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# Some negatively curved manifolds with cusps, mixing and counting

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**Abstract.** Let $X$ be a Hadamard manifold whose sectional curvature $K$ satisfies $-b^2 \leqq K \leqq -1$. We consider a family of free isometry groups $\Gamma$ acting properly discontinuously on $X$ and containing parabolic transformations of divergence type. We show that such groups are of divergent type, we describe the dynamic properties of the map $T$ induced by the action of $\Gamma$ on the boundary of $X$ and we explore the spectrum of the transfer operator associated with $T$. As applications, we establish a mixing property for the geodesic flow on the unit tangent bundle of $X/\Gamma$ and we describe the behaviour as $a$ goes to $+\infty$ of the number of primitive closed geodesics on $X/\Gamma$ whose length is not larger than $a$.

## Introduction

A *Hadamard manifold* is a complete simply connected Riemannian manifold of non positive curvature $K$. Let $X$ be such a manifold, assume that $-b^2 \leqq K \leqq -1$ and denote by $\partial X$ its visual boundary relatively to a reference point 0. Fix two integers $N_1$, $N_2$ such that $N_1 + N_2 \geqq 2$ and $N_2 \geqq 1$ and consider $N_1$ hyperbolic isometries $\alpha_1, \dots, \alpha_{N_1}$ and $N_2$ parabolic ones $\alpha_{N_1+1}, \dots, \alpha_{N_1+N_2}$ satisfying the following conditions:

(1) *For $1 \leqq i \leqq N_1$ there exist in $\partial X$ a compact neighbourhood $C_{\alpha_i}$ of the attracting point $x_{\alpha_i}$ of $\alpha_i$ and a compact neighbourhood $C_{\alpha_i^{-1}}$ of the reppeling point $x_{\alpha_i^{-1}}$ such that*

$$\alpha_i(\partial X - C_{\alpha_i^{-1}}) \subset C_{\alpha_i}.$$

(2) *For $N_1 + 1 \leqq i \leqq N_1 + N_2$ there exists in $\partial X$ a compact neighbourhood $C_{\alpha_i}$ of the unique fixed point $x_{\alpha_i}$ of $\alpha_i$ such that*

$$\forall n \in \mathbb{Z}^* \quad \alpha_i^n(\partial X - C_{\alpha_i}) \subset C_{\alpha_i}.$$

(3) *The $2N_1 + N_2$ neighbourhoods introduced in* (1) *and* (2) *are pairwise disjoint.*

(4) *The elementary groups $\langle \alpha_i \rangle$ for $N_1 + 1 \leqq i \leqq N_1 + N_2$ are of divergence type.*

Such families of isometries can be obtained by taking some powers of a finite number of parabolic or hyperbolic transformations of divergence type which have no fixed points in common. Note that if $N_1 = 0$ we only consider $N_2 \geqq 2$ parabolic transformations satisfying conditions (2) and (3).

Transformations $\alpha_1, \ldots, \alpha_{N_1+N_2}$ generate a free group $\Gamma$ which acts properly discontinuously and freely on $X$. If $N_2 = 0$ the group $\Gamma$ is a *Schottky* group; it acts on the convex hull of its limit set with compact fundamental domain, in other words $\Gamma$ is *convex cocompact*. The geometry of convex cocompact groups is well known ([7], [19], [33]); many results are proved using the thermodynamic formalism which may be applied in this case.

Throughout the present paper we assume that $N_2 \geqq 1$ and we will say that $\Gamma$ is an *extended Schottky* group. Note that any parabolic transformation of $\Gamma$ is conjugated to some power of a parabolic generator; this is the simplest example of a non convex-cocompact group.

In the case where $X = \mathbb{H}^n_{\mathbb{R}}$, there is a well known ergodic theory for geometrically finite discret groups $G$ even if it contains parabolic transformations: for example the Patterson-Sullivan measure associated with $G$ has no atomic part, $G$ is of divergence type [31] and the geodesic flow on the unit tangent bundle of $X/\Gamma$ is topologically mixing [28].

In the non constant curvature case, there are not many results about groups with parabolic transformations and several problems are still open: Are such groups of divergence type? Does there exist an atomic part in the Patterson-Sullivan measure? Is the geodesic flow mixing relatively to the geometrical Patterson-Sullivan measure?

Let us now state the main results of the present paper. Denote by $d$ the Riemannian distance on $X$ and $\delta_\Gamma$ the exponent of convergence of the Poincaré series associated with $\Gamma$. Since the sectional curvature of $X$ is lower bounded, $\delta_\Gamma$ is finite. Let $g$ be a non elliptic isometry, denote by $\delta_g$ the exponent of convergence of the series $\sum_{n \in \mathbb{Z}} e^{-sd(0, g^n 0)}$. If $g$ is hyperbolic then $\delta_g = 0$; if $g$ is parabolic then $\delta_g \geqq \frac{1}{2}$ (cf. § III) We prove the

**Theorem III.1.** *Let $G$ be a non elementary group of isometries of $X$. For any $g \in G$ such that $\sum_{n \in \mathbb{Z}} e^{-\delta_g d(0, g^n 0)} = +\infty$, one has $\delta_G > \delta_g$.*

When $g$ is hyperbolic one just obtains the well known result $\delta_G > 0$. If $G$ contains parabolic transformations then $\delta_G > 1/2$.

The following results are stated for extended Schootky groups $\Gamma$; using the coding of the limit set of these groups we prove the

**Theorem IV.2.** *The Patterson-Sullivan measure $\sigma$ associated with $\Gamma$ has no atomic part.*

As a direct consequence we obtain

**Corollary IV.3.** *The group $\Gamma$ is of divergence type, that is* $\sum_{\gamma \in \Gamma} e^{-\delta_\Gamma d(0,\gamma 0)} = +\infty$.

Since $\Gamma$ contains parabolic transformations, we code the points of its radial limit set $\Lambda^0$ with an infinite alphabet; by geometrical arguments, we show in § V that the boundary map $T$ on $\Lambda^0$ induced by the classical shift on the associated symbolic space is expanding and we construct a $T$-invariant probability measure $\nu$ on $\Lambda^0$.

Denote by $\Lambda$ the limit set of $\Gamma$ and by $G\Lambda$ the set of pairs $(\xi, x)$ where $\xi$ is a geodesic on $X$ with endpoints in $\Lambda$ and $x \in \xi$. The Patterson-Sullivan measure induces a natural measure $\overline{\mu \otimes l}$ on the non wandering set $G\Lambda/\Gamma$ of the geodesic flow $(\bar{g}_t)_{t\in\mathbb{R}}$ on the unit tangent bundle of $X$ which is $(\bar{g}_t)_{t\in\mathbb{R}}$ invariant. We show the

**Theorem VI.2.** *The geodesic flow $(\bar{g}_t)_{t\in\mathbb{R}}$ on $G\Lambda/\Gamma$ is mixing relatively to $\overline{\mu \otimes l}$.*

The proof of this result is based on a renewal theorem for transient Markov walk on $\Lambda \times \mathbb{R}$ [14] and requires a precise investigation of the spectrum of the adjoint operator $P$ and its Fourier transforms $P_\lambda$ associated with $T$ and $\nu$ (cf. § VIII); the fact that $\Gamma$ contains parabolic isometries is essential to describe the top of the spectrum of $P_\lambda$, $\lambda \in \mathbb{R}$. Note that, as far as we know, theorem VI.2 is not proved (or not yet published) in the case where the sectional curvature of $X$ is non constant and $\Gamma$ is convex cocompact, even if $\Gamma$ is a Schottky group.

For any $a > 0$ denote by $\pi(a)$ the number of primitive closed geodesics on $X/\Gamma$ with length not larger than $a$; since $\Gamma$ is not purely hyperbolic the set of closed geodesics on $X/\Gamma$ is not relatively compact. In § VII we prove the following

**Theorem VII.1.** *The function $a \to \pi(a)$ is equivalent to $e^{a\delta_\Gamma}/a\delta_\Gamma$ as a goes to $+\infty$.*

To prove this theorem we use a probabilistic method introduced by S. Lalley [21] and already developped in further directions ([2], [6], and [22]). First we code closed geodesics on $X/\Gamma$ and establish a connection between $\pi(a)$ and the harmonic potential of a certain Markov walk on $\mathbb{R}$. Theorem VII.1 thus appears as a direct consequence of a harmonic renewal theorem for a transient Markov walk on $\mathbb{R}$.

When $M = X/\Gamma$ is compact, theorem VII.1 is due to Selberg [29] ($K = -1$), Margulis [23] ($K$ variable) and has been extended to periodic orbits of Axiom A flows in [25]. When $M$ is not compact and $K = -1$ this theorem is well known if $M$ is a surface ([13], [17], [32]); when $\dim M \geqq 3$ a similar result holds under the following hypotheses: the volume of $M$ is finite [32], or $\pi_1(M)$ is convex cocompact ([21], [25]) or $\pi_1(M)$ is a *Ping-Pong group* [10]. The case where $M$ is not compact and the curvature $K$ is not constant is quite open; in [6] we solve it for small perturbations of the Poincaré metric on the modular surface.

**Remark.** All the results of the present paper are valid and the proofs are rigorously the same if one replaces $X$ by a CAT($-1$)-space whose boundary has a finite visual Hausdorff dimension [27] and admits hyperbolic and parabolic isometries; unfortunately, we have no interesting example of such spaces except pinched Hadamard manifolds.

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## I. Geometry on $X$

Denote by $d$ the Riemannian distance on $X$. Since the sectional curvature is not larger than $-1$ the metric space $(X, d)$ is a CAT$(-1)$-space [4].

Throughout this paper we fix a reference point $0 \in X$. Consider two geodesic rays $t \to r(t)$ and $t \to s(t)$ based at 0; one says that $r$ and $s$ are equivalent if the Hausdorff distance between $r$ and $s$ is bounded. Denote by $\partial X$ the quotient of the set of geodesic rays based at 0 by this equivalence relation; equipped with the quotient topology induced by uniform convergence on compacts, $X \cup \partial X$ is compact. Note that every pair of distinct points in $X \cup \partial X$ determines a unique geodesic on $X$ [4].

A metric $d_{\partial X}$ on $\partial X$ is called a *visual t metric* (with $t > 1$) if there exists $C \geqq 1$ such that for every $x, y \in \partial X$ one has $\frac{1}{C} t^{-d(0,(xy))} \leqq d_{\partial X}(x, y) \leqq C t^{-d(0,(xy))}$ where $d(0, (xy))$ denotes the distance from 0 to the geodesic $(xy)$ joining $x$ and $y$. Such a metric does exist on the boundary of any Gromov hyperbolic space [7], hence in particular on the boundary of a CAT$(-1)$-space. Let $x \in \partial X$ and $t \to r(t)$ be a geodesic ray joigning 0 and $x$; for every $z_1, z_2 \in X$ the limit as $t$ goes to $+\infty$ of the difference $d(z_1, r(t)) - d(z_2, r(t))$ exists and is denoted by $B_x(z_1, z_2)$. Geometrically, $B_x(z_1, z_2)$ represents the algebraic horospherical distance between $z_1$ and $z_2$ relatively to $x$; moreover, if $x_1, x_2 \in \partial X$ and $z$ belongs to the geodesic with extremities $x_1$ and $x_2$ then $(x_1|x_2) = \dfrac{B_{x_1}(0, z) + B_{x_2}(0, z)}{2}$ does not depend on the choice of $z$. One has the

**Theorem I.1** ([4]). *The mapping $D : \partial X \times \partial X \to \mathbb{R}^+$ defined by $D(x_1, x_2) = e^{-(x_1|x_2)}$ if $x_1 \neq x_2$ and $D(x_1, x_1) = 0$ is a visual e-metric on $\partial X$.*

For every isometry $\gamma$ on $X$ and every point $x \in \partial X$ set $|\gamma'(x)| = e^{B_x(0, \gamma^{-1}0)}$. Using the equalities $B_{\gamma(x)}(\gamma z_1, \gamma z_2) = B_x(z_1, z_2)$ and $B_x(z_1, z_3) = B_x(z_1, z_2) + B_x(z_2, z_3)$ one obtains the

**Mean values relation.** $\forall x, y \in \partial X \quad D(\gamma x, \gamma y) = \sqrt{|\gamma'(x)||\gamma'(y)|}\, D(x, y).$

Isometries on $X$ are classified according to their fix points. An isometry is *elliptic* if it has a fixed point inside $X$; in the present paper we only consider non elliptic isometries $\gamma$. Thus

– either $\gamma$ fixes a unique point $x_\gamma \in \partial X$; in this case $\gamma$ is said to be *parabolic* and $|\gamma'(x_\gamma)| = 1$;

– or $\gamma$ fixes exactly two distinct points $x_\gamma$ and $x_{\gamma^{-1}}$; in this case, $\gamma$ is said to be *hyperbolic*, the point $x_\gamma$ satisfies the inequality $|\gamma'(x_\gamma)| < 1$ and is called the *attracting point* of $\gamma$, the point $x_{\gamma^{-1}}$ satisfies the equality $|\gamma'(x_{\gamma^{-1}})| = \dfrac{1}{|\gamma'(x_\gamma)|} > 1$ and is thus called the *repelling point* of $\gamma$. Near $x_{\gamma^{-1}}$ the hyperbolic isometry $\gamma$ looks like a homothety expansion with *expansion rate* $|\gamma'(x_{\gamma^{-1}})|$.

**Definition I.2.** Let $\gamma$ be a hyperbolic isometry acting on $X$; the expansion rate of $\gamma$ is the real number $\Phi(\gamma) = |\gamma'(x_{\gamma^{-1}})|$.

The two following lemmas describe the dynamic on $\partial X$ of non elliptic isometries.

**Lemma I.3.** *Let $\gamma$ be a hyperbolic isometry; for every compact set $E \subset \partial X - \{x_\gamma, x_{\gamma^{-1}}\}$ there exists $A \geqq 1$ depending on $\gamma$ and $E$ such that*

$$(1)\ \forall x \in E,\ \forall n \in \mathbb{Z}^* \quad \frac{\Phi(\gamma)^{-|n|}}{A} \leqq |(\gamma^n)'(x)| \leqq A\Phi(\gamma)^{-|n|},$$

$$(2)\ \forall x, y \in E,\ \forall n \in \mathbb{Z}^* \quad \big||(\gamma^n)'(x)| - |(\gamma^n)'(y)|\big| \leqq A\Phi(\gamma)^{-|n|}D(x, y).$$

*Proof.* Assume $n \geqq 1$; if $n \leqq -1$ it suffices to replace $x_{\gamma^{-1}}$ by $x_\gamma$ in the proof.

