\varphi(a, u, b) \leq C \frac{a^\varepsilon}{\|u\|^{2\varepsilon}} \phi(b),$$

it follows that for any (α, β) in $\mathbb{R}^{*+} \times \mathbb{R}^N$

$$\begin{aligned} & \mathbb{E}\left[[T_{\mathcal{A}} > l]; \varphi\left(\frac{A_1^l}{\alpha}, \frac{u + U_1^l}{\alpha}, \beta + B_1^l\right)\right] \\ & \leq C \alpha^\varepsilon \mathbb{E}\left[[a_1 \leq 1] \cap \left[\max\{A_2^2, \dots, A_2^l\} \leq \frac{1}{a_1}\right]; \frac{(A_1^l)^\varepsilon}{\|u + U_1^l\|^{2\varepsilon}} \phi(\beta + B_1^l)\right] \\ & \leq C \alpha^\varepsilon \int_G \mathbb{E}\left[\frac{(A_2^l)^\varepsilon}{\max\{A_2^2, \dots, A_2^l\}^{2\varepsilon}} \phi(\beta + b + B_2^l)\right] \frac{\mu(da dv db)}{a^\varepsilon \|v\|^{2\varepsilon}}, \end{aligned}$$

the last inequality being a consequence of the fact that $\|u + U_1^l\| \geq \|u_1\|$ \mathbb{P} -almost surely and

$$\mathbf{1}_{\{\max\{A_2^2, \dots, A_2^l\} \leq 1/a_1\}} \leq \frac{1}{a_1^{2\varepsilon} \max\{A_2^2, \dots, A_2^l\}^{2\varepsilon}}.$$

By Theorem 2.6 one obtains

$$l^{(N+3)/2} \mathbb{E} \left[[T_{\mathcal{A}} > l]; \varphi \left(\frac{A_1^l}{\alpha}, \frac{u + U_1^l}{\alpha}, \beta + B_1^l \right) \right] \leq C_1(\varphi) \alpha^\varepsilon.$$

The same upperbound holds under hypotheses G1, G2 and G'3 (see [7, Lemma 3.1]).

It readily follows that

$$\begin{aligned} n^{(N+3)/2} \sum_{k=i}^{[n/2]} I_{n,k}(\varphi) &\leq 2^{(N+3)/2} \sum_{k=i}^{[n/2]} (n-k)^{(N+3)/2} I_{n,k}(\varphi) \\ &\leq C_1(\varphi) \sum_{k=i}^{[n/2]} \mathbb{E} [[\tilde{T}_{\mathcal{A}'} > k]; (\tilde{A}_1^k)^\varepsilon] \\ &\leq C_2(\varphi) \sum_{k=i}^{[n/2]} \frac{1}{k^{3/2}} \end{aligned}$$

and so

$$\limsup_{i \rightarrow +\infty} \limsup_{n \rightarrow +\infty} n^{(N+3)/2} \sum_{k=i}^{[n/2]} I_{n,k}(\varphi) = 0.$$

The control of the sum $\sum_{k=[n/2]}^{n-i} I_{n,k}(\varphi)$ goes along the same lines.

Second step. *Convergence of the sequence*

$$l^{(N+3)/2} \mathbb{E} \left[[T_{\mathcal{A}} > l]; \varphi \left(\frac{A_1^l}{\alpha}, \frac{u + U_1^l}{\alpha}, \beta + B_1^l \right) \right]$$

for any $(\alpha, u, \beta) \in]0, 1] \times (\mathbb{R}^{*+})^d \times \mathbb{R}^N$.

It is the more technical part of the proof and it uses an idea due to Afanasev [1]. Without loss of generality, one may suppose $\alpha = 1$, $u = 0$ and $\beta = 0$. For any $n \geq 1$, set

$$\mathbb{E}_n(\varphi) = n^{(N+3)/2} \mathbb{E} [[T_{\mathcal{A}} > n]; \varphi(A_1^n, U_1^n, B_1^n)].$$

Fix $i \in \mathbb{N}$ such that $1 \leq i \leq n/2$ and consider

$$\mathbb{E}_n(\varphi, i) = n^{(N+3)/2} \mathbb{E} [[T_A > n]; \varphi(A_1^n, U_1^i + A_1^{n-i} U_{n-i+1}^n, B_1^n)].$$

To obtain the claim, it suffices to prove that

$$\text{a) } \limsup_{i \rightarrow +\infty} \limsup_{n \rightarrow +\infty} |\mathbb{E}_n(\varphi) - \mathbb{E}_n(\varphi, i)| = 0,$$

b) for any fixed $n \in \mathbb{N}$, the sequence $\{\mathbb{E}_n(\varphi, i)\}_{n \geq 1}$ converges to a finite limit.

