Conditioned random walk in Weyl chambers and renewal theory

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Abstract We present here the main result from [7] and explain how to use Kashiwara crystal basis theory to associate a random walk to each minuscule irreducible representation V of a simple Lie algebra; the generalized Pitman transform defined in [1] for similar random walks with uniform distributions yields yet a Markov chain when the crystal attached to V is endowed with a probability distribution compatible with its weight graduation. The main probabilistic argument in our proof is a quotient version of a renewal theorem that we state in the context of general random walks in a lattice [7]. We present some explicit examples, which can be computed using insertion schemes on tableaux described in [8].

1 Introduction

1.1 The Pitman transform for the Brownian motion

Let $(B(t))_{t\geq 0}$ be a standard Brownian motion on $\mathbb{R}$ starting at 0. We denote by $m(t)$ the minimum process defined by $m(t) := \inf_{0\leq s\leq t} B(s)$. The *Pitman transform* of $(B(t))_{t\geq 0}$ is given by

$$\mathcal{P}B(t) := B(t) - 2m(t).$$

The reader will find a proof of the following statement in [11]:

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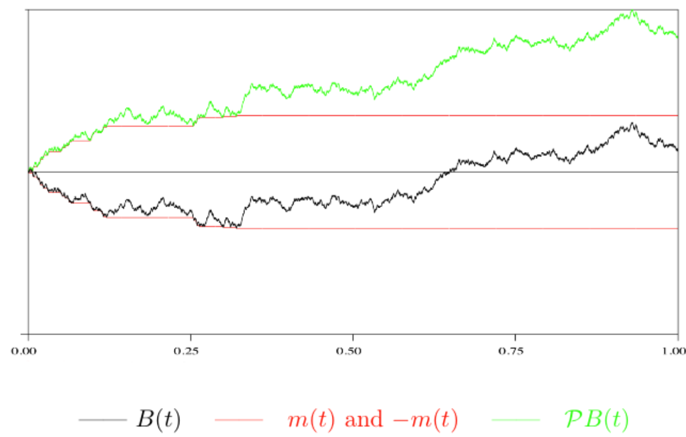
Theorem 1. *The process $\mathcal{P}B(t)$ is a 3-dimensional Bessel process; in particular, it has the same law as the brownian motion on $]0, +\infty[$ conditioned to stay positive.*



A Brownian motion trajectory $B(t)$ and its Pitman transform $\mathcal{P}B(t)$

There exists a multi-dimensionnal generalization of this theorem, called the *generalised Pitman transform* (see [1]): it corresponds for instance to the motion of the eigenvalues of some hermitian brownian motion in $SU(2)$.

1.2 The Pitman transform for the simple random walk

We consider the simple random walk $S_k := X_1 + \cdots + X_k$ on the set $\mathbb{Z}$ with steps ± 1:

$$\mathbb{P}(X_k = 1) = \mathbb{P}(X_k = -1) = \frac{1}{2}.$$

The *Pitman transform* of this random walk is the process $(\mathcal{P}_k := S_k - 2m_k)_{k \geq 0}$ where $m_k = \min(0, S_1, \cdots, S_n)$; it is a Markov chain on $\mathbb{N}$ with transition probabilities

$$\forall a \in \mathbb{N} \qquad p(a, a+1) = \frac{a+2}{2(a+1)} \quad \text{and} \quad p(a, a-1) = \frac{a}{2(a+1)}.$$

By a straightforward computation, one gets

$$\forall a \in \mathbb{N} \qquad p(a, a \pm 1) = \lim_{k \to +\infty} \mathbb{P}(S_1 = a \pm 1 / S_0 = a, S_1 \geq 0, \cdots, S_k \geq 0).$$

To obtain this equality, one may notice for instance that for any $a, k \geq 0$ one gets

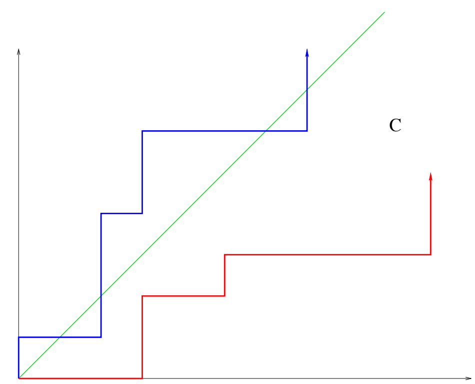
$$\mathbb{P}(S_1 = a+1 / S_0 = a, S_1, \cdots, S_k \geq 0) = \frac{\mathbb{P}(m_{k-1} \geq -a-1)}{\mathbb{P}(m_{k-1} \geq -a-1) + \mathbb{P}(m_{k-1} \geq -a+1)}$$

and use classical estimations of the probability $\mathbb{P}(m_{k-1} \geq -a-1)$ for the simple random walk.

