

ON THE RATE OF CONVERGENCE IN THE WEAK INVARIANCE PRINCIPLE FOR DEPENDENT RANDOM VARIABLES WITH APPLICATIONS TO MARKOV CHAINS

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ABSTRACT. We prove an invariance principle for non-stationary random processes and establish a rate of convergence under a new type of mixing condition. The dependence is exponentially decaying in the gap between the past and the future and is controlled by an assumption on the characteristic function of the finite dimensional increments of the process. The distinct feature of the new mixing condition is that the dependence increases exponentially in the dimension of the increments. The proposed mixing property is particularly suited for processes whose behavior can be described in terms of spectral properties of some related family of operators. Several examples are discussed. We also work out explicit expressions for the constants involved in the bounds. When applied to Markov chains our result specifies the dependence of the constants on the properties of the underlying Banach space and on the initial state of the chain.

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1. INTRODUCTION

Let $(X_k)_{k \geq 1}$ be a sequence of real valued random variables (r.v.'s) defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $S_N(t) = N^{-1/2} \sum_{k=1}^{\lfloor Nt \rfloor} X_k$, $t \in [0, 1]$. The (weak) invariance principle states that the process $\frac{1}{\sqrt{N}} (S_N(t))_{0 \leq t \leq 1}$ converges weakly to the Brownian process $(W(t))_{0 \leq t \leq 1}$ and is a powerful tool for various applications in probability and statistics. It extends the scope of the central limit theorem to continuous functionals of the stochastic process $(S_N(t))_{0 \leq t \leq 1}$, such as, for example, the maxima or the L^2 -norm of the trajectory of the process, considered in the appropriate functional spaces. The rates of convergence in the (weak) invariance principle, for independent r.v.'s under the existence of the moments of order $2 + 2\delta$, with $\delta > 0$, have been obtained in Prokhorov [28], Borovkov [3], Komlós, Major and Tusnády [22], Einmahl [10], Sakhanenko [31], [32], Zaitsev [38], [39] among others. In the case of martingale-differences, for $\delta \leq \frac{1}{2}$, the rates are essentially the same as in the independent case (see, for instance Hall and Heyde [20], Kubilius [23], Grama [11]). The almost sure invariance principle is a reinforcement of the weak invariance principle which states that the trajectories of a process are approximated with the trajectories of the Brownian motion a.s. in the sense that within a particular negligible error $r_N \rightarrow 0$ it holds $\sup_{0 \leq t \leq 1} \left| \frac{1}{\sqrt{N}} S_N(t) - W(t) \right| = O(r_N)$ a.s. There are many recent results concerning the rates

of convergence in the strong invariance principle for weakly dependent r.v.'s under various conditions. We refer to Wu [37], Zhao and Woodroffe [40], Liu and Lin [24], Cuny [5], Merlevède and Rio [25], Dedecker, Doukhan and Merlevède [7] and to the references therein. However, in contrast to the case of independent r.v.'s where it is found that the optimal rate is of order $N^{-\frac{\delta}{2+2\delta}}$ for the strong invariance principle and $N^{-\frac{\delta}{3+2\delta}}$ for the weak invariance principle, the problem of obtaining the best rate of convergence in both the weak and strong invariance principles for dependent variables is not yet settled completely.

Gouëzel [15] has introduced a new type of mixing condition which is tied to spectral properties of the sequence $(X_k)_{k \geq 1}$. Consider the vectors $\bar{X}_1 = (X_{J_1}, \dots, X_{J_{M_1}})$ and $\bar{X}_2 = (X_{k_{gap}+J_{M_1+1}}, \dots, X_{k_{gap}+J_{M_1+M_2}})$, called the past and the future, respectively, where $X_{k+J_m} = \sum_{l \in J_m} X_{k+l}$, $J_m = [j_{m-1}, j_m)$, $j_0 \leq \dots \leq j_{M_1+M_2}$, and k_{gap} is a gap between \bar{X}_1 and \bar{X}_2 . Roughly speaking, the condition used in [15] suppose that the characteristic function of (\bar{X}_1, \bar{X}_2) is exponentially closed to the product of the characteristic functions of the past \bar{X}_1 and the future \bar{X}_2 , with an error term of the form $A \exp(-\lambda k_{gap})$, where λ is some non negative constant and A is exponential in terms of the size of the blocks. This mixing property is particularly suited for systems whose behavior can be described in terms of spectral properties of some related family of operators, as initiated by Nagaev [26], [27] and Guivarc'h [16]. Examples are Markov chains, whose perturbed transition probability operators $(\mathbf{P}_t)_{|t| \leq \varepsilon_0}$ exhibit a spectral gap and enough regularity in t , and dynamical systems, whose characteristic functions can be coded by a family of operators $(\mathcal{L}_t)_{|t| \leq \varepsilon_0}$ satisfying similar properties. Gouëzel proves in [15] an almost sure invariance principle with a rate of convergence of order $N^{-\frac{\delta}{2+4\delta}}$.

The scope of the present paper is to improve on the results of Gouëzel by quantifying the rate of convergence in the (weak) invariance principle for dependent r.v.'s under the mixing condition introduced above. Although the strong and weak invariance principles are closely related, it seems that the rate of convergence in the (weak) invariance principle is less studied under weak dependence constraints. We refer to Doukhan, Leon and Portal [6], Merlevède and Rio [25] and Grama and Neumann [14]. However, these results rely on mixing conditions which are not verified in the present setting. Under the above mentioned mixing and some further mild conditions including the moment assumption $\sup_{k \geq 1} \mathbb{E} |X_k|^{2+2\delta} < \infty$ we obtain a bound of the order $N^{-\frac{1+\alpha}{1+2\alpha} \frac{\alpha}{3+2\alpha}}$, for any $\alpha < \delta$. Moreover, we give explicit expressions of some constants involved in the rate of convergence; for instance, in the case of Markov walks we are able to figure out the dependence of the rate of convergence on the properties of the Banach space related to the corresponding family of perturbed transition operators $(\mathbf{P}_t)_{|t| \leq \varepsilon_0}$ and on the initial state $X_0 = x$ of the associated Markov chain. When compared with the rate $N^{-\frac{1}{2} \frac{\alpha}{1+2\alpha}}$ in the almost sure invariance principle of [15] ours appears with a loss in the power of multiple $\frac{2+2\alpha}{3+2\alpha} < 1$. This loss in the power is exactly the same as in the case of

independent r.v.'s, when we compare the almost sure invariance principle (rate $N^{-\frac{\delta}{2+2\delta}}$) and the (weak) invariance principle (rate $N^{-\frac{\delta}{3+2\delta}}$).

As in the paper [15] our proof relies on a progressive blocking technique (see Bernstein [2]) coupled with a triadic Cantor like scheme and on the Komlós, Major and Tusnády approximation type results for independent r.v.'s (see [22], [10], [38]), which is in contrast to approaches usually based on martingale methods.

As a potential application of the obtained results we point out the asymptotic equivalence of statistical experiments as developed in [12], [13], [14], whose scope can be extended to various models under weak dependence constraints.

Our paper is organized as follows. In Sections 2 and 3 we formulate our main results and give an application to the case of Markov chains. In Section 4 we introduce the notations to be used in the proofs of the main results. Proofs of the main results are given in Sections 5, 6 and 7. In Section 8 we prove some bounds for the L^p norm of the increments of the process $(X_k)_{k \geq 1}$ and, finally, in Section 9 we collect some auxiliary assertions and general facts.

We conclude this section by setting some notations to be used all over the paper. For any $x \in \mathbb{R}^d$, denote by $\|x\|_\infty = \sup_{1 \leq i \leq d} |x_i|$ the supremum norm. For any $p > 0$, the L^p norm of a random variable X is denoted by $\|X\|_{L^p}$. The equality in distribution of two stochastic processes $(Z'_i)_{i \geq 1}$ and $(Z''_i)_{i \geq 1}$, possibly defined on two different probability spaces $(\Omega', \mathcal{F}', \mathbb{P}')$ and $(\Omega'', \mathcal{F}'', \mathbb{P}'')$, will be denoted $\mathcal{L}((Z'_i)_{i \geq 1} | \mathbb{P}') \stackrel{d}{=} \mathcal{L}((Z''_i)_{i \geq 1} | \mathbb{P}'')$. The generalized inverse of a distribution function F on a real line is denoted by F^{-1} , i.e. $F^{-1}(y) = \inf \{x : F(x) > y\}$. By c, c', c'', \dots , possibly supplied with indices $1, 2, \dots$, we denote absolute constants whose values may vary from line to line. The notations $c_{\alpha_1, \dots, \alpha_r}, c'_{\alpha_1, \dots, \alpha_r}, \dots$ will be used to stress that the constants depend only on the parameters indicated in their indices: for instance $c'_{\alpha, \beta}$ denotes a constant depending only on the constants α, β . All other constants will be specifically indicated. As usual, we shall use the shortcut "standard normal r.v." for a normal random variable of mean 0 and variance 1.

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2. MAIN RESULTS

Assume that on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we are given a sequence $(X_i)_{i \geq 1}$ of dependent r.v.'s with values in the real line \mathbb{R} . The expectation with respect to \mathbb{P} is denoted by \mathbb{E} .

The following condition will be used to ensure that the process $(X_i)_{i \geq 1}$ has almost independent increments. Given natural numbers $k_{gap}, M_1, M_2 \in \mathbb{N}$ and a sequence $j_0 \leq \dots \leq j_{M_1+M_2}$ denote $X_{k+J_m} = \sum_{l \in J_m} X_{k+l}$, where $J_m = [j_{m-1}, j_m)$, $m = 1, \dots, M_1 + M_2$ and $k \geq 0$. Consider the vectors $\bar{X}_1 = (X_{J_1}, \dots, X_{J_{M_1}})$ and $\bar{X}_2 = (X_{k_{gap}+J_{M_1+1}}, \dots, X_{k_{gap}+J_{M_1+M_2}})$. Let

$\phi(t_1, t_2) = \mathbb{E}e^{it_1\bar{X}_1+it_2\bar{X}_2}$, $\phi_1(t_1) = \mathbb{E}e^{it_1\bar{X}_1}$ and $\phi_2(t_2) = \mathbb{E}e^{it_2\bar{X}_2}$ be the characteristic functions of (\bar{X}_1, \bar{X}_2) , \bar{X}_1 and \bar{X}_2 respectively. We ask that the dependence between the two vectors \bar{X}_1 (the past) and \bar{X}_2 (the future) decreases exponentially as a function of the size of the gap k_{gap} in the following sense.

Condition C1 *There exist positive constants $\varepsilon_0 \leq 1$, $\lambda_0, \lambda_1, \lambda_2$ such that for any k_{gap} , $M_1, M_2 \in N$, any sequence $j_0 < \dots < j_{M_1+M_2}$ and any $t_1 \in R^{M_1}, t_2 \in R^{M_2}$ satisfying $\|(t_1, t_2)\|_\infty \leq \varepsilon_0$,*

$$|\phi(t_1, t_2) - \phi_1(t_1)\phi_2(t_2)| \leq \lambda_0 \exp(-\lambda_1 k_{gap}) \left(1 + \max_{m=1, \dots, M_1+M_2} \text{card}(J_m)\right)^{\lambda_2(M_1+M_2)}.$$

All over the paper we suppose that the following moment conditions hold true.

Condition C2 *There exist two constants $\delta > 0$ and $\mu_\delta > 0$ such that*

$$\sup_{i \geq 1} \|X_i\|_{L^{2+2\delta}} \leq \mu_\delta < \infty.$$

We suppose also that the sequence $(X_i)_{i \geq 1}$ possesses the following asymptotic homogeneity property.

Condition C3 *There exist a constant $\tau > 0$ and a positive number $\sigma > 0$ such that, for any $\gamma > 0$ and any $n \geq 1$,*

$$\sup_{k \geq 0} \left| n^{-1} \text{Var}_{\mathbb{P}} \left(\sum_{i=k+1}^{k+n} X_i \right) - \sigma^2 \right| \leq \tau n^{-1+\gamma}.$$

The main result of the paper is the following theorem. Denote $\mu_i = \mathbb{E}X_i$, for $i \geq 1$.

Theorem 2.1. *Assume Conditions C1, C2 and C3. Let $0 < \alpha < \delta$. Then, on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ there exist a sequence of random variables $(\tilde{X}_i)_{i \geq 1}$ such that $\mathcal{L}((\tilde{X}_i)_{i \geq 1} | \tilde{\mathbb{P}}) \stackrel{d}{=} \mathcal{L}((X_i)_{i \geq 1} | \mathbb{P})$ and a sequence of independent standard normal random variables $(W_i)_{i \geq 1}$, such that for any $0 < \rho < \frac{\alpha}{2(1+2\alpha)}$ and $N \geq 1$,*

$$\tilde{\mathbb{P}} \left(N^{-1/2} \sup_{k \leq N} \left| \sum_{i=1}^k (\tilde{X}_i - \mu_i - \sigma W_i) \right| > 6N^{-\rho} \right) \leq C_0 N^{-\alpha \frac{1+\alpha}{1+2\alpha} + \rho(2+2\alpha)},$$

where $C_0 = c_{\lambda_1, \lambda_2, \alpha, \delta, \sigma} (1 + \lambda_0 + \mu_\delta + \sqrt{\tau})^{2+2\delta}$ and $c_{\lambda_1, \lambda_2, \alpha, \delta, \sigma}$ depends only on the constants indicated in its indices.

Letting $\rho = \frac{\alpha}{3+2\alpha} \frac{1+\alpha}{1+2\alpha}$ from Theorem 2.1 we get

$$(2.1) \quad \tilde{\mathbb{P}} \left(N^{-1/2} \sup_{k \leq N} \left| \sum_{i=1}^k (\tilde{X}_i - \mu_i - \sigma W_i) \right| > 6N^{-\frac{\alpha}{3+2\alpha} \frac{1+\alpha}{1+2\alpha}} \right) \leq C_0 N^{-\frac{\alpha}{3+2\alpha} \frac{1+\alpha}{1+2\alpha}},$$

where $C_0 = c_{\lambda_1, \lambda_2, \alpha, \delta, \sigma} (1 + \lambda_0 + \mu_\delta + \sqrt{\tau})^{2+2\delta}$ and $c_{\lambda_1, \lambda_2, \alpha, \delta, \sigma}$ depends only on the constants indicated in its indices. Compared with the optimal rate of convergence for independent r.v.'s $N^{-\frac{\alpha}{3+2\alpha}}$ the loss in the power is within the factor $\frac{1+\alpha}{1+2\alpha}$. As $\alpha \rightarrow \infty$ we obtain the

limiting power $\frac{1}{4}$ which is twice worse than the optimal power $\frac{1}{2}$ in the independent case. In particular, if $\alpha = \frac{1}{2}$ (which corresponds to $p = 2 + 2\alpha = 3$) we obtain the rate of convergence $N^{-\frac{\alpha}{3+2\alpha} \frac{1+\alpha}{1+2\alpha}} = N^{-\frac{3}{32}}$, while in the independent case we have $N^{-\frac{1}{8}}$, which represents a loss of the power of the order $\frac{1}{8} - \frac{3}{32} = \frac{1}{32}$.

We would like to point out that in Theorem 2.1 we figure out the explicit dependence of the constant C_0 on the constants λ_0, μ_δ and τ involved in Conditions **C1**, **C2** and **C3**. In the next section we show that Theorem 2.1 can be applied to Markov walks under spectral gap type assumptions on the associate Markov chain. It is important to stress that our result is the first one to figure out the dependence of the constants involved in the bounds on the initial state of the Markov chain. The results of the paper can be also applied to a large class of dynamical systems, however this stays beyond the scope of the present paper. For a discussion of these type of applications we refer to [15].

For the proof of Theorem 2.1, without loss of generality, we shall assume that $\mu_i = 0, i \geq 1$ and $\sigma = 1$, since the general case can be reduced to this one by centering and renormalizing the variables X_i , i.e. by replacing X_i by $X'_i = (X_i - \mu_i) / \sigma$. It is easy to see that Conditions **C1**, **C2**, **C3** are satisfied for the new random variables X'_i with the same λ_0 and with μ_δ, τ replaced by $2\mu_\delta / \sigma, \tau / \sigma^2$.

3. APPLICATIONS TO MARKOV WALKS

Consider a Markov chain $(X_k)_{k \geq 0}$ with values in the measurable state space $(\mathbb{X}, \mathcal{X})$ with the transition probability $\mathbf{P}(x, \cdot), x \in \mathbb{X}$. For every $x \in \mathbb{X}$ denote by \mathbb{P}_x and \mathbb{E}_x the probability measure and expectation generated by the finite dimensional distributions

$$\mathbb{P}_x(X_0 \in B_0, \dots, X_n \in B_n) = 1_{B_0}(x) \int_{B_1} \dots \int_{B_n} \mathbf{P}(x, dx_1) \dots \mathbf{P}(x_{n-1}, dx_n),$$

for any $B_k \in \mathcal{X}, k = 1, \dots, n, n = 1, 2, \dots$ on the space of trajectories $(\mathbb{X}, \mathcal{X})^{\mathbb{N}}$. In particular $\mathbb{P}_x(X_0 = x) = 1$.

Let f be a real valued function defined on the state space \mathbb{X} of the Markov chain $(X_k)_{k \geq 0}$. Let \mathcal{B} be a Banach space of real valued functions on \mathbb{X} endowed with the norm $\|\cdot\|_{\mathcal{B}}$ and let $\|\cdot\|_{\mathcal{B} \rightarrow \mathcal{B}}$ be the operator norm on \mathcal{B} . Denote by $\mathcal{B}' = \mathcal{L}(\mathcal{B}, \mathbb{C})$ the topological dual of \mathcal{B} equipped with the norm $\|\cdot\|_{\mathcal{B}'}$. The unit function on \mathbb{X} is denoted by $e : e(x) = 1$ for $x \in \mathbb{X}$. The Dirac measure at $x \in \mathbb{X}$ is denoted by $\delta_x : \delta_x(g) = g(x)$ for any $g \in \mathcal{B}$.

Introduce the following hypotheses.

Hypothesis M1 (Banach space):

- a) *The unit function e belongs to \mathcal{B} .*
- b) *For every $x \in X$ the Dirac measure δ_x belongs to \mathcal{B}' .*
- c) *$\mathcal{B} \subseteq L^1(\mathbf{P}(x, \cdot))$ for every $x \in X$.*
- d) *There exists a constant $\varepsilon_0 \in (0, 1)$ such that for any $g \in \mathcal{B}$ the function $e^{itf}g \in \mathcal{B}$ for any t satisfying $|t| \leq \varepsilon_0$.*

Note that, for any $x \in \mathbb{X}$ and $g \in L^1(\mathbf{P}(x, \cdot))$, the quantity $\mathbf{P}g(x) := \int_{\mathbb{X}} g(y) \mathbf{P}(x, dy)$ is well defined. In particular, under the hypothesis **M1** c), $\mathbf{P}g(x)$ exists when $g \in \mathcal{B}$. We thus consider the following hypothesis:

Hypothesis M2 (Spectral gap):

- a) The map $g \mapsto \mathbf{P}g$ is a bounded operator on \mathcal{B}
- b) There exist constants $C_Q > 0$ and $\kappa \in (0, 1)$ such that

$$(3.1) \quad \mathbf{P} = \Pi + Q,$$

where Π is a one dimensional projector and Q is an operator on \mathcal{B} satisfying $\Pi Q = Q \Pi = 0$ and $\|Q^n\|_{\mathcal{B} \rightarrow \mathcal{B}} \leq C_Q \kappa^n$.

Notice that, since the image of Π is generated by the unit function e , there exists a linear form $\nu \in \mathcal{B}'$ such that for any $g \in \mathcal{B}$,

$$(3.2) \quad \Pi g = \nu(g) e.$$

When hypotheses **M1** and **M2** hold, we set $\mathbf{P}_t g = \mathbf{P}(e^{itf} g)$ for any $g \in \mathcal{B}$ and $t \in [-\varepsilon_0, \varepsilon_0]$. Notice that $\mathbf{P} = \mathbf{P}_0$.

Hypothesis M3 (Perturbed transition operator):

- a) For any $|t| \leq \varepsilon_0$ the map $g \in \mathcal{B} \rightarrow \mathbf{P}_t g \in \mathcal{B}$ is a bounded operator on \mathcal{B} ,
- b) There exists a constant $C_{\mathbf{P}} > 0$ such that, for all $n \geq 1$ and $|t| \leq \varepsilon_0$,

$$(3.3) \quad \|\mathbf{P}_t^n\|_{\mathcal{B} \rightarrow \mathcal{B}} \leq C_{\mathbf{P}}.$$

Hypothesis M4 (Moment conditions): There exists $\delta > 0$ such that for any $x \in X$,

$$\mu_{\delta}(x) := \sup_{k \geq 1} \left(\mathbb{E}_x |f(X_k)|^{2+2\delta} \right)^{\frac{1}{2+2\delta}} = \sup_{k \geq 1} \left(\left(\mathbf{P}^k |f|^{2+2\delta} \right)(x) \right)^{\frac{1}{2+2\delta}} < \infty.$$

We show first that under the hypotheses **M1**, **M2**, **M3** and **M4**, conditions **C1**, **C2** and **C3** are satisfied. As in the previous section let $k_{gap}, M_1, M_2 \in \mathbb{N}$ and $j_0 \leq \dots \leq j_{M_1+M_2}$ be natural numbers. Denote $Y_{k+J_m} = \sum_{l \in J_m} f(X_{k+l})$, where $J_m = [j_{m-1}, j_m)$, $m = 1, \dots, M_1 + M_2$ and $k \geq 0$. Consider the vectors $\bar{Y}_1 = (Y_{J_1}, \dots, Y_{J_{M_1}})$ and $\bar{Y}_2 = (Y_{k_{gap}+J_{M_1+1}}, \dots, Y_{k_{gap}+J_{M_1+M_2}})$. Denote by $\phi_x(t_1, t_2) = \mathbb{E} e^{it_1 \bar{Y}_1 + it_2 \bar{Y}_2}$, $\phi_{x,1}(t_1) = \mathbb{E}_x e^{it_1 \bar{Y}_1}$ and $\phi_{x,2}(t_2) = \mathbb{E}_x e^{it_2 \bar{Y}_2}$ the characteristic functions of (\bar{Y}_1, \bar{Y}_2) , \bar{Y}_1 and \bar{Y}_2 respectively.

Proposition 3.1. *Assume that the Markov chain $(X_n)_{n \geq 1}$ and the function f satisfy the hypotheses **M1**, **M2**, **M3** and **M4**. Then Condition **C1** is satisfied, i.e. there exists a positive constant $\varepsilon_0 \leq 1$ such that for any $k_{gap}, M_1, M_2 \in \mathbb{N}$, any sequence $j_0 < \dots < j_{M_1+M_2}$ and any $t_1 \in \mathbb{R}^{M_1}, t_2 \in \mathbb{R}^{M_2}$ satisfying $\|(t_1, t_2)\|_{\infty} \leq \varepsilon_0$,*

$$|\phi_x(t_1, t_2) - \phi_{x,1}(t_1) \phi_{x,2}(t_2)| \leq \lambda_0(x) \exp(-\lambda_1 k_{gap}) \times \left(1 + \max_{m=1, \dots, M_1+M_2} \text{card}(J_m) \right)^{\lambda_2(M_1+M_2)},$$

where $\lambda_0(x) = 2C_Q (\|\nu\|_{\mathcal{B}'} + \|\delta_x\|_{\mathcal{B}'}) \|e\|_{\mathcal{B}}$, $\lambda_1 = |\ln \kappa|$ and $\lambda_2 = \max\{1, \log_2 C_{\mathbf{P}}\}$.

Proposition 3.2. *Assume that the Markov chain $(X_n)_{n \geq 1}$ and the function f satisfy the hypotheses **M1**, **M2**, **M3** and **M4**. Then:*

a) *There exists a constant μ such that for any $x \in \mathbb{X}$ and $k \geq 1$*

$$(3.4) \quad |\mathbb{E}_x f(X_k) - \mu| \leq c_\delta A_1(x) \kappa^{k\gamma/4-1},$$

for any positive constant γ satisfying $0 < \gamma \leq \min\{1, 2\delta\}$, where $A_1(x) = 1 + \mu_\delta(x)^{1+\gamma} + \|\delta_x\|_{\mathcal{B}'} \|e\|_{\mathcal{B}} C_{\mathbf{P}} C_Q$. Moreover

$$(3.5) \quad \sum_{k=0}^{\infty} |\mathbb{E}_x f(X_k) - \mu| \leq \bar{\mu}(x) = c_{\delta, \kappa, \gamma} A_1(x).$$

b) *There exists a constant $\sigma \geq 0$ such that for any $x \in \mathbb{X}$,*

$$(3.6) \quad \sup_{m \geq 0} \left| \text{Var}_{\mathbb{P}_x} \left(\sum_{i=m+1}^{m+n} f(X_i) \right) - n\sigma^2 \right| \leq \tau(x) = c_{\delta, \kappa, \gamma} A_2(x),$$

where

$$A_2(x) = 1 + \mu_\delta(x)^{2+\gamma} + (1 + \|\delta_x\|_{\mathcal{B}'} \|e\|_{\mathcal{B}} (C_{\mathbf{P}}^2 C_Q (1 + C_Q) + C_{\mathbf{P}} C_Q (1 + \|\nu\|_{\mathcal{B}'} C_{\mathbf{P}})).$$

Note that the constants μ and σ do not depend on the initial state x .