(1) One has $D(\gamma^n x, x_{\gamma^{-1}}) = \sqrt{|(\gamma^n)'(x)|\,|\gamma'(x_{\gamma^{-1}})|^n}\,D(x, x_{\gamma^{-1}})$. Suppose first that for every $k \geqq 1$ there exist $x_k \in E$ and $p(k) \in \mathbb{N}^*$ such that $D(\gamma^{p(k)}x_k, x_{\gamma^{-1}}) \leqq 1/k$; therefore $\lim_{k \to +\infty} \gamma^{p(k)}(x_k) = x_{\gamma^{-1}}$. On the other hand

$$\frac{D(\gamma^{p(k)}x_k, x_{\gamma^{-1}})}{D(\gamma^{p(k)}x_k, x_\gamma)} = \Phi(\gamma)^{p(k)}\frac{D(x_k, x_{\gamma^{-1}})}{D(x_k, x_\gamma)} \geqq \frac{D(E, x_{\gamma^{-1}})}{\|D\|_\infty}$$

which implies $\lim_{k \to +\infty} \gamma^{p(k)}(x_k) = x_\gamma$; this contradicts the fact that $x_\gamma \neq x_{\gamma^{-1}}$. Consequently, there exists $B > 0$ such that for every $x \in E$ and $n \geqq 1$ one has $D(\gamma^n x, x_{\gamma^{-1}}) \geqq B$. By the mean values relation it follows

$$\left(\frac{B}{\|D\|_\infty}\right)^2 \Phi(\gamma)^{-n} \leqq |(\gamma^n)'(x)| \leqq \left(\frac{\|D\|_\infty}{D(E, x_{\gamma^{-1}})}\right)^2 \Phi(\gamma)^{-n}.$$

(2) Let $x$ and $y$ be in $\partial X$. Set $\lambda(x, y) = \big||(\gamma^n)'(x)| - |(\gamma^n)'(y)|\big|$; one has

$$\begin{aligned}\lambda(x, y) &= \Phi(\gamma)^{-n}\left|\frac{D^2(\gamma^n x, x_{\gamma^{-1}})}{D^2(x, x_{\gamma^{-1}})} - \frac{D^2(\gamma^n y, x_{\gamma^{-1}})}{D^2(y, x_{\gamma^{-1}})}\right| \\ &\leqq \frac{\Phi(\gamma)^{-n}}{D^2(x, x_{\gamma^{-1}})}|D^2(\gamma^n x, x_{\gamma^{-1}}) - D^2(\gamma^n y, x_{\gamma^{-1}})| \\ &\quad + \Phi(\gamma)^{-n}D^2(\gamma^n y, x_{\gamma^{-1}})\left|\frac{1}{D^2(x, x_{\gamma^{-1}})} - \frac{1}{D^2(y, x_{\gamma^{-1}})}\right|.\end{aligned}$$

If $x$ and $y$ belong to $E$ one obtains

$$\begin{aligned}\lambda(x, y) &\leqq \Phi(\gamma)^{-n} \frac{2\|D\|_\infty}{D^2(E, x_{\gamma^{-1}})} D(\gamma^n x, \gamma^n y)\\ &+ \Phi(\gamma)^{-n} \frac{2\|D\|_\infty^3}{D^4(E, x_{\gamma^{-1}})} D(x, y)\\ &\leqq \Phi(\gamma)^{-n} \frac{2\|D\|_\infty}{D^2(E, x_{\gamma^{-1}})} \sqrt{|(\gamma^n)'(x)|\,|(\gamma^n)'(y)|}\, D(x, y)\\ &+ \Phi(\gamma)^{-n} \frac{2\|D\|_\infty^3}{D^4(E, x_{\gamma^{-1}})} D(x, y).\end{aligned}$$

Using inequality (1) one thus obtains the existence of a constant $A$ such that

$$0 \leqq \lambda(x, y) \leqq A\Phi(\gamma)^{-n} D(x, y). \quad \square$$

The dynamic of parabolic isometries is a little different. One has the

**Lemma I.4.** *Let $\gamma$ be a parabolic isometry; for every compact set $E \subset \partial X - \{x_\gamma\}$ and every $y_0 \in E$ there exists $A \geqq 1$ depending on $\gamma$ and $E$*

$$(1)\ \forall x \in E, \forall n \in \mathbb{Z}^* \quad \frac{|(\gamma^n)'(y_0)|}{A} \leqq |(\gamma^n)'(x)| \leqq A|(\gamma^n)'(y_0)|,$$

$$(2)\ \forall x, y \in E, \forall n \in \mathbb{Z}^* \quad \big||(\gamma^n)'(x)| - |(\gamma^n)'(y)|\big| \leqq A|(\gamma^n)'(y_0)|D(x, y).$$

*Furthermore one has* $\lim_{n \to \pm\infty} |(\gamma^n)'(y_0)|^{1/|n|} = 1$.

*Proof.* (1) Since $|\gamma'(x_\gamma)| = 1$ one has $D(\gamma^n x, x_\gamma) = \sqrt{|(\gamma^n)'(x)|}\, D(x, x_\gamma)$ for any $x \in \partial X$; so $\frac{1}{\|D\|_\infty^2} D^2(\gamma^n x, x_\gamma) \leqq |(\gamma^n)'(x)| \leqq \frac{1}{D^2(x_\gamma, E)} D^2(\gamma^n x, x_\gamma)$. For every $x, y \in \partial X - \{x_\gamma\}$ set $\Delta(x, y) = \frac{D(x, y)}{D(x, x_\gamma) D(y, x_\gamma)}$; using the fact that $\Delta(\gamma x, \gamma y) = \Delta(x, y)$ one obtains $\left|\frac{1}{D(\gamma^n x, x_\gamma)} - \frac{1}{D(\gamma^n y, x_\gamma)}\right| \leqq \Delta(x, y)$ for any $n \in \mathbb{Z}^*$. Now, fix $y_0 \in E$; we have $\|\Delta\|_\infty = \sup_{x \in E} \Delta(x, y_0) < +\infty$ and so

$$\forall x \in E \quad \frac{D(\gamma^n y_0, x_\gamma)}{1 + \|\Delta\|_\infty D(\gamma^n y_0, x_\gamma)} \leqq D(\gamma^n x, x_\gamma) \leqq \frac{D(\gamma^n y_0, x_\gamma)}{1 - \|\Delta\|_\infty D(\gamma^n y_0, x_\gamma)}.$$

Since $\lim_{n \to \pm\infty} D(\gamma^n y_0, x_\gamma) = 0$ there exists $C \geqq 1$ such that

$$\frac{1}{C} D(\gamma^n y_0, x_\gamma) \leqq D(\gamma^n x, x_\gamma) \leqq C D(\gamma^n y_0, x_\gamma)$$

and so

$$\frac{D^2(y_0, x_\gamma)}{\|D\|_\infty^2 C^2} |(\gamma^n)'(y_0)| \leqq |(\gamma^n)'(y)| \leqq \frac{C^2 D^2(y_0, x_\gamma)}{D^2(x_\gamma, E)} |(\gamma^n)'(y_0)| .$$

(2) Let $x$ and $y$ in $\partial X$ and set $\lambda(x, y) = ||(\gamma^n)'(x)| - |(\gamma^n)'(y)||$; one has

$$\begin{aligned}\lambda(x, y) &= \left| \frac{D^2(\gamma^n x, x_\gamma)}{D^2(x, x_\gamma)} - \frac{D^2(\gamma^n y, x_\gamma)}{D^2(y, x_\gamma)} \right| \\ &\leqq \frac{1}{D^2(x, x_\gamma)} |D^2(\gamma^n x, x_\gamma) - D^2(\gamma^n y, x_\gamma)| \\ &\quad + D^2(\gamma^n y, x_\gamma) \left| \frac{1}{D^2(x, x_\gamma)} - \frac{1}{D^2 y, x_\gamma)} \right| .\end{aligned}$$

If $x$ and $y$ belong to $E$ one obtains

$$\begin{aligned}\lambda(x, y) &\leqq \frac{2\|D\|_\infty}{D^2(E, x_\gamma)} D(\gamma^n x, \gamma^n y) \\ &\quad + \frac{2\|D\|_\infty}{D^4(E, x_\gamma)} D^2(\gamma^n y, x_\gamma) D(x, y) \\ &\leqq \frac{2\|D\|_\infty}{D^2(E, x_\gamma)} \sqrt{|(\gamma^n)'(x)||(\gamma^n)'(y)|} D(x, y) \\ &\quad + \frac{2\|D\|_\infty^3}{D^4(E, x_\gamma)} |(\gamma^n)'(y)| D(x, y) .\end{aligned}$$

Using inequality (1) one thus obtains the existence of a constant $A \geqq 1$ such that $0 \leqq \lambda(x, y) \leqq A |(\gamma^n)'(y_0)| D(x, y)$.

Finally, since $\lim_{|n| \to +\infty} \frac{|(\gamma^{n+1})'(y_0)|}{|(\gamma^n)'(y_0)|} = \lim_{|n| \to +\infty} |\gamma'(\gamma^n y_0)| = 1$ we have

$$\lim_{|n| \to +\infty} |(\gamma^n)'(y_0)|^{1/|n|} = 1. \quad \square$$

The following result will be useful in the sequel; its proof is similar to the one of proposition 8, chap. 8 in [12].

**Lemma I.5.** *Let $(\gamma_n)_{n \geqq 1}$ be a sequence of pairwise distinct isometries of $X$ such that $(\gamma_n(z_0))_{n \geqq 1}$ converges to a point $x \in \partial X$ for some $z_0 \in X$. Then for every $z$ in $X$ one has $\lim_{n \to +\infty} \gamma_n(z) = x$.*

The action of the sequence $(\gamma_n)_{n \geqq 1}$ on $\partial X$ is a little different; for example if $\gamma$ is a hyperbolic isometry, $x_{\gamma^{-1}}$ is fixed by $\gamma$ and $\lim_{n \to +\infty} \gamma^n(x) = x_\gamma$ for all $x \neq x_{\gamma^{-1}}$. More generally one has

**Corollary I.6.** *Let $(\gamma_n)_{n\geqq 1}$ be a sequence of pairwise distinct isometries of $X$ such that $(\gamma_n(z_0))_{n\geqq 1}$ converges to a point $x \in \partial X$ for some $z_0 \in X$. Then, for any couple $(y, y')$ of distinct points in $\partial X$ one has* $\lim_{n\to+\infty} \inf(D(\gamma_n y, x), D(\gamma_n y', x)) = 0$.

## II. Extended Schottky groups

Fix two integers $N_1$ and $N_2$ such that $N_1 + N_2 \geqq 2$ and $N_2 \geqq 1$ and consider $N_1$ hyperbolic isometries $\alpha_1, \ldots, \alpha_{N_1}$ and $N_2$ parabolic ones $\alpha_{N_1+1}, \ldots, \alpha_{N_1+N_2}$ satisfying the conditions (1), (2), (3) and (4) given in the introduction. The group $\Gamma$ generated by $\alpha_1, \ldots, \alpha_{N_1+N_2}$ is called an *extended Schottky group.*

**Notations.** Denote by $\mathscr{A} = \{\alpha_1, \ldots, \alpha_{N_1+N_2}\}$ and $\mathscr{A}^{\pm} = \{\alpha_1, \alpha_1^{-1}, \ldots, \alpha_{N_1+N_2}, \alpha_{N_1+N_2}^{-1}\}$. For every $1 \leqq i \leqq N_1$ set $C_{\alpha_i^{\pm}} = C_{\alpha_i} \cup C_{\alpha_i^{-1}}$ and for every $N_1 + 1 \leqq i \leqq N_1 + N_2$ set $C_{\alpha_i^{\pm}} = C_{\alpha_i}$.

Using conditions (1), (2) and (3) one shows by induction over $n$ the

**Property II.1** (Ping-Pong property). *Let $a_1, \ldots, a_n \in \mathscr{A}^{\pm}$ such that $a_{i+1} \neq a_i^{-1}$ for $1 \leqq i < n$. Then $a_1 \cdots a_n(\partial X - C_{a_n^{-1}}) \subset C_{a_1}$.*

Consequently $\mathscr{A}$ is a free system of generators of $\Gamma$. Furthermore one has

**Corollary II.2.** *The group $\Gamma$ acts properly discontinuously on $X$.*

*Proof.* Assume that $\Gamma$ does not act properly discontinuously on $X$. So there exists a sequence $(\gamma_n)_{n\geqq 1}$ of distinct elements of $\Gamma$ such that $(\gamma_n(0))_{n\geqq 1}$ remains bounded. Fix two distinct points $x$ and $y$ in $\partial X - \bigcup_{a\in\mathscr{A}} C_{a^{\pm}}$ and choose $z$ and $z'$ on the geodesic $(xy)$; since $(\gamma_n(z))_{n\geqq 1}$ and $(\gamma_n(z'))_{n\geqq 1}$ remain bounded, there exists a subsequence $(\gamma_{n_k})_{k\geqq 1}$ of $(\gamma_n)_{n\geqq 1}$ such that $(\gamma_{n_k}(z))_{k\geqq 1}$ and $(\gamma_{n_k}(z'))_{k\geqq 1}$ converge in $X$. Set $g_k = \gamma_{n_{k+1}}^{-1}\gamma_{n_k}$; one has $\lim_{k\to+\infty} g_k(z) = z$ and $\lim_{k\to+\infty} g_k(z') = z'$ so that $\lim_{k\to+\infty} g_k(x) = x$ and $\lim_{k\to+\infty} g_k(y) = y$. By the Ping-Pong property it follows that $g_k = \mathrm{Id}$ for $k$ large enough which contradicts the hypothesis. □

Recall that $\Phi(\gamma) = |\gamma'(x_{\gamma^{-1}})|$ for every isometry $\gamma$.

**Corollary II.3.** *Let $(\gamma_n)_{n\geqq 1}$ be a sequence of distinct hyperbolic isometries of $\Gamma$ such that $\gamma_n$ and $\gamma_m$ are not conjugated for every $n \neq m$. Then* $\lim_{n\to+\infty} \Phi(\gamma_n) = +\infty$.

*Proof.* Suppose that there exist $B > 0$ and a subsequence (denoted also $(\gamma_n)_{n\geqq 1}$) such that $\Phi(\gamma_n) \leqq B$ for every $n \geqq 1$. Since $\mathscr{A}$ is finite and $\Phi(g\gamma_n g^{-1}) = \Phi(\gamma_n)$ for any isometry $g$, one can suppose that for $n$ large enough, either $\gamma_n = \alpha_i^{k_n}$ for some $1 \leqq i \leqq N_1$, or $\gamma_n = a_{n_1} \cdots a_{nk_n}$ with $a_{n_i} \in \mathscr{A}^{\pm}$, $a_{n(i+1)} \neq a_{ni}^{-1}$ for $1 \leqq i < k_n$ and $a_{n_1} = \alpha$, $a_{nk_n} = \beta$, $\alpha \neq \beta$.