We may represent the process $(\mathcal{P}_k)_{k\geq 0}$ as a process in the plane. We fix the standart basis $\{\overrightarrow{\imath}, \overrightarrow{\jmath}\}$ in $\mathbb{R}^2$; the vector $\overrightarrow{\imath}$ corresponds to the step $+1$ and $\overrightarrow{\jmath}$ to the step -1. We consider for instance the following trajectory

time k	0	1	2	3	4	5	6	7	$\cdots$
X_k		-1	1	1	-1	-1	-1	1	$\cdots$
S_k	0	-1	0	1	0	-1	-2	-1	$\cdots$
path in $\mathbb{Z}^2$	$\vec{0}$	$\vec{\jmath}$	$\vec{\imath}+\vec{\jmath}$	$2\vec{\imath}+\vec{\jmath}$	$2\vec{\imath}+2\vec{\jmath}$	$2\vec{\imath}+3\vec{\jmath}$	$2\vec{\imath}+4\vec{\jmath}$	$3\vec{\imath}+4\vec{\jmath}$	$\cdots$
m_k	0	-1	-1	-1	-1	-1	-2	-2	$\cdots$
$\mathcal{P}_k = S_k - 2m_k$	0	1	2	3	2	1	2	3	$\cdots$
Pitman path in $\mathbb{N}^2$	$\vec{0}$	$\vec{\imath}$	$2\vec{\imath}$	$3\vec{\imath}$	$3\vec{\imath}+\vec{\jmath}$	$3\vec{\imath}+2\vec{\jmath}$	$4\vec{\imath}+2\vec{\jmath}$	$5\vec{\imath}+2\vec{\jmath}$	$\cdots$

and its geometrical representation in the plane



2 The ballot problem in $\mathbb{R}^n, n \geq 2$

We have just seen that the Pitman transform of the simple random walk on $\mathbb{Z}$ can be seen a transform of some process on $\mathbb{N}^2$, the so-called "Bertrand's ballot problem" in combinatoric. We generalize here this correspondence in any dimension.

2.1 Cones and paths

We fix a basis $\mathcal{B}=\{\overrightarrow{e}_1,\cdots,\overrightarrow{e}_n\}$ of $\mathbb{R}^n$ and introduce the following cone

$$\mathcal{C}=\{x\in\mathbb{R}^n \mid x_1\geq\cdots\geq x_n\geq 0\};$$

we denote by $\mathring{\mathcal{C}}:=\{x\in\mathbb{R}^n \mid x_1>\cdots>x_n>0\}$ its interior. We will consider the collection of paths in $\mathbb{Z}^n=\bigoplus_{i=1}^n\mathbb{Z}\overrightarrow{e}_i$ starting at 0 and with steps $\overrightarrow{e}_1,\ldots,\overrightarrow{e}_n$. A *path* of length ℓ in $\mathbb{Z}^n$ is a *word* $w=x_1\cdots x_\ell$ on the alphabet $\{1,\ldots,n\}$. The *weight* of w is $\mathrm{wt}(w)=(\mu_1,\ldots,\mu_n)$ where μ_i is the number of letters i in w. The path w remains inside $\mathcal{C}$ iff $\mathrm{wt}(w)\in\mathcal{C}$ and w satisfies the following condition: *for any $k\geq 1$ and $i\in\{1,\cdots,n-1\}$ the number of i in $x_1\ldots x_k$ is greater than the number of $i+1$.*
Example: The word $w=112321231$ has weight $(4,3,2)$ and the corresponding path remains in $\mathcal{C}$.
We now may ask the following questions in combinatorial theory:

- For any $\mu=(\mu_1,\cdots,\mu_n)\in\mathcal{C}\cap\mathbb{N}^n$, **what's the number c_μ of paths between 0 and μ which stay inside $\mathcal{C}$?**
- For any $\lambda=(\lambda_1,\cdots,\lambda_n)$ and $\mu=(\mu_1,\cdots,\mu_n)\in\mathcal{C}\cap\mathbb{N}^n$ such that $\lambda\leq\mu$ (i-e $\lambda_i\leq\mu_i$ for any $i=1,\cdots,n$), **what's the number $c_{\mu/\lambda}$ of paths between λ and μ which stay inside $\mathcal{C}$?**