The main result of this section is the following theorem. Let $\tilde{\Omega} = \mathbb{R}^\infty \times \mathbb{R}^\infty$. For any $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2) \in \tilde{\Omega}$ denote by $\tilde{Y}_i = \tilde{\omega}_{1,i}$ and $W_i = \tilde{\omega}_{2,i}$, $i \geq 1$, the coordinate processes in $\tilde{\Omega}$.

Theorem 3.3. *Assume that the Markov chain $(X_n)_{n \geq 0}$ and the function f satisfy the hypotheses **M1**, **M2**, **M3** and **M4**, with $\sigma > 0$. Let $0 < \alpha < \delta$. Then there exists a Markov transition kernel $x \rightarrow \tilde{\mathbb{P}}_x(\cdot)$ from $(\mathbb{X}, \mathcal{X})$ to $(\tilde{\Omega}, \mathcal{B}(\tilde{\Omega}))$ such that $\mathcal{L} \left(\left(\tilde{Y}_i \right)_{i \geq 1} \mid \tilde{\mathbb{P}}_x \right) \stackrel{d}{=} \mathcal{L} \left((f(X_i))_{i \geq 1} \mid \mathbb{P}_x \right)$, the $W_i, i \geq 1$, are independent standard normal r.v.'s under $\tilde{\mathbb{P}}_x$ and for any $0 < \rho < \frac{\alpha}{2(1+2\alpha)}$*

$$(3.7) \quad \tilde{\mathbb{P}}_x \left(N^{-\frac{1}{2}} \sup_{k \leq N} \left| \sum_{i=1}^k \left(\tilde{Y}_i - \mu - \sigma W_i \right) \right| > 6N^{-\rho} \right) \leq C(x) N^{-\alpha \frac{1+\alpha}{1+2\alpha} + \rho(2+2\alpha)},$$

with

$$C(x) = C_1 (1 + \mu_\delta(x) + \|\delta_x\|_{\mathcal{B}'})^{2+2\delta},$$

where C_1 is a constant depending only on $\delta, \alpha, \kappa, C_{\mathbf{P}}, C_Q, \|e\|_{\mathcal{B}}$ and $\|\nu\|_{\mathcal{B}'}$.

Note that only the probability $\tilde{\mathbb{P}}_x$ depends on the initial state x while the processes $(\tilde{Y}_k)_{k \geq 0}$ and $(W_k)_{k \geq 0}$ do not depend on it.

As in the previous section, letting $\rho = \frac{\alpha}{3+2\alpha} \frac{1+\alpha}{1+2\alpha}$, under the conditions of Theorem 3.3 we obtain,

$$(3.8) \quad \tilde{\mathbb{P}}_x \left(N^{-\frac{1}{2}} \sup_{k \leq N} \left| \sum_{i=1}^k \left(\tilde{Y}_i - \mu - \sigma W_i \right) \right| > 6N^{-\frac{\alpha}{3+2\alpha} \frac{1+\alpha}{1+2\alpha}} \right) \leq C(x) N^{-\frac{\alpha}{3+2\alpha} \frac{1+\alpha}{1+2\alpha}}.$$

Compared to the rate $N^{-\frac{\alpha}{3+2\alpha}}$ which is optimal in the independent case the rate of convergence $N^{-\frac{\alpha}{3+2\alpha} \frac{1+\alpha}{1+2\alpha}}$ in (3.8) is slower by the factor $N^{\frac{\alpha}{3+2\alpha} \frac{\alpha}{1+2\alpha}}$. As $\alpha \rightarrow \infty$ we obtain $N^{-\frac{1}{4}}$ which is the best rate in the invariance principle that is known for dependent random variables.

In Theorem 3.3 we do not suppose the existence of the stationary measure. Assume that there exists a stationary probability measure ν on \mathbb{X} ; it thus coincides with the linear form ν introduced in (3.2). Let \mathbb{P}_ν and \mathbb{E}_ν the probability measure and expectation generated by the finite dimensional distributions of the chain under the stationary measure ν . Note that, the means $\mathbb{E}_\nu X_k$ and the covariances $\text{Cov}_{\mathbb{P}_\nu}(f(X_l), f(X_{l+k}))$ with respect to ν may not exist, under the hypotheses **M1**, **M2**, **M3** and **M4**. To ensure their existence, we require the following additional condition (where generally $|f|^2 \notin \mathcal{B}$).

Hypothesis M5 (Stationary measure): *On the state space \mathbb{X} there exists a stationary probability measure ν such that $\nu(\sup_{k \geq 0} \mathbf{P}^k(|f|^2)) < \infty$.*

Under Hypothesis **M5** for μ and σ we find the usual expressions of the means and of the variance in the central limit theorem for dependent r.v.'s.

Theorem 3.4. *Assume that the Markov chain $(X_n)_{n \geq 0}$ and the function f satisfy the hypotheses **M1**, **M2**, **M3**, **M4** and **M5**. Assume also that $\sigma_\nu > 0$. Then Proposition 3.2 holds true with $\mu = \nu(f)$ and $\sigma = \sigma_\nu$, where*

$$\nu(f) = \int f(x) \nu(dx)$$

and

$$\sigma_\nu^2 = \text{Var}_{\mathbb{P}_\nu}(f(X_0)) + 2 \sum_{k=1}^{\infty} \text{Cov}_{\mathbb{P}_\nu}(f(X_0), f(X_k)).$$

Moreover, if $\sigma_\nu > 0$ the assertions of Theorem 3.3 and (3.8) hold true with $\mu = \nu(f)$ and $\sigma = \sigma_\nu$.

From Theorem 3.4 one can derive a bound when the Markov chain $(X_n)_{n \geq 0}$ is in the stationary regime. If we assume that $\nu(\sup_{k \geq 0} \mathbf{P}^k(|f|^{2+2\delta})) \leq c_{\nu, \delta} < \infty$ and $\int \|\delta_x\|_{\mathcal{B}'}^{2+2\delta} \nu(dx) \leq c_{\mathcal{B}', \delta} < \infty$, then integrating (3.7) with respect to ν we obtain

$$\tilde{\mathbb{P}}_\nu \left(N^{-\frac{1}{2}} \sup_{k \leq N} \left| \sum_{i=1}^k (\tilde{Y}_i - \mu - \sigma W_i) \right| > 6N^{-\rho} \right) \leq CN^{-\alpha \frac{1+\alpha}{1+2\alpha} + \rho(2+2\alpha)},$$

where C is a constant depending on $\delta, \alpha, \kappa, C_{\mathbf{P}}, C_Q, \|e\|_{\mathcal{B}}, \|\nu\|_{\mathcal{B}'}$ and $c_{\nu, \delta}, c_{\mathcal{B}', \delta}$.

The hypotheses **M1**, **M2**, **M4** and **M5** formulated above can be easily verified by standard methods. As to **M3** it can be verified using two approaches. The first approach is based on the assumption that the family of operators $(\mathbf{P}_t)_{|t| \leq \varepsilon_0}$ is continuous in t at $t = 0$. In this case, **M3** is satisfied by classical perturbation theory (see, for instance Dunford and Schwartz [9]). The second approach is based on a weaker form of continuity of the family $(\mathbf{P}_t)_{|t| \leq \varepsilon_0}$ as developed in Keller and Liverani [21].

We end this section by giving three examples where these hypotheses are satisfied.

3.1. Example 1 (Markov chains with finite state spaces). Suppose that $(X_n)_{n \geq 0}$ is an irreducible ergodic aperiodic Markov chain with finite state space. It is easy to verify that the hypotheses **M1**, **M2**, **M3**, **M4** and **M5** are satisfied and that there exists a unique invariant probability measure ν . Then the conclusions of Theorem 3.4 hold true.

3.2. Example 2 (autoregressive random walk with Bernoulli noise). Consider the autoregressive model $x_{n+1} = \alpha x_n + b_n$, $n \geq 0$, where α is a constant satisfying $\alpha \in (-1, 1)$ and $(b_n)_{n \geq 0}$ are i.i.d. Bernoulli r.v.'s with $P(b = 1) = P(b = -1) = \frac{1}{2}$ and $x_0 = x$. Consider the Banach space $\mathcal{B} = \mathcal{L}$ of continuous functions f on \mathbb{R} such that $\|f\| = |f| + [f] < \infty$, where

$$|f| = \sup_{x \in \mathbb{R}} \frac{|f(x)|}{1 + x^2}, \quad [f] = \sup_{\substack{x, y \in \mathbb{R} \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|(1 + x^2)(1 + y^2)}.$$

Since $\alpha \in (-1, 1)$, the invariant measure ν exists and coincides with the law of the random variable $Z = \sum_{i=1}^{\infty} \alpha^{i-1} b_i$. It is easy to verify that hypotheses **M1**, **M2**, **M3**, **M4** and **M5** are satisfied for the function $f(x) = x$. For the mean $\nu(f)$ we have

$$\nu(f) = \int x \nu(dx) = \mathbb{E}Z = \sum_{i=1}^{\infty} \alpha^{i-1} \mathbb{E}b_1 = \frac{\mathbb{E}b_1}{1 - \alpha}.$$

Since $\mathbb{E}b_1 = 0$, one gets $\nu(f) = 0$ and the variance is computed as follows:

$$\sigma_{\nu}^2 = \lim_{n \rightarrow \infty} \mathbb{E} \left(\sum_{i=1}^n \alpha^{i-1} b_i \right)^2 = \lim_{n \rightarrow \infty} \sum_{i=1}^n \alpha^{2(i-1)} \mathbb{E}b_1^2 = \frac{1}{1 - \alpha^2}.$$

Thus the conclusions of Theorem 3.4 hold true with $\nu(f) = 0$ and $\sigma_{\nu}^2 = \frac{1}{1 - \alpha^2}$.

3.3. Example 3 (stochastic recursion). On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ consider the stochastic recursion

$$x_{n+1} = a_{n+1}x_n + b_{n+1}, \quad n \geq 0,$$

where $(a_n, b_n)_{n \geq 0}$ are i.i.d. r.v.'s with values in $(0, \infty) \times \mathbb{R}$ of the same distribution $\hat{\mu}$ and $x_0 = x$. Following Guivarc'h and Le Page [17], we assume the conditions:

H1 *There exists $\alpha > 2$ such that $\varphi(\alpha) := \int |a|^{\alpha} \hat{\mu}(da, db) < 1$ and $\int |b|^{\alpha} \hat{\mu}(da, db) < +\infty$.*

H2 *$\hat{\mu}(\{(a, b) : ax_0 + b = x_0\}) < 1$ for any $x_0 \in \mathbb{R}$.*

H3 *The set $\{\ln |a| : (a, b) \in \text{supp } \hat{\mu}\}$ generates a dense subgroup of \mathbb{R} .*

Let $\varepsilon \in (0, 1)$, θ and c be positive such that $\alpha - 1 < c + \varepsilon < \theta \leq 2c < \alpha - \varepsilon$. Consider the Banach space $\mathcal{B} = \mathcal{L}_{\varepsilon, c, \theta}$ of continuous functions f on \mathbb{R} such that $\|f\| = |f| + [f] < \infty$, where

$$|f| = \sup_{x \in \mathbb{R}} \frac{|f(x)|}{1 + |x|^{\theta}}, \quad [f] = \sup_{\substack{x, y \in \mathbb{R} \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^{\varepsilon} (1 + |x|^c)(1 + |y|^c)}.$$

The transition probability $\mathbf{P}(x, \cdot)$ of the Markov chain $(x_n)_{n \geq 0}$ is defined by

$$\int h(y) \mathbf{P}(x, dy) = \int h(ax + b) \widehat{\mu}(da, db),$$

for any bounded Borel measurable function $h : \mathbb{R} \rightarrow \mathbb{R}$ and $x \in \mathbb{R}$. For any $x \in \mathbb{R}$ denote by \mathbb{P}_x and \mathbb{E}_x the corresponding probability measure and expectation generated by the finite dimensional distributions on the space of trajectories. It is proved in [17] (Proposition 1) that the series $\sum_{i=1}^{\infty} a_1 \dots a_{i-1} b_i$ is \mathbb{P} -a.s. convergent and the Markov chain $(x_n)_{n \geq 0}$ has a unique invariant probability measure ν which coincides with the law of $Z = \sum_{i=1}^{\infty} a_1 \dots a_{i-1} b_i$. Moreover, it holds $\int |x|^t \nu(dx) < \infty$ for any $t \in [0, \alpha)$.

We shall verify that hypotheses **M1**, **M2**, **M3**, **M4** and **M5** are satisfied with the function $f(x) = x$. Hypothesis **M1** is obvious and hypotheses **M2** and **M3** follow from Theorem 1 and Proposition 4 in [17]. If $\delta > 0$ is such that $2 + 2\delta \leq \alpha$, by simple calculations we obtain

$$\left(\mathbb{E}_x |x_n|^{2+2\delta} \right)^{\frac{1}{2+2\delta}} \leq \varphi(2 + 2\delta)^{\frac{n}{2+2\delta}} |x| + \frac{\|b_1\|_{2+2\delta}}{1 - \varphi(2 + 2\delta)^{\frac{1}{2+2\delta}}}.$$

Taking the sup in $n \geq 1$, we get

$$\begin{aligned} \mu_\delta(x) &= \sup_{n \geq 1} \left(\mathbb{E}_x |f(x_n)|^{2+2\delta} \right)^{\frac{1}{2+2\delta}} \\ &\leq \varphi(2 + 2\delta)^{\frac{1}{2+2\delta}} |x| + \frac{\|b_1\|_{2+2\delta}}{1 - \varphi(2 + 2\delta)^{\frac{1}{2+2\delta}}}, \end{aligned}$$

which proves that hypothesis **M4** is satisfied. Finally, hypothesis **M5** is verified since

$$\int \mu_\delta(x)^2 \nu(dx) \leq 2 \left(\varphi(2 + 2\delta)^{\frac{1}{1+\delta}} \int x^2 \nu(dx) + \left(\frac{\|b_1\|_{2+2\delta}}{1 - \varphi(2 + 2\delta)^{\frac{1}{2+2\delta}}} \right)^2 \right) < \infty.$$

The mean is given by $\nu(f) = \mathbb{E}Z = \sum_{i=1}^{\infty} (\mathbb{E}a_1)^{i-1} \mathbb{E}b_1 = \frac{\mathbb{E}b_1}{1 - \mathbb{E}a_1}$. Without loss of generality we can assume that $\nu(f) = 0$, i.e. that $\mathbb{E}b_1 = 0$; then the variance is

$$\sigma_\nu^2 = \text{Var}_{\mathbb{P}}(Z) = \lim_{n \rightarrow \infty} \mathbb{E} \left(\sum_{i=1}^n a_1 \dots a_{i-1} b_i \right)^2 = \lim_{n \rightarrow \infty} \sum_{i=1}^n (\mathbb{E}a_1^2)^{i-1} \mathbb{E}b_1^2 = \frac{\mathbb{E}b_1^2}{1 - \mathbb{E}a_1^2}.$$

Then the conclusions of Theorem 3.4 hold true with $\mu = \nu(f) = 0$ and $\sigma = \sigma_\nu^2 = \frac{\mathbb{E}b_1^2}{1 - \mathbb{E}a_1^2}$.

A multivariate version of the stochastic recursion has been considered in Guivarc'h and Le Page [18], [19] and can be treated in the same manner.

4. PARTITION OF THE SET \mathbb{N} AND NOTATIONS

In the sequel $\varepsilon, \beta \in (0, 1)$ will be such that $\varepsilon + \beta < 1$ (all over the paper ε is supposed to be very small, while β will be optimized). Denote for simplicity $[a, b) = \{l \in \mathbb{N} : a \leq l < b\}$. Let $k_0 \geq 1$ be a natural number. We start by splitting the set \mathbb{N} into subsets $[2^k, 2^{k+1})$, $k = k_0, k_0 + 1, \dots$ called blocks. Consider the k -th block $[2^k, 2^{k+1})$. We leave a large gap $J_{k,1}$ of length $2^{\lfloor \varepsilon k \rfloor + \lfloor \beta k \rfloor}$ at the left end of the k -th block. Then, following a triadic Cantor-like

scheme, we split the remaining part $[2^k + 2^{[\varepsilon k] + [\beta k]}, 2^{k+1})$ into subsets $I_{k,j}$ and $J_{k,j}$ called islands and gaps as explained below. At the resolution level 0 a gap of size $2^{[\varepsilon k] + [\beta k]}/2$ is put in the middle of the interval $[2^k + 2^{[\varepsilon k] + [\beta k]}, 2^{k+1})$. This yields two intervals of equal length. At the resolution level 1 two additional gaps of length $2^{[\varepsilon k] + [\beta k]}/2^2$ are put in the middle of the each obtained interval which yields four intervals of equal length. Continuing in the same way, at the resolution level $[\beta k]$ we obtain $2^{[\beta k]}$ intervals $I_{k,j}$, $j = 1, \dots, 2^{[\beta k]}$ called islands and the same number of gaps $J_{k,j}$, $j = 1, \dots, 2^{[\beta k]}$ which we index from left to right (recall that $J_{k,1} = J_{k,2^0}$ denotes the large gap at the left end of the k -th block). It is obvious that $[2^k, 2^{k+1})$ is the union of the constructed island and gaps, so that

$$(4.1) \quad [2^k, 2^{k+1}) = J_{k,1} \cup I_{k,1} \cup \dots \cup J_{k,2^{[\beta k]}} \cup I_{k,2^{[\beta k]}}.$$

Note that in the block k there are one gap of the length $2^{[[\varepsilon k] + [\beta k]]}$ and 2^l gaps of length $2^{[[\varepsilon k] + [\beta k]] - l - 1}$, where $l = 0, \dots, [\beta k] - 1$. The length of the finest gap (for example $J_{k,2^{[\beta k]}}$) is $2^{[\varepsilon k]}$. The total length of the gaps in the block k is

$$L_k^{gap} = 2^{[[\varepsilon k] + [\beta k]]} + \sum_{l=0}^{[\beta k] - 1} 2^l 2^{[[\varepsilon k] + [\beta k]] - l - 1} = (2 + [\beta k]) 2^{[[\varepsilon k] + [\beta k]] - 1}.$$

Recall that, according to the construction, the islands of the k -th block have the same length

$$\begin{aligned} |I_{k,j}| &= (2^{k+1} - 2^k - (2 + [\beta k]) 2^{[[\varepsilon k] + [\beta k]] - 1}) / 2^{[\beta k]} \\ &= 2^{k - [\beta k]} - (1 + [\beta k]) 2^{[[\varepsilon k]] - 1}. \end{aligned}$$

An obvious upper bound is $|I_{k,j}| \leq 2^{k - [\beta k]}$. Since $\varepsilon < 1 - \beta$ we have $|I_{k,j}| \geq 2^{k - [\beta k]} - 2^{[[\varepsilon k]] - c'_{\beta, \varepsilon} \ln k} \geq c_{\varepsilon, \beta} 2^{k(1 - \beta)}$, with some $c_{\varepsilon, \beta} \in (0, \frac{1}{2})$. Since the length of the k -th block is 2^k , the total length of the islands in this block is equal to

$$L_k^{isl} = 2^k - 2^{[[\varepsilon k] + [\beta k]] - 1} (2 + [\beta k]).$$

Note that, for some constant $c_\beta > 0$,

$$(4.2) \quad c_\beta 2^k \leq L_k^{isl} \leq 2^k.$$

Denote by \mathcal{K} the set of double indices (k, j) , with $k = 1, 2, \dots$ being the index of the block and $j = 1, \dots, 2^{[\beta k]}$ being the index of the island in the block k . The set \mathcal{K} will be endowed with lexicographical order \preceq . Then the sets $I_{k,j}$ and $J_{k,j}$, $(k, j) \in \mathcal{K}$, will be also endowed with the lexicographical order. Let $N \in \mathbb{N}$. From (4.1), there exists a unique $(n, m) \in \mathcal{K}$ such that $2^n \leq N < 2^{n+1}$ and $N \in J_{n,m} \cup I_{n,m}$, where the dependence of n and m on N is suppressed from the notation; denote $\mathcal{K}_N = \{(k, j) : (k, j) \preceq (n, m)\}$.

For ease of reading we recall the notations and properties that will be used throughout the paper:

P1: ε and β are positive numbers such that $\varepsilon + \beta < 1$. Later on, the constant ε will be chosen to be small enough.

P2: $\mathcal{K} = \{(k, j) : k = 1, 2, \dots, j = 1, \dots, 2^{[\beta k]}\}$.

P3: For any $N \in \mathbb{N}$ the unique couple $(n, m) \in \mathcal{K}$ is such that $N \in J_{n,m} \cup I_{n,m}$.

P4: $\mathcal{K}_N = \{(k, j) : (k, j) \preceq (n, m)\}$.

P5: $I_{k,j}$, $j = 1, \dots, 2^{[\beta k]}$ are the islands and $J_{k,j}$, $j = 1, \dots, 2^{[\beta k]}$ are the gaps in the k -th block.

P6: The number of islands and the number of gaps in the k -th block are both equal to $m_k = 2^{[\beta k]}$. Set $m_{k,n} = m_k + \dots + m_n$.

P7: The islands in k -th block have the same length $|I_{k,j}| = 2^{k-[\beta k]} - (1 + [\beta k] 2^{[[\varepsilon k]]-1}) \leq 2^{k-[\beta k]}$. This implies $|I_{k,j}| \geq c_{\varepsilon,\beta} 2^{k(1-\beta)}$ for some constant $c_{\varepsilon,\beta} \in (0, \frac{1}{2})$.

P8: The length of the finest gap in the k -th block is $|J_{k,j}| = 2^{[[\varepsilon k]]}$. This implies $|J_{k,j}| \geq 2^{[[\varepsilon k]]}$.

P9: The length $|J_{k,1}|$ of the gap at the left end of the k -th block is $2^{[\varepsilon k]+[\beta k]}$.

P10: For each pair $(k, j) \in \mathcal{K}$, we denote $X_{(k,j)} = \sum_{i \in I_{k,j}} X_i$ and $W_{(k,j)} = \sum_{i \in I_{k,j}} W_i$.

P11: $\mathcal{L}_{X_1, \dots, X_d}$ denotes the probability law of the random vector (X_1, \dots, X_d) .

5. AUXILIARY RESULTS

Without loss of generality we assume that on the initial probability space there is a sequence of independent r.v.'s $(Y_{(k,j)})_{(k,j) \in \mathcal{K}}$ such that $Y_{(k,j)} \stackrel{d}{=} X_{(k,j)}$, $(k, j) \in \mathcal{K}$. Let $k_0 \in \mathbb{N}_+$ and $n > k_0$. Suppose that on the same probability space there is an i.i.d. sequence of \mathbb{R}^1 -valued r.v.'s $(V_{(k,j)})_{(k,j) \in \mathcal{K}}$ with means 0 whose characteristic function has as support the interval $[-\varepsilon_0, \varepsilon_0]$ and such that $\mathbb{E} |V_{(k,j)}|^{r_0} < \infty$ for any $r_0 > 0$. We suppose that the sequence $(V_{(k,j)})_{(k,j) \in \mathcal{K}}$ is independent of $(X_{(k,j)})_{(k,j) \in \mathcal{K}}$ and $(Y_{(k,j)})_{(k,j) \in \mathcal{K}}$. Denote $X_{(k)} = (X_{(k,1)}, \dots, X_{(k,m_k)})$, $Y_{(k)} = (Y_{(k,1)}, \dots, Y_{(k,m_k)})$ and $V_{(k)} = (V_{(k,1)}, \dots, V_{(k,m_k)})$. In the sequel π denotes the Prokhorov distance (for details see Section 9.1 of the Appendix).

Assume conditions **C1** and **C2**. The main result of this section is the following proposition, which is of independent interest.

Proposition 5.1. *There exists a constant $c_{\varepsilon,\beta,\lambda_1,\lambda_2}$ such that, for any $k_0 = 1, 2, \dots$ and $n > k_0$,*

$$\pi \left(\mathcal{L}_{X_{(k_0)+V_{(k_0)}}, \dots, X_{(n)+V_{(n)}}, \mathcal{L}_{Y_{(k_0)+V_{(k_0)}}, \dots, Y_{(n)+V_{(n)}} \right) \leq c_{\varepsilon,\beta,\lambda_1,\lambda_2} (1 + \lambda_0 + \mu_\delta) \exp \left(-\frac{\lambda_1}{4} 2^{\frac{\varepsilon}{2} k_0} \right).$$

Proof. Without loss of generality we assume that there exists a sequence of independent random vectors $R_{(k)}$, $k = 1, \dots, n$ such that $R_{(k)} \stackrel{d}{=} X_{(k)} + V_{(k)}$ and such that the sequence $(R_{(k)})_{k=1, \dots, n}$ is independent of the sequences $(X_{(k)} + V_{(k)})_{k=1, \dots, n}$, $(Y_{(k,j)})_{(k,j) \in \mathcal{K}}$ and $(V_{(k,j)})_{(k,j) \in \mathcal{K}}$.