In the first case one has $\Phi(\gamma_n) = \Phi(\alpha_i)^{|k_n|}$ with $\Phi(\alpha_i) > 1$ and corollary II.3 follows. In the second case $x_{\gamma_n^{-1}} \in C_{\beta^{\pm}}$ and $\gamma_n C_{\alpha^{\pm}} \subset C_{\alpha^{\pm}}$. For every $x \in C_{\alpha^{\pm}}$ one thus has

$|(\gamma_n)'(x)| \geqq \frac{D^2(C_{\alpha^\pm}, C_{\beta^\pm})}{B\|D\|_\infty^2}$ and therefore $\inf_{n\geqq 1} D(\gamma_n x, \gamma_n x') > 0$ for every pair $(x, x')$ of distinct points in $C_{\alpha^\pm}$, which contradicts corollary I.6. □

The following lemma is important in order to code some limit points of $\Gamma$. For every subset $E \subset \partial X$ denote by $\operatorname{diam} E = \sup_{\substack{x,y\in E\\ x\neq y}} D(x,y)$. One has

**Lemma II.4.** *Let $(a_i)_{i\geqq 1} \in (\mathscr{A}^\pm)^{\mathbb{N}^*}$ such that $a_{i+1} \neq a_i^{-1}$ for every $i \geqq 1$. Then*

$$\lim_{n\to+\infty} \operatorname{diam} a_1 \cdots a_n \Big( \bigcup_{a\in\mathscr{A}-\{a_n^{-1}\}} C_a \Big) = 0 .$$

*Proof.* Since the sequence $\big(a_1 \cdots a_n \big( \bigcup_{a\in\mathscr{A}-\{a_n^{-1}\}} C_a\big)\big)_{n\geqq 1}$ is decreasing, it suffices to show the lemma for some subsequence. For every $n \geqq 1$ set $\gamma_n = a_1 \cdots a_n$ and consider a subsequence $(\gamma_{n_k})_{k\geqq 1}$ such that $a_{n_k} = \alpha \neq a_1^{\pm 1}$ (if such a subsequence does not exist replace the initial sequence $a_1, a_2, \ldots$ by $a_1', a_2, \ldots$ with $a_1' \neq a_1^{\pm 1}$ and $a_1' \neq a_2^{-1}$); transformations $\gamma_{n_k}$ are thus hyperbolic and are not mutually conjugated. One has

$$\begin{aligned}
\operatorname{diam} \gamma_{n_k}\Big( \bigcup_{a\in\mathscr{A}-\{a^{-1}\}} C_a \Big) &\leqq \sup_{x\in \bigcup_{a\neq\alpha^{-1}} C_a} |\gamma_{n_k}'(x)| \;\; \|D\|_\infty \\
&\leqq \sup_{x\in \bigcup_{a\neq\alpha^{-1}} C_a} \frac{D^2(\gamma_{n_k} x, x_{\gamma_{n_k}^{-1}})}{\Phi(\gamma_{n_k}) D^2(x, x_{\gamma_{n_k}^{-1}})} \;\; \|D\|_\infty \\
&\leqq \frac{\|D\|_\infty^3}{\Phi(\gamma_{n_k}) D^2\big( \bigcup_{a\in\mathscr{A}-\{\alpha^{-1}\}} C_a, C_{\alpha^{-1}}\big)} .
\end{aligned}$$

By corollary II.3 one has $\lim_{n\to+\infty} \Phi(\gamma_{n_k}) = +\infty$ which finishes the proof. □

Denote by $\Lambda$ the limit set of $\Gamma$; by definition $\Lambda = \overline{\Gamma 0} \cap \partial X$ and it is the least $\Gamma$-invariant closed subset of $\partial X$. Let us introduce the

**Notations.** For every $a \in \mathscr{A}^\pm$ set $\Lambda_a = \Lambda \cap C_a$ and $\Lambda_{a^\pm} = \Lambda \cap C_{a^\pm}$.

Let $\Lambda^0$ be the limit set $\Lambda$ minus the $\Gamma$-orbit of the fixed points of $\alpha_1, \ldots, \alpha_{N_1+N_2}$ and set $\Lambda_a^0 = \Lambda^0 \cap C_a$ and $\Lambda_{a^\pm}^0 = \Lambda^0 \cap C_{a^\pm}$.

Fix $x_0 \in \partial X - \bigcup_{a\in\mathscr{A}} C_{a^\pm}$. Let $x \in \Lambda^0$; since $x$ is a limit point there exists a sequence $(\gamma_n)_{n\geqq 1}$ of $\Gamma$ such that $\lim_{n\to+\infty} \gamma_n(x_0) = x$. Since $\mathscr{A}$ is finite and $x \notin \Gamma x_\alpha$ for any $\alpha \in \mathscr{A}^\pm$ there exists a subsequence $(\gamma_{n_k})_{k\geqq 1}$ of $(\gamma_n)_{n\geqq 1}$ such that $\gamma_{n_k} = a_1^{n_1} \cdots a_{l(k)}^{n_{l(k)}}$ with $a_i \in \mathscr{A}$, $n_i \in \mathbb{Z}^*$ and $a_{i+1} \neq a_i$. The unicity of the sequence $(a_i^{n_i})_{i\geqq 1}$ is a direct consequence of the Ping-Pong property. We therefore have

**Property II.5** (coding property). *Fix $x_0 \in \partial X - \bigcup_{a\in\mathscr{A}} C_{a^\pm}$. For every $x \in \Lambda^0$ there exists an unique sequence $\omega(x) = (a_i^{n_i})_{i\geqq 1}$ with $a_i \in \mathscr{A}$, $n_i \in \mathbb{Z}^*$ and $a_{i+1} \neq a_i$ such that $\lim_{k\to+\infty} a_1^{n_1} \cdots a_k^{n_k} x_0 = x$.*

## III. The critical gap property

Let $g$ be a non-elliptic isometry on $X$ and denote by $\delta_g$ the exponent of convergence of the Poincaré series $\sum_{n\in\mathbb{Z}} e^{-sd(0,g^n0)}$. If $g$ is hyperbolic, replace 0 by a point $z$ which belongs to the axis of $g$; one obtains $\sum_{n\in\mathbb{Z}} e^{-sd(z,g^nz)} = \sum_{n\in\mathbb{Z}} e^{-snB_{x_g}(0,g0)}$. Since $B_{x_g}(0,g0)>0$, one has $\delta_g=0$ and the group generated by $g$ is of divergence type.

If $g$ is parabolic, the estimate of $\delta_g$ is much more complicated. For example, a direct computation shows that $\delta_g=1/2$ when $X$ is the real hyperbolic half space and $\delta_g\in\{1/2,1\}$ when $X$ is the complex hyperbolic half space (*we thank here M. Bourdon who has pointed out this fact to us*). In the general case where $X$ is a Hadamard manifold with sectional curvature $-b^2\leqq K\leqq -1$, denote by $\mathscr{H}$ the horosphere through 0 and with center $x_g$ and by $h$ the distance in $\mathscr{H}$ with respect to the induced metric; for any $p,q\in\mathscr{H}$ one has $2\sinh\frac{d(p,q)}{2}\leqq h(p,q)\leqq\frac{2}{b}\sinh\frac{b}{2}d(p,q)$ ([16], Thm. 4.6). Since $g^n(0)$ belongs to $\mathscr{H}$ for any $n\in\mathbb{Z}$ one has

$$\begin{aligned} d(0,g^n0) &\leqq 2\operatorname{Log}\big(h(0,g^n0)+1\big)\\ &\leqq 2\operatorname{Log}\big(h(0,g0)+\cdots+h(g^{n-1}0,g^n0)+1\big)\\ &\leqq 2\operatorname{Log}\big((|n|+1)\,c_n(g)\big)\end{aligned}$$

with

$$c_n(g)=\sup_{1\leqq k\leqq n} 1+h(g^{k-1}0,g^k0)\leqq \sup_{1\leqq k\leqq n} 1+\frac{2}{b}\sinh\frac{2}{b}d(g^{k-1}0,g^k0)\leqq 1+\frac{2}{b}\sinh\frac{2}{b}d(0,g0).$$

It readily follows that $\sum_{n\in\mathbb{Z}^*} e^{-sd(0,g^n0)}\geqq A^{2s}\sum_{n\in\mathbb{Z}^*}\frac{1}{n^{2s}}$ for some positive constant $A$ which implies $\delta_g\geqq 1/2$.

If $X$ is a symmetric space of rank one, the group generated by any parabolic isometry $g$ is of divergence type (see for example [8]); this is also the case if there exists a horoball centered at $x_g$ isometric to a horoball of a symmetric space of rank one. Note that there exist Hadamard manifolds of pinched curvature which possess parabolic isometries of convergent type ([9]).

**Theorem III.1.** *Let $G$ be a non elementary group of isometries of $X$. For any $g\in G$ such that $\sum_{n\in\mathbb{Z}} e^{-\delta_g d(0,g^n0)}=+\infty$, one has $\delta_G>\delta_g$.*

If $G$ is purely hyperbolic one obtains $\delta_\Gamma>0$; this fact is not new and it holds even if $X$ is a general Gromov hyperbolic space [7]. If $G$ contains parabolic transformations one obtains in particular $\delta_\Gamma>1/2$; this inequality was already proved in constant curvature ([3], [26]).

*Proof.* One adapts here a Beardon's argument [3]. Fix $g\in G$ such that the series $\sum_{n\in\mathbb{Z}^*} e^{-sd(0,g^n0)}$ diverges at its critical exponent $\delta_g$; since $G$ is non elementary there exists an hyperbolic isometry $h$ such that the group $\langle g,h\rangle$ generated by $g$ and $h$ is an extended Schottky group. One has

$$\begin{aligned}\sum_{\gamma\in\langle g,h\rangle} e^{-sd(0,\gamma 0)} &\geqq \sum_{k\geqq 1}\sum_{n_1,\dots,n_k\in\mathbb{N}^*} e^{-sd(0,g^{n_1}hg^{n_2}h\dots g^{n_k}h0)}\\ &\geqq \sum_{k\geqq 1}\sum_{n_1,\dots,n_k\in\mathbb{N}^*} e^{-s(d(0,g^{n_1}0)+\dots+d(0,g^{n_k}0)+kd(0,h0))}\\ &\geqq \sum_{k\geqq 1}\Big(e^{-sd(0,h0)}\sum_{n\geqq 1} e^{-sd(0,g^n0)}\Big)^k.\end{aligned}$$

Since $\lim_{\substack{s\to\delta_g\\ s>\delta_g}}\sum_{n\geqq 1} e^{-sd(0,g^n0)}=+\infty$, one can choose $s>\delta_g$ such that $e^{-sd(0,h0)}\sum_{n\geqq 1} e^{-sd(0,g^n0)}>1$; this implies that the series $\sum_{\gamma\in\langle g,h\rangle} e^{-sd(0,\gamma 0)}$ diverges for some $s>\delta_g$ so that $\delta_G\geqq\delta_{\langle g,h\rangle}>\delta_g$. □

## IV. The Patterson-Sullivan measure on $\Lambda$

Denote by $\delta_\Gamma$ the exponent of convergence of the Poincaré series $\sum_{\gamma\in\Gamma} e^{-sd(0,\gamma 0)}$ and $\sigma$ the Patterson-Sullivan measure on $\Lambda$ [19], [26]. Since the sectional curvature of $X$ is lower bounded, $\delta_\Gamma$ is finite. For any isometry $\gamma\in\Gamma$ one has

$$(*)\qquad \frac{d(\gamma^{-1}\sigma)}{d\sigma}(x)=|\gamma'(x)|^{\delta_\Gamma}\quad \sigma(dx)\text{-a.s.}$$

Recall the *shadow* $\theta(z,r)$ on $\partial X$ of the ball $\mathbb{B}(z,r)$ with center $z\in X$ and radius $r>0$ is the set of points $x\in\partial X$ such that the geodesic ray joining 0 and $x$ meets $\mathbb{B}(z,r)$. Let us recall the following

**Lemma IV.1** (Sullivan's shadow lemma [7]). *There exist $C\geqq 1$ and $d_0>0$ such that for every $r\geqq d_0$ and $\gamma\in\Gamma$ one has*

$$\frac{1}{C}e^{-\delta_\Gamma d(0,\gamma 0)}\leqq\sigma(\theta(\gamma 0,r))\leqq Ce^{-\delta_\Gamma d(0,\gamma 0)+2r\delta_\Gamma}.$$

When the curvature is constant it is well known that the Patterson-Sullivan measure on $\Lambda$ has no atom [31]. By the Ping-Pong property and the dynamic of generators on $\partial X$ we show that the same property holds for any extended Schottky group acting on $X$. The following theorem has been obtained with J.P. Otal.

**Theorem IV.2.** *The Patterson-Sullivan measure $\sigma$ associated with $\Gamma$ has no atomic part.*

*Proof.* Since $\Gamma$ is geometrically finite ([5]), $\Lambda$ is the disjoint union of the radial limit set and the fixed points of parabolic transformations of $\Gamma$. Radial limit points cannot be atoms of $\sigma$ (see for example [33]), thus one just has to check that $\sigma\{x_\alpha\}=0$ for any parabolic generator $\alpha\in\mathscr{A}$.

For $s > \delta_\Gamma$ and $y \in X$ set $g_s(y) = \sum_{\gamma \in \Gamma} e^{-sd(y,\gamma y)} h(d(y,\gamma y))$ where $h$ is an increasing function of arbitrary small exponential growth which makes the series $g_s(y)$ diverge at $s = \delta_\Gamma$; more precisely for any $\varepsilon > 0$ there exists $d_\varepsilon > 0$ such that $h(d+t) \leqq e^{\varepsilon t} h(d)$ for any $t \geqq 0$ and $d \geqq d_\varepsilon$. By theorem III.1 one can fix $\varepsilon$ such that $\delta_\alpha + \varepsilon < \delta_\Gamma$; consequently, the series $\sum_{n \in \mathbb{Z}} e^{(-\delta_\Gamma + \varepsilon) d(0, \alpha^n 0)}$ converges.

Set $\Gamma' = \{\gamma = a_1 \cdots a_n \in \Gamma \mid a_i \neq a_{i+1}^{-1} \text{ and } a_1 \neq \alpha\}$; since the limit points of $\Gamma' y$ belong to $\Lambda - \Lambda_\alpha$ there exists a closed cone $C_z$ of vertex $z \in X$ such that $\Gamma' y \subset C_z$ but $x_\alpha \notin C_z$. Without loss of generality one may suppose that the distance between the origin 0 and the cone $C_z$ is greater than $d_\varepsilon$ and that the horosphere centered at $x_\alpha$ containing 0 is included in a cone of vertex 0 which does not intersect $C_z$; for every $k \geqq 1$ there thus exists a cone $C_k$ of vertex 0 and axis $[0, x_\alpha)$ such that $C_k \cap \alpha^n(C_z) = \emptyset$ when $|n| \leqq k$.