2.2 The simple random walk on $\mathbb{N}^n$

We fix a probability vector $p=(p_1,\cdots,p_n)$ in $\mathbb{R}^n$ (that is $p_i\geq 0$ for any $1\leq i\leq n$ and $p_1+\cdots+p_n=1$) and consider a sequence $(X_\ell)_{\ell\geq1}$ of i. i. d. random variables defined on a probability space $(\Omega,\mathcal{T},\mathbb{P})$ such that

$$\forall i\in\{1,\cdots,n\}\quad \mathbb{P}(X_\ell=\overrightarrow{e}_i)=p_i.$$

The random walk $(S_\ell=X_1+\cdots+X_\ell)_{\ell\geq0}$ has the transition probability matrix

$$\Pi(\alpha,\beta)=\begin{cases}p_i & \text{if} \qquad \beta-\alpha=\overrightarrow{e}_i\in\mathcal{B},\\ 0 & \text{otherwise.}\end{cases}$$

If $\beta:=\alpha+\ell_1\overrightarrow{e}_1+\cdots+\ell_n\overrightarrow{e}_n$, with $\alpha\in\mathbb{N}^n,\ell_1\cdots,\ell_n\geq0$, all the paths joigning α to β have length $\ell=\ell_1+\cdots+\ell_n$ and the same probability $p_1^{\ell_1}\times\cdots\times p_n^{\ell_n}$; then

$$\Pi^\ell(\alpha,\beta)=\frac{\ell!}{\ell_1!\cdots\ell_n!}p_1^{\ell_1}\times\cdots\times p_n^{\ell_n}.$$

2.3 The conditioned random walk in $\mathcal{C}$

Let $\Pi_{\mathcal{C}}$ be the restriction of Π to the cone $\mathcal{C}$. One gets the

Proposition 1. *If $m := \mathbb{E}(X_\ell) \in \mathring{\mathcal{C}}$ (or equivalently $p_1 > \cdots > p_n$) then*

$$\forall \lambda \in \mathcal{C} \quad \mathbb{P}_\lambda\Big(S_\ell \in \mathcal{C}, \forall \ell \geq 0\Big) > 0.$$

Moreover, the function $h : \lambda \mapsto \mathbb{P}_\lambda\Big(S_\ell \in \mathcal{C}, \forall \ell \geq 0\Big)$ is $\Pi_{\mathcal{C}}$-harmonic.

The transition matrix $P_{\mathcal{C}}$ of the random walk $(S_\ell)_{\ell \geq 0}$ conditioned to stay inside $\mathcal{C}$ is the h-Doob transform of $\Pi_{\mathcal{C}}$ given by:

$$\forall \lambda, \mu \in \mathcal{C} \quad P_{\mathcal{C}}(\lambda, \mu) = \frac{h(\mu)}{h(\lambda)} \Pi_{\mathcal{C}}(\lambda, \mu).$$

The aim of this work is to explain how to compute $P_{\mathcal{C}}$ and the value of $h(\lambda), \lambda \in \mathcal{C}$, when $m = (p_1, \cdots, p_n) \in \mathring{\mathcal{C}}$. Following N. O'Connell [10], we will use the theory of representation and generalize the Pitman transform in this discrete context.

2.4 The representation theory of $\mathfrak{sl}_n(\mathbb{C})$

2.4.1 Weights of $\mathfrak{sl}_n(\mathbb{C})$

The *weights lattice* associated with $\mathfrak{sl}_n(\mathbb{C})$ is $P := \mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}\overrightarrow{e}_i$ and the cone of *dominant weights* is $P_+ := \displaystyle\bigoplus_{i=1}^n \mathbb{N}\overrightarrow{e}_i$. The set of *root* is $R := \{\pm(\overrightarrow{e}_i - \overrightarrow{e}_j)/1 \leq i < j \leq n\}$; the set of *positive roots* R_+ is the one of vectors $\overrightarrow{e}_i - \overrightarrow{e}_j, 1 \leq i < j \leq n$, the *simple roots* are the $n-1$ vectors $\overrightarrow{e}_i - \overrightarrow{e}_{i+1}, 1 \leq i \leq n-1$.

We denote by $\mathcal{I}$ the set of irreducible finite dimensional representations of $\mathfrak{sl}_n(\mathbb{C})$. It is a classical fact that the elements of $\mathcal{I}$ are labelled by the dominant weights: for any $\lambda \in P_+$, we denote by $V(\lambda)$ the corresponding irreducible finite dimensional representation and the map $\lambda \longleftrightarrow V(\lambda)$ is a one-to-one correspondence between P_+ and $\mathcal{I}$. For instance, the natural representation $\mathbb{C}^n$ of $\mathfrak{sl}_n(\mathbb{C})$ is labelled $V(1, \underbrace{0, \cdots, 0}_{n-1 \text{ times}})$, or simply $V(1)$ [1].

For any $\ell \in \mathbb{N}$, one gets the decomposition $V(1)^{\otimes \ell} = \displaystyle\bigoplus_{\mu \in P_+} V(\mu)^{\oplus f_\mu}$.