The further proof is slit into two parts a) and b). In the part a) we give a bound for the Prokhorov distance between the vectors $(X_{(k_0)} + V_{(k_0)}, \dots, X_{(n)} + V_{(n)})$ and $(R_{(k_0)}, \dots, R_{(n)})$, while in the part b) we give a bound for the Prokhorov distance between the vectors $(R_{(k_0)}, \dots, R_{(n)})$ and $(Y_{(k_0)} + V_{(k_0)}, \dots, Y_{(n)} + V_{(n)})$. Proposition 5.1 follows from (5.1) and (5.9) by triangle inequality.

Part a). We show that there exists a constant $c_{\varepsilon, \beta, \lambda_1, \lambda_2}$ such that, for any $k_0 = 1, 2, \dots$ and $n > k_0$,

$$(5.1) \quad \pi \left(\mathcal{L}_{X_{(k_0)+V_{(k_0)}, \dots, X_{(n)+V_{(n)}}}, \mathcal{L}_{R_{(k_0)}, \dots, R_{(n)}} \right) \leq c_{\varepsilon, \beta, \lambda_1, \lambda_2} (1 + \lambda_0 + \mu_\delta) \exp \left(-\frac{\lambda_1}{4} 2^{\frac{\varepsilon}{2} k_0} \right).$$

For $k = k_0, \dots, n$, define the vectors $Z_{(k)} = (X_{(k_0)} + V_{(k_0)}, \dots, X_{(k)} + V_{(k)})$ and $\tilde{Z}_{(k)} = (Z_{(k-1)}, R_{(k)})$. By Lemma 9.3, one gets

$$(5.2) \quad \pi \left(\mathcal{L}_{Z_{(n)}}, \mathcal{L}_{R_{(k_0)}, \dots, R_{(n)}} \right) \leq \sum_{k=k_0}^n \pi \left(\mathcal{L}_{Z_{(k)}}, \mathcal{L}_{\tilde{Z}_{(k)}} \right).$$

Let $\phi_{(k)}$ (resp. $\tilde{\phi}_{(k)}$) be the characteristic function of the vector $Z_{(k)}$ (resp. $\tilde{Z}_{(k)}$) and $m_{k_0, k} = m_{k_0} + \dots + m_k$. Then, by Lemma 9.5, for any $T > 0$,

$$(5.3) \quad \begin{aligned} \pi \left(\mathcal{L}_{Z_{(k)}}, \mathcal{L}_{\tilde{Z}_{(k)}} \right) &\leq (T/\pi)^{m_{k_0, k}/2} \left(\int_{t \in \mathbb{R}^{m_{k_0, k}}} \left| \phi_{(k)}(t) - \tilde{\phi}_{(k)}(t) \right|^2 dt \right)^{1/2} \\ &+ \mathbf{P} \left(\max_{k_0 \leq l \leq k} \max_{1 \leq j \leq m_l} |X_{(l, j)}| > T \right). \end{aligned}$$

Denote by $\varphi_{(k)}$ and $\psi_{(k)}$ the characteristic functions of the vectors $X_{(k)}$ and $(X_{(k_0)}, \dots, X_{(k)})$ respectively. Since $V_{(k_0)}, \dots, V_{(k)}$ are independent of $X_{(k_0)}, \dots, X_{(k)}$ and $Y_{(k_0)}, \dots, Y_{(k)}$, we have

$$(5.4) \quad \begin{aligned} &\int_{t \in \mathbb{R}^{m_{k_0, k}}} \left| \phi_{(k)}(t) - \varphi_{(k)}(t) \right|^2 dt \\ &= \int_{t_1 \in \mathbb{R}^{m_{k_0}}} \dots \int_{t_k \in \mathbb{R}^{m_k}} \left| \phi_{(k)}(t_{k_0}, \dots, t_k) - \varphi_{(k)}(t_{k_0}, \dots, t_k) \right|^2 dt_{k_0} \dots dt_k \\ &\leq I_1 \\ &\equiv \int_{t_1 \in \mathbb{R}^{m_{k_0}}} \dots \int_{t_k \in \mathbb{R}^{m_k}} \left| \psi_{(k)}(t_{k_0}, \dots, t_k) - \psi_{(k-1)}(t_{k_0}, \dots, t_{k-1}) \varphi_{(k)}(t_k) \right|^2 dt_{k_0} \dots dt_k. \end{aligned}$$

To bound the right-hand side of (5.4), note that $m_{k_0, k} = (2^{[\beta k_0]} + \dots + 2^{[\beta k]}) \leq 2^{[\beta k]+1}$ and, according to the construction, the length of the gap between the vectors $X_{(k-1)}$ and $X_{(k)}$ is $k_{gap} = 2^{[\varepsilon k]+[\beta k]}$. Note also that $|I_{k, j}| \leq 2^{k-[\beta k]}$ and $|\varepsilon_0| \leq 1$. Let us remind that the characteristic functions of the r.v. $V_{(k, j)}$ have support the interval $[-\varepsilon_0, \varepsilon_0]$ and that the sequence $(V_{(k, j)})_{(k, j) \in \mathcal{K}}$ is independent of $(X_{(k, j)})_{(k, j) \in \mathcal{K}}$; it readily implies that the integrals above are in fact over the set $[-\varepsilon_0, \varepsilon_0]^{m_{k_0, k}}$. Using condition **C1**, with $M_1 = m_{k_0, k-1}$ and $M_2 = m_k$, one may thus write

$$(5.5) \quad \begin{aligned} I_1 &\leq \lambda_0 \left(1 + \max_{l \leq k, j \leq m_k} |I_{l, j}| \right)^{\lambda_2 (M_1 + M_2)} \exp(-\lambda_1 k_{gap}) \varepsilon_0^{m_{k_0, k}} \\ &\leq \lambda_0 (1 + 2^{k-[\beta k]})^{\lambda_2 2^{[\beta k]+1}} \exp(-\lambda_1 k_{gap}) \\ &\leq \lambda_0 \exp(-\lambda_1 2^{[\varepsilon k]+[\beta k]} + \lambda_2 2^{[\beta k]+1} \ln(1 + 2^{k-[\beta k]})) \\ &\leq c_{\varepsilon, \beta, \lambda_1, \lambda_2} \lambda_0 \exp \left(-\frac{\lambda_1}{2} 2^{[\varepsilon k]+[\beta k]} \right). \end{aligned}$$

Putting together (5.3), (5.4) and (5.5), we get

$$(5.6) \quad \begin{aligned} \pi \left(\mathcal{L}_{Z_{(k)}}, \mathcal{L}_{\tilde{Z}_{(k)}} \right) &\leq c_{\varepsilon, \beta, \lambda_1, \lambda_2} \lambda_0 (T/\pi)^{m_{k_0, k/2}} \exp \left(-\frac{\lambda_1}{2} 2^{[\varepsilon k] + [\beta k]} \right) \\ &+ \sum_{k_0 \leq l \leq k} \sum_{1 \leq j \leq m_l} \mathbb{P} \left(|X_{(l, j)}| > T \right). \end{aligned}$$

Since $|I_{(l, j)}| \leq 2^l$, by Markov's inequality and condition **C2**,

$$\mathbb{P} \left(|X_{(l, j)}| > T \right) \leq T^{-1} \mathbb{E} |X_{(l, j)}| \leq T^{-1} 2^l \max_i \mathbb{E} |X_i| \leq \mu_\delta T^{-1} 2^l.$$

Choosing $T = \exp \left(2^{[\varepsilon k]/2} \right)$, one gets

$$(5.7) \quad \begin{aligned} \sum_{k_0 \leq l \leq k} \sum_{1 \leq j \leq m_l} \mathbb{P} \left(|X_{(l, j)}| > T \right) &\leq \mu_\delta T^{-1} \sum_{k_0 \leq l \leq k} m_l 2^l \\ &\leq \mu_\delta \exp \left(-2^{[\varepsilon k]/2} \right) \sum_{k_0 \leq l \leq k} 2^{[\beta l]} 2^l \\ &\leq c_\beta \mu_\delta \exp \left(-2^{[\varepsilon k]/2} / 2 \right). \end{aligned}$$

Since $m_{k_0, k} \leq 2^{\beta k}$, one gets

$$(5.8) \quad (T/\pi)^{m_{k_0, k/2}} \leq \exp \left(\frac{1}{2} 2^{[\varepsilon k]/2 + \beta k} \right).$$

From (5.6), (5.7) and (5.8), we deduce

$$\begin{aligned} \pi \left(\mathcal{L}_{Z_{(k)}}, \mathcal{L}_{\tilde{Z}_{(k)}} \right) &\leq c_{\varepsilon, \beta, \lambda_1, \lambda_2} \lambda_0 \exp \left(\frac{1}{2} 2^{[\varepsilon k]/2 + [\beta k]} \right) \exp \left(-\frac{\lambda_1}{2} 2^{[\varepsilon k] + [\beta k]} \right) \\ &+ c_\beta \mu_\delta \exp \left(-2^{[\varepsilon k]/2} / 2 \right) \\ &\leq (1 + \lambda_0 + \mu_\delta) c_{\varepsilon, \beta, \lambda_1, \lambda_2} \exp \left(-\frac{\lambda_1}{4} 2^{\varepsilon k/2} \right). \end{aligned}$$

Using (5.2)

$$\begin{aligned} \pi \left(\mathcal{L}_{Z_{(n)}}, \mathcal{L}_{(R_{(k_0)}, \dots, R_{(n)})} \right) &\leq (1 + \lambda_0 + \mu_\delta) c_{\varepsilon, \beta, \lambda_1, \lambda_2} \sum_{k=k_0}^n \exp \left(-\frac{\lambda_1}{4} 2^{[\varepsilon k]/2} \right) \\ &\leq (1 + \lambda_0 + \mu_\delta) c'_{\varepsilon, \beta, \lambda_1, \lambda_2} \exp \left(-\frac{\lambda_1}{4} 2^{[\varepsilon k]_0/2} \right). \end{aligned}$$

This concludes the proof of the part a).

Part b). We show that exists a constant $c_{\varepsilon, \beta, \lambda_1, \lambda_2}$ such that, for any $k_0 = 1, 2, \dots$ and $n > k_0$,

$$(5.9) \quad \pi \left(\mathcal{L}_{R_{(k_0)}, \dots, R_{(n)}}, \mathcal{L}_{Y_{(k_0)+V_{(k_0)}}, \dots, Y_{(n)+V_{(n)}}} \right) \leq c_{\varepsilon, \beta, \lambda_1, \lambda_2} (1 + \lambda_0 + \mu_\delta) \exp \left(-\frac{\lambda_1}{8} 2^{[\varepsilon k]_0/2} \right).$$

By Lemma 9.4, since $R_{(k_0)}, \dots, R_{(n)}$ and $Y_{(k_0)+V_{(k_0)}}, \dots, Y_{(n)+V_{(n)}}$ are independent r.v.'s, one may write

$$(5.10) \quad \pi \left(\mathcal{L}_{R_{(k_0)}, \dots, R_{(n)}}, \mathcal{L}_{Y_{(k_0)+V_{(k_0)}}, \dots, Y_{(n)+V_{(n)}}} \right) = \sum_{k=k_0}^n \pi \left(\mathcal{L}_{R_{(k)}}, \mathcal{L}_{Y_{(k)+V_{(k)}}} \right)$$

and it suffices to prove that, for any $k = 1, 2, \dots$, one gets

$$(5.11) \quad \pi \left(\mathcal{L}_{R_{(k)}}, \mathcal{L}_{Y_{(k)} + V_{(k)}} \right) \leq (1 + \lambda_0 + \mu_\delta) c'_{\varepsilon, \beta, \lambda_1, \lambda_2} \exp \left(-\frac{\lambda_1}{8} 2^{[\varepsilon k]/2} \right).$$

For this, recall that, according to the construction in Section 4, at the resolution level 0, a gap of length $2^{[[\varepsilon k] + [\beta k]]/2}$ in the middle of the block $R_{(k)}^{0,0} = R_{(k)}$ splits it into two vectors $\tilde{R}_{(k)}^{0,1}$ and $\tilde{R}_{(k)}^{0,2}$; let $R_{(k)}^{0,1}$ and $R_{(k)}^{0,2}$ be independent versions of $\tilde{R}_{(k)}^{0,1}$ and $\tilde{R}_{(k)}^{0,2}$ respectively. Next, at the resolution level 1, for any $j \in \{1, 2\}$, a gap of length $2^{[(\varepsilon + \beta)k]}/4$ in the middle of the block $R_{(k)}^{0,j}$ splits it into two vectors $\tilde{R}_{(k)}^{1,2j-1}$ and $\tilde{R}_{(k)}^{1,2j}$; let $R_{(k)}^{1,2j-1}$ and $R_{(k)}^{1,2j}$ be independent versions of $\tilde{R}_{(k)}^{1,2j-1}$ and $\tilde{R}_{(k)}^{1,2j}$ respectively. Assuming that at the resolution level $l \in \{1, \dots, [\beta k]\}$ the independent r.v.'s $R_{(k)}^{l,j}$, $j \in \{1, \dots, 2^l\}$, are already constructed, we shall perform the construction at the resolution level $l + 1$. Note that, at the resolution level l for any $j \in \{1, \dots, 2^l\}$, a gap of length $2^{(\varepsilon + \beta)k}/2^{l+1}$ in the middle of the block $R_{(k)}^{l,j}$ splits it into two vectors $\tilde{R}_{(k)}^{l+1,2j-1}$ and $\tilde{R}_{(k)}^{l+1,2j}$; it is enough to set $R_{(k)}^{l+1,2j-1}$ and $R_{(k)}^{l+1,2j}$ to be independent versions of $\tilde{R}_{(k)}^{l+1,2j-1}$ and $\tilde{R}_{(k)}^{l+1,2j}$ respectively. It is easy to see that at the final level $l_k = [\beta k]$ we have $R_{(k)}^{l_k, j} \stackrel{d}{=} Y_{(k,j)} + V_{(k,j)}$, for $j = 1, \dots, m_k = 2^{[\beta k]}$.

Let $l \in \{0, \dots, [\beta k]\}$. For $j \in \{1, \dots, 2^l\}$, denote by $\psi_k^{l,2j-1}$ and $\psi_k^{l,2j}$ the characteristic functions of $R_{(k)}^{l,2j-1}$ and $R_{(k)}^{l,2j}$. Using Lemma 9.5 and the independence of the vectors $\tilde{R}_{(k)}^{l,2j-1}$ and $\tilde{R}_{(k)}^{l,2j}$, we get

$$(5.12) \quad \begin{aligned} & \pi \left(\mathcal{L}_{R_{(k)}^{l,j}}, \mathcal{L}_{R_{(k)}^{l+1,2j-1}, R_{(k)}^{l+1,2j}} \right) \\ & \leq \left((T/\pi)^{2^{-l}m_k} \int_{(t,s) \in \mathbb{R}^{2^{-l}m_k}} \left| \psi_k^{l,j}(t,s) - \psi_k^{l+1,2j-1}(t) \psi_k^{l+1,2j}(s) \right|^2 dt ds \right)^{1/2} \\ & + \sum_{1 \leq j \leq 2^{-l}m_k} \mathbb{P} \left(|X_{(k,j)} + V_{(k,j)}| > T \right). \end{aligned}$$

By Condition **C1** with $N = M = \frac{m_k}{2} 2^{-l}$ and $k_{gap} = 2^{[\varepsilon k] + [\beta k] - l - 1}$, we obtain

$$(5.13) \quad \begin{aligned} & \int_{(t,s) \in \mathbb{R}^{m_k 2^{-l}}} \left| \psi_k^{l,j}(t,s) - \psi_k^{l+1,2j-1}(t) \psi_k^{l+1,2j}(s) \right|^2 dt ds \\ & = \int_{(t,s) \in \mathbb{R}^{m_k 2^{-l}}, \|t\|_\infty \leq \varepsilon_0, \|s\|_\infty \leq \varepsilon_0} \left| \psi_k^{l,j}(t,s) - \psi_k^{l+1,2j-1}(t) \psi_k^{l+1,2j}(s) \right|^2 dt ds \\ & \leq \lambda_0 \exp \left(\lambda_2 m_k 2^{-l} \ln(1 + 2^{k - [\beta k]}) - \lambda_1 2^{[\varepsilon k] + [\beta k] - 1 - l} \right) (2\varepsilon_0)^{m_k 2^{-l}} \\ & \leq \lambda_0 c''_{\varepsilon, \beta, \lambda_1, \lambda_2} \exp \left(-\frac{\lambda_1}{4} 2^{[\varepsilon k] + [\beta k] - l} \right). \end{aligned}$$

We will thus take $T = \exp(\lambda_1 2^{[[\varepsilon k]]/2})$ so that

$$(T/\pi)^{2^{-l}m_k} \leq \exp(\lambda_1 2^{-l} m_k 2^{[[\varepsilon k]]/2}) \leq \exp(\lambda_1 2^{[[\varepsilon k]]/2 + [\beta k] - l}).$$

In order to control the terms $\mathbb{P}(|X_{(k,j)} + V_{(k,j)}| > T)$, we use Markov's inequality, condition **C2** and the fact that $|I_{k,j}| \leq 2^k$; it readily follows that

$$\begin{aligned} \mathbb{P}(|X_{(k,j)} + V_{(k,j)}| > T) &\leq T^{-1} (\mathbb{E}|X_{(k,j)}| + \mathbb{E}|V_{(k,j)}|) \\ &\leq T^{-1} \left(2^k \max_i \mathbb{E}|X_i| + c2^k \right) \\ &\leq (1 + \mu_\delta) c2^k \exp(-\lambda_1 2^{-\lceil \varepsilon k \rceil / 2}). \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{1 \leq j \leq 2^{-l} m_k} \mathbb{P}(|X_{(k,j)} + V_{(k,j)}| > T) &\leq 2^{-l} m_k (1 + \mu_\delta) c2^k \exp(-2^{\lceil \varepsilon k \rceil / 2}) \\ &\leq (1 + \mu_\delta) \exp(-\lambda_1 2^{\lceil \varepsilon k \rceil / 2}) 2^{-l} 2^{2[\beta k] + k} \\ (5.14) \quad &\leq (1 + \mu_\delta) c_{\varepsilon, \beta, \lambda_1, \lambda_2} \exp\left(-\frac{\lambda_1}{2} 2^{\lceil \varepsilon k \rceil / 2}\right). \end{aligned}$$

From (5.12), (5.13) and (5.14), we get

$$\begin{aligned} &\pi \left(\mathcal{L}_{R_{(k)}^{l,j}}, \mathcal{L}_{R_{(k)}^{l+1,2j-1}}, R_{(k)}^{l+1,2j} \right) \\ &\leq (1 + \lambda_0 + \mu_\delta) c_{\varepsilon, \beta, \lambda_1, \lambda_2} \\ &\quad \times \left[\exp(\lambda_1 2^{-l} 2^{\lceil \varepsilon k \rceil / 2 + [\beta k]}) \exp\left(-\frac{\lambda_1}{2} 2^{-l} 2^{\lceil \varepsilon k \rceil + [\beta k]}\right) + c \exp\left(-\frac{\lambda_1}{2} 2^{\lceil \varepsilon k \rceil / 2}\right) \right] \\ (5.15) \quad &\leq (1 + \lambda_0 + \mu_\delta) c_{\varepsilon, \beta, \lambda_1, \lambda_2} \exp\left(-\frac{\lambda_1}{4} 2^{\lceil \varepsilon k \rceil / 2}\right). \end{aligned}$$

Since $R_{(k)}^{l,j}$, $j = 1, \dots, 2^l$ are independent r.v.'s, by triangle inequality, one gets

$$\begin{aligned} \pi \left(\mathcal{L}_{R_{(k)}}, \mathcal{L}_{Y_{(k)} + V_{(k)}} \right) &= \pi \left(\mathcal{L}_{R_{(k)}^{0,0}}, \mathcal{L}_{Y_{(k)} + V_{(k)}} \right) \\ &\leq \pi \left(\mathcal{L}_{R_{(k)}^{0,0}}, \mathcal{L}_{R_{(k)}^{0,1}, R_{(k)}^{0,2}} \right) + \pi \left(\mathcal{L}_{R_{(k)}^{0,1}, R_{(k)}^{0,2}}, \mathcal{L}_{Y_{(k)} + V_{(k)}} \right) \\ &\leq \pi \left(\mathcal{L}_{R_{(k)}^{0,0}}, \mathcal{L}_{R_{(k)}^{0,1}, R_{(k)}^{0,2}} \right) + \pi \left(\mathcal{L}_{R_{(k)}^{0,1}, R_{(k)}^{0,2}}, \mathcal{L}_{R_{(k)}^{1,1}, \dots, R_{(k)}^{1,4}} \right) \\ &\quad + \pi \left(\mathcal{L}_{R_{(k)}^{1,1}, \dots, R_{(k)}^{1,4}}, \mathcal{L}_{Y_{(k)} + V_{(k)}} \right) \\ &\quad \dots \\ (5.16) \quad &\leq \sum_{l=0}^{[\beta k]-1} \pi \left(\mathcal{L}_{R_{(k)}^{l,1}, \dots, R_{(k)}^{l,2^l}}, \mathcal{L}_{R_{(k)}^{l+1,1}, \dots, R_{(k)}^{l+1,2^{l+1}}} \right). \end{aligned}$$

By Lemma 9.4 and (5.15) one gets

$$\begin{aligned} \pi \left(\mathcal{L}_{R_{(k)}^{l,1}, \dots, R_{(k)}^{l,2^l}}, \mathcal{L}_{R_{(k)}^{l+1,1}, \dots, R_{(k)}^{l+1,2^{l+1}}} \right) &\leq \sum_{j=1}^{2^l} \pi \left(\mathcal{L}_{R_{(k)}^{l,j}}, \mathcal{L}_{R_{(k)}^{l+1,2j-1}, R_{(k)}^{l+1,2j}} \right) \\ (5.17) \quad &\leq c_{\varepsilon, \beta, \lambda_1, \lambda_2} 2^l (1 + \lambda_0 + \mu_\delta) \\ (5.18) \quad &\quad \times \exp\left(-\frac{\lambda_1}{4} 2^{\lceil \varepsilon k \rceil / 2}\right). \end{aligned}$$

From (5.16) and (5.18), it follows

$$\begin{aligned} \pi \left(\mathcal{L}_{R_{(k)}}, \mathcal{L}_{Y_{(k)}+V_{(k)}} \right) &\leq \sum_{l=0}^{[\beta k]-1} 2^l (1 + \lambda_0 + \mu_\delta) c_{\varepsilon, \beta, \lambda_1, \lambda_2} \exp \left(-\frac{\lambda_1}{4} 2^{[\varepsilon k]/2} \right) \\ &\leq 2^{[\beta k]} (1 + \lambda_0 + \mu_\delta) c_{\varepsilon, \beta, \lambda_1, \lambda_2} \exp \left(-\frac{\lambda_1}{4} 2^{[\varepsilon k]/2} \right) \\ &\leq (1 + \lambda_0 + \mu_\delta) c'_{\varepsilon, \beta, \lambda_1, \lambda_2} \exp \left(-\frac{\lambda_1}{8} 2^{[\varepsilon k]/2} \right). \end{aligned}$$

Finally, using (5.11) finishes the proof of the part b). \square

6. PROOF OF THEOREM 2.1

This section is devoted to the proof of Theorem 2.1; it is separated in several steps. We first construct the coupling with independent r.v.'s. (Section 6.1) and thus with independent normal r.v.'s. (Section 6.2). In Section 6.3, we explicit the construction of the sequences $(\tilde{X}_i)_{1 \leq i \leq N}$ and $(W_i)_{1 \leq i \leq N}$ and in Sections 6.4, 6.5, 6.6 and 6.7 we put together and optimize the bounds.

6.1. Coupling with independent r.v.'s. Assume conditions **C1** and **C2**. The following proposition shows that the partial sums $\sum_{(l,i) \preceq (k,j)} X_{(l,i)}$ can be coupled with high probability with the partial sums $\sum_{(l,i) \preceq (k,j)} Y_{(l,i)}$.

Proposition 6.1. *Let $\alpha < \delta$, $\beta > \frac{1}{2}$ and $0 < \rho < \frac{1-\beta}{2}$. Then, for any $N \in \mathbb{N}$, on some extension of the initial probability space there is a version $(X'_{(k,j)})_{(k,j) \in \mathcal{K}_N}$ of the sequence $(X_{(k,j)})_{(k,j) \in \mathcal{K}_N}$ and a version $(Y'_{(k,j)})_{(k,j) \in \mathcal{K}_N}$ of the sequence $(Y_{(k,j)})_{(k,j) \in \mathcal{K}_N}$ such that*

$$\begin{aligned} &\mathbb{P} \left((2^n)^{-\frac{1}{2}} \sup_{(k,j) \in \mathcal{K}_N} \left| \sum_{(l,i) \preceq (k,j)} (X'_{(l,i)} - Y'_{(l,i)}) \right| \geq (2^n)^{-\rho} \right) \\ &\leq C_1 (2^n)^{-1-\alpha+(\varepsilon+\rho)(2+2\alpha)}, \end{aligned}$$

where $\varepsilon \in (0, \frac{1}{2})$ is arbitrary chosen and $C_1 = c_{\varepsilon, \beta, \lambda_1, \lambda_2, \alpha, \rho} (1 + \lambda_0 + \mu_\delta)^{2+2\delta}$ for some positive constant $c_{\varepsilon, \beta, \lambda_1, \lambda_2, \alpha, \rho}$.