Recall that a Patterson-Sullivan measure $\sigma$ is a weak limit as $s \to \delta_\Gamma^+$ (along a subsequence if necessary) of the family of measures $\sigma_s = \dfrac{1}{g_s(y)} \sum_{\gamma \in \Gamma} e^{-sd(0,\gamma y)} h(d(0,\gamma y)) \delta_{\gamma y}$ where $\delta_{\gamma y}$ is the Dirax mass at $\gamma y$. By the choice of the cones $C_k$ one has

$$\sigma_s(C_k) \leqq \frac{1}{g_s(y)} \sum_{|n|>k} \sum_{\gamma' \in \Gamma'} e^{-sd(0,\alpha^n \gamma' y)} h(d(0,\alpha^n \gamma' y)).$$

By hyperbolic geometrical arguments (see for example [8], lemma 3.1) there exists a constant $K > 0$ such that $d(0,\alpha^n 0) + d(0,\gamma' y) - K \leqq d(0,\alpha^n \gamma' y) \leqq d(0,\alpha^n 0) + d(0,\gamma' y)$ for any $n \in \mathbb{Z}^*$ and $\gamma' \in \Gamma'$; consequently, for $s > \delta_\Gamma$ one has, up to multiplicative constants

$$\begin{aligned}\sigma_s(C_k) &\leqq \frac{1}{g_s(y)} \sum_{|n|>k} e^{(-s+\varepsilon) d(0,\alpha^n 0)} \sum_{\gamma' \in \Gamma'} e^{-sd(0,\gamma' y)} h(d(0,\gamma' y)) \\ &\leqq \left( \sum_{|n|>k} e^{(-\delta_\Gamma + \varepsilon) d(0,\alpha^n 0)} \right) \sigma_s(C_z).\end{aligned}$$

Letting $s \to \delta_\Gamma$ one obtains $\sigma\{x_\alpha\} \leqq \sigma(C_k \cap \partial X) \leqq \left( \sum_{|n|>k} e^{(-\delta_\Gamma + \varepsilon) d(0,\alpha^n 0)} \right) \sigma(C_z \cap \partial X)$ for any integer $k \geqq 1$. Letting $k \to +\infty$ one obtains $\sigma\{x_\alpha\} = 0$. □

**Corollary IV.3.** *The group $\Gamma$ is of divergence type, i.e.* $\sum_{\gamma \in \Gamma} e^{-\delta_\Gamma d(0,\gamma 0)} = +\infty$.

*Proof.* The following argument is classical (see for example [31] or [33]). Set $\Gamma = \{g, n \geqq 1\}$ and assume $\sum_{n \geqq 1} e^{-\delta_\Gamma d(0, g_n(0))} < +\infty$; the lemma III.1 thus implies $\sum_{n \geqq 1} \sigma(\theta(g_n 0, A)) < +\infty$ for $A$ large enough. By the Borel-Cantelli lemma one obtains $\sigma(\limsup_{n \to +\infty} \theta(g_n 0, A)) = 0$. The inclusion $\Lambda^0 \subset \bigcup_{\substack{A \in \mathbb{N} \\ A \geqq d_0}} \bigcap_{N \geqq 1} \bigcup_{n \geqq N} \theta(g_n 0, A)$ leads to $\sigma(\Lambda^0) = 0$ and so $\sigma(\Lambda - \Lambda^0) = 1$; since $\Lambda - \Lambda^0$ is countable, this last equality contradicts the fact that $\sigma$ has no atomic part. □

When $X$ is the real hyperbolic half space, this divergence property is satisfied for any non elementary geometrically finite discret group [31]; recently this result has been extended by K. Corlette and A. Iozzi [8] to the case where $X$ is a symmetric space of rank one.

## V. The geodesic flow and the boundary map

Let $G\Lambda$ be the set of pairs $(\xi, x)$ where $\xi$ is an oriented geodesic on $X$ whose endpoints $\xi^- = \xi(-\infty)$ and $\xi^+ = \xi(+\infty)$ belong to $\Lambda$ and $x \in \xi$. Denote by $0_\xi$ the intersection of $\xi$ with the horosphere based at $\xi^+$ passing through the origin 0 and $\partial^2\Lambda$ the set $\Lambda \times \Lambda$-diagonal. The map $\pi : G\Lambda \to \partial^2\Lambda \times \mathbb{R}$ defined by $\pi(\xi, x) = (\xi^-, \xi^+, B_{\xi^+}(0_\xi, x))$ is bijective. The group $\Gamma$ acts on $\partial^2\Lambda \times \mathbb{R}$ in the following way:

$$\gamma(x_-, x, s) = (\gamma(x_-), \gamma(x), s - B_x(0, \gamma^{-1}0))$$

for any $\gamma \in \Gamma$ and $(x_-, x, s) \in \partial^2\Lambda \times \mathbb{R}$. Denote $(g_t)_{t\in\mathbb{R}}$ the geodesic flow on $\partial^2\Lambda \times \mathbb{R}$ defined by $g_t(x_-, x, s) = (x_-, x, s+t)$.

Set $\partial^2\Lambda^0 = \bigcup_{\substack{\alpha,\beta\in\mathscr{A}\\ \alpha\neq\beta}} \Lambda^0_{\alpha^\pm} \times \Lambda^0_{\beta^\pm}$. For any $(x_-, x) \in \partial^2\Lambda^0$ such that $a^n$ is the first term of the sequence $\omega(x)$, set $f(x) = B_x(0, a^n 0)$ and $\bar{T}(x_-, x) = (a^{-n}x_-, a^{-n}x)$; the action of $\Gamma$ on $\partial^2\Lambda \times \mathbb{R}$ induces a map $\bar{T}_f$ on $\partial^2\Lambda \times \mathbb{R}$ defined by

$$\bar{T}_f(x_-, x, s) = (\bar{T}(x_-, x), s - f(s)).$$

Remark that $\bar{T}_f$ is invertible with inverse $\bar{T}_f^{-1}(y_-, y, t) = (x_-, x, t + f(x))$ where $(y_-, y) = \bar{T}(x_-, x)$. Denote $G\Lambda^0$ the set of pairs $(\xi, x) \in G\Lambda$ such that endpoints of $\xi$ belong to $\Lambda^0$. The quotient $G\Lambda^0/\Gamma$ is identified with $\partial^2\Lambda^0 \times \mathbb{R}/\langle \bar{T}_f \rangle$.

Let $\mu$ be the measure on $\partial^2\Lambda$ defined by $\mu(dx_- dx) = \dfrac{\sigma(dx_-)\sigma(dx)}{D(x_-, x)^{2\delta_\Gamma}}$ and let $l$ be the Lebesgue measure on $\mathbb{R}$; since $D(\Lambda^0_{\alpha^\pm}, \Lambda^0_{\beta^\pm}) > 0$ for every $\alpha, \beta \in \mathscr{A}$, $\alpha \neq \beta$, the restriction $\mu_0$ of $\mu$ to the set $\partial^2\Lambda^0$ is finite. Furthermore, the measure $\mu \otimes l$ is invariant under the action of $\Gamma$ and of the geodesic flow $(g_t)_{t\in\mathbb{R}}$. In paragraph VIII we will prove that $0 < \nu(f) < +\infty$ which readily implies that $\mu_0 \otimes l$ induces on $G\Lambda^0/\Gamma$ a finite measure $\overline{\mu_0 \otimes l}$ invariant under the geodesic flow $(\bar{g}_t)_{t\in\mathbb{R}}$ induced by $(g_t)_{t\in\mathbb{R}}$.

There are close connections between the geodesic flow $(\bar{g}_t)_{t\in\mathbb{R}}$ and the action of $\Gamma$ on $\Lambda$; in particular, if $G$ is a geometrically finite discret group of divergent type, the geodesic flow on the unit tangent bundle of $X/\Gamma$ is ergodic relatively to $\overline{\mu \otimes l}$ ([18], [31]). In paragraph VI we prove that if $\Gamma$ is an extended Schottky group then $(\bar{g}_t)_{t\in\mathbb{R}}$ is mixing relatively to $\overline{\mu \otimes l}$; to show this we first have to control the action of $\Gamma$ on $\Lambda^0$.

Let $T$ be the *boundary map* on $\Lambda^0$ induced by $\bar{T}$ and defined by $T(x) = a^{-n}x$ where $a^n$ is the first term of $\omega(x)$; this mapping is the geometrical interpretation of the shift operator on the symbolic space $\{\omega(x), x \in \Lambda^0\}$ and its properties will play an important rule in the sequel.

**Proposition V.1.** *There exists $N \in \mathbb{N}^*$ such that* $\inf_{x \in \Lambda^0} |(T^N)'(x)| > 1$.

Set $B_0 = \inf_{x \in \Lambda^0} |(T^N)'(x)| > 1$; using the mean values relation one obtains the

**Corollary V.2.** *Let $x, y \in \Lambda^0$ such that the sequences $\omega(x)$ and $\omega(y)$ have the same $N$ first terms. Then $D(T^N x, T^N y) \geqq B_0 D(x, y)$.*

*Proof of proposition* V.1. Fix $B > 0$ and suppose that for any $n \geqq 0$ there exists $x_n \in \Lambda^0$ such that $|(T^n)'(x_n)| \leqq B$. Set $\omega(x_n) = (a_{nk}^{p_{nk}})_{k \geqq 1}$; without loss of generality one can suppose $a_{n1} = \alpha$, $a_{nn} = \beta_1$ and $a_{n(n+1)} = \beta_2$ for any integer $n \geqq 1$. If $\alpha = \beta_1$, let $a \in \mathscr{A} - \{\beta_1\}$ and set $X_n = a x_{n-1}$ for any $n \geqq 1$; one has $|(T^n)'(X_n)| = |(a^{-1})'(X_n)|\,|(T^{n-1})'(x_{n-1})|$ and so $|(T^n)'(X_n)| \leqq B \sup_{x \in \Lambda^0} |(a^{-1})'(x)|$ which proves that $(x_n)_{n \geqq 1}$ and $(X_n)_{n \geqq 1}$ satisfy a similar condition. Hence, without loss of generality, one may suppose $\alpha \neq \beta_1^{\pm 1}$.

Set $\gamma_n = a_{n1}^{p_{n1}} \cdots a_{nn}^{p_{nn}}$ and $y_n = \gamma_n^{-1} x_n$. Since $\beta_1 \neq \beta_2$ we have $D(y_n, x_{\gamma_n}^-) \geqq D(C_{\beta_2^\pm}, C_{\beta_1^\pm}) > 0$ so that

$$D(x_n, x_{\gamma_n}^-) = \sqrt{|\gamma_n'(y_n)|\,|\gamma_n'(x_{\gamma_n}^-)|}\, D(y_n, x_{\gamma_n}^-) \geqq \sqrt{\frac{\Phi(\gamma_n)}{B}}\, D(C_{\beta_2^\pm}, C_{\beta_1^\pm}).$$

Condition $\alpha \neq \beta_1$ implies that the transformations $\gamma_n$ are not pairwise conjugated; by corollary II.3 one obtains $\lim_{n \to +\infty} \Phi(\gamma_n) = +\infty$ so that $\lim_{n \to +\infty} D(x_n, x_{\gamma_n}^-) = +\infty$ which contradicts the compactness of $(\partial X, D)$. □

Now we construct a $T$-invariant probability measure on $\Lambda^0$. By equality $(*)$ of paragraph IV and by the mean values relation, the measure $\mu_0(dx_- \, dx) = \dfrac{\sigma(dx_-)\sigma(dx)}{D(x_-, x)^{2\delta_\Gamma}}$ on $\partial^2 \Lambda^0$ is $\bar{T}$-invariant. Set $\mu_0(\partial^2 \Lambda^0) = 1/C$ and let $p: \partial^2 \Lambda^0 \to \Lambda^0$ be the projection on the second coordinate; one has

**Proposition V.3.** *The measure $\nu = Cp(\mu_0)$ is a $T$-invariant probability measure on $\Lambda^0$, absolutely continuous with respect to $\sigma$ with density $h$ given by*

$$\forall \alpha \in \mathscr{A},\ \forall x \in \Lambda^0_{\alpha^\pm} \quad h(x) = C \int_{\Lambda^0 - \Lambda^0_{\alpha^\pm}} \frac{\sigma(dy)}{D(x, y)^{2\delta_\Gamma}}.$$

Let us now introduce the transfer operator $P$ associated with $(T, \nu)$. Denote by $\mathbb{L}^1(\Lambda^0, \nu)$ (resp. $\mathbb{L}^\infty(\Lambda^0, \nu)$) the standard completion of the space of Borel functions from $\Lambda^0$ into $\mathbb{R}$ which are integrable (resp. bounded) with respect to $\nu$. Since $\nu$ is $T$-invariant, the transformation $T$ induces an isometry on $\mathbb{L}^1(\Lambda^0, \nu)$ defined by $T(\psi) = \psi \circ T$ for any $\psi \in \mathbb{L}^1(\Lambda^0, \nu)$. For any $\varphi \in \mathbb{L}^\infty(\Lambda^0, \nu)$, let $P\varphi$ be the function in $\mathbb{L}^\infty(\Lambda^0, \nu)$ such that

$$\forall \psi \in \mathbb{L}^1(\Lambda^0, \nu) \int_{\Lambda^0} \varphi(x)(T\psi)(x)\nu(dx) = \int_{\Lambda^0} P\varphi(x)\psi(x)\nu(dx).$$

One has $P\varphi(x) = \sum_{y \in \Lambda^0/Ty=x} \frac{h(y)}{h(x)} e^{-\delta_\Gamma f(y)} \varphi(y) = \sum_{\substack{\alpha \in \mathscr{A} \\ n \in \mathbb{Z}^*}} 1_{\Lambda^0 - \Lambda^0_{\alpha^\pm}}(x) \frac{h(\alpha^n x)}{h(x)} |(\alpha^n)'(x)|^{\delta_\Gamma} \varphi(\alpha^n x)$ for $\nu$-almost all $x$ in $\Lambda^0$. A priori, $P$ acts on $\mathbb{L}^\infty(\Lambda^0, \nu)$; nevertheless it is possible to define $P\varphi(x)$ in $\bar{\mathbb{R}}^+$ for any positive Borel function $\varphi$ on $\Lambda$ in the following way:

**Definition V.4.** For every Borel function $\varphi$ from $\Lambda$ into $\mathbb{R}^+$ and every point $x \in \Lambda$, set

$$P\varphi(x) = \sum_{\substack{\alpha \in \mathscr{A} \\ n \in \mathbb{Z}^*}} p_{\alpha^n}(x) \varphi(\alpha^n x)$$

with $p_{\alpha^n}(x) = 1_{\Lambda - \Lambda_{\alpha^\pm}}(x) \dfrac{h(\alpha^n x)}{h(x)} |(\alpha^n)'(x)|^{\delta_\Gamma}$.