[1] in order to simplify the notations, we will omit the (last) coordinates 0 which appear in $\lambda \in P_+$

More generally, for any $\lambda \in P_+$, one gets $V(\lambda) \otimes V(1)^{\otimes \ell} = \bigoplus_{\mu \in P_+} V(\mu)^{\oplus f_{\mu/\lambda}}$.

We have the

Proposition 2. *For any $\lambda, \mu \in \mathcal{C}$ such that $\lambda \leq \mu$, one gets*

$$c_\mu = f_\mu \quad \text{and} \quad c_{\mu/\lambda} = f_{\mu/\lambda}.$$

Consequently, in order to compute the exact values of c_μ and $c_{\mu/\lambda}$, we may use the representation theory.

2.4.2 The notion of crystals

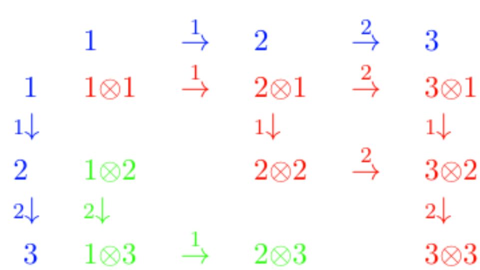
One may associate to each $V(\lambda) \in \mathcal{I}$ its *Kashiwara crystal* $B(\lambda)$. This is the combinatoric skeleton of the $U_q(\mathfrak{sl}_n(\mathbb{C}))$-module with dominant weight $\lambda \in P_+$: it has a structure of a colored and oriented graph (see [5], [6]).

Example: *The crystal of $V(1) = \mathbb{C}^n$ is*

$$B(1) : 1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{3} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n.$$

The crystal $B(\lambda) \otimes B(\mu)$ associated with $V(\lambda) \otimes V(\mu)$ may be constructed with $B(\lambda)$ and $B(\mu)$; its set of vertices is the direct product of the ones of $B(\lambda)$ and $B(\mu)$, the crystal structure (that is the choice of the arrows between vertices) being given by some technical rules presented for instance in [7], Theorem 5.1. One important property of the crystals theory is that the irreducible components of $V(\lambda) \otimes V(\mu)$ are in one-to-one correspondence with the connected components of $B(\lambda) \otimes B(\mu)$.

Example: The crystals $B(1)$ and $B(1)^{\otimes 2}$ for $\mathfrak{sl}_3(\mathbb{C})$
The crystal $B(1)$ of $V(1) = \mathbb{C}^3$ is $B(1) : 1 \xrightarrow{1} 2 \xrightarrow{2} 3$.
The crystal $B(1)^{\otimes 2}$ associated with $V(1)^{\otimes 2}$ is



The two connected components are labelled by their root vertex, namely $1 \otimes 1$ and $1 \otimes 2$.

The letters which appear in the vertex $1 \otimes 1$ *are both equal to* 1*, this vertex thus corresponds to the irreducible component* $V(2,0,0) \simeq V(2)$*; in the same way, the vertex* $1 \otimes 2$ *corresponds to* $V(1,1,0) \simeq V(1,1)$*; so*

$$V(1)^{\otimes 2} \simeq V(2) \oplus V(1,1).$$

This means that the only ones "allowed" paths of lenght 2 in $\mathcal{C} := \{(x,y,z) \in \mathbb{N}^3 / x \geq y \geq z\}$ *are* $2\overrightarrow{e_1}$ *and* $\overrightarrow{e_1} + \overrightarrow{e_2}$.

2.4.3 Relation between the crystal and the set of words

The word $w = x_1 \cdots x_\ell$ on the alphabet $\{1, \ldots, n\}$ may be identified with the vertex

$$b = x_1 \otimes \cdots \otimes x_\ell \in B(1)^{\otimes \ell}$$

We denote by $B(b)$ the connected component of $B(1)^{\otimes \ell}$ which contains b. The *Pitman transform* will be the map $\mathcal{P}$ defined by

$$\begin{aligned} \mathcal{P} \ : \ B(1)^{\otimes \ell} &\to \mathcal{C} \\ b &\mapsto \text{ highest weight of } B(b). \end{aligned}$$

2.4.4 The probability distribution on the crystal

The probability of the letter i is p_i; it will be the probability of the vertex $i \in B(1)$. The word $x_1 \cdots x_\ell$ has probability $p^{\mu_1} \cdots p^{\mu_n}$ where $(\mu_1, \cdots, \mu_n)$ is the weight of this word; this is also the probability of the vertex $b = x_1 \otimes \cdots \otimes x_\ell \in B(1)^{\otimes \ell}$. Finally, we have fixed a probability p on $B(1)$, endowed $B(1)^{\otimes \mathbb{N}}$ with $p^{\otimes \mathbb{N}}$ and set

$$(S_\ell) := \text{the sequence of weights of the corresponding process on } B(1)^{\otimes \mathbb{N}}.$$

The *Pitman process* $(\mathcal{H}_\ell)_\ell$ is the sequence of weights defined as the images by $\mathcal{P}$ of the k-vectors $(S_\ell)_{1 \leq \ell \leq k}, k \geq 1$.