Proof. For the sake of brevity, it is convenient to set $k_0 = [\varepsilon n]$, $X_{k_0, n} = (X_{(k_0)}, \dots, X_{(n)})$, $Y_{k_0, n} = (Y_{(k_0)}, \dots, Y_{(n)})$ and $V_{k_0, n} = (V_{(k_0)}, \dots, V_{(n)})$; the variables $\tilde{X}_{k_0, n} = X_{k_0, n} + V_{k_0, n}$ and $\tilde{Y}_{k_0, n} = Y_{k_0, n} + V_{k_0, n}$ are the smoothed versions of $X_{k_0, n}$ and $Y_{k_0, n}$. By Proposition 5.1, with $k_0 = [\varepsilon n]$, there exists a constant $c_{\varepsilon, \beta, \lambda_1, \lambda_2}$ such that

$$(6.1) \quad \pi \left(\mathcal{L}_{\tilde{X}_{k_0, n}}, \mathcal{L}_{\tilde{Y}_{k_0, n}} \right) \leq \Delta = (1 + \lambda_0 + \mu_\delta) c_{\varepsilon, \beta, \lambda_1, \lambda_2} \exp \left(-\frac{\lambda_1}{4} 2^{\varepsilon^2 n/2} \right).$$

Using Strassen-Dudley's theorem (see Lemma 9.1), we conclude that on some extension of the initial probability space there are random vectors $\tilde{S}_{k_0,n} = (S_{(k_0)}, \dots, S_{(n)})$ and $\tilde{T}_{k_0,n} = (T_{(k_0)}, \dots, T_{(n)})$ such that $\tilde{S}_{k_0,n} \stackrel{d}{=} \tilde{X}_{k_0,n}$, $\tilde{T}_{k_0,n} \stackrel{d}{=} \tilde{Y}_{k_0,n}$ and

$$(6.2) \quad \mathbb{P} \left(\left\| \tilde{S}_{k_0,n} - \tilde{T}_{k_0,n} \right\|_{\infty} \geq \Delta \right) \leq \Delta.$$

We shall remove the smoothing from the vectors $\tilde{S}_{k_0,n}$ and $\tilde{T}_{k_0,n}$. Without loss of generality we may assume that there is a random vector U with uniform distribution on $[0, 1]^{m_{k_0,n}}$ and independent of $(\tilde{S}_{k_0,n}, \tilde{T}_{k_0,n})$. We thus consider the transition kernels $G_1(x|y) := \mathbb{P}(X_{k_0,n} \leq x | \tilde{X}_{k_0,n} = y)$ and $G_2(x|y) := \mathbb{P}(Y_{k_0,n} \leq x | \tilde{Y}_{k_0,n} = y)$ and set $X'_{k_0,n} := G_1^{-1}(U | \tilde{S}_{k_0,n})$, $V'_{k_0,n} := \tilde{S}_{k_0,n} - X'_{k_0,n}$, $Y'_{k_0,n} := G_2^{-1}(U | \tilde{T}_{k_0,n})$ and $V''_{k_0,n} := \tilde{T}_{k_0,n} - Y'_{k_0,n}$. The two sequences $X'_{k_0,n}$ and $Y'_{k_0,n}$ are such that $\tilde{S}_{k_0,n} = X'_{k_0,n} + V'_{k_0,n}$, $\tilde{T}_{k_0,n} = Y'_{k_0,n} + V''_{k_0,n}$ and $X'_{k_0,n} \stackrel{d}{=} X_{k_0,n}$, $Y'_{k_0,n} \stackrel{d}{=} Y_{k_0,n}$, $V'_{k_0,n} \stackrel{d}{=} V''_{k_0,n} \stackrel{d}{=} V_{k_0,n}$. The coordinates of the vectors $X'_{k_0,n}$ and $Y'_{k_0,n}$ are denoted by $X'_{(k,j)}$ and $Y'_{(k,j)}$, $(k,j) \in \mathcal{K}$. Since $\tilde{S}_{(k,j)} = X'_{(k,j)} + V'_{(k,j)}$ and $\tilde{T}_{(k,j)} = Y'_{(k,j)} + V''_{(k,j)}$, we have, for any $x \geq 1$,

$$R = \mathbb{P} \left(\sup_{k_0 \leq k, (k,j) \in \mathcal{K}_N} \left| \sum_{(l,i) \preceq (k,j)} (X'_{(l,i)} - Y'_{(l,i)}) \right| \geq 2x \right) \leq R_1 + R_2,$$

where

$$R_1 = \mathbb{P} \left(\sup_{k_0 \leq k, (k,j) \in \mathcal{K}_N} \left| \sum_{(l,i) \preceq (k,j)} \tilde{S}_{(l,i)} - \tilde{T}_{(l,i)} \right| \geq x \right),$$

$$R_2 = \mathbb{P} \left(\sup_{k_0 \leq k, (k,j) \in \mathcal{K}_N} \left| \sum_{(l,i) \preceq (k,j)} (V'_{(l,i)} - V''_{(l,i)}) \right| \geq x \right).$$

First, we shall give a control for R_1 . Note that $\text{card } \mathcal{K}_N \leq c2^{\beta n}$. For any sequence of positive numbers $(\alpha_{(k,j)})_{(k,j) \in \mathcal{K}}$ such that $\sum_{(k,j) \in \mathcal{K}} \alpha_{(k,j)} \leq 1$, it holds

$$\begin{aligned} & \left\{ \sup_{k_0 \leq k, (k,j) \in \mathcal{K}_N} \left| \sum_{k_0 \leq l, (l,i) \preceq (k,j)} (\tilde{S}_{(l,i)} - \tilde{T}_{(l,i)}) \right| \geq x \right\} \\ & \subseteq \bigcup_{k_0 \leq k, (k,j) \in \mathcal{K}_N} \left\{ \left| \sum_{k_0 \leq l, (l,i) \preceq (k,j)} (\tilde{S}_{(l,i)} - \tilde{T}_{(l,i)}) \right| \geq x \right\} \\ & \subseteq \bigcup_{k_0 \leq k, (k,j) \in \mathcal{K}_N} \bigcup_{k_0 \leq l, (l,i) \preceq (k,j)} \left\{ \left| \tilde{S}_{(l,i)} - \tilde{T}_{(l,i)} \right| \geq x \alpha_{(l,i)} \right\} \\ & = \bigcup_{k_0 \leq k, (k,j) \in \mathcal{K}_N} \left\{ \left| \tilde{S}_{(k,j)} - \tilde{T}_{(k,j)} \right| \geq x \alpha_{(k,j)} \right\}, \end{aligned}$$

which implies that

$$R_1 \leq \sum_{k_0 \leq k, (k,j) \in \mathcal{K}_N} \mathbb{P} \left(\left| \tilde{S}_{(k,j)} - \tilde{T}_{(k,j)} \right| \geq x \alpha_{(k,j)} \right).$$

Let $p = 2 + 2\alpha < 2 + 2\delta$. By Chebyshev's inequality

$$R_1 \leq x^{-p} \sum_{k_0 \leq k, (k,j) \in \mathcal{K}_N} \alpha_{(k,j)}^{-p} \mathbb{E} \left| \tilde{S}_{(k,j)} - \tilde{T}_{(k,j)} \right|^p.$$

By a truncation argument, with Δ from (6.1) and (6.2),

$$\begin{aligned} R_1 &\leq x^{-p} \Delta^p \sum_{k_0 \leq k, (k,j) \in \mathcal{K}_N} \alpha_{(k,j)}^{-p} \\ &\quad + x^{-p} \sum_{k_0 \leq k, (k,j) \in \mathcal{K}_N} \alpha_{(k,j)}^{-p} \mathbb{E} \left| \tilde{S}_{(k,j)} - \tilde{T}_{(k,j)} \right|^p \mathbf{1} \left(\left| \tilde{S}_{(k,j)} - \tilde{T}_{(k,j)} \right| \geq \Delta \right). \end{aligned}$$

Let $\eta \in (0, \delta - \alpha)$, $p' = p + 2\eta$ and $\gamma = \frac{2\eta}{p+2\eta} \leq \eta$. Applying Hölder's inequality one may write

$$\begin{aligned} &\left\| \left| \tilde{S}_{(k,j)} - \tilde{T}_{(k,j)} \right| \mathbf{1} \left(\left| \tilde{S}_{(k,j)} - \tilde{T}_{(k,j)} \right| \geq \Delta \right) \right\|_{L^p} \\ &\leq \left\| \tilde{S}_{(k,j)} - \tilde{T}_{(k,j)} \right\|_{L^{p'}} \mathbb{P} \left(\left| \tilde{S}_{(k,j)} - \tilde{T}_{(k,j)} \right| > \Delta \right)^{\frac{\gamma}{p}}. \end{aligned}$$

By Condition **C2**, for some constant $c > 0$, we get

$$\left\| \tilde{S}_{(k,j)} - \tilde{T}_{(k,j)} \right\|_{L^{p'}} \leq 2 \|X_{(k,j)}\|_{L^{p'}} + 2 \|V'_{(k,j)}\|_{L^{p'}} \leq c(1 + \mu_\delta) |I_{k,j}|;$$

consequently, using (6.2)

$$\begin{aligned} R_1 &\leq x^{-p} \Delta^p \sum_{k_0 \leq k, (k,j) \in \mathcal{K}_N} \alpha_{(k,j)}^{-p} \\ &\quad + c(1 + \mu_\delta)^p x^{-p} \sum_{k_0 \leq k, (k,j) \in \mathcal{K}_N} \alpha_{(k,j)}^{-p} |I_{k,j}|^p \left(\mathbf{P} \left(\left| \tilde{S}_{(k,j)} - \tilde{T}_{(k,j)} \right| \geq \Delta \right) \right)^\gamma \\ &\leq x^{-p} \Delta^p \sum_{k_0 \leq k, (k,j) \in \mathcal{K}_N} \alpha_{(k,j)}^{-p} \\ &\quad + c(1 + \mu_\delta)^p x^{-p} \Delta^\gamma \sum_{k_0 \leq k, (k,j) \in \mathcal{K}_N} \alpha_{(k,j)}^{-p} 2^{(k - [\beta k])p} \\ &\leq c_{\varepsilon, \beta, \lambda_1, \lambda_2, \eta} (1 + \lambda_0 + \mu_\delta)^{p+\gamma} \exp \left(-\frac{\lambda_1}{4} \gamma 2^{\varepsilon^2 n/2} \right) x^{-p} \sum_{k_0 \leq k \leq n} \sum_{j \leq 2^{[\beta k]}} \alpha_{(k,j)}^{-p} 2^{kp}. \end{aligned}$$

Now, choosing $\alpha_{(k,j)} = 2^{-k} j^{-2}$, we obtain

$$\begin{aligned} \sum_{k_0 \leq k \leq n} \sum_{j \leq 2^{[\beta k]}} \alpha_{(k,j)}^{-p} 2^{kp} &\leq \sum_{k_0 \leq k \leq n} \sum_{j \leq 2^{[\beta k]}} 2^{2kp} j^{2p} \leq 2^{2np} \sum_{k_0 \leq k \leq n} \sum_{j \leq 2^{[\beta k]}} j^{2p} \\ &\leq 2^{2np} \sum_{k_0 \leq k \leq n} 2^{(2p+1)[\beta k]} \leq 2^{2np} 2^{(2p+1)[\beta n]} n \leq 2^{nc_{\alpha, \beta}}, \end{aligned}$$

which implies that

$$R_1 \leq c_{\varepsilon, \beta, \lambda_1, \lambda_2, \eta} (1 + \lambda_0 + \mu_\delta)^{p+\gamma} \exp\left(-\frac{1}{4}\gamma\lambda_1 2^{\varepsilon^2 n/2}\right) 2^{nc_{\alpha, \beta}} x^{-p}.$$

Since $\gamma = \frac{2\eta}{p+2\eta} \leq \eta \leq p\eta$ and $x \geq 1$, we conclude that

$$(6.3) \quad R_1 \leq A' \exp\left(-\frac{1}{4}\gamma\lambda_1 (2^n)^{\varepsilon^2/2}\right)$$

for some $A' = c'_{\varepsilon, \beta, \lambda_1, \lambda_2, \alpha, \eta} (1 + \lambda_0 + \mu_\delta)^{p(1+\eta)}$. Now we give a control for R_2 . Using Doob's inequality, for any $\lambda > 2$,

$$(6.4) \quad \begin{aligned} R_2 &\leq 2\mathbb{P}\left(\sup_{k_0 \leq k, (k,j) \in \mathcal{K}_N} \left| \sum_{k_0 \leq l, (l,i) \preceq (k,j)} V'_{(l,i)} \right| \geq x\right) \\ &\leq 2x^{-\lambda} \mathbb{E}\left(\sum_{k_0 \leq l, (l,i) \in \mathcal{K}_N} |V'_{(l,i)}|^\lambda\right)^{1/\lambda}. \end{aligned}$$

By Rosenthal's inequality

$$(6.5) \quad \begin{aligned} \left(\mathbb{E}\left(\sum_{k_0 \leq l, (l,i) \in \mathcal{K}_N} |V'_{(l,i)}|^\lambda\right)\right)^{1/\lambda} &\leq c_\lambda \left(\sum_{k_0 \leq l, (l,i) \in \mathcal{K}_N} \mathbb{E}\left(|V'_{(l,i)}|^2\right)\right)^{1/2} \\ &\quad + c_\lambda \left(\sum_{k_0 \leq l, (l,i) \in \mathcal{K}_N} \mathbb{E}\left(|V'_{(l,i)}|^\lambda\right)\right)^{1/\lambda} \\ &\leq c'_\lambda (2^{\beta n})^{1/2}. \end{aligned}$$

From (6.3), (6.4) and (6.5) we obtain

$$\begin{aligned} &\mathbb{P}\left(\sup_{k_0 \leq k, (k,j) \in \mathcal{K}_N} \left| \sum_{(l,i) \preceq (k,j)} (X'_{(l,i)} - Y'_{(l,i)}) \right| \geq 2x\right) \\ &\leq A' \exp\left(-\frac{1}{4}\gamma\lambda_1 (2^n)^{\varepsilon^2/2}\right) + c_\lambda (2^{\beta n})^{\lambda/2} x^{-\lambda}. \end{aligned}$$

Choosing $x = \frac{1}{2} (2^n)^{\frac{1}{2}-\rho}$, we find

$$(6.6) \quad \begin{aligned} &\mathbb{P}\left((2^n)^{-\frac{1}{2}} \sup_{k_0 \leq k, (k,j) \in \mathcal{K}_N} \left| \sum_{(l,i) \preceq (k,j)} (X'_{(l,i)} - Y'_{(l,i)}) \right| \geq (2^n)^{-\rho}\right) \\ &\leq A' \exp\left(-\frac{1}{4}\gamma\lambda_1 (2^n)^{\varepsilon^2/2}\right) + c_\lambda (2^n)^{-\frac{1}{2}\lambda(1-\beta-2\rho)}. \end{aligned}$$

So far we performed the construction for $k \geq k_0$. It remains to construct the sequences $X'_{(k,j)}$ and $Y'_{(k,j)}$ for $(k,j) \preceq (k_0 - 1, m_{k_0-1})$. This construction can be performed by any method such that the sequences $(X'_{(k,j)})$ and $(Y'_{(k,j)})$, where $(k,j) \preceq (k_0 - 1, m_{k_0-1})$, are

independent and $Y'_{(k,j)} \stackrel{d}{=} X_{(k,j)}$ for the same (k, j) . Indeed, let $F_{X|Y_1, \dots, Y_k}(x|y_1, \dots, y_k)$ be the conditional distribution of X given $[Y_1 = y_1, \dots, Y_k = y_k]$ and let $(U_{(k,j)})$ be a sequence of independent r.v.'s uniformly distributed on $(0, 1)$. Denote for brevity the constructed part by $\mathbf{X}'_{k_0} = (X'_{(k,j)})_{k_0 \leq k, (k,j) \in \mathcal{K}_N}$. Define $X'_{(k_0-1,1)}$ as the conditional quantile transform

$$X'_{(k_0-1,1)} = F_{X_{(k_0-1,1)}|\mathbf{X}'_{k_0}}^{-1}(U_{(k_0-1,1)}|\mathbf{X}'_{k_0}),$$

where $\mathbf{X}'_{k_0} = (X_{(k,j)})_{k_0 \leq k, (k,j) \in \mathcal{K}_N}$. We continue by setting

$$X'_{(k_0-1,j)} = F_{X_{(k_0-1,j)}|\mathbf{X}'_{k_0}}^{-1}(U_{(k_0-1,j-1)}|X'_{(k_0-1,1)}, \dots, X'_{(k_0-1,j-1)}|\mathbf{X}'_{k_0}),$$

for $j = 2, \dots, m_{k_0-1}$. In the same way we extend the construction to all $X'_{(k,j)}$ with $1 \leq k < k_0 - 1$. The construction of the sequence $(Y'_{(k,j)})$, for $(k, j) \preceq (k_0 - 1, m_{k_0-1})$ is similar.

Since the sequence $(X_k)_{k \geq 1}$ satisfies Condition **C1**, so does the sequence $(X'_{(k,j)})$. Using the maximal inequality stated in Proposition 8.1 and noting that the cardinality of the set $\{(k, j) : (k, j) \preceq (k_0 - 1, m_{k_0-1})\}$ is less or equal to $2^{\beta k_0} \leq 2^{\varepsilon n}$, we obtain, for any $\eta' \in (0, \frac{\delta - \alpha}{(2 + \alpha + \delta)^2})$,

$$\mathbb{E} \left(\sup_{(k,j) \preceq (k_0-1, m_{k_0-1})} \left| \sum_{(l,i) \preceq (k,j)} X'_{(l,i)} \right|^p \right) \leq A'' (2^{\varepsilon n})^{\frac{1}{2}p},$$

for some constant $A'' = c''_{\varepsilon, \beta, \lambda_1, \lambda_2, \delta, \alpha, \eta} (1 + \lambda_0 + \mu_\delta)^{p(1+\eta')}$. By Chebyshev's inequality, for any $x > 0$ we get

$$\begin{aligned} \mathbb{P} \left(\sup_{(k,j) \preceq (k_0-1, m_{k_0-1})} \left| \sum_{(l,i) \preceq (k,j)} X'_{(l,i)} \right| \geq x \right) &\leq x^{-p} \mathbb{E} \sup_{(k,j) \preceq (k_0-1, m_{k_0-1})} \left| \sum_{(l,i) \preceq (k,j)} X'_{(l,i)} \right|^p \\ &\leq A'' x^{-p} (2^{\varepsilon n})^{\frac{1}{2}p}. \end{aligned}$$

Substituting $x = (2^n)^{\frac{1}{2}-\rho}$ yields

$$\mathbb{P} \left((2^n)^{-\frac{1}{2}} \sup_{(k,j) \preceq (k_0-1, m_{k_0-1})} \left| \sum_{(l,i) \preceq (k,j)} X'_{(l,i)} \right| \geq (2^n)^{-\rho} \right) \leq A'' (2^n)^{-\frac{p}{2} + p(\rho + \frac{1}{2}\varepsilon)}.$$

A similar inequality can be proved with $Y'_{(l,i)}$ instead of $X'_{(l,i)}$. Combining this with (6.6) we obtain,

$$\begin{aligned} &\mathbb{P} \left((2^n)^{-\frac{1}{2}} \sup_{(k,j) \in \mathcal{K}_N} \left| \sum_{(l,i) \preceq (k,j)} (X'_{(l,i)} - Y'_{(l,i)}) \right| \geq 2(2^n)^{-\rho} \right) \\ (6.7) \quad &\leq A''' \left(\exp \left(-\frac{1}{4} \gamma \lambda_1 (2^n)^{\varepsilon^2/2} \right) + (2^n)^{-\frac{1}{2}\lambda(1-\beta-2\rho)} + (2^n)^{-\frac{p}{2} + p(\rho + \frac{1}{2}\varepsilon)} \right), \end{aligned}$$

for some $A''' = c'''_{\varepsilon, \beta, \lambda_1, \lambda_2, \delta, \delta', \eta, \lambda} (1 + \lambda_0 + \mu_\delta)^{p(1+\eta'+\eta)}$. Recall that $p = 2 + 2\alpha$, $\alpha < \delta$, $\beta > \frac{1}{2}$ and $\rho < \frac{1-\beta}{2}$. Taking $\lambda = \frac{2+2\alpha}{1-\beta-2\rho} > p$, the right-hand side of (6.7) does not exceed

$A''' (2^n)^{-1-\alpha+(2+2\alpha)(\rho+\varepsilon)}$. It remains to choose a sufficiently small $\eta+\eta'$ such that $p(1+\eta+\eta') \leq 2+2\delta$, which implies

$$A''' \leq c_{\varepsilon,\beta,\lambda_1,\lambda_2,\alpha,\rho}''' (1+\lambda_0+\mu_\delta)^{2+2\delta}.$$

The assertion of the proposition follows. \square

6.2. Coupling with independent normal r.v.'s. Assume conditions **C1**, **C2** and **C3**. Without loss of generality we can consider that $\mu_i = 0$, $i \geq 1$ and $\sigma = 1$. The following proposition shows that with high probability the partial sums $\sum_{(l,i) \preceq (k,j)} X_{(l,i)}$ can be coupled with the partial sums of some normal r.v.'s. Note the presence of two terms in the upper bound below. One of them called *dependence error* comes from replacing dependent blocks by independent ones; the second one called *Sakhanenko's error* is due to the use of Sakhanenko's strong approximation result for the independent blocks.

Proposition 6.2. *Let $\alpha < \delta$, $\beta > \frac{1}{2}$ and $0 < \rho < \frac{1-\beta}{2}$. Then for any $N \in \mathbb{N}$, on some extension of the initial probability space there exists a sequence of independent standard normal r.v.'s $(W'_{(k,j)})_{(k,j) \in \mathcal{K}_N}$ and a version $(X'_{(k,j)})_{(k,j) \in \mathcal{K}_N}$ of the sequence $(X_{(k,j)})_{(k,j) \in \mathcal{K}_N}$ such that*

$$(6.8) \quad \begin{aligned} & \mathbb{P} \left((2^n)^{-\frac{1}{2}} \sup_{(k,j) \in \mathcal{K}_N} \left| \sum_{(l,i) \preceq (k,j)} (X'_{(l,i)} - \sigma_{l,i} W'_{(l,i)}) \right| \geq 2(2^n)^{-\rho} \right) \\ & \leq C_2 (2^n)^{-1-\alpha+(\varepsilon+\rho)(2+2\alpha)} \quad (\text{dependence error}) \\ & \quad + C_2 (2^n)^{-\beta\alpha+\rho(2+2\alpha)} \quad (\text{Sakhanenko's error}) \end{aligned}$$

where $\sigma_{l,i}^2 = \text{Var}(X_{(l,i)})$ and $C_2 = c_{\varepsilon,\beta,\lambda_1,\lambda_2,\alpha,\rho} (1+\lambda_0+\mu_\delta)^{2+2\delta}$.