Note that for every $x \in \Lambda^0$ and every $n \geqq 1$ one has

$$P^n\varphi(x) = \sum_{y \in \Lambda^0/T^n y = x} \frac{h(y)}{h(x)} e^{-\delta_\Gamma S_n f(y)} \varphi(y).$$

In the same way the mapping $\bar{T}_f$ induces a transformation $T_f$ on $\Lambda^0 \times \mathbb{R}$ defined by $T_f(x, s) = (Tx, s - f(x))$ for every $(x, s) \in \Lambda^0 \times \mathbb{R}$. In some sense, the set $\partial^2\Lambda^0$ is a section for the geodesic flow on $\partial^2\Lambda \times \mathbb{R}/\Gamma$ and $T_f$ is the first return map for this flow on this section; the transformation $T_f$ memorizes the "travel time" between two consecutive passages through $\partial^2\Lambda^0$. Let us introduce the operator $\tilde{P}$ associated with $T_f$:

**Definition V.5.** For every Borel function $\psi$ from $\Lambda \times \mathbb{R}$ into $\mathbb{R}^+$ and every $(x, t) \in \Lambda \times \mathbb{R}$ set

$$\tilde{P}\psi(x, t) = \sum_{\substack{\alpha \in \mathscr{A} \\ n \in \mathbb{Z}^*}} p_{\alpha^n}(x) \psi(\alpha^n x, t + f(\alpha^n x)).$$

Note that for every $(x, t) \in \Lambda^0 \times \mathbb{R}$ one has

$$\tilde{P}\psi(x, t) = \sum_{y \in \Lambda^0/Ty = x} \frac{h(y)}{h(x)} e^{-\delta_\Gamma f(y)} \psi(y, t + f(y))$$

and

$$\forall n \geqq 1 \quad \tilde{P}^n\psi(x, t) = \sum_{y \in \Lambda^0/T^n y = x} \frac{h(y)}{h(x)} e^{-\delta_\Gamma S_n f(y)} \psi(y, t + S_n f(y)).$$

## VI. A renewal theorem to prove that the geodesic flow is mixing

**Notation.** From this section on, $\Gamma$ is an extended Schottky group and $\delta$ is the exponent of convergence of the Poincaré series $\sum_{\gamma \in \Gamma} e^{-sd(0, \gamma 0)}$.

We state here a classical renewal theorem which describes the behaviour as $a$ goes to $\pm\infty$ of the potential $\sum_{n=0}^{+\infty} \tilde{P}^n((x, a), dy\, dt)$ and we show how one may deduce the mixing property of the geodesic flow $(\bar{g}_t)_{t\in\mathbb{R}}$. First let us introduce a functional space $L$ on which $P$ acts.

**Notation.** Let $L$ be the space of functions $\varphi$ from $\Lambda$ into $\mathbb{C}$ such that

$$\|\varphi\| = |\varphi|_\infty + m(\varphi) < +\infty$$

where $|\cdot|_\infty$ is the norm of uniform convergence on $\Lambda$ and

$$m(\varphi) = \sup_{\alpha\in\mathscr{A}} \sup_{\substack{x,y\in\Lambda_{\alpha^\pm}\\ x\neq y}} \frac{|\varphi(x)-\varphi(y)|}{D(x,y)^{\delta_0}} \quad \text{with } \delta_0 = \inf\{1, \delta\}.$$

For every $\lambda\in\mathbb{R}$ let $P_\lambda$ be the operator defined by $P_\lambda\varphi = P(e^{i\lambda f}\varphi)$. In paragraph VIII we will prove the following facts.

**Properties VI.1** (properties R).

(R1) *The operator $P$ acts on $(L, \|\,.\,\|)$.*

(R2) *One has $0 < \nu(f) < +\infty$ and $\sup_{x\in X} Pf^n(x) < +\infty$ for any $n \geqq 1$.*

(R3) *For any real number $\lambda$ the operator $P_\lambda$ acts on $L$; moreover, the mapping $\lambda\mapsto P_\lambda$ is analytic from $(\mathbb{R}, |\,.\,|)$ into the Banach space $(\mathscr{L}(L), \|\,.\,\|_{\mathscr{L}(L)})$ of continuous linear applications on $(L, \|\,.\,\|)$ with the usual norm.*

(R4) *One has $P1 = 1$, the eigenvalue 1 is simple and isolated in the spectrum of $P$ and $\nu$ is the projection on the associated eigenspace $\mathbb{C}\,1_\Lambda$.*

(R5) *For every $\lambda \neq 0$ the spectral radius of $P_\lambda$ on $(L, \|\,.\,\|)$ is strictly less than 1.*

Note that property (R5) is closely related to the fact that $\Gamma$ contains parabolic transformations; if $\Gamma$ is a Schottky group this property is not proved.

Using arguments developped in [1], [14] one proves the following theorem:

**A renewal theorem.** *Assume that $(P, f)$ satisfies properties R. Then for any compact set $K\subset\Lambda\times\mathbb{R}$, for any bounded Borel function $\varphi:\Lambda\to\mathbb{R}$ whose discontinuity points are a $\nu$-negligeable set and for any continuous function $u:\mathbb{R}\to\mathbb{R}$ with compact support one has*

$$\lim_{a\to+\infty} \sup_{(x,s)\in K} \left| \sum_{n=0}^{+\infty} \tilde{P}^n(\varphi\otimes u)(x, s-a) - \frac{\nu(\varphi)\int_{\mathbb{R}} u(t)\,dt}{\nu(f)} \right| = 0$$

*and*

$$\lim_{a\to+\infty} \sup_{(x,s)\in K} \left| \sum_{n=0}^{+\infty} \tilde{P}^n(\varphi\otimes u)(x, s+a) \right| = 0.$$

Throughout this paragraph we will assume that properties R hold. Recall that the inequality $0 < \nu(f) < +\infty$ implies that $\overline{\mu \otimes l}$ is finite on $GA/\Gamma$ and that $\overline{\mu \otimes l}(GA/\Gamma) = \nu(f)$.

**Theorem VI.2.** *Let $\Gamma$ be an extended Schottky group. Then the geodesic flow $(\bar{g}_t)_{t \in \mathbb{R}}$ on $GA/\Gamma$ is mixing relatively to $\overline{\mu \otimes l}$.*

*Proof.* We adapt here Y. Guivarc'h and J. Hardy's proof of the mixing property for a special flow constructed with a Hölder continuous function over a subshift of finite type [14]. One has to show that for every functions $\Phi$ and $\Psi$ in $\mathbb{L}^2(GA/\Gamma, \overline{\mu \otimes l})$

$$\lim_{t \to +\infty} I_t(\Phi, \Psi) = \frac{1}{\nu(f)} \overline{\mu \otimes l}(\Phi) \overline{\mu \otimes l}(\Psi)$$

with $I_t(\Phi, \Psi) = \int_{\partial^2 A \times \mathbb{R}/\Gamma} \Phi(x_-, x, s) \Psi \circ \bar{g}_t(x_-, x, s) \overline{\mu \otimes l}(dx_- \, dx ds)$. By proposition III.2 the measure $\sigma$ has no atomic part; the same holds for $\mu \otimes l$ and so, without loss of generality, one may suppose that $\Phi$ and $\Psi$ are defined on $GA^0$; one identifies $GA^0/\Gamma$ with a suitable fundamental domain $S \subset \partial^2 A^0 \times \mathbb{R}$ for the action of the group $\langle \bar{T}_f \rangle$ and we also denote by $(\bar{g}_t)_{t \in \mathbb{R}}$ the corresponding flow on $S$. Using a density argument consider $\Phi = \varphi \otimes u$ where $\varphi$ is Hölder continuous on $\partial^2 A^0$, $u$ is continuous on $\mathbb{R}$ and the support of $\Phi$ is included in $S$. In the same way, it suffices to consider a Hölder continuous function $\psi$ on $\partial^2 A^0$ and a continuous function on $\mathbb{R}$ whose supports are compact and such that

$$\forall (x_-, x, s) \in S \quad \Psi(x_-, x, s) = \sum_{n \in \mathbb{Z}} \psi \otimes v\big(\bar{T}_f^n(x_-, x, s)\big).$$

One thus obtains $I_t(\Phi, \Psi) = I_t^+(\Phi, \Psi) + I_t^-(\Phi, \Psi)$ with

$$I_t^+(\Phi, \Psi) = \sum_{n \geq 0} \int_{A^0 \times \mathbb{R}} \varphi(x_-, x) u(s) \psi \otimes v\big(\bar{T}_f^n(x_-, x, s+t)\big) \mu(dx_- \, dx) ds$$

and

$$I_t^-(\Phi, \Psi) = \sum_{n \geq 1} \int_{A^0 \times \mathbb{R}} \varphi(x_-, x) u(s) \psi \otimes v\big(\bar{T}_f^{-n}(x_-, x, s+t)\big) \mu(dx_- \, dx) ds.$$

Let us first deal with the term $I_t^+(\Phi, \Psi)$. Identify $(x_-, x)$ with a bilateral sequence $(a_i^{p_i})_{i \in \mathbb{Z}}$ with $\omega(x) = (a_i^{p_i})_{i \geq 1}$ and $\omega(x_-) = (a_i^{-p_i})_{i \leq 0}$. Using again a density argument, one may suppose that $\varphi$ and $\psi$ do only depend on the unilateral sequence $(a_i^{p_i})_{i \geq -l}$ with $l \geq 0$; moreover, one may suppose $l = 0$ because the measure $\mu$ is $\bar{T}$-invariant. Then, for a suitable choice of $\Phi = \varphi \otimes u$ and $\Psi = \psi \otimes v$ one has

$$\begin{aligned} I_t^+(\Phi, \Psi) &= \sum_{n=0}^{+\infty} \int_{A^0 \times \mathbb{R}} \frac{\varphi(x)}{h(x)} u(s) \psi\big(T^n(x)\big) v\big(s + t - S_n f(x)\big) \nu(dx) ds \\ &= \sum_{n=0}^{+\infty} \int_{A^0 \times \mathbb{R}} \tilde{P}^n\left(\frac{\varphi}{h} \otimes u\right)(x, s-t) \psi(x) v(s) \nu(dx) ds. \end{aligned}$$

On the other hand, since $\mu \otimes l$ is $\bar{T}_f$-invariant, one has

$$I_t^-(\Phi,\Psi)=\sum_{n=1}^{+\infty}\int_{\Lambda^0\times\mathbb{R}}\varphi\otimes u(\overline{T}_f^n(x_-,x,s))\,\psi(x_-,x)\,v(s+t)\,\mu(dx_-\,dx)\,ds$$

and for a suitable choice of functions $\Phi$ and $\Psi$ one has

$$I_t^-(\Phi,\Psi)=\sum_{n=1}^{+\infty}\int_{\Lambda^0\times\mathbb{R}}\varphi(x)\,u(s)\,\tilde{P}^n\left(\frac{\psi}{h}\otimes v\right)(x,s+t)\,\nu(dx)\,ds\,.$$

By the renewal theorem one obtains

$$\lim_{t\to+\infty}\ \sup_{(x,s)\in\mathrm{Supp}(\Psi)}\left|\sum_{n=0}^{+\infty}\tilde{P}^n\left(\frac{\varphi}{h}\otimes u\right)(x,s-t)-\frac{\nu(\varphi)\int_{\mathbb{R}}u(y)\,dy}{\nu(f)}\right|=0$$

and

$$\lim_{t\to+\infty}\sum_{n=1}^{+\infty}\tilde{P}^n(\psi\otimes v)(x,s+t)=0\,.$$

The Lebesgue dominated convergence theorem allows us to conclude

$$\lim_{t\to+\infty}I_t(\Phi,\Psi)=\frac{1}{\nu(f)}\,\overline{\mu\otimes l}(\Phi)\,\overline{\mu\otimes l}(\Psi)$$

which finishes the proof. □

## VII. An harmonic renewal theorem to count closed geodesics on $X/\Gamma$

Consider the equivalence relation $\sim$ on $\Gamma$ defined by $\gamma_1\sim\gamma_2$ if and only if $\gamma_1$ and $\gamma_2$ are conjugated in $\Gamma$. For any class $c\in\Gamma/\sim$ choose $\gamma_c=a_1^{n_1}\cdots a_k^{n_k}$ in $c$ such that $a_i\in\mathscr{A}$, $a_{i+1}\neq a_i$ for $1\leqq i<k$ and $a_1\neq a_k$. Denote by $\mathscr{C}_0$ the set of $\gamma_c$ which are primitive (i.e. $\gamma_c\neq\gamma^n$ for all $\gamma\in\Gamma$) and hyperbolic. Let $\gamma_0=a_1^{n_1}\cdots a_k^{n_k}\in\mathscr{C}_0$; the expansion $\omega(x_{\gamma_0})$ of the attractive fixed point $x_{\gamma_0}$ of $\gamma_0$ is periodic with period $a_1^{n_1},\dots,a_k^{n_k}$. Set $l(\gamma_0)=d(z,\gamma_0(z))$ where $z$ belongs to the axis of $\gamma_0$; one has $l(\gamma_0)=B_{x_{\gamma_0}}(0,\gamma_0(0))$. Recall that $T:\Lambda^0\to\Lambda^0$ and $f:\Lambda^0\to\mathbb{R}$ are defined by $T(x)=a^{-n}(x)$ and $f(x)=B_x(0,a^n0)$ where $a^n$ is the first term of $\omega(x)$. For any $k\geqq 1$ set $S_kf(x)=f(x)+\cdots+f(T^{k-1}x)$. Using the fact that $B_{\gamma x}(\gamma z_1,\gamma z_2)=B_x(z_1,z_2)$ for every isometry $\gamma$ and that $B_x(z_1,z_2)=B_x(z_1,z)+B_x(z,z_2)$ one obtains $l(\gamma_0)=S_kf(x_{\gamma_0}^+)$ for any $\gamma_0=a_1^{n_1}\cdots a_k^{n_k}\in\mathscr{C}_0$.

Conversely, consider a $T$-periodic point $x\in\partial X$ and let $k$ be its least period; one has $\omega(x)=a_1^{n_1},\dots,a_k^{n_k},a_1^{n_1},\dots,a_k^{n_k},\dots$. Set $\gamma=a_1^{n_1}\cdots a_k^{n_k}$; since $a_k\neq a_1$ we have $\lim_{p\to+\infty}\gamma^p0=x$ which shows that $\gamma$ is hyperbolic and $x$ is its attractive fixed point. Moreover $\gamma\in\mathscr{C}_0$ since $k$ is the least period of $x$.

Finally, the number $\pi(a)$ of $\gamma$ in $\mathscr{C}_0-\{\alpha_1,\dots,\alpha_{N_1}\}$ such that $l(\gamma)\leqq a$ is given by

$$\pi(a)=\sum_{k=1}^{+\infty}\frac{1}{k}\,\#\{x\in\Lambda^0\,|\,x\text{ is }T\text{ periodic, }k\text{ is the least period of }x\text{ and }S_kf(x)\leqq a\}\,.$$

By corollary II.3 the number $\pi(a)$ is finite for every $a > 0$. In this paragraph we prove the

**Theorem VII.1.** *Let $\Gamma$ be an extended Schottky group and $\delta$ be the exponent of convergence of the Poincaré series associated with $\Gamma$. Then, the function $a \to \pi(a)$ is equivalent to $\frac{e^{a\delta}}{a\delta}$ when $a$ goes to $+\infty$.*

To prove this theorem, one approximates $\pi(a)$ by harmonic potentials $\sum_{n \geqq 1} \frac{1}{n} \tilde{P}^n$; in paragraph VIII we prove that $(P, f)$ satisfies Properties R given in the previous section which allows us to state the following result.