2.4.5 The character and the Schur functions

We consider the triangular decomposition $\mathfrak{g} := \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$ of the Lie algebra $\mathfrak{g}$; any representation M of $\mathfrak{g} =$ may be decomposed in *weight spaces*

$$M := \bigoplus_{\mu \in P} M_\mu$$

with $M_\mu := \{v \in M / h(v) = \mu(v)v \text{ for any } h \in \mathfrak{h}\}$. The *character function of* M is the Laurent polynomial s_M defined by

$$\forall x \in \mathbb{C}^n \ s_M(x) := \sum_{\mu \in P} \dim M_\mu \ x^\mu$$

When M is an irreducible representation $V(\lambda)$, the character function is called the *Schur function* and denoted s_λ.
Example: The Schur function of the natural representation of $\mathfrak{sl}(n, \mathbb{C})$.

For any $\lambda = (\lambda_1, \cdots, \lambda_n) \in P_+$ and $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we denote by $a_\lambda(x)$ the Vandermonde function

$$a_\lambda(x) := \det(x_i^{\lambda_j}) = \begin{vmatrix} x_1^{\lambda_1} & x_1^{\lambda_2} & \cdots & x_1^{\lambda_n} \\ x_2^{\lambda_1} & x_2^{\lambda_2} & \cdots & x_2^{\lambda_n} \\ \vdots & \vdots & \vdots & \vdots \\ x_n^{\lambda_1} & x_n^{\lambda_2} & \cdots & x_n^{\lambda_n} \end{vmatrix}.$$

For $\delta = (n-1, n-2, \cdots, 0)$, one gets

$$a_\delta(x) := \begin{vmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & 1 \end{vmatrix} = \prod_{1 \le i < j \le n} (x_i - x_j).$$

For any $\lambda \in P_+$, the Schur function s_λ of $V(\lambda)$ is given by

$$s_\lambda(x) := \frac{a_{\lambda+\delta}(x)}{a_\delta(x)}; \tag{1}$$

in particular, the Schur function of $V(1) = V(1, 0, \cdots, 0) = \mathbb{C}^n$ is

$$s_1(x) := \frac{a_{(1,0,\cdots,0)+\delta}(x)}{a_\delta(x)} = \frac{1}{a_\delta(x)} \times \begin{vmatrix} x_1^n & x_1^{n-2} & \cdots & 1 \\ x_2^n & x_2^{n-2} & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_n^n & x_n^{n-2} & \cdots & 1 \end{vmatrix} = x_1 + \cdots + x_n. \tag{2}$$

One may now state the following

Theorem 2. *([10])*

- *The process $(\mathcal{H}_\ell)_{\ell \ge 0}$ is a Markov chain on $\mathcal{C}$ with transition probability*

$$P_{\mathcal{H}}(\lambda, \mu) = \frac{s_\lambda(p_1, \cdots, p_n)}{s_\mu(p_1, \cdots, p_n)} 1_B(\mu - \lambda).$$

- *The transition matrix $P_{\mathcal{C}}$ of the r.w. $(S_\ell)_{\ell \ge 0}$ conditionned to stay inside $\mathcal{C}$ is equal to $P_{\mathcal{H}}$.*
- *In particular, one gets* $\mathbb{P}_0(S_\ell \in \mathcal{C}, \forall \ell \ge 0) = \prod_{\alpha \in R_+}(1 - p^{-\alpha})$.

3 The probabilistic argument

3.1 The Markov chain $(\mathcal{H}_\ell)_{\ell\geq 0}$

Using the crystal basis theory, one may check that $(\mathcal{H}_\ell)_{\ell\geq 0}$ is a Markov chain with transition matrix

$$P_{\mathcal{H}}(\lambda,\mu)=f_{\mu/\lambda}\frac{s_\mu(p)}{s_\lambda(p)s_1(p)}$$

with $f_{\mu/\lambda}\in\{0,1\}$; in the present case, we have $P_{\mathcal{H}}(\lambda,\mu)=f_{\mu/\lambda}s_\mu(p)/s_\lambda(p)$ since $s_1(p)=p_1+\cdots+p_n=1$. We denote by $\Pi_{\mathcal{C}}$ the restriction of Π to the cone $\mathcal{C}$; the matrix $P_{\mathcal{H}}$ is the ψ-Doob transform of the substochastic matrix $\Pi_{\mathcal{C}}$ with $\psi(\lambda):=\frac{s_\lambda(p)}{p^\lambda}$. The question is thus to prove that ψ coincides up to a multiplicative constant with the function h given in Proposition 1.