Proof. Let $p = 2 + 2\alpha$. Since $|I_{k,j}| \leq 2^{k-[\beta k]}$, we obtain, using Proposition 8.2

$$\mathbb{E} |X_{(k,j)}|^p \leq A |I_{k,j}|^{\frac{p}{2}} \leq A (2^{k-[\beta k]})^{\frac{p}{2}},$$

where $A = c_{\lambda_1,\lambda_2,\delta',\eta} (1+\lambda_0+\mu_\delta)^{p(1+\eta)}$ and $\eta > 0$ is arbitrary. Taking into account that $m_k = 2^{[\beta k]} \leq 2^{\beta k}$, we have

$$(6.9) \quad \begin{aligned} \sum_{(k,j) \in \mathcal{K}_N} \mathbb{E} |X_{(k,j)}|^p &= \sum_{k=1}^n \sum_{j=1}^{m_k} \mathbb{E} |X_{(k,j)}|^p \\ &\leq \sum_{k=1}^n m_k A (2^{k-[\beta k]})^{\frac{p}{2}} \\ &\leq A 2^{\frac{p}{2}} \sum_{k=1}^n 2^{k(\beta+\frac{p}{2}(1-\beta))} \\ &\leq c_{\alpha,\beta} A (2^n)^{\beta+\frac{p}{2}(1-\beta)}. \end{aligned}$$

By (9.2) in Section 9.2, on some probability space $(\Omega'', \mathcal{F}'', \mathbb{P}'')$, there exist a version $(Y''_{(k,j)})_{(k,j) \in \mathcal{K}_N}$ of the sequence $(Y_{(k,j)})_{(k,j) \in \mathcal{K}_N}$ and independent standard normal r.v.'s $(W'_{(k,j)})_{(k,j) \in \mathcal{K}_N}$ such

that

$$\mathbb{P}'' \left(\sup_{(k,j) \in \mathcal{K}_N^0} \left| \sum_{(l,i) \preceq (k,j)} (Y''_{(l,i)} - \sigma_{l,i} W'_{(l,i)}) \right| \geq a \right) \leq \frac{c_p}{a^p} \sum_{(k,j) \in \mathcal{K}_N^0} \mathbb{E} |X_{(k,j)}|^p.$$

Choosing $a = (2^{\beta n})^{\frac{1}{2}-\rho}$ and taking into account (6.9) we obtain

$$\begin{aligned} & \mathbb{P}'' \left((2^n)^{-\frac{1}{2}} \sup_{(k,j) \in \mathcal{K}_N^0} \left| \sum_{(l,i) \preceq (k,j)} (Y''_{(l,i)} - \sigma_{l,i} W'_{(l,i)}) \right| \geq (2^n)^{-\rho} \right) \\ & \leq c_p (2^n)^{-\frac{p}{2} + \rho p} c_{\alpha,\beta} A (2^n)^{\beta + \frac{p}{2}(1-\beta)} \\ & \leq c'_{\alpha,\beta} A (2^n)^{-\beta\alpha + \rho(2+2\alpha)}. \end{aligned}$$

By Berkes-Philip's lemma (Lemma 2.1 of [1]) we can reconstruct on some new probability space the sequences $(X'_{(k,j)})_{(k,j) \in \mathcal{K}_N}$, $(Y'_{(k,j)})_{(k,j) \in \mathcal{K}_N}$, $(Y''_{(k,j)})_{(k,j) \in \mathcal{K}_N}$ and $(W'_{(k,j)})_{(k,j) \in \mathcal{K}_N}$ in such a way that $Y'_{(k,j)} = Y''_{(k,j)}$ a.s. for any $(k,j) \in \mathcal{K}_N$. Without loss of generality we shall consider this new probability space as an extension of the initial probability space. Using Proposition 6.1 we obtain

$$\begin{aligned} & \mathbb{P} \left(2^{-\frac{n}{2}} \sup_{(k,j) \in \mathcal{K}_N} \left| \sum_{(l,i) \preceq (k,j)} (X'_{(l,i)} - \sigma_{l,i} W'_{(l,i)}) \right| \geq 2 (2^n)^{-\rho} \right) \\ & \leq C_1 (2^n)^{-1-\alpha+(\varepsilon+\rho)(2+2\alpha)} \quad (\text{dependence error}) \\ & \quad + c'_{\delta,\beta} A (2^n)^{-\beta\alpha + \rho(2+2\alpha)} \quad (\text{Sakhanenko's error}) \end{aligned}$$

with C_1 defined by Proposition 6.1. Taking into account that $p = 2+2\alpha$, $\alpha < \delta$ and choosing η sufficiently small we get $p(1+\eta) \leq 2+2\delta$, which implies $c'_{\alpha,\beta} A \leq c'_{\lambda_1, \lambda_2, \alpha, \eta} (1 + \lambda_0 + \mu_\delta)^{2+2\delta}$. This concludes the proof. \square

6.3. Construction of the sequences $(\tilde{X}_i)_{1 \leq i \leq N}$ and $(W_i)_{1 \leq i \leq N}$. As before, we suppose that $\mu_i = 0$, $i \geq 1$ and $\sigma^2 = 1$. Let $(X'_{(k,j)})_{(k,j) \in \mathcal{K}_N}$ and $(W'_{(k,j)})_{(k,j) \in \mathcal{K}_N}$ be the sequences which appear in Proposition 6.2.

First we shall construct the sequence $(W_i)_{1 \leq i \leq N}$. Note that, by condition **C3**, the variances $\sigma_{k,i}^2 = \text{Var}(X_{(k,j)})$ can be approximated by $\sigma^2 |I_{k,j}| = |I_{k,j}|$, but in general do not coincide with $|I_{k,j}|$. Therefore to perform our construction we have to replace each of the non identically distributed normal random variables $\sigma_{k,j}^2 W'_{(k,j)}$ by some sums of independent identically distributed standard normal random variables. Let $(W_i)_{1 \leq i \leq N}$ be a sequence of independent standard normal r.v.'s; let $\xi_{k,j}$ be an extra standard normal random variable. Set $I_{k,j} := \{i_1, \dots, i_{|I_{k,j}|}\}$ with $i_1 \preceq \dots \preceq i_{|I_{k,j}|}$ and let $i_{k,j}^*$ be the maximal index $j \in \{i_1, \dots, i_{|I_{k,j}|}\}$ for which the variance of the partial sum $\sum_{i=i_1}^j W_i$ does not exceed $\sigma_{k,j}^2$, i.e. $i_{k,j}^* = i_{m_{k,j}^*}$, where $m_{k,j}^* = \min \{|I_{k,j}|, \lceil \sigma_{k,j}^2 \rceil\}$.

It is easy to check that $W''_{(k,j)} := \sum_{i=i_1}^{i_{k,j}^*} W_i + \xi_{k,j} f_{k,j}$ where $f_{k,j}^2 = |\sigma_{k,j}^2 - i_{k,j}^*|$ is a normal random variable with mean 0 and variance $\sigma_{k,j}^2$; furthermore, we may consider $W''_{(k,j)}$

as a new version of $\sigma_{k,j}W'_{(k,j)}$. The random variable $\sum_{k \in I_{k,j}} W_k - W''_{(k,j)}$, which is equal to $\sum_{i_{k,j}^*+1 \leq k \leq |I_{k,j}|} W_k - \xi_{k,j}f_{k,j}$, has also a normal random variable of mean 0 and variance $(|I_{k,j}| - i_{k,j}^*) + f_{k,j}^2 = |\sigma_{k,j}^2 - |I_{k,j}||$. By Berkes-Philip's lemma (Lemma 2.1 of [1]), without loss of generality, we can reconstruct the sequences $(X'_{(k,j)})_{(k,j) \in \mathcal{K}_N}$, $(\sigma_{k,j}W'_{(k,j)})_{(k,j) \in \mathcal{K}_N}$, $(W''_{(k,j)})_{(k,j) \in \mathcal{K}_N}$ and $(W''_{(k,j)})_{(k,j) \in \mathcal{K}_N}$ on the same probability space in such way that a.s. $(\sigma_{k,j}W'_{(k,j)})_{(k,j) \in \mathcal{K}_N} = (W''_{(k,j)})_{(k,j) \in \mathcal{K}_N}$. We shall consider that this probability space is an extension of the initial probability space. Thus we have constructed the W_i 's when i belongs to the union of all islands $I = \cup_{(k,j) \in \mathcal{K}_N} I_{k,j}$ with the property that the $\eta_{k,j} = \sum_{i \in I_{k,j}} W_i - \sigma_{k,j}W'_{(k,j)}$ are independent normal and centered random variables with variances $v_{k,j}^2 = |\sigma_{k,j}^2 - |I_{k,j}|| \leq \tau |I_{k,j}|^\gamma$, for any $\gamma > 0$. Therefore the sum $\sum_{(l,i) \leq (n,m)} \eta_{l,i}$ is normal with mean 0 and variance $\sum_{(l,i) \leq (n,m)} v_{k,i}^2 \leq c_\beta \tau 2^{(\beta+\gamma)n}$; by Doob's inequality, with $p = 2 + 2\alpha$, it follows

$$\begin{aligned} \mathbb{P} \left((2^n)^{-\frac{1}{2}} \sup_{(k,j) \in \mathcal{K}_N} \left| \sum_{(l,i) \leq (k,j)} \eta_{l,i} \right| \geq (2^n)^{-\rho} \right) &\leq (2^n)^{-p/2+\rho p} \mathbb{E} \left(\left| \sum_{(l,i) \leq (n,m)} \eta_{l,i} \right|^p \right) \\ &\leq c_{\alpha,\beta} (2^n)^{-p/2+\rho p} (\tau 2^{(\beta+\gamma)n})^{p/2} \\ &= c_{\alpha,\beta} \tau^{1+\alpha} (2^n)^{-(1-\beta)(1+\alpha)+(\rho+\gamma/2)(2+2\alpha)} \end{aligned}$$

(6.10)

where $\gamma > 0$ is arbitrary. When i belongs to the union of gaps $J = \cup_{(k,j) \in \mathcal{K}_N} J_{k,j}$ the variables W_i can be taken as any independent standard normal random variables independent of the sequence $(W_k)_{k \in I}$.

So far we have constructed the variables $(X'_{(k,j)})_{(k,j) \in \mathcal{K}_N}$ corresponding to sums over the islands. Now we proceed to construct the components of the sequence $(\tilde{X}_i)_{1 \leq i \leq N}$. First, we proceed with the components belonging to all islands. For each $(k,j) \in \mathcal{K}_N$, we construct a sequence $(\tilde{X}_i)_{i \in I_{k,j}}$ such that $\sum_{i \in I_{k,j}} \tilde{X}_i = X'_{(k,j)}$ and $(\tilde{X}_i)_{i=1,\dots,N} \stackrel{d}{=} (X_i)_{i=1,\dots,N}$. Denote by $F_{X|Y_1,\dots,Y_k}(x|y_1,\dots,y_k)$ the conditional distribution of X given $[Y_1 = y_1, \dots, Y_k = y_k]$. Without loss of generality, on the initial probability space, there exists a sequence U_1, \dots, U_N of independent r.v.'s uniformly distributed on $(0, 1)$. Let $i_1, \dots, i_{|I_{k,j}|}$ be the indices in the set $I_{k,j}$. The required construction is performed in the standard way by defining first \tilde{X}_{i_1} as the conditional quantile transform

$$F_{X_{i_1}|X_{(k,j)}}^{-1}(U_{k,i_1}|X'_{(k,j)})$$

and then by setting, for $l = 2, \dots, |I_{k,j}|$,

$$\tilde{X}_{i_l} = F_{X_{i_l}|X_{i_1},\dots,X_{i_{l-1}},X_{(k,j)}}^{-1}(U_{i_l}|\tilde{X}_{i_1}, \dots, \tilde{X}_{i_{l-1}}, X'_{(k,j)}).$$

Thus we have constructed the vector $\tilde{X}_I = (\tilde{X}_i)_{i \in I}$, where $I = \cup_{(k,j) \in \mathcal{K}_N} I_{k,j}$ is the union of all islands, such that $\tilde{X}_I \stackrel{d}{=} X_I = (X_i)_{i \in I}$. In the same way we construct the \tilde{X}_i when i belongs to the union of gaps $J = \cup_{(k,j) \in \mathcal{K}_N} J_{k,j} := \{j_1, \dots, j_{|J|}\}$: set $X_{j_1} = F_{X_{j_1}|X_I}^{-1}(U_{k,j_1}|\tilde{X}_I)$ and subsequently $\tilde{X}_{j_l} = F_{X_{j_l}|X_{j_1}, \dots, X_{j_{l-1}}, X_I}^{-1}(U_{j_l}|\tilde{X}_{j_1}, \dots, \tilde{X}_{j_{l-1}}, \tilde{X}_I)$, for $l = 2, \dots, |J|$.

6.4. Putting together the bounds. Denote by $r_{k,j}$ the right end of the island $I_{k,j}$ and let $\mathcal{L}_N = \{1\} \cup \{r_{k,j} : (k,j) \in \mathcal{K}_N\}$ be the set of $r_{k,j}$, ordered with respect to lexicographical order \preceq . For any $r = r_{k,j} \in \mathcal{L}_N$ denote by r^{next} be the next element in the set \mathcal{L}_N , i.e. $r^{next} = \inf \{r' : r' \in \mathcal{L}_N, r \preceq r'\}$.

Let $(\tilde{X}_i)_{1 \leq i \leq N}$ and $(W_i)_{1 \leq i \leq N}$ be the sequences constructed in Section 6.3. Recall that by construction, for any $r = r_{k,j} \in \mathcal{L}_N$, we have $\{1, \dots, r\} = \sum_{(l,i) \preceq (k,j)} J_{l,j} \cup I_{k,j}$. First we replace in the statement of Theorem 2.1 the sup over the set $\{j : 1 \leq j \leq N\}$ by the sup over the grid \mathcal{L}_N and the sup of the oscillation term: in other words, the random variable

$\sup_{1 \leq j \leq N} \left| \sum_{i \leq j} (\tilde{X}_i - W_i) \right|$ is bounded by

$$(6.11) \quad \underbrace{\sup_{r \in \mathcal{L}_N} \left| \sum_{i \leq r} (\tilde{X}_i - W_i) \right|}_{(\text{sup over the grid } \mathcal{L}_N)} + \underbrace{\sup_{r \in \mathcal{L}_N} \sup_{r \leq r' \leq \min\{r^{next}-1, N\}} \left| \sum_{r \leq i \leq r'} (\tilde{X}_i - W_i) \right|}_{(\text{oscillation term})}.$$

For any $r = r_{k,j}$, we have

$$(6.12) \quad \sum_{1 \leq i \leq r} \tilde{X}_i = \sum_{(l,h) \preceq (k,j)} \left(\sum_{i \in I_{l,h}} \tilde{X}_i + \sum_{i \in J_{l,h}} \tilde{X}_i \right) = \sum_{(l,h) \preceq (k,j)} (\underline{X}_{(l,h)} + \bar{X}_{(l,h)}),$$

where $\underline{X}_{(l,h)} = \sum_{i \in I_{l,h}} \tilde{X}_i$ and $\bar{X}_{(l,h)} = \sum_{i \in J_{l,h}} \tilde{X}_i$. In the same way

$$(6.13) \quad \sum_{1 \leq i \leq r} W_i = \sum_{(l,h) \preceq (k,j)} \left(\sum_{i \in I_{l,h}} W_i + \sum_{i \in J_{l,h}} W_i \right) = \sum_{(l,h) \preceq (k,j)} (\underline{W}_{(l,h)} + \bar{W}_{(l,h)})$$

where $\underline{W}_{(l,h)} = \sum_{i \in I_{l,h}} W_i$ and $\overline{W}_{(l,h)} = \sum_{i \in J_{l,h}} W_i$. From (6.11), (6.12) and (6.13) we obtain

$$\begin{aligned}
(6.14) \quad & \sup_{1 \leq j \leq N} \left| \sum_{i \leq j} (\tilde{X}_i - W_i) \right| \\
& \leq \sup_{(k,j) \in \mathcal{K}_N} \left| \sum_{(l,h) \leq (k,j)} \underline{X}_{(l,h)} - \underline{W}_{(l,h)} \right| \quad (\text{sup over islands}) \\
& \quad + \sup_{(k,j) \in \mathcal{K}_N} \left| \sum_{(l,h) \leq (k,j)} (\overline{X}_{(l,h)} - \overline{W}_{(l,h)}) \right| \quad (\text{sup over gaps}) \\
& \quad + \sup_{r \in \mathcal{L}_N} \sup_{r' \leq \min\{r^{next}-1, N\}} \left| \sum_{l \leq i \leq r'} (\tilde{X}_i - W_i) \right| \quad (\text{oscillation term})
\end{aligned}$$

where the term "sup over islands" is bounded by the sum

$$\begin{aligned}
(6.15) \quad & \sup_{(k,j) \in \mathcal{K}_N} \left| \sum_{(l,h) \leq (k,j)} \underline{X}_{(l,h)} - \sigma_{l,h} W'_{(l,h)} \right| \quad (\text{normal approximation}) \\
& \quad + \sup_{(k,j) \in \mathcal{K}_N} \left| \sum_{(l,h) \leq (k,j)} (\sigma_{l,h} W'_{(l,h)} - \underline{W}_{(l,h)}) \right|. \quad (\text{variance homogenization})
\end{aligned}$$

The control of the term "normal approximation" has been yet given in Proposition 6.2 where it is bounded by two terms "dependence error" and "Sakhanenko's error". The "variance homogenization" term is controlled by (6.10). As to terms "sup over gaps" and "oscillation term" they will be considered in (6.17) and (6.18) below.

6.5. Bound for the partial sums over gaps. Let $p = 2 + 2\alpha$, where $\alpha < \delta$. Since the blocks are indexed by $l = k_0, \dots, n$ and the total length of the gaps in the block l is less than $(2 + [\beta l]) 2^{[\beta l] + [\varepsilon l] - 1}$ the total length L^{gap} of all gaps satisfies

$$L^{gap} = \sum_{(l,i) \leq (n,m)} \sum_{h \in J_{l,i}} = \sum_{k_0 \leq l \leq n} (2 + [\beta l]) 2^{[\beta l] + [\varepsilon l] - 1} \leq c_{\varepsilon, \beta} 2^{(\beta + \varepsilon)k}.$$

By Proposition 8.1, we have, for any $\eta > 0$,

$$\begin{aligned}
\left\| \sup_{(k,j) \leq (n,m)} \left\| \sum_{(l,i) \leq (k,j)} \overline{X}_{(l,i)} \right\| \right\|_{L^p} & \leq c_{\lambda_1, \lambda_2, \alpha, \delta, \eta} (1 + \lambda_0 + \mu_\delta)^{1+\eta} \left(\sum_{(l,h) \leq (n,m)} \sum_{h \in J_{l,i}} \right)^{\frac{1}{2}} \\
& \leq c_{\varepsilon, \beta, \lambda_1, \lambda_2, \alpha, \delta, \eta} (1 + \lambda_0 + \mu_\delta)^{1+\eta} (2^{(\beta + \varepsilon)n})^{\frac{1}{2}}.
\end{aligned}$$

Using Chebyshev's inequality with $x = (2^n)^{\frac{1}{2}-\rho}$, we get

$$\begin{aligned}
\mathbb{P} \left(\sup_{(k,j) \in \mathcal{K}_N} \left| \sum_{(l,i) \leq (k,j)} \bar{X}_{(l,i)} \right| \geq x \right) &\leq \frac{1}{x^p} \mathbb{E} \left(\sup_{(k,j) \leq (n,m)} \left| \sum_{(l,i) \leq (k,j)} \bar{X}_{(l,i)} \right|^p \right) \\
&\leq A (2^n)^{-p/2+\rho p} (2^{(\beta+\varepsilon)n})^{p/2} \\
(6.16) \qquad \qquad \qquad &\leq A (2^n)^{-(1-\beta-\varepsilon)p/2+\rho p},
\end{aligned}$$

where $A = c_{\varepsilon,\beta,\lambda_1,\lambda_2,\alpha,\delta,\eta} (1 + \lambda_0 + \mu_\delta)^{p(1+\eta)}$. A similar bound can be established with $\bar{W}_{(l,i)}$ instead of $\bar{X}_{(l,i)}$. Using this bound and (6.16) it follows that, for any $0 < \rho < \frac{1}{4}$,

$$\begin{aligned}
&\mathbb{P} \left((2^n)^{-\frac{1}{2}} \sup_{(k,j) \in \mathcal{K}_N} \left| \sum_{(l,i) \leq (k,j)} (\bar{X}_{(l,i)} - \bar{W}_{(l,i)}) \right| \geq (2^n)^{-\rho} \right) \\
&\leq A' (2^n)^{-(1-\beta-\varepsilon)p/2+\rho p} \\
(6.17) \qquad \qquad \qquad &\leq A' (2^n)^{-(1-\beta)(1+\alpha)+(\rho+\varepsilon/2)(2+2\alpha)}
\end{aligned}$$

where $A' = c'_{\varepsilon,\beta,\lambda_1,\lambda_2,\alpha,\delta} (1 + \lambda_0 + \mu_\delta)^{2+2\delta}$.

6.6. Bound for the oscillation term. Denote for brevity $r^+ = \min \{r^{next} - 1, N\}$. First note that

$$r^+ - r \leq \max_{(k,j) \in \mathcal{K}_N} (|I_{k,j}| + |J_{k,j}|) \leq c_{\varepsilon,\beta} (2^{(\beta+\varepsilon)n} + 2^{(1-\beta)n}).$$

Let $p = 2 + 2\alpha$ where $\alpha < \delta$. By Proposition 8.1 of Section 8, for any $\eta > 0$, we have

$$\left\| \sup_{r \in \mathcal{L}_N} \sup_{r \leq l \leq r^+} \left\| \sum_{r \leq i \leq l} \tilde{X}_i \right\| \right\|_{L^p}^p \leq A \left(\sup_{r \in \mathcal{L}_N} (r^+ - r) \right)^{p/2} \leq c_{\varepsilon,\beta} A (2^{(\beta+\varepsilon)n} + 2^{(1-\beta)n})^{p/2},$$

where $A = c_{\lambda_1,\lambda_2,\alpha,\delta,\eta} (1 + \lambda_0 + \mu_\delta)^{p(1+\eta)}$. Therefore, by Chebyshev's inequality, with $x = \frac{1}{2} (2^n)^{\frac{1}{2}-\rho}$ and $\rho > 0$,

$$\begin{aligned}
\mathbb{P} \left(\sup_{r \in \mathcal{L}_N} \sup_{r \leq l \leq r^+} \left| \sum_{r \leq i \leq l} \tilde{X}_i \right| \geq x \right) &\leq x^{-p} \mathbb{E} \left(\sup_{r \in \mathcal{L}_N} \sup_{r \leq l \leq r^+} \left| \sum_{r \leq i \leq l} \tilde{X}_i \right|^p \right) \\
&\leq c_{\varepsilon,\beta} A x^{-p} (2^{(\beta+\varepsilon)n} + 2^{(1-\beta)n})^{p/2} \\
&\leq c'_{\varepsilon,\beta} A 2^{p+n\rho p} \left((2^n)^{-(1+\alpha)(1-\beta-\varepsilon)} + (2^n)^{-(1+\alpha)\beta} \right).
\end{aligned}$$

Choosing η small enough we have $p(1+\eta) \leq 2 + 2\delta$ and therefore

$$c_{\varepsilon,\beta} 2^p A \leq A' = c'_{\varepsilon,\beta,\lambda_1,\lambda_2,\alpha,\delta} (1 + \lambda_0 + \mu_\delta)^{2+2\delta}.$$

Since a similar bound can be established with W_i instead of X_i , we obtain the following bound for the oscillation term:

$$(6.18) \quad \begin{aligned} & \mathbb{P} \left((2^n)^{-\frac{1}{2}} \sup_{r \in \mathcal{L}_N} \sup_{r \leq l \leq r^+} \left| \sum_{r \leq i \leq l} (\tilde{X}_i - W_i) \right| \geq 2 (2^n)^{-\rho} \right) \\ & \leq 2A' (2^n)^{(2+2\alpha)(\rho+\varepsilon/2)} \left((2^n)^{-(1+\alpha)(1-\beta)} + (2^n)^{-(1+\alpha)\beta} \right). \end{aligned}$$

6.7. Optimizing the bounds . Let $\alpha < \delta$, $\beta > \frac{1}{2}$ and $0 < \rho < \frac{1-\beta}{2}$. Using (6.14), we may decompose the quantity $\mathbb{P} \left((2^n)^{-\frac{1}{2}} \sup_{1 \leq j \leq N} \left| \sum_{i=1}^j (\tilde{X}_i - W_i) \right| \geq 6 (2^n)^{-\rho} \right)$ in 3 terms, the first one “sup over islands” being itself decomposed in two terms (see (6.15)); consequently, this quantity is decomposed in four terms listed below:

- the first term “normal approximation” is controlled with Proposition 6.2, it is bounded by two terms named “dependence error” and “Sakhanenko’s error”,
- the second term “variance homogenization” is controlled in 6.10 with $\gamma = \varepsilon$
- the term “sup over gaps” is controlled in (6.17),
- the term “oscillation term” is controlled in (6.18).

Putting these bounds altogether, we obtain

$$\begin{aligned} & \mathbb{P} \left((2^n)^{-\frac{1}{2}} \sup_{1 \leq j \leq N} \left| \sum_{i=1}^j (X_i - W_i) \right| \geq 6 (2^n)^{-\rho} \right) \\ & \leq A (2^n)^{-(1+\alpha)+(\rho+\varepsilon)(2+2\alpha)} \quad (\text{dependence error}) \\ & \quad + A (2^n)^{-\beta\alpha+\rho(2+2\alpha)} \quad (\text{Sakhanenko's error}) \\ & \quad + A (2^n)^{-(1-\beta)(1+\alpha)+(\rho+\varepsilon/2)(2+2\alpha)} \quad (\text{variance homogenization error}) \\ & \quad + A (2^n)^{-(1-\beta)(1+\alpha)+(\rho+\varepsilon/2)(2+2\alpha)} \quad (\text{gaps error}) \\ & \quad + A \left((2^n)^{-(1-\beta)(1+\alpha)} + (2^n)^{-\beta(1+\alpha)} \right) (2^n)^{(\rho+\varepsilon/2)(2+2\alpha)} \quad (\text{oscillation error}) \end{aligned}$$

where $A = c_{\varepsilon, \beta, \lambda_1, \lambda_2, \alpha} (1 + \tau^{1+\alpha} + (1 + \lambda_0 + \mu_\delta)^{2+2\alpha})$.