**A harmonic renewal theorem.** *For any bounded Borel function $\varphi : \Lambda \to \mathbb{R}$ whose discontinuity points form a $\nu$-negligeable set in $\Lambda$ and for any continuous function $u : \mathbb{R} \to \mathbb{R}$ with compact support one has*

$$\lim_{a \to +\infty} \sup_{x \in \Lambda} \left| a \sum_{n=1}^{+\infty} \frac{1}{n} \tilde{P}^n(\varphi \otimes u)(x, -a) - \nu(\varphi) \int_{\mathbb{R}} u(t)\, dt \right| = 0 .$$

*In particular, if $\nu(\varphi) > 0$, and if $u$ is a non decreasing continuous function from $\mathbb{R}^+$ to $\mathbb{R}^+$ one has*

$$\lim_{a \to +\infty} \sup_{x \in \Lambda} \left| a \frac{\sum_{n=1}^{+\infty} \frac{1}{n} \tilde{P}^n(\varphi \otimes u 1_{[0,a]})(x, 0)}{\frac{\nu(\varphi)}{a} \int_0^a u(t)\, dt} - 1 \right| = 0 .$$

**Notation.** Let $a > 0$ and $A$ be a Borel set included in $\Lambda$. Set $g_A = \frac{1_A}{h}$ and $\xi_a(t) = e^{\delta t} 1_{[0,a]}(t)$. For any $(x, t) \in \Lambda \times \mathbb{R}$ set $N_{(x,a)}(A) = \sum_{n=1}^{+\infty} \frac{1}{n} \tilde{P}^n(g_A \otimes \xi_a)(x, 0)$.

Applying the above harmonic renewal theorem with $\varphi = g_A$ and $u 1_{[0,a]} = \xi_a$ one thus obtains the following

**Corollary VII.2.** *For every Borel set $A \subset \Lambda$ such that $\sigma(A) > 0$ and with $\nu$-negligeable boundary, $N_{(x,a)}(A)$ is equivalent to $h(x) \frac{e^{a\delta}}{a\delta} \sigma(A)$ as $a$ goes to $+\infty$, uniformly in $x \in \Lambda$.*

*Proof of theorem* VII.1. Set

$$N(a) = \sum_{n=1}^{+\infty} \frac{1}{n} \# \{x \in \Lambda^0 \mid T^n x = x \text{ and } S_n f(x) \leqq a\} .$$

One has $N(a) - N(a/2) \leqq \pi(a) \leqq N(a)$; to prove theorem VII.1 it suffices to show that $\lim_{a \to +\infty} \frac{N(a)}{e^{a\delta}/a\delta} = 1$. The following proposition is proved at the end of the current paragraph.

**Proposition VII.3** (perturbation of $T$-periodic points). *There exists $k_0 \in \mathbb{N}^*$ such that for any $k \geqq k_0$ there exist a countable partition $(\Lambda_{ki}^0)_{i\geqq 1}$ of open sets in $\Lambda$, a sequence of positive constants $(C_{ki})_{i\geqq 1}$ with $\sum_{i\geqq 1} C_{ki} < +\infty$ and $\theta_k \in \mathbb{R}^{*+}$ with $\lim_{k\to+\infty} \theta_k = 0$ such that*

(i) $\lim_{k\to+\infty} \sigma(\Lambda_{ki}^0) = 0$,

(ii) *for any $x_{ki}$ in $(\Lambda_{ki}^0)$ one has*

$$\sum_{i=1}^{+\infty} h(x_{ki}) N_{(x_{ki}, a-\theta_k)}(\Lambda_{ki}^0) \leqq N(a) \leqq \sum_{i=1}^{+\infty} h(x_{ki}) N_{(x_{ki}, a+\theta_k)}(\Lambda_{ki}^0),$$

(iii) $N_{(x_{ki}, a)}(\Lambda_{ki}^0) \leqq C_{ki} \dfrac{e^{a\delta}}{a\delta}$ *for any $a > 0$.*

Applying Fatou's lemma in the left inequality (ii) of this proposition one has by corollary VII.2

$$e^{-\delta\theta_k} \sum_{i=1}^{+\infty} h(x_{ki}) \sigma(\Lambda_{ki}^0) \leqq \liminf_{a\to+\infty} \frac{a\delta}{e^{a\delta}} N(a).$$

In the same way, Lebesgue dominated convergence theorem and inequality (iii) give

$$\limsup_{a\to+\infty} \frac{a\delta}{e^{a\delta}} N(a) \leqq e^{\delta\theta_k} \sum_{i=1}^{+\infty} h(x_{k,i}) \sigma(\Lambda_{k,i}^0)$$

so that

$$e^{-\delta\theta_k} \sum_{i=1}^{+\infty} h(x_{ki}) \sigma(\Lambda_{ki}^0) \leqq \liminf_{a\to+\infty} \frac{a\delta}{e^{a\delta}} N(a) \leqq \limsup_{a\to+\infty} \frac{a\delta}{e^{a\delta}} N(a) \leqq e^{\delta\theta_k} \sum_{i=1}^{+\infty} h(x_{ki}) \sigma(\Lambda_{ki}^0).$$

Since $\sigma(\Lambda - \Lambda^0) = 0$ and $\lim_{k\to+\infty} \sigma(\Lambda_{ki}^0) = 0$ one has $\lim_{k\to+\infty} \sum_{i=1}^{+\infty} h(x_{ki}) \sigma(\Lambda_{ki}^0) = \int_\Lambda h(x) \sigma(dx) = 1$. Letting $k \to +\infty$ one obtains $\lim_{a\to+\infty} \dfrac{a\delta}{e^{a\delta}} N(a) = 1$. □

*Proof of proposition* VII.3. Let $k \in \mathbb{N}^*$ and consider the equivalence relation $\mathcal{R}_k$ on $\Lambda^0$ defined by $x \mathcal{R}_k y$ if and only if $\omega(x)$ and $\omega(y)$ have the same $k$ first terms. This relation induces a partition $(\Lambda_{ki}^0)_{i\geqq 1}$ on $\Lambda^0$.

(i) Let us first prove that $\lim_{k\to+\infty} \sigma(\Lambda_{ki}^0) = 0$. Fix $i \geqq 1$ and for any $x \in \Lambda_{ki}^0$ set $\omega(x) = (a_{kj}^{p_{kj}})_{j\geqq 1}$ and $\gamma_{kj} = a_{k1}^{p_{k1}} \cdots a_{kk}^{p_{kk}}$; one has $\sigma(\Lambda_{ki}^0) = \int_{\Lambda^0 - \Lambda_{a_{kk}}^0} |(\gamma_{kk}')(y)|^{2\delta} \sigma(dy)$. Without loss of generality one may suppose $a_{k1} \neq a_{kk}^{\pm 1}$ for any $k \geqq 1$ (if this condition does not hold it suffices to replace $\gamma_{kk}$ with $a_1 \gamma_{kk}$ where $a_1 \neq a_{kk}^{\pm 1}$). Then $\gamma_{kk}$ is hyperbolic and $\lim_{k\to+\infty} \Phi(\gamma_{kk}) = +\infty$ by corollary II.3. Consequently

$$|\gamma_{kk}'(y)| = \frac{D^2(\gamma_{kk} y, x_{\gamma_{kk}}^-)}{\Phi(\gamma_{kk}) D^2(y, x_{\gamma_{kk}}^-)} \geqq \frac{\|D\|_\infty^2}{\Phi(\gamma_{kk}) D^2(\Lambda_{a_{kk}}, \Lambda_{a_{kk}})}$$

so that $\lim_{k\to+\infty} \sigma(\Lambda_{ki}^0) = 0$.

(ii) For every $i \geqq 1$ fix $x_{ki} \in \Lambda^0_{ki}$. Let $x \in \Lambda^0_{ki}$ such that $T^n x = x$ for some $n \geqq 1$, denote by $\tilde{x}$ the unique point of $\Lambda^0_{ki}$ such that $T^n \tilde{x} = x_i$ and $\omega(x)$, $\omega(\tilde{x})$ have the same $n$ first terms. There is a bijection between the sets $\{x \in \Lambda^0 \mid T^n x = x\} \cap \Lambda^0_{ki}$ and $T^{-n}(\{x_{ki}\}) \cap \Lambda^0_{ki}$; the following lemma whose proof is given further allows us to control the difference $S_n f(x) - S_n f(\tilde{x})$.

**Lemma VII.4.** *There exist $k_0 \in \mathbb{N}^*$, $A, B \in \mathbb{R}^{*+}$ and $0 < r < 1$ such that for any $l \geqq k_0$ and any $x, y \in \Lambda^0$ whose associated sequences $\omega(x)$ and $\omega(y)$ have the same $l$ first terms, one has $|f(x) - f(y)| \leqq AD(Tx, Ty) \leqq Br^l$.*

Now fix $k \geqq k_0$; if $T^n x = x$ then $\omega(x)$ and $\omega(\tilde{x})$ have the same $n + k$ first terms so that $|S_n f(x) - S_n f(\tilde{x})| \leqq \theta_k$ with $\theta_k = B \sum_{l=k}^{+\infty} r^l$; inequalities (ii) of proposition VII.3 follow immediately.

(iii) For any Borel set $A \subset \Lambda^0$ set $N_{(x_{ki},a)}(A) = N'_{(x_{ki},a)}(A) + N''_{(x_{ki},a)}(A)$ with

$$N'_{(x_{ki},a)}(A) = \sum_{n=1}^{k} \frac{1}{n} \sum_{y/T^n y = x_{ki}} 1_A(y) 1_{[0,a]}(S_n f(y))$$

and

$$N''_{(x_{ki},a)}(A) = \sum_{n=k+1}^{+\infty} \frac{1}{n} \sum_{y/T^n y = x_{ki}} 1_A(y) 1_{[0,a]}(S_n f(y)).$$

By lemmas I.3 and I.4 there exist $C > 1$ and sequences $(K_{\alpha^n})_{n \in \mathbb{Z}^*}$, $\alpha \in \mathscr{A}$ such that for any $y \in \Lambda^0 - \Lambda^0_\alpha$ one has $|(\alpha^n)'(y)| \leqq CK_{\alpha^n}$. Fix $i \geqq 1$ and let $a_1^{p_1}, \ldots, a_k^{p_k}$ be the $k$ first terms of $\omega(x_{ki})$; for any $1 \leqq n \leqq k$ there is a unique $y \in \Lambda^0_{ki}$ such that $T^n y = x_{ki}$ and one has $S_n f(y) \geqq |\operatorname{Log} K_{a_1^{p_1}} \cdots K_{a_n^{p_n}}| - n \operatorname{Log} C$. So

$$N'_{(x_{ki},a)}(\Lambda^0_{ki}) = \sum_{n=1}^{k} \frac{1}{n} 1_{[0,a+n\operatorname{Log}C]}(|\operatorname{Log} K_{a_1^{p_1}} \cdots K_{a_n^{p_n}}|).$$

Since $x \mapsto \dfrac{e^x}{1+x}$ is increasing on $\mathbb{R}^+$, for any $b > 0$ we have $1_{[0,b]}(x) \leqq \dfrac{e^{b\delta}}{1 + b\delta} \dfrac{1 + x\delta}{e^{x\delta}}$ and so $N'_{(x_{ki},a)}(\Lambda^0_{ki}) \leqq C'_i \dfrac{e^{a\delta}}{a\delta}$ with $C'_i = \sum_{n=1}^{k} \dfrac{C^{n\delta}}{n} (1 + \delta |\operatorname{Log} K_{a_1^{p_1}} \cdots K_{a_n^{p_n}}|)(K_{a_1^{p_1}} \cdots K_{a_n^{p_n}})^\delta$.

On the other hand

$$N''_{(x_{ki},a)}(\Lambda^0_{ki}) = \sum_{n=k+1}^{+\infty} \frac{1}{n} \sum_{z/T^{n-k}z = x_{ki}} \sum_{y/T^k y = z} 1_{\Lambda^0_{ki}}(y) 1_{[0,a]}(S_{n-k} f(z) + S_k f(y)).$$

If $y \in \Lambda^0$, the $k$ first terms of $\omega(y)$ are the ones of $\omega(x_{ki})$ and so $S_k f(y) \geqq A_{ki}$ with $A_{ki} = \operatorname{Log} K_{a_1^{p_1}} \cdots K_{a_k^{p_k}}$. We thus have $N''_{(x_{ki},a)}(\Lambda^0_{ki}) \leqq N_{(x_{ki},a-A_{ki})}(\Lambda^0)$; so $N''_{(x_{ki},a-A_{ki})}(\Lambda^0) = 0$ if $a \leqq A_{ki}$ and, by corollary VII.2, $N''_{(x_{ki},a)}(\Lambda^0_{ki}) \leqq C'' \dfrac{e^{\delta(a-A_{ki})}}{1 + \delta(a - A_{ki})}$ if $a > A_{ki}$. Finally $N''_{(x_{ki},a)}(\Lambda^0_{ki}) \leqq C''_i \dfrac{e^{a\delta}}{a\delta}$ with $C''_i = C'' e^{-\delta A_{ki}}$.

By the following lemma, the sums

$$\sum_{\substack{a_1,\ldots,a_k\in\mathscr{A}\\ a_{i+1}\neq a_i}}\ \sum_{p_1,\ldots,p_k\in\mathbb{Z}^*}(K_{a_1^{p_1}}\cdots K_{a_n^{p_n}})^\delta \text{ and } \sum_{\substack{a_1,\ldots,a_k\in\mathscr{A}\\ a_{i+1}\neq a_i}}\ \sum_{p_1,\ldots,p_k\in\mathbb{Z}^*}|\mathrm{Log}\,K_{a_1^{p_1}}\cdots K_{a_n^{p_n}}|(K_{a_1^{p_1}}\cdots K_{a_n^{p_n}})^\delta$$

are finite so that $\sum_{i\geqq 1} C_i'$ and $\sum_{i\geqq 1} C_i''$ converge. □

**Lemma VII.5.** *There exists $\varepsilon>0$ such that $\sum_{n\in\mathbb{Z}^*} K_{\alpha^n}^{\delta-\varepsilon}<+\infty$ for every $\alpha\in\mathscr{A}$.*

*Proof of lemma* VII.4. Fix $x, y$ in $\Lambda^0$ such that the sequences $\omega(x)$ and $\omega(y)$ have the same $l$ first terms. By proposition V.1, there exist $N\geqq 1$ and $B_0>1$ such that $D(T^N x, T^N y)\geqq B_0 D(x,y)$; consequently $D(x,y)\leqq K_1 r^l$ for some $0<r<1$ and $K_1>0$.