PROOF OF CONVERGENCE a). We use the equality

$$U_1^n = U_1^i + A_1^i U_{i+1}^{n-i} + A_1^{n-i} U_{n-i+1}^n,$$

without loss of generality one may suppose that φ is continuously differentiable, and so, for any $\varepsilon > 0$ there exists $C > 0$ and a positive function ϕ with compact support on \mathbb{R}^N such that

$$|\varphi(a, u, b) - \varphi(a, v, b)| \leq C a^\varepsilon \|u - v\|^\varepsilon \phi(b),$$

consequently

$$\begin{aligned} & |\mathbb{E}_n(\varphi) - \mathbb{E}_n(\varphi, i)| \\ & \leq C n^{(N+3)/2} \mathbb{E} [[T_A > n]; (A_1^n)^\varepsilon (A_1^i)^\varepsilon \|U_{i+1}^{n-i}\|^\varepsilon \phi(B_1^n)] \\ & \leq C n^{(N+3)/2} \sum_{k=i+1}^{n-i} \mathbb{E} [[T_A > n]; (A_1^n)^\varepsilon (A_1^{k-1})^\varepsilon \|u_k\|^\varepsilon \phi(B_1^n)]. \end{aligned}$$

Note that for $i \leq k \leq [n/2]$ one has

$$\begin{aligned} & \mathbb{E} [[T_A > n]; (A_1^n)^\varepsilon (A_1^{k-1})^\varepsilon \|u_k\|^\varepsilon \phi(B_1^n)] \\ & \leq \mathbb{E} [[T_A > k-1] \cap \left[\max \{A_{k+1}^{k+1}, \dots, A_{k+1}^n\} \leq \frac{1}{A_1^k} \right]]; \\ & \quad (A_1^n)^\varepsilon (A_1^{k-1})^\varepsilon \|u_k\|^\varepsilon \phi(B_1^n)] \\ & \leq \mathbb{E} [[T_A > k-1]; (A_1^{k-1})^{\varepsilon/2} a_k^{-\varepsilon/2} \|u_k\|^\varepsilon \\ & \quad \cdot \max \{A_{k+1}^{k+1}, \dots, A_{k+1}^n\}^{-3\varepsilon/2} (A_{k+1}^n)^\varepsilon \phi(B_1^n)]. \end{aligned}$$

By Theorem 2.6,

$$(n-k)^{(N+3)/2} \mathbb{E} [\max \{A_{k+1}^{k+1}, \dots, A_{k+1}^n\}^{-3\varepsilon/2} (A_{k+1}^n)^\varepsilon \phi(\beta + B_{k+1}^n)]$$

is bounded, uniformly in $\beta \in \mathbb{R}^N$ and so

$$(n-k)^{(N+3)/2} E [[T_A > n]; (A_1^n)^\varepsilon (A_1^{k-1})^\varepsilon \|u_k\|^\varepsilon \phi(B_1^n)] \leq \frac{C_1}{k^{3/2}} .$$

When $[n/2] \leq k \leq n-i$ one obtains by a similar argument

$$k^{(N+3)/2} \mathbb{E} [[T_A > n]; (A_1^n)^\varepsilon (A_1^{k-1})^\varepsilon \|u_k\|^\varepsilon \phi(B_1^n)] \leq \frac{C_2}{(n-k)^{3/2}} .$$

Finally one has

$$|\mathbb{E}_n(\varphi) - \mathbb{E}_n(\varphi, i)| \leq C_3 \frac{1}{\sqrt{i}} ,$$

convergence a) follows.

PROOF OF CONVERGENCE b). Fix an integer i ; we have

$$\begin{aligned} & \mathbb{E}_n(\varphi, i) \\ &= \int_G E_n(\varphi, g, h_1, h_2, \dots, h_i) P_{\mathcal{A}}^i(e, dg) \mu(dh_1) \mu(dh_2) \cdots \mu(dh_i) \end{aligned}$$

with

$$\begin{aligned} & E_n(\varphi, g, h_1, h_2, \dots, h_i) \\ &= \mathbb{E} \left[\left[\max \{A_{i+1}^{i+1}, \dots, A_{i+1}^{n-i}\} \leq \frac{1}{a(g)} \right] \right. \\ & \quad \cap \left[A_{i+1}^{n-i} \leq \min \left\{ \frac{1}{a(g)}, \frac{1}{a(g)a(h_1)}, \dots, \frac{1}{a(g)a(h_1) \cdots a(h_i)} \right\} \right]; \\ & \quad \varphi(a(g) A_{i+1}^{n-i} a(h_1) \cdots a(h_i), u(g) + a(g) A_{i+1}^{n-i} u(h_1 \cdots h_i), \\ & \quad B_1^{n-i} + b(h_1) + \cdots + b(h_i)) . \end{aligned}$$

Using Theorem 2.6, one may see that, for any $g, h_1, \dots, h_i \in G$, the sequence

$$\{n^{(N+3)/2} E_n(\varphi, g, h_1, h_2, \dots, h_i)\}_{n \geq 1}$$

converges to a finite limit. To obtain the convergence b), we have to use Lebesgue dominated convergence theorem and therefore, we have to obtain an appropriate upperbound for $n^{(N+3)/2} E_n(\varphi, g, h_1, h_2, \dots, h_i)$.