For the moment let us ignore the factors containing ε which have a small contribution to the bound. The term (dependence error) is negligible with respect to all other terms; equating the powers of the term (Sakhanenko’s error) and the term (gaps error) (or equivalently variance homogenization error) we get $\beta\alpha = (1 - \beta)(1 + \alpha)$ i.e. $\beta = \frac{1+\alpha}{1+2\alpha}$. Implementing

$\beta = \frac{1+\alpha}{1+2\alpha}$ in the above inequality yields

$$\begin{aligned} & \mathbb{P} \left((2^n)^{-\frac{1}{2}} \sup_{1 \leq l \leq N} \left| \sum_{i=1}^l (\tilde{X}_i - W_i) \right| \geq 6(2^n)^{-\rho} \right) \\ & \leq A(2^n)^{-1-\delta+(\rho+\varepsilon)(2+2\alpha)} \quad (\text{dependence error}) \\ & \quad + 6A(2^n)^{-\frac{\alpha(1+\alpha)}{1+2\alpha}+(\rho+\varepsilon)(2+2\alpha)} \quad (\text{Sakhanenko's error} \\ & \quad \quad \quad + \text{variance homogenization error} \\ & \quad \quad \quad + \text{gaps error} + \text{oscillation error}). \end{aligned}$$

Taking into account that $\alpha < \delta$ and $2^n \leq N < 2^{n+1}$ we obtain

$$(6.19) \quad \mathbb{P} \left(N^{-\frac{1}{2}} \sup_{1 \leq l \leq N} \left| \sum_{i=1}^l (\tilde{X}_i - W_i) \right| \geq 6N^{-\rho} \right) \leq A' N^{-\frac{\alpha(1+\alpha)}{1+2\alpha}+(\rho+\varepsilon)(2+2\alpha)}$$

where $A' = c_{\varepsilon, \lambda_1, \lambda_2, \alpha, \delta} (1 + \lambda_0 + \mu_\delta + \sqrt{\tau})^{2+2\delta}$ and ρ satisfies $0 < \rho < \frac{1-\beta}{2} = \frac{\alpha}{2(1+2\alpha)}$.

Note that the function $g(\alpha) = \frac{\alpha(1+\alpha)}{1+2\alpha} - \rho(2+2\alpha)$ is strictly increasing on \mathbb{R}^+ when $\rho < \frac{1}{4}$. Therefore we can get rid of the constant ε in the bound by choosing $\alpha' < \alpha$. If we let $\Delta = g(\alpha) - g(\alpha') > 0$ and choose ε sufficiently small, we obtain $(2^n)^{-g(\alpha)+\varepsilon(2+2\alpha)} = (2^n)^{-g(\alpha')-\Delta+\varepsilon(2+2\alpha)} \leq (2^n)^{-g(\alpha')}$. Since α and α' are arbitrary satisfying $\alpha' < \alpha < \delta$, the assertion of Theorem 2.1 follows with α' replacing α .

We have performed a construction of the sequences $\tilde{X}^{(N)} = (\tilde{X}_i)_{1 \leq i \leq N}$ and $W^{(N)} = (W_i)_{1 \leq i \leq N}$ for each fixed $N \geq 1$, where for each N the constructed sequences in general are different. Below we show how to obtain a construction of the entire sequences $(\tilde{X}_i)_{i \geq 1}$ and $(W_i)_{i \geq 1}$.

Let $\Omega^{(N)} = \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$. Without loss of generality, for any $\omega = (\omega_1, \omega_2) \in \Omega^{(N)}$, the sequences $\tilde{X}^{(N)}$ and $W^{(N)}$ can be reconstructed on $\Omega^{(N)}$ so that $\tilde{X}_i = \omega_{1,i}$, $W_i = \omega_{2,i}$ and their joint distribution, say $\mathbb{P}^{(N)}$, is preserved. Each measure $\mathbb{P}^{(N)}$ can be extended (arbitrarily) on the space $\mathbb{R}^\infty \times \mathbb{R}^\infty$. From the bound (6.19) it follows that the sequence of measures $\mathbb{P}^{(N)}$ is tight. Therefore there is a weak limit which satisfies (6.19) and thus provides the desired construction.

7. PROOF OF THE RESULTS OF SECTION 3

Throughout this section we assume that the Markov chain $(X_n)_{n \geq 0}$ and the function f satisfy the hypotheses **M1**, **M2**, **M3** and **M4**.

7.1. Proof of Proposition 3.1. First, we establish the following bound for the characteristic functions $\phi_{x,1}$, $\phi_{x,2}$ and ϕ_x involved in Proposition 3.1.

Lemma 7.1. *For any $k_{gap}, M_1, M_2 \in \mathbb{N}$, any sequence $j_0 < \dots < j_{M_1+M_2}$ and any $t = (t_i)_i \in \mathbb{R}^{M_1}$, $s = (s_i)_i \in \mathbb{R}^{M_2}$ satisfying $\|(t, s)\|_\infty \leq \varepsilon_0$,*

$$|\phi_x(t, s) - \phi_{x,1}(t) \phi_{x,2}(s)| \leq 2C_Q C_{\mathbf{P}}^{M_1+M_2} \left(\|\nu\|_{\mathcal{B}'} + \|\delta_x\|_{\mathcal{B}'} \right) \|e\|_{\mathcal{B}} \kappa^{k_{gap}}.$$

Proof. Set for brevity $\phi_1 = \phi_{x,1}$, $\phi_2 = \phi_{x,2}$ and $\phi = \phi_x$. The characteristic function ϕ can be rewritten in the following form:

$$\phi(t, s) = \left(\mathbf{P}^{j_0} \mathbf{P}_{t_1}^{|J_1|} \dots \mathbf{P}_{t_{M_1}}^{|J_{M_1}|} \mathbf{P}^{k_{gap}} \mathbf{P}_{s_{M_1+1}}^{|J_{M_1+1}|} \dots \mathbf{P}_{s_{M_1+M_2}}^{|J_{M_1+M_2}|} e \right) (x).$$

Since $\mathbf{P} = \Pi + Q$ we get $\mathbf{P}^k = \Pi + Q^k$, and thus

$$(7.1) \quad \phi(t, s) = \phi_{\Pi}(t, s) + \phi_Q(t, s)$$

with

$$(7.2) \quad \phi_{\Pi}(t, s) := \left(\mathbf{P}^{j_0} \mathbf{P}_{t_1}^{|J_1|} \dots \mathbf{P}_{t_{M_1}}^{|J_{M_1}|} \Pi^{k_{gap}} \mathbf{P}_{s_{M_1+1}}^{|J_{M_1+1}|} \dots \mathbf{P}_{s_{M_1+M_2}}^{|J_{M_1+M_2}|} e \right) (x)$$

and

$$(7.3) \quad \phi_Q(t, s) := \left(\mathbf{P}^{j_0} \mathbf{P}_{t_1}^{|J_1|} \dots \mathbf{P}_{t_{M_1}}^{|J_{M_1}|} Q^{k_{gap}} \mathbf{P}_{s_{M_1+1}}^{|J_{M_1+1}|} \dots \mathbf{P}_{s_{M_1+M_2}}^{|J_{M_1+M_2}|} e \right) (x).$$

First, since $\Pi^{k_{gap}} \mathbf{P}_{s_{M_1+1}}^{|J_{M_1+1}|} \dots \mathbf{P}_{s_{M_1+M_2}}^{|J_{M_1+M_2}|} e = \nu \left(\mathbf{P}_{s_{M_1+1}}^{|J_{M_1+1}|} \dots \mathbf{P}_{s_{M_1+M_2}}^{|J_{M_1+M_2}|} e \right) e$, we may write, setting

$$\psi_2(s) := \nu \left(\mathbf{P}_{s_{M_1+1}}^{|J_{M_1+1}|} \dots \mathbf{P}_{s_{M_1+M_2}}^{|J_{M_1+M_2}|} e \right)$$

$$\begin{aligned} \phi_{\Pi}(t, s) &= \psi_2(s) \left(\mathbf{P}^{j_0} \mathbf{P}_{t_1}^{|J_1|} \dots \mathbf{P}_{t_{M_1}}^{|J_{M_1}|} e \right) (x) \\ &= \psi_2(s) \phi_1(t). \end{aligned}$$

Notice that $\phi_2(s) = \left(\mathbf{P}^{k_{gap}+j_{M_1}} \mathbf{P}_{s_{M_1+1}}^{|J_{M_1+1}|} \dots \mathbf{P}_{s_{M_1+M_2}}^{|J_{M_1+M_2}|} e \right) (x)$; using the equality $\nu \mathbf{P} = \nu$, one

gets $\psi_2(s) = \nu \left(\mathbf{P}^{k_{gap}+j_{M_1}} \mathbf{P}_{s_{M_1+1}}^{|J_{M_1+1}|} \dots \mathbf{P}_{s_{M_1+M_2}}^{|J_{M_1+M_2}|} e \right)$ which allows us to control the difference between ψ_2 and ϕ_2 , namely

$$\begin{aligned} \psi_2(s) - \phi_2(s) &= (\nu - \delta_x) \left(\mathbf{P}^{k_{gap}+j_{M_1}} \mathbf{P}_{s_{M_1+1}}^{|J_{M_1+1}|} \dots \mathbf{P}_{s_{M_1+M_2}}^{|J_{M_1+M_2}|} e \right) \\ &= (\nu - \delta_x) \left(\Pi \mathbf{P}_{s_{M_1+1}}^{|J_{M_1+1}|} \dots \mathbf{P}_{s_{M_1+M_2}}^{|J_{M_1+M_2}|} e \right) \\ &\quad + (\nu - \delta_x) \left(Q^{k_{gap}} \mathbf{P}^{j_{M_1}} \mathbf{P}_{s_{M_1+1}}^{|J_{M_1+1}|} \dots \mathbf{P}_{s_{M_1+M_2}}^{|J_{M_1+M_2}|} e \right) \\ &= (\nu - \delta_x) (e) \nu \left(\mathbf{P}_{s_{M_1+1}}^{|J_{M_1+1}|} \dots \mathbf{P}_{s_{M_1+M_2}}^{|J_{M_1+M_2}|} e \right) \\ &\quad + (\nu - \delta_x) \left(Q^{k_{gap}} \mathbf{P}^{j_{M_1}} \mathbf{P}_{s_{M_1+1}}^{|J_{M_1+1}|} \dots \mathbf{P}_{s_{M_1+M_2}}^{|J_{M_1+M_2}|} e \right) \end{aligned}$$

with $(\nu - \delta_x) (e) = 0$; consequently

$$\begin{aligned} \left| \psi_2(s) - \phi_2(s) \right| &= \left| (\nu - \delta_x) \left(Q^{k_{gap}+j_{M_1}} \mathbf{P}_{s_{M_1+1}}^{|J_{M_1+1}|} \dots \mathbf{P}_{s_{M_1+M_2}}^{|J_{M_1+M_2}|} e \right) \right| \\ &\leq C_Q C_{\mathbf{P}}^{1+M_2} \kappa^{k_{gap}} (\|\nu\|_{\mathcal{B}'} + \|\delta_x\|_{\mathcal{B}'}) \|e\|_{\mathcal{B}}. \end{aligned}$$

On the other hand, one easily gets

$$(7.4) \quad \left| \phi_Q(t, s) \right| \leq C_Q C_{\mathbf{P}}^{1+M_1+M_2} \kappa^{k_{gap}} \|e\|_{\mathcal{B}} \|\delta_x\|_{\mathcal{B}'}. \quad \square$$

Writing $\phi(t, s) = \phi_1(t)\phi_2(s) + \phi_1(t)(\psi_2(s) - \phi_2(s)) + \phi_Q(t, s)$ and using the previous inequalities, one finally gets

$$\left| \phi(t, s) - \phi_1(t)\phi_2(s) \right| \leq 2C_Q C_{\mathbf{P}}^{1+M_1+M_2} \left(\|\nu\|_{\mathcal{B}'} + \|\delta_x\|_{\mathcal{B}'} \right) \kappa^{k_{gap}} \|e\|_{\mathcal{B}}.$$

Lemma is proved. □

To prove Proposition 3.1, set $k_0 = \max\{1, \log_2 C_{\mathbf{P}}\}$ so that $C_{\mathbf{P}} \leq 2^{k_0}$.

Since $\max_{m=1, \dots, M_1+M_2} \text{card}(J_m) \geq 1$, one gets

$$C_{\mathbf{P}}^{M_1+M_2} \leq 2^{k_0(M_1+M_2)} \leq \left(1 + \max_{m=1, \dots, M_1+M_2} \text{card}(J_m) \right)^{k_0(M_1+M_2)}.$$

Now, Proposition 3.1 follows from Lemma 7.1.

7.2. Proof of Proposition 3.2. We need two auxiliary lemmas.

Lemma 7.2. *For any $l, k = 0, 1, \dots$*

$$(7.5) \quad \left| \text{Cov}_{\mathbb{P}_x}(f(X_l), f(X_{l+k})) \right| \leq A(x) \kappa^{k\gamma/4},$$

for any positive constant γ satisfying $0 < \gamma \leq \min\{1, 2\delta\}$, where

$$A(x) = c_{\delta} \left(1 + C_Q C_{\mathbf{P}}^2 (\|\nu\|_{\mathcal{B}'} + \|\delta_x\|_{\mathcal{B}'}) \|e\|_{\mathcal{B}} + \mu_{\delta}^{2+\gamma}(x) \right).$$

Proof. We give a proof involving Lemma 7.1. Let V and V' be two independent identically distributed r.v.'s of mean 0, independent of X_l and X_{l+m} and whose common characteristic function is supported in the interval $[-\varepsilon_0, \varepsilon_0]$, for some $\varepsilon_0 > 0$. Set $Y_l = f(X_l) + V$ and $Y'_{l+k} = f(X_{l+k}) + V'$.

Let $\tilde{\phi}_1$ (resp. $\tilde{\phi}_2, \tilde{\phi}(t, u)$) be the characteristic function of Y_l (resp. $Y'_{l+k}, (Y_l, Y'_{l+k})$). Set $g_T(x) = x 1_{(|x| \leq T)}$ and $h_T(x, y) = g_T(x) g_T(y)$ for $x, y \in \mathbb{R}$. Let \hat{g}_T (resp. \hat{h}_T) be the Fourier transform of the function g_T (resp. h_T) defined by

$$\hat{g}_T(t) = \int e^{itx} g_T(x) dx,$$

$$\text{(resp. } \hat{h}_T(t, u) = \int \int e^{i(tx+uy)} h_T(x, y) dx dy = \hat{g}_T(t) \hat{g}_T(u)).$$

For any $T > 0$ and $l \geq 1, k \geq 0$, one gets

$$(7.6) \quad \mathbb{E}_x f(X_l) f(X_{l+k}) = \mathbb{E}_x Y_l Y'_{l+k} = \mathbb{E}_x h_T(Y_l, Y'_{l+k}) + R_0$$

with

$$(7.7) \quad |R_0| \leq \mathbb{E}_x |Y_l Y'_{l+k}| 1_{(|Y_l| > T)} + \mathbb{E}_x |Y_l Y'_{l+k}| 1_{(|Y'_{l+k}| > T)}.$$

By the inverse Fourier transform, one may write

$$\mathbb{E}_x f(X_l) f(X_{l+k}) = \frac{1}{(2\pi)^2} \int \int \overline{\widehat{h}_T(t, u)} \widetilde{\phi}(t, u) dt du + R_0.$$

Analogously

$$(7.8) \quad \mathbb{E}_x f(X_l) = \mathbb{E}_x Y_l = \mathbb{E}_x g_T(Y_l) + R_1 = \frac{1}{2\pi} \int \overline{\widehat{g}_T(t)} \widetilde{\phi}_1(t) dt + R_1$$

and

$$\mathbb{E}_x f(X_{l+k}) = \mathbb{E}_x Y'_{l+k} = \mathbb{E}_x g_T(Y'_{l+k}) + R_2 = \frac{1}{2\pi} \int \overline{\widehat{g}_T(u)} \widetilde{\phi}_2(u) du + R_2,$$

where

$$(7.9) \quad R_1 := \mathbb{E}_x Y_l 1_{(|Y_l| > T)} \quad \text{and} \quad R_2 := \mathbb{E}_x Y'_{l+k} 1_{(|Y'_{l+k}| > T)}.$$

This gives

$$(7.10) \quad \begin{aligned} \text{Cov}_{\mathbb{P}_x}(f(X_l), f(X_{l+k})) &= \mathbb{E}_x f(X_l) f(X_{l+k}) - \mathbb{E}_x f(X_l) \mathbb{E}_x f(X_{l+k}) \\ &= \frac{1}{(2\pi)^2} \int \int \overline{\widehat{h}_T(t, u)} \left(\widetilde{\phi}(t, u) - \widetilde{\phi}_1(t) \widetilde{\phi}_2(u) \right) dt du + R, \end{aligned}$$

where

$$(7.11) \quad R = R_0 + R_1 \mathbb{E}_x g_T(Y'_{l+k}) + R_2 \mathbb{E}_x g_T(Y_l) + R_1 R_2.$$

Note that

$$\left| \int \int \overline{\widehat{h}_T(t, u)} \left(\widetilde{\phi}(t, u) - \widetilde{\phi}_1(t) \widetilde{\phi}_2(u) \right) dt du \right| \leq \|\widehat{h}_T\|_{L^2} \|\widetilde{\phi} - \widetilde{\phi}_1 \widetilde{\phi}_2\|_{L^2}.$$

Since V, V' are independent of X_l, X_{l+k} , we have $\widetilde{\phi}(t, u) = \phi(t, u) \mathbb{E}_x e^{itV} \mathbb{E}_x e^{iuV'}$ and $\widetilde{\phi}_1(t) = \phi_1(t) \mathbb{E}_x e^{itV}$, $\widetilde{\phi}_2(u) = \phi_2(u) \mathbb{E}_x e^{iuV'}$, where

$$\phi(t, u) := \mathbb{E}_x e^{itf(X_l) + iuf(X_{l+k})} = (\mathbf{P}^{l-1} \mathbf{P}_t \mathbf{P}^{k-1} \mathbf{P}_u e)(x)$$

and $\phi_1(t) = \mathbb{E}_x e^{itf(X_l)} = (\mathbf{P}^{l-1} \mathbf{P}_t e)(x)$, $\phi_2(u) = \mathbb{E}_x e^{iuf(X_{l+k})} = (\mathbf{P}^{l-1} \mathbf{P}_t e)(x)$. Since the support of the characteristic functions of V and V' is the interval $[-\varepsilon_0, \varepsilon_0]$ the function $\widetilde{\phi} - \widetilde{\phi}_1 \widetilde{\phi}_2$ vanishes outside the square $[-\varepsilon_0, \varepsilon_0]^2$. Then, by Lemma 7.1,

$$(7.12) \quad \begin{aligned} \|\widetilde{\phi} - \widetilde{\phi}_1 \widetilde{\phi}_2\|_{L^2} &\leq 2\varepsilon_0 \sup_{|t| \leq \varepsilon_0, |u| \leq \varepsilon_0} |\phi(t, u) - \phi_1(t) \phi_2(u)| \\ &\leq 4\varepsilon_0 C_Q C_{\mathbf{P}}^3 \kappa^k \left(\|\nu\|_{\mathcal{B}'} + \|\delta_x\|_{\mathcal{B}'} \right) \|e\|_{\mathcal{B}}. \end{aligned}$$

Using the inequality

$$(7.13) \quad \|\widehat{h}_T\|_{L^2}^2 = \int \int h_T^2(x, y) dx dy = \left(\int g_T^2(x) dx \right)^2 \leq \frac{4}{9} T^6,$$

one obtains

$$(7.14) \quad |\text{Cov}_{\mathbb{P}_x}(f(X_l), f(X_{l+k}))| \leq \frac{2}{3\pi^2} T^3 \varepsilon_0 C_Q C_{\mathbf{P}}^3 \kappa^k \left(\|\nu\|_{\mathcal{B}'} + \|\delta_x\|_{\mathcal{B}'} \right) \|e\|_{\mathcal{B}} + |R|.$$

Now we shall give a bound for $|R|$. By Hölder's inequality, with $q_\delta = \frac{1+\delta}{\delta} > 1$,

$$\mathbb{E}_x |Y_l| |Y'_{l+k}| 1_{(|Y_l| > T)} \leq \left(\mathbb{E}_x |Y_l|^{2+2\delta} \right)^{\frac{1}{2+2\delta}} \left(\mathbb{E}_x |Y'_{l+k}|^{2+2\delta} \right)^{\frac{1}{2+2\delta}} \mathbb{P}_x (|Y_l| > T)^{\frac{1}{q_\delta}}.$$

Using hypothesis **M4**, we have

$$\left(\mathbb{E}_x |Y_l|^{2+2\delta} \right)^{\frac{1}{2+2\delta}} \leq \left(\mathbb{E}_x |f(X_l)|^{2+2\delta} \right)^{\frac{1}{2+2\delta}} + \left(\mathbb{E}_x |V|^{2+2\delta} \right)^{\frac{1}{2+2\delta}} \leq c_\delta A_0(x),$$

with $A_0(x) = \mu_\delta(x) + 1$. Similarly $\left(\mathbb{E}_x |Y'_{l+k}|^{2+2\delta} \right)^{\frac{1}{2+2\delta}} \leq c_\delta A_0(x)$. On the other hand, for any $\gamma \in (0, 2\delta]$, one gets

$$\mathbb{P}_x (|Y_l| > T) \leq \frac{1}{T^{\gamma q_\delta}} \mathbb{E}_x |Y_l|^{\gamma q_\delta} \leq \frac{c_\delta}{T^{\gamma q_\delta}} A_0^{\gamma q_\delta}(x).$$

Putting together these bounds gives

$$(7.15) \quad \mathbb{E}_x |Y_l| |Y'_{l+k}| 1_{(|Y_l| > T)} \leq c_\delta T^{-\gamma} A_0^{2+\gamma}(x).$$

In the same way we obtain, for any $\gamma \in (0, 2\delta]$,

$$(7.16) \quad \mathbb{E}_x |Y_l| |Y'_{l+k}| 1_{(|Y'_{l+k}| > T)} \leq c_\delta T^{-\gamma} A_0^{2+\gamma}(x).$$

From (7.7), (7.15), (7.16), it follows that

$$(7.17) \quad |R_0| \leq c_\delta T^{-\gamma} A_0^{2+\gamma}(x).$$

From (7.16), taking $k = 0$ we get, for any $\gamma \in (0, 2\delta]$,

$$(7.18) \quad \max \{R_1, R_2\} \leq \sup_{l \geq 0} \left(\mathbb{E}_x Y_l^2 1_{(|Y_l| > T)} \right)^{\frac{1}{2}} \leq c_\delta T^{-\gamma/2} A_0^{1+\gamma/2}(x).$$

Since

$$|\mathbb{E}_x g_T(Y_l)| \leq \left(\mathbb{E}_x \left(|Y_l|^{2+2\delta} \right) \right)^{\frac{1}{2+2\delta}} \leq c_\delta A_0(x)$$

and

$$|\mathbb{E}_x g_T(Y'_{l+k})| \leq c_\delta A_0(x),$$

from (7.17), (7.18), it follows that,

$$(7.19) \quad |R| \leq c_\delta T^{-\gamma/2} A_0^{2+\gamma}(x),$$

for any $\gamma \in (0, 2\delta]$, where we assume without loss of generality that $A_0(x) \geq 1$. The inequalities (7.14) and (7.19) yield, for any $\gamma \in (0, 2\delta]$,

$$\begin{aligned} |\text{Cov}_{\mathbb{P}_x}(f(X_l), f(X_{l+k}))| &\leq \frac{2}{3\pi^2} T^3 \varepsilon_0 C_Q C_{\mathbf{P}}^3 \kappa^k (\|\nu\|_{\mathcal{B}'} + \|\delta_x\|_{\mathcal{B}'}) \|e\|_{\mathcal{B}} \\ &\quad + c_\delta T^{-\gamma/2} A_0^{2+\gamma}(x). \end{aligned}$$

Choosing $T = \kappa^{-k/4}$ and taking into account that $A_0^{2+\gamma}(x) \leq c_\delta(1 + \mu_\delta^{2+\gamma}(x))$, it follows that

$$|\text{Cov}_{\mathbb{P}_x}(f(X_l), f(X_{l+k}))| \leq A(x) c_\delta \kappa^{k \min\{1, \gamma/2\}/4},$$

which finishes the proof Lemma 7.2. \square

Lemma 7.3. *Let $0 < \gamma \leq \min\{1, 2\delta\}$. Then:*

a) *There exists a real number μ not depending on x such that, for any $k \geq 1$,*

$$|\mathbb{E}_x f(X_k) - \mu| \leq c_\delta A_1(x) \kappa^{k\gamma/4-1}$$

where $A_1(x) = 1 + \mu_\delta(x)^{1+\gamma} + \|\delta_x\|_{\mathcal{B}'} \|e\|_{\mathcal{B}} C_{\mathbf{P}} C_Q$. Moreover

$$\sum_{k=0}^{\infty} |\mathbb{E}_x f(X_k) - \mu| \leq \bar{\mu}(x) = c_{\gamma, \kappa, \delta} A_1(x).$$

b) *There exists a sequence of (possibly complex) numbers $(s_k)_{k \geq 0}$ not depending on x such that*

$$(7.20) \quad |\text{Cov}_{\mathbb{P}_x}(f(X_l), f(X_{l+k})) - s_k| \leq c_\delta A_2(x) \kappa^{l\gamma/4-1}$$

where

$$\begin{aligned} A_2(x) &= 1 + \mu_\delta(x)^{2+\gamma} \\ &\quad + \|\delta_x\|_{\mathcal{B}'} \|e\|_{\mathcal{B}} (C_{\mathbf{P}}^2 C_Q (\|\nu\|_{\mathcal{B}'} \|e\|_{\mathcal{B}} + C_Q) + C_{\mathbf{P}} C_Q (1 + \|\nu\|_{\mathcal{B}'} C_{\mathbf{P}})). \end{aligned}$$