3.2 The Doob theorem

Let E be a countable set and Q sub-stochastic matrix transition on E. Let G be the the Green kernel associated with Q. Fix an origin $x^*\in E$ such that $0<G(x^*,y)<+\infty$ for any $y\in E$ and let K be the Martin kernel defined by

$$\forall x,y\in E\quad K(x,y)=\frac{G(x,y)}{G(x^*,y)}.$$

Let $\mathbf{h}$ be an strictly positive and Q-harmonic function on E, let $Q_{\mathbf{h}}$ be the $\mathbf{h}$-Doob transform of Q and consider a Markov chain $(Y_\ell^{\mathbf{h}})_{\ell\geq 0}$ on E with transition matrix $Q_{\mathbf{h}}$. One gets the classical following result :

Theorem 3. *(Doob, [3])Let* $\mathbf{f}:E\to\mathbb{R}$ *such that*

$$\forall x\in E\quad \lim_{\ell\to+\infty}K(x,Y_\ell^{\mathbf{h}}(\omega))=\mathbf{f}(x)\quad \mathbb{P}(d\omega)-\text{a.s.}$$

Then there exits $c>0$ *such that* $\mathbf{f}=c\mathbf{h}$.

In our case, we take $E=\mathcal{C}$ with origin $x^*=0$, the sub-stochastic matrix Q is $\Pi_{\mathcal{C}}$ and $\mathbf{h}(\lambda)=h(\lambda)=\mathbb{P}_\lambda(S_\ell\in\mathcal{C},\forall\ell\geq 0)$. By the Strong Law of Large Numbers, one gets

$$S_\ell\sim\ell m+o(\ell)\quad \mathbb{P}-\text{a.s.}$$

N. O' Connell directly checks, using the explicit expression of the Schur function s_λ given in (1), that for m inside the cone $\mathcal{C}$ and any sequence $\mu_\ell=\ell m+o(\ell)$

$$K(\lambda, \mu_\ell) = p^{-\lambda} \frac{f_{\mu_\ell/\lambda}}{f_{\mu_\ell}} \to \frac{s_\lambda(p)}{p^\lambda} \quad \text{as} \quad \ell \to +\infty.$$

Unfortunately, such an explicit formula for the Schur function does not exists in the more general situation we want to consider and we avoid his approach as follows: using the theory of crystalin bases, we may decompose the Martin kernel and write

$$K(\lambda, \mu_\ell) = \frac{1}{p^\lambda} \sum_{\gamma \text{ weight of } V(\lambda)} f_{\gamma/\lambda} \times p^\gamma \times \frac{G(0, \mu_\ell - \gamma)}{G(0, \mu_\ell)}$$

for any λ and $\mu_\ell = \ell m + o(\ell) \in \mathcal{C}$ with ℓ large enough. It remains to prove that, for any $\gamma \in \mathcal{C}$

$$\frac{G(0, \mu_\ell - \gamma)}{G(0, \mu_\ell)} \to 1 \quad \text{when} \quad \ell \to +\infty.$$

3.3 A quotient renewal theorem in the cone

The central argument of our approach is the following

Theorem 4. *(Lecouvey C., Lesigne E. & P.M. (2011), [7], [8]) Assume the random variables X_ℓ are almost surely bounded and that the mean vector $m := \mathbb{E}(X_\ell]$ lies inside the cone $\mathcal{C}$. Let $\alpha < 2/3$ and $(\mu_\ell)_\ell, (h_\ell)_\ell$ be two sequences in $\mathbb{Z}^n$ such that $\lim \ell^{-\alpha}\|\mu_\ell - \ell m\| = 0$ and $\lim \ell^{-1/2}\|h_\ell\| = 0$. Then, when ℓ tends to infinity, we have*

$$\sum_{k\geq 1} \mathbb{P}\left(S_1 \in \mathcal{C}, \ldots, S_k \in \mathcal{C}, S_k = \mu_\ell + h_\ell\right) \sim \sum_{k\geq 1} \mathbb{P}\left(S_1 \in \mathcal{C}, \ldots, S_k \in \mathcal{C}, S_k = \mu_\ell\right).$$

The first ingredient of the proof is the following

Lemma 1. *(R. Garbit (2008) [4]) Assume the random variables X_ℓ are square integrable and centered. Then, for any $\alpha > \frac{1}{2}$, there exists $c = c_\alpha > 0$ such that, for all ℓ large enough and $\mu \in \mathcal{C}$*

$$\mathbb{P}\left(S_1 \in \mathcal{C}, \ldots, S_\ell \in \mathcal{C}, S_\ell = \mu\right) \geq \exp\left(-c\ell^\alpha\right).$$