Now let $\alpha\in\mathscr{A}$ and $n\in\mathbb{Z}^*$ such that $\alpha^n$ is the first term of the sequences $\omega(x)$ and $\omega(y)$; $T(x)$ and $T(y)$ do not belong to $\Lambda_\alpha$ and by lemmas I.3 and I.4 (applied with $\gamma=\alpha$ and $E=\Lambda-\Lambda_{\alpha^\pm}$) there exists $K_2>0$ such that

$$\frac{\|(\alpha^n)'(Tx)|-|(\alpha^n)'(Ty)\|}{|(\alpha^n)'(Ty)|}\leqq K_2 D(Tx,Ty).$$

One concludes using the local expansion $|\mathrm{Log}(1+u)|\leqq 2|u|$ for $|u|$ small enough. □

*Proof of lemma* VII.5. Fix $\alpha\in\mathscr{A}$; by lemmas I.3 and I.4, there exist $A_\alpha\geqq 1$ and $(K_{\alpha^n})_{n\in\mathbb{Z}^*}$ such that

$$\forall x\in\Lambda-\Lambda_{\alpha^\pm}\quad \frac{K_{\alpha^n}}{A_\alpha}\leqq|(\alpha^n)'(x)|\leqq A_\alpha K_{\alpha^n}.$$

Let $\beta$ be in $\mathscr{A}-\{\alpha\}$; one has $\sum_{n\in\mathbb{Z}^*}\frac{\sigma(\Lambda_\beta)}{A_\alpha^\delta}K_{\alpha^n}^\delta\leqq\sigma\big(\bigcup_{n\in\mathbb{Z}^*}\alpha^n(\Lambda_\beta)\big)<+\infty$ so that the series $\sum_{n\in\mathbb{Z}^*}K_{\alpha^n}^\delta$ converges. The same argument holds if one replaces $\Gamma$ with a subgroup $\Gamma_1$ containing $\alpha^k$ and $\beta^k$ for some $k\geqq 1$ and whose exponent of convergence is strictly less than $\delta$ (this is possible by the critical gap property). This readily ensures that the sum $\sum_{n\in\mathbb{Z}^*}K_{\alpha^n}^{\delta-\varepsilon}$ is finite for some $\varepsilon>0$ small enough. □

## VIII. Spectrum of Fourier operators $P_\lambda$, $\lambda\in\mathbb{R}$

Recall that $P$ is the transfer operator associated with $(T,\nu)$; for every Borel function $\varphi$ from $\Lambda$ into $\mathbb{R}^+$ one has

$$\forall x\in\Lambda\quad P\varphi(x)=\sum_{\alpha\in\mathscr{A}}\sum_{n\in\mathbb{Z}^*}p_{\alpha^n}(x)\,\varphi(\alpha^n(x))$$

with $p_{\alpha^n}(x)=1_{\Lambda-\Lambda_{\alpha^\pm}}(x)\,\dfrac{h(\alpha^n x)}{h(x)}\,|(\alpha^n)'(x)|^\delta$ and $h(x)=\displaystyle\int_{\Lambda-\Lambda_{\alpha^\pm}}\frac{\sigma(dy)}{D(x,y)^{2\delta}}$ for every $x\in\Lambda_{\alpha^\pm}$. Furthermore, for every $\lambda\in\mathbb{R}$ one has $P_\lambda(\varphi)=P(e^{i\lambda f}\varphi)$.

We consider the space $L$ of functions $\varphi$ from $\Lambda$ into $\mathbb{C}$ such that $\|\varphi\|=|_\infty+m(\varphi)<+\infty$ where $|.|_\infty$ is the norm of uniform convergence on $\Lambda$ and $m(\varphi)=\sup_{\alpha\in\mathscr{A}}\sup_{\substack{x,y\in\Lambda_\alpha\\ x\neq y}}\dfrac{|\varphi(x)-\varphi(y)|}{D(x,y)^{\delta_0}}$ with $\delta_0=\inf\{1,\delta\}$. We have $1\in L$, $\forall\varphi\in L\quad|\varphi|_\infty\leqq\|\varphi\|$

and $L$ is dense in the space of continuous functions on $\Lambda$ normed with $|\,.\,|_\infty$. Moreover, $(L, \|\,.\,\|)$ is a Banach space and, by Ascoli's theorem, the canonical one-to-one map from $(L, \|\,.\,\|)$ into $(L, |\,.\,|_\infty)$ is compact.

In the present paragraph we will use intensively the following result which is a consequence of lemmas I.3, I.4 and VII.5.

**Lemma VIII.1** (dynamic of generators). *For every $\alpha \in \mathscr{A}$ there exists $A_\alpha \geqq 1$ and a sequence $(K_{\alpha^n})_{n \in \mathbb{Z}^*}$ such that*

$$(1)\ \forall x \in \Lambda - \Lambda_{\alpha^\pm}, \forall n \in \mathbb{Z}^* \quad \frac{K_{\alpha^n}}{A_\alpha} \leqq |(\alpha^n)'(x)| \leqq A_\alpha K_{\alpha^n},$$

$$(2)\ \forall x, y \in \Lambda - \Lambda_{\alpha^\pm}, \forall n \in \mathbb{Z}^* \quad \|(\alpha^n)'(x)| - |(\alpha^n)'(y)\| \leqq A_\alpha K_{\alpha^n} D(x, y).$$

*Moreover if $\alpha$ is hyperbolic one has $K_{\alpha^n} = 1/\Phi(\alpha)^{|n|}$ with $\Phi(\alpha) > 1$ and if $\alpha$ is parabolic one has $\lim\limits_{n \to +\infty} K_{\alpha^n}^{1/n} = 1$ and $\sum\limits_{n \in \mathbb{Z}^*} K_{\alpha^n}^{\delta - \varepsilon} < +\infty$ for some $\varepsilon > 0$.*

Let us now show that properties VI.1 (properties R) hold.

**Property (R1).**

**Proposition VIII.2.** *The operator $P$ acts on $L$.*

*Proof.* Since $D(\Lambda_{\alpha^\pm}, \Lambda_{\beta^\pm}) > 0$ for $\alpha, \beta \in \mathscr{A}, \alpha \neq \beta$, the function $h$ belongs to $L$ and is non negative. Proposition VIII.2 is thus a direct consequence of the following result:

**Lemma VIII.3.** *For every $\alpha \in \mathscr{A}$ and every $n \in \mathbb{Z}^*$ the function $p_{\alpha^n}$ belongs to $L$; furthermore, the sequence $\left(\dfrac{\|p_{\alpha^n}\|}{K_{\alpha^n}^\delta}\right)_{n \in \mathbb{Z}^*}$ is bounded.*

*Proof.* By lemma VIII.1, there exists $A > 0$ such that for every $n \in \mathbb{Z}^*$ and every $\alpha \in \mathscr{A}$ one has $|(\alpha^n)'(x)| \leqq A K_{\alpha^n}$ and $\big||(\alpha^n)'(x)| - |(\alpha^n)'(y)|\big| \leqq A K_{\alpha^n} D(x, y)$ for every $x, y \in \Lambda - \Lambda_{\alpha^\pm}$; this readily implies $D(\alpha^n x, \alpha^n y) \leqq A K_{\alpha^n} D(x, y)$. Consequently $\|p_{\alpha^n}\|_\infty \leqq A |h|_\infty \left|\dfrac{1}{h}\right|_\infty K_{\alpha^n}^\delta$ and for any $x, y \in \Lambda - \Lambda_{\alpha^\pm}$ we have

$$\begin{aligned}
|p_{\alpha^n}(x) - p_{\alpha^n}(y)| &\leqq \frac{|h(\alpha^n x) - h(\alpha^n y)|}{h(x)} |(\alpha^n)'(x)|^\delta + \left|\frac{1}{h(x)} - \frac{1}{h(y)}\right| h(\alpha^n y) |(\alpha^n)'(x)|^\delta \\
&\quad + \frac{h(\alpha^n y)}{h(y)} \big||(\alpha^n)'(x)|^\delta - |(\alpha^n)'(y)|^\delta\big| \\
&\leqq m(h) \left|\frac{1}{h}\right|_\infty A^{\delta + \delta_0} K_{\alpha^n}^{\delta + \delta_0} D(x, y)^{\delta_0} + m\left(\frac{1}{h}\right) |h|_\infty A^\delta K_{\alpha^n}^\delta D(x, y)^{\delta_0} \\
&\quad + |h|_\infty \left|\frac{1}{h}\right|_\infty (1 + \delta) A^\delta K_{\alpha^n}^\delta D(x, y)^{\delta_0}.
\end{aligned}$$

The sequence $\left(\dfrac{\|p_{\alpha^n}\|}{K_{\alpha^n}^{\delta}}\right)_{n\in\mathbb{Z}^*}$ is thus bounded. □

**Property (R2).** Recall that the function $f$ from $\Lambda^0$ into $\mathbb{R}$ is defined by $f(x) = B_x(0, a^n 0)$ where $a^n$ is the first term of $\omega(x)$; we extend the definition of $f$ in the following way

$$\forall \alpha \in \mathscr{A},\ \forall n \in \mathbb{Z}^*,\ \forall x \in \Lambda - \Lambda_{\alpha^\pm} \quad f(\alpha^n x) = B_{\alpha^n x}(0, \alpha^n 0).$$

**Proposition VIII.4.** *One has*

(i) $0 < \nu(f) < +\infty$,

(ii) $\forall n \geqq 1 \quad \sup\limits_{x\in\Lambda} P(f^n)(x) < +\infty$.

To prove this proposition we have to estimate functions $f \circ \alpha^n$, $n \in \mathbb{Z}^*$, on each set $\Lambda - \Lambda_{\alpha^\pm}$, $\alpha \in \mathscr{A}$. The following lemma is a direct corollary of lemma VIII.1:

**Lemma VIII.5.** *There exists $K > 0$ such that for any $\alpha \in \mathscr{A}$ and $n \in \mathbb{Z}^*$ one has*

$$\sup_{x\in\Lambda-\Lambda_{\alpha^\pm}} |f(\alpha^n x)| \leqq K|\mathrm{Log}(K_{\alpha^n})|,$$

$$\sup_{\substack{x,y\in\Lambda-\Lambda_{\alpha^\pm}\\ x\neq y}} \frac{|f(\alpha^n x) - f(\alpha^n y)|}{D(x,y)^{\delta_0}} \leqq K.$$

*Proof of proposition* VIII.4. (i) One has

$$\begin{aligned}
\nu(f) &\leqq |h|_\infty \sum_{\substack{\alpha\in\mathscr{A}\\ n\in\mathbb{Z}^*}} \sigma(f 1_{\alpha^n(\Lambda-\Lambda_{\alpha^\pm})})\\
&\leqq |h|_\infty \sum_{\substack{\alpha\in\mathscr{A}\\ n\in\mathbb{Z}^*}} \int_{\Lambda-\Lambda_{\alpha^\pm}} f(\alpha^n x)|(\alpha^n)'(x)|^{\delta}\sigma(dx)\\
&\leqq K \sum_{\substack{\alpha\in\mathscr{A}\\ n\in\mathbb{Z}^*}} |\mathrm{Log} K_{\alpha^n}| K_{\alpha^n}^{\delta} \sigma(\Lambda - \Lambda_{\alpha^\pm}) \quad \text{by lemmas VIII.1 and VIII.5}\\
&< +\infty \quad \text{by lemma VIII.1.}
\end{aligned}$$

The fact that $\nu(f) > 0$ is a consequence of the expanding property of $T$: since there exists $N \in \mathbb{N}^*$ such that $|(T^N)'(x)| \geqq B_0 > 1$ for every $x \in \Lambda^0$ one has $S_N f(x) \geqq \mathrm{Log} B_0 > 0$ for every $x \in \Lambda^0$ so that $\nu(S_N f) = N\nu(f) > 0$.

(ii) By lemmas VIII.1 and VIII.5, there exists $C > 0$ such that for any $n \in \mathbb{Z}^*$

$$\|p_{\alpha^n}(f\circ\alpha^n)^l\| \leqq \|p_{\alpha^n}\|\ \|f\circ\alpha^n\|^l \leqq C^l K_{\alpha^n}^{\delta}|\mathrm{Log}\, K_{\alpha^n}|^l.$$

By lemma VIII.1 there exists $\varepsilon > 0$ such that $\sum\limits_{n\geqq 1} K_{\alpha^n}^{\delta-\varepsilon} < +\infty$ for $\alpha \in \mathscr{A}$ which ensures that $\sum\limits_{l\geqq 0}\sum\limits_{n\geqq 1} \dfrac{|t|^l}{l!}|\mathrm{Log}(K_{\alpha^n})|^l K_{\alpha^n}^{\delta} < +\infty$ as soon as $|t| < \varepsilon$. □

**Property (R3).** Recall that for any real number $\lambda \in \mathbb{R}$ the operator $P_\lambda$ is defined by $P_\lambda \varphi = P(e^{i\lambda f}\varphi)$ for every bounded Borel function $\varphi$ from $\Lambda$ into $\mathbb{R}$. We show here that $P_\lambda$ acts on $L$ and we describe the regularity of the map $\lambda \mapsto P_\lambda$.

**Proposition VIII.6.** *For any $\lambda \in \mathbb{R}$ the operator $P_\lambda$ acts on $L$; moreover the mapping $\lambda \mapsto P_\lambda$ is analytic from $\mathbb{R}$ into the space $(\mathscr{L}(L), \|\,.\,\|_{\mathscr{L}(L)})$ of continuous linear applications on $(L, \|\,.\,\|)$ with the usual norm.*

*Proof.* Using lemmas VIII.3 and VIII.5 one obtains

$$\|e^{i\lambda f\circ\alpha^n}\| \leqq |\lambda| m(f\circ\alpha^n) + 1 \leqq K|\lambda| + 1 < +\infty$$

so that $e^{i\lambda f\circ\alpha^n} \in L$; it readily follows that $P_\lambda$ acts on $L$. Now, fix $\lambda_0 \in \mathbb{R}$; using lemmas VIII.3 and VIII.5 one can see that for $t$ small enough the series

$$\sum_{l\geqq 0} \frac{|t|^l}{l!} \sum_{\substack{\alpha\in\mathscr{A}\\ n\in\mathbb{Z}^*}} \|p_{\alpha^n}\|\,\|e^{i\lambda_0 f\circ\alpha^n}\|\,\|f\circ\alpha^n\|^l = \sum_{\substack{\alpha\in\mathscr{A}\\ n\in\mathbb{Z}^*}} \|p_{\alpha^n}\|\,\|e^{i\lambda_0 f\circ\alpha^n}\|\,e^{|t|\,\|f\circ\alpha^n\|}$$

converges; consequently one has

$$\left\| P_{\lambda_0+t}(.) - \sum_{l=0}^{N} \frac{(it)^l}{l!} P_{\lambda_0}(f^l.) \right\| \leqq \sum_{l=N+1}^{+\infty} \sum_{\substack{\alpha\in\mathscr{A}\\ n\in\mathbb{Z}^*}} \frac{|t|^l}{l!} \|p_{\alpha^n}\|\,\|e^{i\lambda_0 f\circ\alpha^n}\|\,\|f\circ\alpha^n\|^l \to 0 \quad \text{as} \quad N\to+\infty$$

so that the map $\lambda \mapsto P_\lambda$ is analytic on $\mathbb{R}$. □

**Property (R4).**

**Proposition VIII.7.** *One has $P1_\Lambda = 1_\Lambda$, the eigenvalue 1 simple and isolated in the spectrum of $P$ and the corresponding eigenspace is $\mathbb{C}\,1_\Lambda$.*

*Proof.* Since the probability $\nu$ is $T$-invariant one has $P1_\Lambda(x) = 1_\Lambda(x)$ for *$\nu$-almost all $x$* in $\Lambda$. This property holds in fact for *every* point in $\Lambda$ because $P$ acts on $L$.