Moreover, for $k \geq 0$,

$$|s_k| \leq A_2(x) \kappa^{k\gamma/4-1}$$

and

$$|s_0| + 2 \sum_{k=1}^{\infty} |s_k| \leq c_{\gamma, \kappa, \delta} A_2(x).$$

Proof. To avoid repetitions we prove first the part b). We keep the notations from the proof of Lemma 7.2. Denote $\tilde{\phi}_0(t, u) = \tilde{\phi}(t, u) - \tilde{\phi}_1(t) \tilde{\phi}_2(u)$. By (7.10), for any $l = 0, 1, \dots$

$$\text{Cov}_{\mathbb{P}_x}(f(X_l), f(X_{l+k})) = \frac{1}{(2\pi)^2} \int \int \overline{\tilde{h}_T(t, u)} \tilde{\phi}_0(t, u) dt du + R,$$

with R defined by (7.11). Since V, V' are independent of X_l, X_{l+k} ,

$$\begin{aligned} \tilde{\phi}(t, u) &= \left(\mathbb{E}_x e^{i0 \sum_{j=1}^{l-1} X_j + it X_l + i0 \sum_{j=l+1}^{l+k-1} X_j + iu X_{l+k}} \right) \mathbb{E}_x e^{itV} \mathbb{E}_x e^{iuV'} \\ (7.21) \quad &= (\mathbf{P}^{l-1} \mathbf{P}_t \mathbf{P}^{k-1} \mathbf{P}_u e)(x) \mathbb{E}_x e^{itV} \mathbb{E}_x e^{iuV'}. \end{aligned}$$

Note that, for $k, l \geq 2$,

$$\begin{aligned} (\mathbf{P}^{l-1} \mathbf{P}_t \mathbf{P}^{k-1} \mathbf{P}_u e)(x) &= \delta_x(\mathbf{P}^{l-1} \mathbf{P}_t \mathbf{P}^{k-1} \mathbf{P}_u e) \\ &= \delta_x(\Pi \mathbf{P}_t \mathbf{P}^{k-1} \mathbf{P}_u e) + \delta_x(Q^{l-1} \mathbf{P}_t \mathbf{P}^{k-1} \mathbf{P}_u e) \\ &= \nu(\mathbf{P}_t \mathbf{P}^{k-1} \mathbf{P}_u e) \\ &\quad + \delta_x(Q^{l-1} \mathbf{P}_t \Pi \mathbf{P}_u e) + \nu(Q^{l-1} \mathbf{P}_t Q^{k-1} \mathbf{P}_u e). \end{aligned}$$

Since

$$|\delta_x(Q^{l-1} \mathbf{P}_t \Pi \mathbf{P}_u e)| = |\delta_x(Q^{l-1} \mathbf{P}_t e) \nu(\mathbf{P}_u e)| \leq \kappa^{l-1} C_Q C_{\mathbf{P}}^2 \|\delta_x\|_{\mathcal{B}'} \|\nu\|_{\mathcal{B}'} \|e\|_{\mathcal{B}}^2$$

and

$$|\delta_x(Q^{l-1} \mathbf{P}_t Q^{k-1} \mathbf{P}_u e)| \leq \kappa^{l+k-2} C_Q^2 C_{\mathbf{P}}^2 \|\delta_x\|_{\mathcal{B}'} \|e\|_{\mathcal{B}'},$$

we obtain

$$(7.22) \quad \left| \tilde{\phi}(t, u) - \tilde{\psi}(t, u; k) \right| \leq \kappa^{l-1} C_{\mathbf{P}}^2 C_Q (\|\nu\|_{\mathcal{B}'} \|e\|_{\mathcal{B}} + C_Q) \|\delta_x\|_{\mathcal{B}'} \|e\|_{\mathcal{B}'},$$

where

$$\tilde{\psi}(t, u; k) = \nu(\mathbf{P}_t \mathbf{P}^{k-1} \mathbf{P}_u e) \mathbb{E}_x e^{itV} \mathbb{E}_x e^{iuV'}.$$

Note that $\tilde{\psi}(t, u; k)$ does not depend on the initial state x since V and V' are independent of the Markov chain. In the same way

$$\begin{aligned} \tilde{\phi}_1(t) &= (\mathbf{P}^{l-1} \mathbf{P}_t e)(x) \mathbb{E}_x e^{itV}, \\ \tilde{\phi}_2(u) &= (\mathbf{P}^{l+k-1} \mathbf{P}_u e)(x) \mathbb{E}_x e^{iuV'}, \end{aligned}$$

where, for $m \geq 2$,

$$\begin{aligned} (\mathbf{P}^{m-1} \mathbf{P}_t e)(x) &= \delta_x(\mathbf{P}^{m-1} \mathbf{P}_t e) \\ &= \delta_x(\Pi \mathbf{P}_t e) + \delta_x(Q^{m-1} \mathbf{P}_t e) \\ &= \nu(\mathbf{P}_t e) + \delta_x(Q^{m-1} \mathbf{P}_t e). \end{aligned}$$

Since $|\delta_x(Q^{m-1} \mathbf{P}_t e)| \leq \kappa^{m-1} \|\delta_x\|_{\mathcal{B}'} \|e\|_{\mathcal{B}} C_{\mathbf{P}} C_Q$, we get

$$(7.23) \quad \left| \tilde{\phi}_1(t) - \tilde{\psi}_1(t) \right| \leq \kappa^{l-1} \|\delta_x\|_{\mathcal{B}'} \|e\|_{\mathcal{B}} C_{\mathbf{P}} C_Q,$$

$$(7.24) \quad \left| \tilde{\phi}_2(u) - \tilde{\psi}_1(u) \right| \leq \kappa^{l+k-1} \|\delta_x\|_{\mathcal{B}'} \|e\|_{\mathcal{B}} C_{\mathbf{P}} C_Q,$$

where

$$(7.25) \quad \tilde{\psi}_1(t) = \nu(\mathbf{P}_t e) \mathbb{E}_x e^{itV} = \nu(\mathbf{P}_t e) \mathbb{E}_x e^{itV'}$$

does not depend on the initial state x of the Markov chain. Denote $\tilde{\psi}_0(t, u; k) = \tilde{\psi}(t, u; k) - \tilde{\psi}_1(t) \tilde{\psi}_1(u)$. From (7.23) and (7.24) it follows

$$\begin{aligned} & \left| \tilde{\phi}_0(t, u) - \tilde{\psi}_0(t, u; k) \right| \\ & \leq \left| \tilde{\phi}(t, u) - \tilde{\psi}(t, u; k) \right| + \left| \tilde{\phi}_1(t) \tilde{\phi}_2(u) - \tilde{\psi}_1(t) \tilde{\psi}_1(u) \right| \\ & \leq \left| \tilde{\phi}(t, u) - \tilde{\psi}(t, u; k) \right| + \left| \tilde{\phi}_1(t) - \tilde{\psi}_1(t) \right| + |\nu(\mathbf{P}_t e)| \left| \tilde{\phi}_2(u) - \tilde{\psi}_1(u) \right| \\ & \leq \kappa^{l-1} \|\delta_x\|_{\mathcal{B}'} \|e\|_{\mathcal{B}} C_{\mathbf{P}}^2 C_Q (\|\nu\|_{\mathcal{B}'} \|e\|_{\mathcal{B}} + C_Q) + \kappa^{l-1} \|\delta_x\|_{\mathcal{B}'} C_{\mathbf{P}} C_Q (1 + |\nu(\mathbf{P}_t e)|) \\ & \leq \kappa^{l-1} \|\delta_x\|_{\mathcal{B}'} \|e\|_{\mathcal{B}} (C_{\mathbf{P}}^2 C_Q (\|\nu\|_{\mathcal{B}'} \|e\|_{\mathcal{B}} + C_Q) + C_{\mathbf{P}} C_Q (1 + \|\nu\|_{\mathcal{B}'} C_{\mathbf{P}})) \\ (7.26) & \leq C(x) \kappa^{l-1}, \end{aligned}$$

where $C(x) = \|\delta_x\|_{\mathcal{B}'} \|e\|_{\mathcal{B}} (C_{\mathbf{P}}^2 C_Q (\|\nu\|_{\mathcal{B}'} \|e\|_{\mathcal{B}} + C_Q) + C_{\mathbf{P}} C_Q (1 + \|\nu\|_{\mathcal{B}'} C_{\mathbf{P}}))$. Denote by $s_{k,T}$ the complex number defined by

$$s_{k,T} = \frac{1}{(2\pi)^2} \int \int \overline{\hat{h}_T(t, u)} \tilde{\psi}_0(t, u; k) dt du.$$

Note that $s_{k,T}$ does not depend on the initial state x of the Markov chain since so is $\tilde{\psi}_0(t, u; k)$. With this notation we have

$$\text{Cov}_{\mathbb{P}_x}(f(X_l), f(X_{l+k})) - s_{k,T} = R' + R,$$

where

$$R' = \frac{1}{(2\pi)^2} \int \int \overline{\widehat{h}_T(t, u)} \left(\tilde{\phi}_0(t, u) - \tilde{\psi}_0(t, u; k) \right) dt du.$$

Since $\mathbb{E}_x e^{itV} \mathbb{E}_x e^{iuV'}$ has a support in the square $[-\varepsilon_0, \varepsilon_0]^2$, using (7.13) and (7.26) it follows that

$$(7.27) \quad |R'| \leq \frac{1}{(2\pi)^2} \left\| \widehat{h}_T \right\|_{L^2} \left\| \tilde{\phi}_0 - \tilde{\psi}_0 \right\|_{L^2} \leq \frac{T^3}{3\pi^2} \varepsilon_0^2 C(x) \kappa^{l-1}.$$

From (7.27) and (7.19), for any $\gamma \in (0, 2\delta]$ and any $l, k = 0, 1, \dots$,

$$(7.28) \quad \begin{aligned} |\text{Cov}_{\mathbb{P}_x}(f(X_l), f(X_{l+k})) - s_{k,T}| &\leq C(x) \frac{T^3}{3\pi^2} \varepsilon_0^2 \kappa^{l-1} \\ &\quad + c_\delta T^{-\gamma} A_0^{2+\gamma}(x), \end{aligned}$$

From (7.28), for any $l, l' = 2, 3, \dots$ one obtains

$$\begin{aligned} &|\text{Cov}_{\mathbb{P}_x}(f(X_l), f(X_{l+k})) - \text{Cov}_{\mathbb{P}_x}(f(X_{l'}), f(X_{l'+k}))| \\ &\leq c_\delta T^{-\gamma} A_0^{2+\gamma}(x) + C(x) \frac{2T^3}{3\pi^2} \varepsilon_0^2 \kappa^{\min\{l, l'\}-1}. \end{aligned}$$

Taking $T = \kappa^{-\frac{1}{4} \min\{l, l'\}}$ we get, for any $\gamma \leq \min\{1, 2\delta\}$,

$$(7.29) \quad |\text{Cov}_{\mathbb{P}_x}(f(X_l), f(X_{l+k})) - \text{Cov}_{\mathbb{P}_x}(f(X_{l'}), f(X_{l'+k}))| \leq c_\delta A(x) \kappa^{\min\{l, l'\}\gamma/4-1},$$

where $A(x) = A_0^{2+\gamma}(x) + C(x)$. The sequence $\text{Cov}_{\mathbb{P}_x}(f(X_l), f(X_{l+k}))$, $l = 1, 2, \dots$ is thus Cauchy; denote by $s_k(x)$ its limit as $l \rightarrow \infty$. Taking the limit as $l \rightarrow \infty$ in (7.28), we get that

$$(7.30) \quad |s_k(x) - s_{k,T}| \leq C(x) \frac{T^3}{3\pi^2} \varepsilon_0^2 \kappa^{l-1} + c_\delta T^{-\gamma} A_0^{2+\gamma}(x).$$

Letting $T = T_l = \kappa^{-\frac{l}{4}}$ this implies that $\lim_{l \rightarrow \infty} s_{k, T_l} = s_k(x)$. Since s_{k, T_l} does not depend on x , we conclude that $s_k(x)$ is also a constant not depending on x , say s_k . Taking the limit as $l' \rightarrow \infty$ in (7.29) we obtain (7.20).

The second assertion of the part b) of the lemma follows from (7.20) and Lemma 7.2 setting $l = k$.

The third assertion of the part b) follows immediately from the second one.

Let us now prove the part a). From (7.8), we have

$$|\mathbb{E}_x f(X_l) - m_T| \leq \frac{1}{2\pi} \int \left| \overline{\widehat{g}_T(t)} \right| \left| \tilde{\phi}_1(t) - \tilde{\psi}_1(t) \right| dt + |R_1|,$$

where

$$m_T = \frac{1}{2\pi} \int \overline{\widehat{g}_T(t)} \tilde{\psi}_1(t) dt,$$

R_1 is defined by (7.9) and $\tilde{\psi}_1$ is defined by (7.25). Note that m_T does not depend on x since so is $\tilde{\psi}(t)$. Taking into account the bounds in (7.18) and (7.23), we get

$$|\mathbb{E}_x f(X_l) - m_T| \leq \kappa^{l-1} \|\delta_x\|_{\mathcal{B}'} \|e\|_{\mathcal{B}} C_{\mathbf{P}} C_Q \frac{1}{2\pi} \int \left| \overline{\widehat{g}_T(t)} \right| dt + T^{-\gamma} c_{\delta} A_0^{1+\gamma}(x).$$

Recalling that $g_T(x) = x1(|x| \leq T)$, to bound $\int \left| \overline{\widehat{g}_T(t)} \right| dt$ we use the usual isometry relation

$$\left(\int \left| \overline{\widehat{g}_T(t)} \right| dt \right)^2 \leq \int \left| \widehat{g}_T(t) \right|^2 dt = \int g_T^2(x) dx = \frac{2}{3} T^3.$$

This implies, for any $\gamma \leq \min\{1, 2\delta\}$,

$$|\mathbb{E}_x f(X_l) - m_T| \leq c_{\delta} (\|\delta_x\|_{\mathcal{B}'} \|e\|_{\mathcal{B}} C_{\mathbf{P}} C_Q T^3 \kappa^{l-1} + T^{-\gamma} A_0^{1+\gamma}(x)).$$

Taking $T = \kappa^{-\frac{l}{4}}$, we have

$$(7.31) \quad |\mathbb{E}_x f(X_l) - m_T| \leq c_{\delta} A_1(x) \kappa^{l\gamma/4-1},$$

where $A_1(x) = 1 + A_0^{1+\gamma}(x) + \|\delta_x\|_{\mathcal{B}'} \|e\|_{\mathcal{B}} C_{\mathbf{P}} C_Q$. From this inequality it follows that

$$(7.32) \quad |\mathbb{E}_x f(X_l) - \mathbb{E}_x f(X_k)| \leq c_{\delta} A_1(x) \kappa^{\min\{l,k\}\gamma/4-1},$$

which proves that the sequence $(\mathbb{E}_x f(X_l))_{l \geq 1}$ is Cauchy and therefore has a limit denoted $\mu(x)$. Since m_T does not depend on x , taking \lim as $l \rightarrow \infty$ in (7.31) we conclude that $\mu(x) = \mu$ is not depending on x . Taking $\lim_{k \rightarrow \infty}$ in (7.32), we get

$$|\mathbb{E}_x f(X_l) - \mu| \leq c_{\delta} A_1(x) \kappa^{l \min\{1,\gamma\}/4-1},$$

which proves the first assertion of the part a) of the lemma. The second follows from the first one. \square

The bound (3.5) of Proposition 3.2 follows from the part a) of the previous lemma. It remains to prove the bound (3.6) of Proposition 3.2.

Let $0 < \gamma \leq \min\{1, 2\delta\}$. First note that, from Lemma 7.3 and Lemma 7.2 we obtain, for $k = 0, 1, \dots$,

$$|\text{Cov}_{\mathbb{P}_x}(f(X_l), f(X_{l+k}) - s_k)| \leq A_2(x) c_{\delta, \kappa} \kappa^{c_{\gamma, \kappa} \max\{l, k\}},$$

where $A_2(x)$ is defined in Proposition 3.2. Then, for any $k = 0, 1, \dots$,

$$\begin{aligned} \sum_{l=m}^{m+n-1} \sum_{k=1}^{m+n-l} |\text{Cov}_{\mathbb{P}_x}(f(X_l), f(X_{l+k}) - s_k)| &\leq A_2(x) c_{\delta, \kappa} \sum_{l=m}^{m+n-1} \sum_{k=1}^{m+n-l} e^{-c_{\gamma, \kappa} \max\{l, k\}} \\ &\leq A_2(x) c''_{\delta, \gamma, \kappa}. \end{aligned}$$

Since

$$\text{Var}_{\mathbb{P}_x} \left(\sum_{l=m}^{m+n-1} f(X_l) \right) = \sum_{l=m}^{m+n-1} \text{Var}_{\mathbb{P}_x}(f(X_l)) + 2 \sum_{l=m}^{m+n-1} \sum_{k=1}^{m+n-l} \text{Cov}_{\mathbb{P}_x}(f(X_l), f(X_{l+k}))$$

we get

$$\left| \text{Var}_{\mathbb{P}_x} \left(\sum_{l=m}^{m+n-1} f(X_l) \right) - \left(ns_0 + \sum_{l=m}^{m+n-1} \sum_{k=1}^{m+n-l} (s_k + s_k^*) \right) \right| \leq A_2(x) c_{\delta, \gamma, \kappa}''.$$

Taking into account that, by Lemma 7.3, s_k are not depending on x and $|s_k| \leq A_2(x) \kappa^{k\gamma/4-1}$ we obtain

$$(7.33) \quad \left| \text{Var}_{\mathbb{P}_x} \left(\sum_{l=m}^{m+n-1} f(X_l) \right) - n \left(s_0 + \sum_{k=1}^{\infty} (s_k + s_k^*) \right) \right| \leq A_2(x) c_{\delta, \gamma, \kappa}'''.$$

Dividing by n and taking the limit as $n \rightarrow \infty$ in (7.33), we deduce that $s_0 + \sum_{k=1}^{\infty} (s_k + s_k^*)$ converges to a non-negative number not depending on x , say $\sigma^2 \geq 0$. Now (3.6) follows from (7.33).

7.3. Proof of Theorem 3.3. First note that conditions **C1** and **C3** are satisfied by Propositions 3.1 and 3.2. Condition **C2** is satisfied by hypothesis **M4**. Let $\mu_i(x) = \mathbb{E}_x f(X_i)$. Let $\alpha < \delta$ and $\delta' = \frac{1}{2}(\alpha + \delta)$. Since $\alpha < \delta'$, from 2.1 with δ' replacing δ , it follows that for any $x \in \mathbb{X}$ there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P}_x)$, a sequence of independent standard normal r.v.'s $(W'_i)_{i \geq 1}$ and a sequence of r.v.'s $(Y'_i)_{i \geq 1}$ such that $(Y'_i)_{i \geq 1} \stackrel{d}{=} (f(X_i))_{i \geq 1}$ and for any $0 < \rho < \frac{1}{2} \frac{\alpha}{1+2\alpha}$,

$$(7.34) \quad \mathbb{P}_x \left(N^{-\frac{1}{2}} \sup_{k \leq N} \left| \sum_{i=1}^k (Y'_i - \mu_i(x) - \sigma W'_i) \right| > N^{-\rho} \right) \leq C_0(x) N^{-\alpha \frac{1+\alpha}{1+2\alpha} + \rho(2+2\alpha)},$$

where $C_0(x) = C'_0 \left(1 + \lambda_0(x) + \mu_{\delta'}(x) + \sqrt{\tau(x)} \right)^{2+2\delta'}$ and $\lambda_0(x), \mu_{\delta}(x), \tau(x), \lambda_1, \lambda_2$ and σ^2 are defined in Propositions 3.1 and 3.2. If $\bar{\mu}(x) \leq N^{\frac{1}{2}-\rho}$ (with $\bar{\mu}(x)$ from Proposition 3.2 using (7.34) we have

$$(7.35) \quad \begin{aligned} & \mathbb{P}_x \left(N^{-\frac{1}{2}} \sup_{k \leq N} \left| \sum_{i=1}^k (Y'_i - \mu - \sigma W'_i) \right| > 2N^{-\rho} \right) \\ & \leq \mathbb{P}_x \left(\sup_{k \leq N} \left| \sum_{i=1}^k (Y'_i - \mu_i(x) - \sigma W'_i) \right| > 2N^{\frac{1}{2}-\rho} - \bar{\mu}(x) \right) \\ & \leq C_0(x) N^{-\alpha \frac{1+\alpha}{1+2\alpha} + \rho(2+2\alpha)}. \end{aligned}$$

If $\bar{\mu}(x) > N^{\frac{1}{2}-\rho}$, it is obvious that

$$(7.36) \quad 1 \leq \left(\bar{\mu}(x) N^{-\frac{1}{2}+\rho} \right)^{2\alpha} \leq \bar{\mu}(x)^{2\alpha} N^{-\alpha+2\rho\alpha}.$$

From (7.35) and (7.36) we get

$$\begin{aligned} & \mathbb{P}_x \left(N^{-\frac{1}{2}} \sup_{k \leq N} \left| \sum_{i=1}^k (Y'_i - \mu - \sigma W'_i) \right| > 2N^{-\rho} \right) \\ & \leq (C_0(x) + \bar{\mu}(x)^{2\alpha}) N^{-\alpha \frac{1+\alpha}{1+2\alpha} + \rho(2+2\alpha)}. \end{aligned}$$

Taking into account the expressions for $\lambda_0(x), \mu_\delta(x), \tau(x), \lambda_1, \lambda_2, \bar{\mu}(x)$ and choosing γ small we obtain

$$C_0(x) + \bar{\mu}(x)^{2\alpha} \leq C(x) = C_1(1 + \|\delta_x\|_{\mathcal{B}'} + \mu_\delta(x))^{2+2\delta},$$

where C_1 is a constant depending only on $\delta, \alpha, \kappa, C_{\mathbf{P}}, C_Q, \|e\|_{\mathcal{B}}, \|\nu\|_{\mathcal{B}'}$.

Generally the measure \mathbb{P}_x and the constructed sequence $(Y'_i)_{i \geq 1}$ depend both on the initial state x . It is easy to reconstruct $(Y'_i)_{i \geq 1}$ independently of x . Indeed, on the canonical space $\tilde{\Omega} = \mathbb{R}^\infty \times \mathbb{R}^\infty$ there is a probability measure $\tilde{\mathbb{P}}_x$ which coincides with the joint distribution of the sequence $(Y'_i, W)_{i \geq 1}$. It is enough to redefine $Y'_i = \omega_{1,i}$ and $W_i = \omega_{2,i}$ as the coordinate processes, where $\omega = (\omega_1, \omega_2) \in \tilde{\Omega}$. With this construction only the measure $\tilde{\mathbb{P}}_x$ depends on the initial state x . The measurability of the map $x \in \mathbb{X} \rightarrow \tilde{\mathbb{P}}_x(\cdot)$ follows from the construction.

7.4. Proof of Theorem 3.4. In addition to conditions of Theorem 3.3 assume hypothesis **M5**. First we note that hypothesis **M5** ensures the existence of the mean $\nu(f) = \mathbf{E}_\nu X_k = \int (\mathbb{E}_x X_k) \nu(dx)$ and of the mixed moment $\mathbf{E}_\nu(X_l X_{l+k}) = \int \mathbb{E}_x(X_l X_{l+k}) \nu(dx)$ with respect to the invariant measure. By Proposition 3.2, we have $\lim_{k \rightarrow \infty} \mathbb{E}_x X_k = \mu$, ν -a.s. on \mathbb{X} . Then by Lebesgue theorem on dominated convergence

$$\nu(f) = \mathbf{E}_\nu X_k = \lim_{k \rightarrow \infty} \int (\mathbb{E}_x X_k) \nu(dx) = \int \left(\lim_{k \rightarrow \infty} \mathbb{E}_x X_k \right) \nu(dx) = \mu.$$

Without loss of generality we can assume that $\nu(f) = 0$. Using hypothesis **M5** and $\nu(f) = 0$, we have

$$\begin{aligned} \int \text{Cov}_{\mathbf{P}_x}(X_l, X_{l+k}) \nu(dx) &= \int \mathbb{E}_x(X_l X_{l+k}) \nu(dx) - \int \mathbb{E}_x(X_l) \mathbb{E}_x(X_{l+k}) \nu(dx) \\ &= \mathbf{E}_\nu(X_l X_{l+k}) - \int \mathbb{E}_x(X_l) \mathbb{E}_x(X_{l+k}) \nu(dx) \\ &= \text{Cov}_{\mathbb{P}_\nu}(X_0, X_k) - \int \mathbb{E}_x(X_l) \mathbb{E}_x(X_{l+k}) \nu(dx). \end{aligned}$$

By Proposition 3.2, we have $\lim_{l \rightarrow \infty} \text{Cov}_{\mathbf{P}_x}(X_l, X_{l+k}) = s_k$ and $\lim_{l \rightarrow \infty} \mathbb{E}_x(X_l) = 0$ for any $x \in \mathbb{X}$. As before, integrating with respect to the stationary measure and using Lebesgue theorem on dominated convergence, it follows that $s_k = \text{Cov}_{\mathbb{P}_\nu}(X_0, X_k)$. Thus the conclusions of Theorem 3.3 hold true with $\mu = \nu(f)$ and $\sigma^2 = \sigma_\nu^2$, which proves Theorem 3.4.