The second ingredient is a version of the renewal limit theorem due to H. Carlson and S. Wainger [2]. We assume that $m := \mathbb{E}(X_\ell)$ is nonzero. Let $(\overrightarrow{\epsilon_1}, \ldots, \overrightarrow{\epsilon_{n-1}})$ be an orthonormal basis of the hyperplan $m^\perp$. If $x \in \mathbb{R}^n$, denote by x' its orthogonal projection on $m^\perp$ expressed in this basis and let B be the covariance matrix of the random vector X'_ℓ. Let $\mathcal{N}_B$ be the $(n-1)$-dimensional Gaussian density with covariance matrix B and V be the n-dimensional volume of the fundamental domain of the group generated by the support of the law of X_ℓ. The following result may be deduced from [2], the proof of the present statement is detailed in [9]:

Theorem 5. *We assume the random variables X_ℓ have an exponential moment. Fix $\alpha < 2/3$ and let (μ_ℓ) be a sequence of real numbers such that $\mu_\ell = m\ell + o(\ell^\alpha)$. Then, when ℓ goes to infinity, we have*

$$\sum_{k\geq 0} \mathbb{P}(S_k = \mu_\ell) \sim \frac{V}{\|m\|} \ell^{-(n-1)/2} \mathcal{N}_B\left(\frac{1}{\sqrt{\ell}}\mu'\right).$$

We will apply this result along the sequences $(\mu_\ell)_\ell = (S_\ell(\omega))_\ell$ for almost all $\omega \in \Omega$, which is possible since, for any $\epsilon > 0$, one gets $S_\ell \sim \ell m + o(\ell^{\frac{1}{2}+\epsilon})$ a.s.

4 Generalization: The Pitman transform for minuscule representations

We consider in [7] a representation $V(\delta)$ of a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and endow the associated crystal $B(\delta)$ with a probability distribution $p = (p_b)_{b\in B(\delta)}$ which is compatible with the weight graduation of $B(\delta)$. As above, we may consider a random walk $(S_\ell)_\ell$ in the weight lattice $P = \mathbb{Z}^n$ with independent increments of law p and transition matrix Π; as in the previous section, we also construct a Markov chain $(\mathcal{H}_\ell)_\ell$ in the Weyl chamber $\mathcal{C} \subset \mathbb{Z}^n$, with transition matrix $P_{\mathcal{H}}$, which will play the role of the Pitman process.

We prove that $(\mathcal{H}_\ell)_\ell$ coincides with the ψ-Doob transform of the restriction to $\mathcal{C}$ of the transition matrix of $(S_\ell)_\ell$ (for some explicit function ψ expressed in terms of Schur functions) if and only if $V(\delta)$ is *minuscule* [(2)]. When $V(\delta)$ is minuscule, we also prove that for any m in the interior $\mathring{\mathcal{C}}$ of $\mathcal{C}$, one may choose the probability $p = (p_b)_{b\in B(\delta)}$ on the crystal $B(\delta)$ (and so the random walk $(S_\ell)_{l\geq 0}$ on $\mathbb{Z}^n$) in such a way its drift is m.

The main result of [7] may thus be stated as follows

Theorem 6. *(Lecouvey C., Lesigne E. & P.M. [7]) If the representation $V(\delta)$ is minuscule and $m = \mathbb{E}(X) \in \mathring{\mathcal{C}}$, then the transition matrix of the r.w.*

[2] $V(\delta)$ is minuscule when the orbit of δ under the action of the Weyl group of $\mathfrak{g}$ contains all the weights of $V(\delta)$. The minuscule representations are given in the following table

type	minuscule weights	N	decomposition on the basis B
A_n	$\omega_i, i = 1, \dots, n$	$n+1$	$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$
B_n	ω_n	n	$\omega_n = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n)$
C_n	ω_1	n	$\omega_1 = \varepsilon_1$
D_n	$\omega_1, \omega_{n-1}, \omega_n$	n	$\omega_1 = \varepsilon_1, \omega_{n+t} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n) + t\varepsilon_n,\ t \in \{-1, 0\}$
E_6	ω_1, ω_6	8	$\omega_1 = \frac{2}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6), \omega_6 = \frac{1}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \varepsilon_5$
E_7	ω_7	8	$\omega_7 = \varepsilon_6 + \frac{1}{2}(\varepsilon_8 - \varepsilon_7)$.