Using the fact that for any $\varepsilon > 0$ there exist $C > 0$ and a positive continuous function ϕ with compact support on \mathbb{R}^N such that $|\varphi(a, u, b)| \leq C a^\varepsilon \phi(b)$, one thus obtains

$$n^{(N+3)/2} E_n(\varphi, g, h_1, h_2, \dots, h_i) \leq C_1 a(g)^{-3\varepsilon/2} a(h_1)^\varepsilon \dots a(h_i)^\varepsilon$$

which allows us to use the Lebesgue dominated convergence theorem for ε small enough; convergence b) follows.

Consequently, $\{n^{(N+3)/2} I_{n,0}(\varphi)\}_{n \geq 1}$ converges to a finite limit; furthermore, for any $i \geq 1$ and any compact set $K \subset \mathbb{R}^{*+} \times \mathbb{R}^N$, the dominated convergence theorem ensures the existence of a finite limit as n goes to $+\infty$ for

$$\left\{ n^{(N+3)/2} \sum_{k=0}^i I_{n,k}(\varphi, K) \right\}_{n \geq 1},$$

where

$$\begin{aligned} I_{n,k}(\varphi, K) &= \int_G \mathbf{1}_K(g) \\ &\cdot \left(\int_G \varphi \left(\frac{a(h)}{a(g)}, \frac{u(g) + u(h)}{a(g)}, b(g) + b(h) \right) P_{\mathcal{A}}^{n-k}(e, dh) \right) \\ &\cdot \tilde{P}_{\mathcal{A}'}^k(e, dg). \end{aligned}$$

The following step shows that the indicator function $\mathbf{1}_K$ does not disturb too much the behaviour of these integrals.

Third step. *Control of the residual terms.*

In the first step of the present proof, we have shown that, for any $\varepsilon > 0$ there exists $C_1 > 0$ such that

$$\begin{aligned} (n-k)^{(N+3)/2} \mathbb{E} \left[[T_{\mathcal{A}} > n-k]; \varphi \left(\frac{A_1^{n-k}}{\alpha}, \frac{u + U_1^{n-k}}{\alpha}, \beta + B_1^{n-k} \right) \right] \\ \leq C_1(\varphi) \alpha^\varepsilon. \end{aligned}$$

It follows that for any $0 < \delta < 1$

$$\sum_{k=1}^i \int_{\{g \in G: a(g) \leq \delta\}} \left(\int_G \varphi \left(\frac{a(h)}{a(g)}, \frac{u(g) + u(h)}{a(g)}, b(g) + b(h) \right) P_{\mathcal{A}}^{n-k}(e, dh) \right)$$

$$\begin{aligned}
& \cdot \tilde{P}_{\mathcal{A}'}^k(e, dg) \\
& \leq C_1 \sum_{k=1}^i \frac{1}{(n-k)^{(N+3)/2}} \mathbb{E} [[\tilde{T}_{\mathcal{A}'} > k]; (\tilde{A}_1^k)^\varepsilon] \\
& \leq C_1 \sum_{k=1}^i \frac{1}{(n-k)^{(N+3)/2} k^{3/2}}.
\end{aligned}$$

On the other hand for any fixed $U > 0$, one has

$$\begin{aligned}
& \sum_{k=1}^i \int_{\{g \in G: \|u(g)\| \geq U\}} \left(\int_G \varphi \left(\frac{a(h)}{a(g)}, \frac{u(g)+u(h)}{a(g)}, b(h)+b(g) \right) P_{\mathcal{A}}^{n-k}(e, dh) \right) \\
& \quad \cdot \tilde{P}_{\mathcal{A}'}^k(e, dg) \\
& \leq \frac{C_1}{U^{\varepsilon/2}} \sum_{k=1}^i \frac{1}{(n-k)^{(N+3)/2}} E [[\tilde{T}_{\mathcal{A}'} > k]; (\tilde{A}_1^k)^\varepsilon \|\tilde{U}_k\|^{\varepsilon/2}] \\
& \leq \frac{C_1}{U^{\varepsilon/2}} \sum_{k=1}^i \frac{1}{(n-k)^{(N+3)/2} k^{3/2}}.
\end{aligned}$$

The last inequality being guaranteed by standart estimations. (see [7, Lemma 3.3] for more details).

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Emile Le Page
Institut Mathématique de Rennes
Université de Bretagne Sud
1 Rue de la Loi
Vannes 56000, FRANCE
emile.lepage@univ-ubs.fr

and

Marc Peigné
Institut Mathématique de Rennes
Université de Rennes I
Campus de Beaulieu
35042 Rennes Cedex, FRANCE
peigne@univ-rennes1.fr