The description of the spectrum of $P$ on $L$ is based on the following theorem, due to Ionescu-Tulcea and Marinescu and whose modern formulation can be found in [18].

**Theorem** (Ionescu-Tulcea and Marinescu). *Let $(E, \|\,.\,\|_E)$ be a $\mathbb{C}$-Banach space and $Q$ a linear continuous operator on $(E, \|\,.\,\|_E)$ whose spectral radius is $\leqq 1$. Assume that there exists on $E$ a norm $|\,.\,|$ such that*

(i) *the operator $Q$ is compact from $(E, \|\,.\,\|_E)$ into $(E, |\,.\,|)$,*

(ii) *there exist $0 < r < 1$, $R > 0$ and $N \in \mathbb{N}^*$ such that $Q$ satisfies the following inequality:*

$$\forall \varphi \in E \quad \|Q^N\varphi\|_E \leqq r\|\varphi\|_E + R|\varphi|.$$

*Then $Q$ admits at most a finite number of modulus one eigenvalues, the associated eigenspaces are finite dimensional and the rest of the spectrum of $Q$ on $(E, \|.\|_E)$ is included in a disc of radius strictly less than* 1.

In order to control the spectrum of $P$ on $L$ we need the following

**Lemma VIII.8.** *There exist $0 < r < 1$, $R > 0$ et $N \in \mathbb{N}^*$ such that*

$$\forall \varphi \in L \quad \|P^N \varphi\| \leqq r\|\varphi\| + R|\varphi|_\infty .$$

Iterating this inequality, one proves that $(\|P^n\|)_{n \geqq 1}$ is bounded so that $\lim_{n \to +\infty} \|P^n\|^{1/n} \leqq 1$; consequently, by Ionescu-Tulcea and Marinescu's theorem, the operator $P$ on $L$ has at most a finite number of modulus one eigenvalues, the corresponding eigenspaces are finite dimensional and the rest of the spectrum is included in a disc of radius strictly less than 1. Proposition VIII.7 follows, thanks to the

**Lemma VIII.9.** *Let $\varphi \in L$ such that $P\varphi = e^{i\theta}\varphi$. If $\# \mathscr{A} \geqq 3$ one thus has $e^{i\theta} = 1$ and $\varphi = C 1_\Lambda$, $C \in \mathbb{C}$; otherwise, $\# \mathscr{A} \geqq 2$ and there are two cases*:

- $e^{i\theta} = 1$ *and* $\varphi \in \mathbb{C}\, 1_\Lambda$,
- $e^{i\theta} = -1$ *and* $\varphi \in \mathbb{C}\ (1_{\Lambda_{\alpha_1^\pm}} - 1_{\Lambda_{\alpha_2^\pm}})$.

It remains to establish lemmas VIII.8 and 9.

*Proof of lemma* VIII.8. By proposition V.1 there exists $N \geqq 1$ such that $\inf_{x \in \Lambda^0} |(T^N)'(x)| = B_0 > 1$. For $\alpha \in \mathscr{A}$ set

$$\mathscr{A}_N(\alpha) = \{\bar{a} = (a_1, \ldots, a_N) \in \mathscr{A}^N \,|\, a_{j+1} \neq a_j \text{ for } 1 \leqq j < N \text{ and } a_N \neq \alpha\};$$

for $x, y \in \Lambda_\alpha$, $\bar{a} = (a_1, \ldots, a_N) \in \mathscr{A}_N(\alpha)$ and $\bar{k} = (k_1, \ldots, k_N) \in (\mathbb{Z}^*)^N$ we thus have

$$D\,(a_1^{k_1} \cdots a_N^{k_N} x, a_1^{k_1} \cdots a_N^{k_N} y) \leqq \frac{1}{B_0}\, D(x, y).$$

Set $p_{\bar{a}\bar{k}}(x) = p_{a_1^{k_1}}(a_2^{k_2} \cdots a_N^{k_N} x)\, p_{a_2^{k_2}}(a_3^{k_3} \cdots a_N^{k_N} x) \cdots p_{a_N^{k_N}}(x)$; for $\varphi \in L$ and $x, y \in \Lambda_\alpha$ we have

$$\begin{aligned} |P^N\varphi(x) - P^N\varphi(y)| &\leqq \sum_{\substack{\bar{a} \in \mathscr{A}_N(\alpha) \\ \bar{k} \in (\mathbb{Z}^*)^N}} p_{\bar{a}\bar{k}}(x)\, |\varphi(a_1^{k_1} \cdots a_N^{k_N} x) - \varphi(a_1^{k_1} \cdots a_N^{k_N} y)| \\ &\quad + |\varphi|_\infty \sum_{\substack{\bar{a} \in \mathscr{A}_N(\alpha) \\ \bar{k} \in (\mathbb{Z}^*)^N}} |p_{\bar{a}\bar{k}}(x) - p_{\bar{a}\bar{k}}(y)| \\ &\leqq m(\varphi) \sum_{\substack{\bar{a} \in \mathscr{A}_N(\alpha) \\ \bar{k} \in (\mathbb{Z}^*)^N}} p_{\bar{a}\bar{k}}(x)\, D(a_1^{k_1} \cdots a_N^{k_N} x, a_1^{k_1} \cdots a_N^{k_N} y)^{\delta_0} \\ &\quad + |\varphi|_\infty \sum_{\substack{\bar{a} \in \mathscr{A}_N(\alpha) \\ \bar{k} \in (\mathbb{Z}^*)^N}} |p_{\bar{a}\bar{k}}(x) - p_{\bar{a}\bar{k}}(y)| \\ &\leqq \left( \frac{1}{B^{\delta_0}}\, m(\varphi) + |\varphi|_\infty \sum_{\substack{\bar{a} \in \mathscr{A}_N(\alpha) \\ \bar{k} \in (\mathbb{Z}^*)^N}} \frac{|p_{\bar{a}\bar{k}}(x) - p_{\bar{a}\bar{k}}(y)|}{D(x, y)^{\delta_0}} \right) D(x, y)^{\delta_0}. \end{aligned}$$

Note that $\sum_{\substack{\bar{a}\in\mathscr{A}_N(\alpha)\\ \bar{k}\in(\mathbb{Z}^*)^N}} |p_{\bar{a}\bar{k}}(x)-p_{\bar{a}\bar{k}}(y)| \leqq \sum_{\substack{\bar{a}\in\mathscr{A}_N(\alpha)\\ \bar{k}\in(\mathbb{Z}^*)^N}} \sum_{s=1}^{N} M_{\bar{a}\bar{k}}(s,x,y)$ with

$$\begin{aligned} M_{\bar{a}\bar{k}}(s,x,y) &= p_{a_1^{k_1}\dots a_{s-1}^{k_{s-1}}}(a_s^{k_s}\cdots a_N^{k_N}x) \\ &\quad \times |p_{a_s^{k_s}}(a_{s+1}^{k_{s+1}}\cdots a_N^{k_N}x) - p_{a_s^{k_s}}(a_{s+1}^{k_{s+1}}\cdots a_N^{k_N}y)|\, p_{a_{s+1}^{k_{s+1}}\dots a_N^{k_N}}(y) \\ &\leqq p_{a_1^{k_1}\dots a_{s-1}^{k_{s-1}}}(a_s^{k_s}\cdots a_N^{k_N}x)\, m(p_{a_s^{k_s}})\, C^{N-r} D(x,y)^{\delta_0} p_{a_{s+1}^{k_{s+1}}\dots a_N^{k_N}}(y) \end{aligned}$$

where $C\in[1,+\infty[$ is such that $D(a^n x, a^n y)^{\delta_0} \leqq C D(x,y)^{\delta_0}$ for every $a\in\mathscr{A}$, $x,y\in\Lambda-\Lambda_{a^\pm}$ and $n\in\mathbb{Z}^*$; one thus obtains

$$m(P^N\varphi) \leqq \frac{1}{B_0^{\delta_0}} m(\varphi) + \frac{C^{N+1}}{C-1} \sum_{\alpha\in\mathscr{A}} \sum_{n\in\mathbb{Z}^*} \|p_{\alpha^n}\| |\varphi|_\infty$$

and lemma VIII.8 follows with $r = \frac{1}{B_0^{\delta_0}}$ and $R = 1 + \frac{C^{N+1}}{C-1} \sum_{\alpha\in\mathscr{A}} \sum_{n\in\mathbb{Z}^*} \|p_{\alpha^n}\|$. □

*Proof of lemma* VIII.9. Let $\varphi\in L$ and $\theta\in\mathbb{R}$ such that $P\varphi = e^{i\theta}\varphi$. Equalities $P\varphi = e^{i\theta}\varphi$ and $\nu P = \nu$ imply $P|\varphi|(x) = |\varphi(x)|\,\nu(dx)\,p.s$; since $\varphi$ and $P\varphi$ belong to $L$ and the support of $\nu$ is $\Lambda$, this last equality holds for any $x$ in $\Lambda$.

Suppose that $\#\mathscr{A}\geqq 3$. Let $y, y'$ in $\Lambda$ such that $|\varphi(y)| = \sup_{x\in\Lambda}|\varphi(x)|$, $|\varphi(y')| = \inf_{x\in\Lambda}|\varphi(x)|$ and consider $\alpha\in\mathscr{A}$ such that $y$ and $y'$ do not belong to $\Lambda_{\alpha^\pm}$ (such an $\alpha$ does exist since $\#\mathscr{A}\geqq 3$); by a convexity argument we have $|\varphi(y)| = |\varphi(\alpha^n y)|$ and $|\varphi(y')| = |\varphi(\alpha^n y')|$ for every $n\in\mathbb{Z}^*$. Letting $n\to+\infty$, one obtains $|\varphi(y)| = |\varphi(y')| = |\varphi(x_\alpha)|$ which ensures that $|\varphi|$ is constant on $\Lambda$. Assume $|\varphi|\neq 0$; for every $\alpha\in\mathscr{A}$ and $x\in\Lambda_{\alpha^\pm}$ one has $e^{i\theta} = \sum_{\beta\in\mathscr{A}} \sum_{n\in\mathbb{Z}^*} p_{\beta^n}(x) \frac{\varphi(\beta^n x)}{\varphi(x)}$ and so $\varphi(x) = e^{-i\theta}\varphi(\beta^n x)$ for every $\beta\neq\alpha$ and $n\in\mathbb{Z}^*$; letting $n\to+\infty$ one obtains $\varphi(x) = e^{-i\theta}\varphi(x_\beta)$ which proves that $\varphi$ is constant on $\Lambda$ and that $e^{i\theta}=1$.

Suppose now $\mathscr{A} = \{\alpha_1, \alpha_2\}$. We have $P(1_{\Lambda_{\alpha_1^\pm}}) = 1_{\Lambda_{\alpha_2^\pm}}$ and $P(1_{\Lambda_{\alpha_2^\pm}}) = 1_{\Lambda_{\alpha_1^\pm}}$. Moreover

$$\forall x\in\Lambda_{\alpha_1^\pm} \quad P^2\varphi(x) = \sum_{n,m\in\mathbb{Z}^*} p_{\alpha_2^n}(x)\, p_{\alpha_1^m}(\alpha_2^n x)\, \varphi(\alpha_1^m\alpha_2^n x) = e^{2i\theta}\varphi(x).$$

Let $y_1$ and $y_1'$ in $\Lambda_{\alpha_1^\pm}$ such that $|\varphi(y_1)| = \sup_{y\in\Lambda_{\alpha_1^\pm}}|\varphi(y)|$ and $|\varphi(y_1')| = \inf_{y\in\Lambda_{\alpha_1^\pm}}|\varphi(y)|$; by a convexity argument we have $|\varphi(y_1)| = |\varphi(\alpha_1^m\alpha_2 y_1)|$ and $|\varphi(y_1')| = |\varphi(\alpha_1^m\alpha_2 y_1')|$ for every $m\in\mathbb{Z}^*$. Letting $m\to+\infty$, one obtains that $|\varphi|$ is constant on $\Lambda_{\alpha_1^\pm}$. If $|\varphi|\neq 0$ on $\Lambda_{\alpha_1^\pm}$, one has $\varphi(x) = e^{-2i\theta}\varphi(x_{\alpha_1})$ which proves that $\varphi$ is constant on $\Lambda_{\alpha_1^\pm}$ and $e^{2i\theta}=1$. The same conclusion holds on $\Lambda_{\alpha_2^\pm}$ which finishes the proof. □

**Property (R5).** We describe here the top of the spectrum on $L$ of the operators $P_\lambda$, $\lambda\neq 0$.

**Proposition VIII.10.** *For any $\lambda\in\mathbb{R}^*$, the spectral radius of $P_\lambda$ on $(L, \|.\|)$ is strictly less than* 1.

*Proof.* Let $\varphi \in L$; we have $\forall x \in \Lambda \quad P\varphi(x) = \sum_{\alpha \in \mathscr{A}} \sum_{n \in \mathbb{Z}^*} p_{\alpha^n}(x) e^{i\lambda f(\alpha^n x)} \varphi(\alpha^n x)$. To control the spectrum of $P_\lambda$ we prove that $P_\lambda$ satisfies hypotheses of Ionescu-Tulcea and Marinescu's theorem; if one replaces $p_{\alpha^n}$ with $p_{\alpha^n} e^{i\lambda f \circ \alpha^n}$ in the proof of lemma VIII.8 one shows that there exist $r \in ]0,1[$ and $A, B > 0$ such that

$$\forall \varphi \in L \quad \|P_\lambda^N \varphi\| \leqq r\|\varphi\| + (A|\lambda| + B)|\varphi|_\infty .$$

Iterating this inequality, one obtains that $(P_\lambda^n)_{n \geqq 1}$ is bounded in $(L, \|.\|)$; the spectral radius of $P_\lambda$ is thus $\leqq 1$ and proposition VIII.10 is a consequence of the following

**Lemma VIII.11.** *For $\lambda \neq 0$ the operator $P_\lambda$ does not admit eigenvalues of modulus one.*

*Proof.* The equality $P_\lambda \varphi = e^{i\theta}\varphi$ gives $P|\varphi| = |\varphi|$ and so, by lemma VIII.9, $|\varphi|$ is constant on $\Lambda$. By a convexity argument $e^{i\theta}\varphi(x) = e^{i\lambda f(\alpha^n x)} \varphi(\alpha^n x)$ so that $\lim_{n \to +\infty} e^{i\lambda f(\alpha^n x)}$ does exist for every $\alpha \in \mathscr{A}$ and $x \in \Lambda - \Lambda_{\alpha^\pm}$. Suppose that $\alpha$ is parabolic; one has $\lim_{n \to +\infty} f(\alpha^n x) = +\infty$ and $\lim_{n \to +\infty} f(\alpha^{n+1} x) - f(\alpha^n x) = 0$. Consequently, for any $a > 0$ there exists a sequence $(n_k)_{k \geqq 1}$ of integers such that $\lim_{k \to +\infty} f(\alpha^{n_{k+1}} x) - f(\alpha^{n_k} x) = a$; it follows $e^{i\lambda a} = 1$; and so $\lambda = 0$. (Note that the existence of parabolic transformations in $\Gamma$ is essential in the present proof.) □

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