$(S_\ell)_{\ell\geq 0}$ conditioned to stay inside $\mathcal{C}$ is equal to $P_{\mathcal{H}}$. In particular, for any $\lambda \in P_+$, one gets

$$\mathbb{P}_\lambda(S_\ell \in C, \forall \ell \geq 0) = p^{-\lambda} s_\lambda(p) \prod_{\alpha \in R_+} (1 - p^{-\alpha}).$$

Furthermore, when $\mu^{(\ell)} = \ell m + o(\ell^\alpha)$ with $\alpha < 2/3$, one gets

$$\lim_{\ell\to\infty} \frac{f^\ell_{\mu^{(\ell)}/\lambda}}{f^\ell_{\mu^{(\ell)},\lambda}} = s_\lambda(p).$$

The same result holds for direct sums of distinct minuscule representations and also for some *super Lie algebras*, for instance $\mathfrak{g}(m,n)$ (see [9]).
Example: Case of a C_2 representation: $\mathfrak{sp}(4,\mathbb{C})$.

We consider the representation $V = V(\omega_1)$. The corresponding crystal is

$$B(\omega_1) : 1 \xrightarrow{1} 2 \xrightarrow{1} \overline{2} \xrightarrow{1} \overline{1}.$$

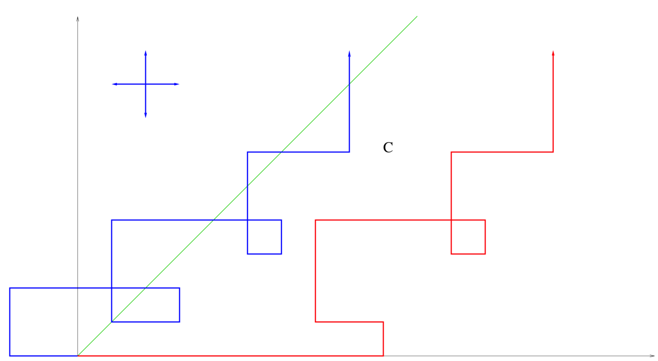
The probability $p = (p_{\vec{e}_1}, p_{-\vec{e}_1}, p_{\vec{e}_2}, p_{-\vec{e}_2})$ is such that

$$p_{\vec{e}_1} \times p_{-\vec{e}_1} = p_{\vec{e}_2} \times p_{-\vec{e}_2}.$$

In this case, one fixes $0 < p_2 < p_1 < 1$ with $p_1 + p_2 < 1$ and sets

$$p_{\vec{e}_1} = p_1, p_{-\vec{e}_1} = \frac{c}{p_1}, p_{\vec{e}_2} = p_2 \text{ and } p_{-\vec{e}_2} = \frac{c}{p_2}$$

with $c = p_1 p_2(\frac{1}{p_1+p_2} - 1)$ (so that $p_1 + p_2 + \frac{c}{p_1} + \frac{c}{p_2} = 1$).



A random path in the plane and its Pitman transform, for the vectorial representation of $\mathfrak{sp}(4,\mathbb{C})$

$$\mathbb{P}_0\Big(S_\ell \in \mathcal{C}, \forall \ell \geq 1\Big) = (1 - \frac{p_2}{p_1})(1 - \frac{c}{p_1 p_2})(1 - \frac{c}{p_1})(1 - \frac{c}{p_2}).$$

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References

1. Biane P., Bougerol P., O'Connell N.: Littelmann paths and Brownian paths, Duke Math. J., **130** (1), 127–167 (2005)
2. Carlsson H., Wainger S.: On the Multi-Dimensional Renewal Theorem, J. Math. Anal. Appl. **100** (1), 316–322 (1984)
3. Doob J.L.: Classical Potential Theory and its Probabilistic Counterpart, Springer, (1984)
4. Garbit G.: Temps de sortie d'un cône pour une marche aléatoire centrée, [Exit time of a centered random walk from a cone] C. R. A.S. Paris **345** (10), 587–591 (2007)
5. Kashiwara M.: On crystal bases, Canadian Mathematical Society, Conference Proceedings **16**, 155–197 (1995)
6. Lecouvey C.: Combinatorics of crystal graphs for the root systems of types A_n, B_n, C_n, D_n and G_2, Combinatorial aspect of integrable systems, MSJ Mem., Math. Soc. Japan, Tokyo **17**, 11–41 (2007)
7. Lecouvey C., Lesigne E., Peigné M.: Random walks in Weyl chambers and crystals, Proc. London Math. Soc. (3) **104** no. 2, 323–358 (2012)
8. Lecouvey C., Lesigne E., Peigné M.: Conditionned ballot problems and combinatorial representation theory, http://arxiv.org/abs/1202.3604 (2012)
9. Lecouvey C., Lesigne E., Peigné M.: Le théorème de renouvellement multi-dimensionnel dans le "cas lattice", http://hal.archives-ouvertes.fr, hal-00522875.2010 (2010)
10. O' Connell N.: A path-transformation for ramdom walks and the Robison-Schensted correspondence, Trans. Amer. Math. Soc., **355**, 669–3697 (2003)
11. Revuz D., Yor M.: Continuous Martingales and Brownian Motion, Springer, Corrected 3rd printing (2005)