8. MAXIMAL INEQUALITIES

In this section we state two bounds which are used repeatedly in the paper. The first one gives a control for the L_p -norm of the maxima of the partial sums of a sequence of dependent r.v.'s. This proposition is a consequence of the second one which give a control of the L_p -norm of the partial sums of a sequence of dependent r.v.'s. It is assumed that conditions **C1** and **C2** hold true.

Proposition 8.1. *Let $\delta' < \delta$ and $\epsilon > 0$. Then, there is a constant $c_{\lambda_1, \lambda_2, \delta, \delta', \epsilon}$ such that for any $m, n \geq 1$ it holds*

$$\left\| \sup_{1 \leq k \leq n} \left\| \sum_{i=m}^{m+k-1} X_i \right\|_{L^{2+2\delta'}} \right\| \leq c_{\lambda_1, \lambda_2, \delta, \delta', \epsilon} (1 + \lambda_0 + \mu_\delta)^{1+\epsilon} n^{\frac{1}{2}}.$$

Proof. Denote for brevity $S_{m,n} = \sum_{i=m}^{m+n-1} X_i$. Let δ'' be such that $\delta' < \delta'' < \delta$. By Proposition 8.2 (which we assume for the moment), for any $m, n \geq 1$ and $\epsilon > 0$ we have $\|S_{m,n}\|_{L^{2+2\delta''}} \leq An^{\frac{1}{2}}$, where $A = c_{\lambda_1, \lambda_2, \delta'', \epsilon} (1 + \lambda_0 + \mu_\delta)^{1+\epsilon}$. Substituting $S'_{m,n} = S_{m,n}/A$, we get $\|S'_{m,n}\|_{L^{2+2\delta''}} \leq n^{\frac{1}{2}}$, for any $m, n \geq 1$. By Theorem A in Serfling [35] (see also Billingsley [4] p. 102), it follows that $\|\sup_{1 \leq k \leq n} S'_{m,n}\|_{L^{2+2\delta''}} \leq n^{\frac{1}{2}} \log_2(4n)$, for any $m, n \geq 1$. Since $\delta' < \delta''$, it follows that

$$\left\| \sup_{1 \leq k \leq n} S'_{m,n} \right\|_{L^{2+2\delta'}} \leq \left\| \sup_{1 \leq k \leq n} S'_{m,n} \right\|_{L^{2+2\delta''}} \leq \left(n^{\frac{1}{2}} \log_2(4n) \right)^{\frac{2+2\delta''}{2+2\delta'}} \leq c_{\delta, \delta'} n^{\frac{1}{2}},$$

from which we deduce $\|\sup_{1 \leq k \leq n} S_{m,n}\|_{L^{2+2\delta'}} \leq Ac_{\delta, \delta'} n^{\frac{1}{2}}$. \square

The following assertion is an adaptation of Proposition 4.1 in Gouëzel [15]. In order to derive an explicit dependence of the constant involved in the bound on some of the constants in conditions **C1** and **C2** we give an independent proof. Tracking this explicit dependence plays a crucial role in the proof of Theorem 3.3 to work out the dependence of the bound on the initial state of the Markov chain $X_0 = x$.

Proposition 8.2. *Let $0 < \delta' < \delta$ and $\epsilon > 0$. Then, $c_{\lambda_1, \lambda_2, \delta', \epsilon}$ such that, for any $m, n \geq 1$,*

$$\left\| \sum_{i=m}^{m+n-1} X_i \right\|_{L^{2+2\delta'}} \leq c_{\lambda_1, \lambda_2, \delta', \epsilon} (1 + \lambda_0 + \mu_\delta)^{1+\epsilon} n^{\frac{1}{2}}.$$

The proof of this proposition is given below. First we state several auxiliary assertions.

8.1. Auxiliary assertions.

Proposition 8.3. *There is a constant $c_{\lambda_1, \lambda_2, \epsilon}$ such that, for any $\epsilon > 0$,*

$$(8.1) \quad \left\| \sum_{i=m}^{m+n-1} X_i \right\|_{L^2} \leq c_{\lambda_1, \lambda_2, \epsilon} (1 + \lambda_0 + \mu_\delta)^{1+\epsilon} n^{\frac{1}{2}}.$$

The proof is based on the following two lemmas.

Lemma 8.4. *Let $u_n = \max_{m \geq 1} \left\| \sum_{i=m}^{m+n-1} X_i \right\|_{L^2}^2$, $n \geq 1$. Then, for any natural numbers $a, b \geq 1$ and any $\alpha \in (0, \frac{1}{2})$, $\gamma \in (0, \delta)$, it holds*

$$u_{a+b} \leq A + u_a + u_b + (c\mu_\delta)^2 (a^{2\alpha} + b^{2\alpha}) + c\mu_\delta (a^\alpha + b^\alpha) (A + u_a + u_b)^{1/2} + cu_a^{1/2} + cu_b^{1/2},$$

where $c > 1$ and $A = c_{\lambda_1, \lambda_2, \gamma, \alpha} (1 + \lambda_0 + \mu_\delta)^{2+\gamma}$.

Proof. Let $m \in \mathbb{N}$. Assume that $a \leq b$ (the case $a > b$ is treated in the same manner). Denote $Y_1 = \sum_{i=m}^{m+a-1} X_i$, $Y_2 = \sum_{i=m+a+\lfloor b^\alpha \rfloor}^{m+a+b-1} X_i$ and $Y_0 = \sum_{i=m}^{m+a+b-1} X_i$, where $\alpha \in (0, \frac{1}{2})$. Note that $Y_0 = Y_1 + Y_2 + Y_{gap}$, where $Y_{gap} = \sum_{i=m+a}^{m+a+\lfloor b^\alpha \rfloor-1} X_i$. Therefore

$$(8.2) \quad \|Y_0\|_{L^2}^2 \leq \|Y_1 + Y_2\|_{L^2}^2 + \|Y_{gap}\|_{L^2}^2 + 2\|Y_1 + Y_2\|_{L^2} \|Y_{gap}\|_{L^2}.$$

In the sequel we shall bound each of the terms in the right-hand side of (8.2).

Let V_1 and V_2 be two independent identically distributed r.v.'s of mean 0, independent of Y_1 and Y_2 with a common characteristic function supported in the interval $[-\varepsilon_0, \varepsilon_0]$, for some $\varepsilon_0 \in (0, 1)$ and such that $\|V_i\|_{L^{2+2\delta}} \leq c$. Denote $\tilde{Y}_1 = Y_1 + V_1$ and $\tilde{Y}_2 = Y_2 + V_2$. Let Z_1 and Z_2 be independent copies of \tilde{Y}_1 and \tilde{Y}_2 . Since $\mathbf{E}e^{itV_1}$ is supported in the interval $[-\varepsilon_0, \varepsilon_0]$, by Lemma 9.5, for any $T > 0$,

$$\begin{aligned} \pi \left(\left(\tilde{Y}_1, \tilde{Y}_2 \right), (Z_1, Z_2) \right) &\leq \frac{T}{\pi} \left(\int_{[-\varepsilon_0, \varepsilon_0]^2} |\phi(t, u) - \psi_1(t) \psi_2(u)|^2 dt du \right)^{1/2} \\ &\quad + \mathbb{P} \left(\max \left\{ \left| \tilde{Y}_1 \right|, \left| \tilde{Y}_2 \right| \right\} > T \right), \end{aligned}$$

where ϕ is the characteristic function of the vector (Y_1, Y_2) and ψ_1 and ψ_2 are the characteristic functions of the r.v.'s Y_1 and Y_2 . Condition **C1** implies that

$$\begin{aligned} |(\phi(t, u) - \psi_1(t) \psi_2(u))| &\leq \lambda_0 (1+b)^{2\lambda_2} \exp(-\lambda_1 [b^\alpha]) \\ &\leq \lambda_0 c_{\lambda_1} (1+b)^{2\lambda_2} \exp(-\lambda_1 b^\alpha). \end{aligned}$$

Let $T = e^{\lambda_1 b^\alpha / 2}$. Taking into account that

$$\begin{aligned} \mathbb{P} \left(\max \left\{ \left| \tilde{Y}_1 \right|, \left| \tilde{Y}_2 \right| \right\} > T \right) &\leq T^{-1} \mathbb{E} \max \left\{ \left| \tilde{Y}_1 \right|, \left| \tilde{Y}_2 \right| \right\} \\ &\leq T^{-1} (\|Y_1 + V_1\|_{L^{2+2\delta}} + \|Y_2 + V_2\|_{L^{2+2\delta}}) \\ &\leq e^{-\frac{\lambda_1}{2} b^\alpha} \left(c + (a+b) \max_{l \geq 0} \|X_l\|_{L^{2+2\delta}} \right) \\ &\leq c_\delta e^{-\frac{\lambda_1}{2} b^\alpha} b (1 + \mu_\delta) \end{aligned}$$

we obtain

$$\begin{aligned} \pi \left(\left(\tilde{Y}_1, \tilde{Y}_2 \right), (Z_1, Z_2) \right) &\leq \frac{1}{\pi} \lambda_0 (1+b)^{2\lambda_2} e^{-\frac{\lambda_1}{2} b^\alpha} + c_\delta b e^{-\frac{\lambda_1}{2} b^\alpha} (1 + \mu_\delta) \\ &\leq \Delta = c_{\delta, \alpha} (1+b)^{2\lambda_2} e^{-\frac{\lambda_1}{2} b^\alpha} (1 + \lambda_0 + \mu_\delta). \end{aligned}$$

By Lemma 9.1 there is a coupling of $(\tilde{Y}_1, \tilde{Y}_2)$ and (Z_1, Z_2) such that

$$P \left(\left\| \left(\tilde{Y}_1, \tilde{Y}_2 \right) - (Z_1, Z_2) \right\|_\infty \geq \Delta \right) \leq \Delta.$$

Let $S = \tilde{Y}_1 + \tilde{Y}_2 - (Z_1 + Z_2)$. Taking into account that $\|V_i\|_{L^{2+2\delta}} \leq c$, we have

$$\begin{aligned}
\|S\|_{L^{2+2\delta}} &= \left\| \tilde{Y}_1 + \tilde{Y}_2 - (Z_1 + Z_2) \right\|_{L^{2+2\delta}} \\
&\leq 2 \left\| \tilde{Y}_1 + \tilde{Y}_2 \right\|_{L^{2+2\delta}} \\
&\leq c(a+b) \left(1 + \max_{l \geq 0} \|X_l\|_{L^{2+2\delta}} \right) \\
(8.3) \quad &\leq cb(1 + \mu_\delta).
\end{aligned}$$

Then, for any $\gamma \in (0, \delta)$,

$$\begin{aligned}
\|S\|_{L^2}^2 &\leq 4\Delta^2 + \mathbb{E}|S|^2 \mathbf{1}(|S| \geq 2\Delta) \\
&\leq 4\Delta^2 + \|S\|_{L^{2+2\gamma}}^2 \mathbb{P}(|S| \geq 2\Delta)^{\frac{\gamma}{1+\gamma}} \\
&\leq 4\Delta^2 + cb^2(1 + \mu_\delta)^2 \Delta^{\frac{\gamma}{1+\gamma}} \\
&\leq 4c^2(1+b)^{4\lambda_2} (1 + \lambda_0 + \mu_\delta)^2 e^{-\lambda_1 b^\alpha} \\
&\quad + 4b^2(1 + \mu_\delta)^2 c_{\delta, \alpha}^{\frac{\gamma}{1+\gamma}} (1+b)^{4\lambda_2 \frac{\gamma}{1+\gamma}} e^{-\frac{\lambda_1}{2} \frac{\gamma}{1+\gamma} b^\alpha} (1 + \lambda_0 + \mu_\delta)^{\frac{\gamma}{1+\gamma}} \\
(8.4) \quad &\leq A',
\end{aligned}$$

where $A' = c'_{\lambda_1, \lambda_2, \gamma, \alpha} (1 + \lambda_0 + \mu_\delta)^{2+\gamma}$. From (8.3) and (8.4), it follows that

$$\begin{aligned}
\left\| \tilde{Y}_1 + \tilde{Y}_2 \right\|_{L^2}^2 &\leq \|S\|_{L^2}^2 + \|Z_1\|_{L^2}^2 + \|Z_2\|_{L^2}^2 \\
(8.5) \quad &\leq A' + \left\| \tilde{Y}_1 \right\|_{L^2}^2 + \left\| \tilde{Y}_2 \right\|_{L^2}^2.
\end{aligned}$$

Since $\|V_i\|_{L^2} \leq c$, we have

$$(8.6) \quad \|Y_1 + Y_2\|_{L^2} \leq \left\| \tilde{Y}_1 + \tilde{Y}_2 \right\|_{L^2} + 2c.$$

Taking into account (8.5) and (8.6), one gets

$$\begin{aligned}
(8.7) \quad &\|Y_1 + Y_2\|_{L^2}^2 \\
&\leq A' + \left\| \tilde{Y}_1 \right\|_{L^2}^2 + \left\| \tilde{Y}_2 \right\|_{L^2}^2 + 4c \left\| \tilde{Y}_1 \right\|_{L^2} + 4c \left\| \tilde{Y}_2 \right\|_{L^2} + 4c^2.
\end{aligned}$$

Since $\left\| \tilde{Y}_k \right\|_{L^2} \leq \|Y_k\|_{L^2} + c$, it holds

$$\begin{aligned}
\|Y_1 + Y_2\|_{L^2}^2 &\leq A' + (\|Y_1\|_{L^2} + c)^2 + (\|Y_2\|_{L^2} + c)^2 \\
&\quad + 4c(\|Y_1\|_{L^2} + c) + 4c(\|Y_2\|_{L^2} + c) + 4c^2 \\
(8.8) \quad &\leq A' + \|Y_1\|_{L^2}^2 + \|Y_2\|_{L^2}^2 + 6c(\|Y_1\|_{L^2} + \|Y_2\|_{L^2}) + 14c^2.
\end{aligned}$$

Since the gap is of size $[b^\alpha]$,

$$(8.9) \quad \sup_{m \geq 1} \|Y_{gap}\|_{L^2} \leq [b^\alpha] \max_{i \geq 1} \|X_i\|_{L^{2+2\delta}} \leq b^\alpha \mu_\delta.$$

From (8.2), (8.8) and (8.9) we obtain

$$(8.10) \quad \begin{aligned} \|Y_0\|_{L^2}^2 &\leq A' + \|Y_1\|_{L^2}^2 + \|Y_2\|_{L^2}^2 + b^{2\alpha} \mu_\delta^2 \\ &\quad + 2b^\alpha \mu_\delta (\|Y_1\|_{L^2} + \|Y_2\|_{L^2}) \\ &\quad + 6c (\|Y_1\|_{L^2} + \|Y_2\|_{L^2}) + 14c^2. \end{aligned}$$

Recall that $u_a = \sup_{m \geq 1} \|Y_1\|_{L^2}^2$, $u_b = \sup_{m \geq 1} \|Y_2\|_{L^2}^2$ and $u_{a+b} = \sup_{m \geq 1} \|Y_0\|_{L^2}^2$. Using (8.9) we have $\|Y_2\|_{L^2} \leq u_b^{1/2} + \|Y_{gap}\|_{L^2} \leq u_b^{1/2} + b^\alpha \mu_\delta$. From this and (8.10) we deduce that

$$\begin{aligned} u_{a+b} &\leq A' + u_a + u_b + b^{2\alpha} \mu_\delta^2 + 2b^\alpha \mu_\delta u_b^{1/2} \\ &\quad + 2b^\alpha \mu_\delta (u_a^{1/2} + u_b^{1/2} + b^\alpha \mu_\delta) \\ &\quad + 6c (u_a^{1/2} + u_b^{1/2} + b^\alpha \mu_\delta) \text{ over } + 14c^2. \end{aligned}$$

Rearranging the terms and taking into account that $A' > 1$, we obtain

$$\begin{aligned} u_{a+b} &\leq A' + 14c^2 + u_a + u_b + 3b^{2\alpha} \mu_\delta^2 \\ &\quad + b^\alpha \mu_\delta (6c + u_a^{1/2} + u_b^{1/2}) + 6cu_a^{1/2} + 6cu_b^{1/2}, \end{aligned}$$

which proves the lemma. \square

Lemma 8.5. *Assume that the sequence $(u_n)_{n \geq 1}$ is such that $u_n > 0$ and*

$$u_{a+b} \leq (u_a + u_b + A) + (a^{2\alpha} + b^{2\alpha}) B^2 + (a^\alpha + b^\alpha) B (u_a + u_b + A)^{1/2} + cu_a^{1/2} + cu_b^{1/2},$$

for all $a, b \geq 1$ and some $A, B > 0$, $\alpha \in (0, \frac{1}{2})$. Then

$$u_n \leq c_\alpha (1 + u_1 + A + B^2) n.$$

Proof. Note that $xy \leq \frac{1}{2}(\varepsilon x^2 + \varepsilon^{-1}y^2)$, for any $x, y, \varepsilon > 0$. Using the assumption of the lemma, we have

$$\begin{aligned} u_{a+b} &\leq (u_a + u_b + A) + (1 + \varepsilon^{-1}) (a^{2\alpha} + b^{2\alpha}) B^2 \\ &\quad + \frac{\varepsilon}{2} (u_a + u_b + A) + \varepsilon^{-1} c^2 + \frac{\varepsilon}{2} u_a + \frac{\varepsilon}{2} u_b \\ &\leq (1 + \varepsilon) (u_a + u_b + A) + \varepsilon^{-1} c^2 + (1 + \varepsilon^{-1}) (a^{2\alpha} + b^{2\alpha}) B^2. \end{aligned}$$

Denote $v_k = \max_{1 \leq n \leq 2^k} u_n$, $k \geq 0$. From the previous equation it follows that

$$v_{k+1} \leq (1 + \varepsilon) (2v_k + A) + \varepsilon^{-1} c^2 + (1 + \varepsilon^{-1}) 2^{2\alpha k+1} B^2.$$

Dividing by $(2 + 2\varepsilon)^{k+1}$ we get

$$\begin{aligned} \frac{v_{k+1}}{(2 + 2\varepsilon)^{k+1}} &\leq \frac{2v_k + A}{2(2 + 2\varepsilon)^k} + (1 + \varepsilon^{-1}) \frac{2^{2\alpha k+1}}{(2 + 2\varepsilon)^{k+1}} (B^2 + c^2) \\ &\leq \frac{v_k}{(2 + 2\varepsilon)^k} + \frac{A}{2(2 + 2\varepsilon)^k} + (1 + \varepsilon^{-1}) \frac{2^{2\alpha k+1}}{(2 + 2\varepsilon)^{k+1}} (B^2 + c^2). \end{aligned}$$

Taking into account that $\alpha < \frac{1}{2}$, by induction, we obtain

$$\begin{aligned} \frac{v_k}{(2+2\varepsilon)^k} &\leq v_0 + \frac{A}{2} \sum_{i=0}^{\infty} \frac{1}{(2+2\varepsilon)^i} + 2(1+\varepsilon^{-1})(B^2+c^2) \sum_{i=1}^{\infty} \frac{2^{2\alpha i}}{(2+2\varepsilon)^{i+1}} \\ &\leq v_0 + \frac{A}{2} \frac{2+2\varepsilon}{1+2\varepsilon} + 2(1+\varepsilon^{-1})(B^2+c^2) \frac{1+\varepsilon}{\varepsilon} \\ &\leq v_0 + c'_\varepsilon (A+B^2+c^2), \end{aligned}$$

where c'_ε depends only on ε . This implies that

$$v_k = \max_{1 \leq n \leq 2^k} u_n \leq C_0 (2+2\varepsilon)^k,$$

where $C_0 = (v_0 + c'_\varepsilon (A+B^2+c^2))$. Once again using the assumption of the lemma it follows that

$$\begin{aligned} &v_{k+1} \\ &\leq (2v_k + A) + 2^{2\alpha k+1} B^2 + 2^{\alpha k+1} B \left(2C_0 (2+2\varepsilon)^k + A \right)^{1/2} + 2cC_0^{1/2} (2+2\varepsilon)^{k/2} \\ &\leq (2v_k + A) + 2^{2\alpha k+1} B^2 + 2^{\alpha k+1} B \left(2C_0^{1/2} (2+2\varepsilon)^{k/2} + A^{1/2} \right) + 2cC_0^{1/2} (2+2\varepsilon)^{k/2}. \end{aligned}$$

Dividing by 2^{k+1} and choosing $\varepsilon = \varepsilon(\alpha)$ so small that $2+2\varepsilon \leq 2^{1+(\frac{1}{2}-\alpha)}$, one gets

$$\begin{aligned} \frac{v_{k+1}}{2^{k+1}} &\leq \frac{v_k}{2^k} + \frac{A}{2^{k+1}} + 2^{(2\alpha-1)k+1} B^2 + 2^{(\alpha-1)k+1} B \left(2C_0^{1/2} 2^{\frac{k}{2}+\frac{k}{2}(\frac{1}{2}-\alpha)} + A^{1/2} \right) \\ &\quad + 2^{-k} cC_0^{1/2} 2^{\frac{k}{2}+\frac{k}{2}(\frac{1}{2}-\alpha)} \\ &\leq \frac{v_k}{2^k} + \frac{A}{2^{k+1}} + 2^{(2\alpha-1)k+1} B^2 + 4BC_0^{1/2} 2^{(\alpha-\frac{1}{2})\frac{k}{2}} + 2^{(\alpha-1)k+1} BA^{1/2} \\ &\quad + cC_0^{1/2} 2^{-\frac{k}{2}(\frac{1}{2}+\alpha)}. \end{aligned}$$

Using induction, this implies

$$\frac{v_k}{2^k} \leq c_\alpha (A + C_0 + B^2),$$

since ε depends only on α . From this we get $u_{2^k} \leq D2^k$, for any $k \geq 1$, where $D = c'_\alpha (1 + v_0 + A + B^2)$. Therefore, for any $2^{k-1} \leq n \leq 2^k$ it holds $u_n \leq D2^k \leq 2D2^{k-1} \leq 2Dn$. Lemma is proved. \square

Let $\alpha = \frac{1}{4}$. In the notations of Lemma 8.4 $u_1 \leq \max_{m \geq 1} \|X_m\|_{L^{2+2\delta}}^2 \leq \mu_\delta^2$. From Lemmas 8.4 and 8.5 with $B = c\mu_\delta$ it follows, for any $\epsilon \in (0, \delta)$,

$$\begin{aligned} \max_{m \geq 1} \left\| \sum_{i=m}^{m+n-1} X_i \right\|_{L^2}^2 &= u_n \leq c_\alpha (u_1 + A + c^2 \mu_\delta^2) n \\ &\leq c \left(c_{\lambda_1, \lambda_2, \epsilon} (1 + \lambda_0 + \mu_\delta)^{2+\frac{\epsilon}{1+\epsilon}} + 2\mu_\delta^2 \right) n \\ &\leq c'_{\lambda_1, \lambda_2, \epsilon} (1 + \lambda_0 + \mu_\delta)^{2+\epsilon} n, \end{aligned}$$

which proves Proposition 8.3.

8.2. Proof of Proposition 8.2. Let $m, n \in \mathbb{N}$ and $a = [n^{1-\alpha}]$ and $b = [n^{\alpha+\rho}]$, where $\alpha > 0$, and $\rho > 0$ are such that $2\alpha + \rho < 1$. Note that $a > b$ and $ba \leq n^{1-\rho}$. Consider the intervals $I_k = [m + (k-1)a, m + ka - b)$, $J_k = [m + ka - b, m + ka)$, for $k = 1, \dots, [n^\alpha]$ and $I_{\text{fin}} = [m + ba, m + n)$, such that $[m, m + n) = \cup_{k=1}^{[n^\alpha]} (I_k \cup J_k) \cup I_{\text{fin}}$. Here $a - b > 0$ and $b > 0$ are interpreted as the length of an island I_k and the length of a gap J_k respectively.

Denote $Y_k = \sum_{i \in I_k} X_i$, $k = 1, \dots, [n^\alpha]$. Let $V_1, \dots, V_{[n^\alpha]}$ be independent identically distributed r.v.'s of mean 0, independent of $Y_1, \dots, Y_{[n^\alpha]}$ with a common characteristic function supported in the interval $[-\varepsilon_0, \varepsilon_0]$, for some $\varepsilon_0 > 0$ and such that $\|V_k\|_{L^{2+2\delta}} \leq c$, $k = 1, \dots, [n^\alpha]$. Denote $\tilde{Y}_k = Y_k + V_k$. Let $Z_1, \dots, Z_{[n^\alpha]}$ be independent copies of $\tilde{Y}_1, \dots, \tilde{Y}_{[n^\alpha]}$. By Lemma 9.3

$$(8.11) \quad \pi \left(\left(\tilde{Y}_1, \dots, \tilde{Y}_{[n^\alpha]} \right), (Z_1, \dots, Z_{[n^\alpha]}) \right) \leq \sum_{k=1}^{[n^\alpha]} \pi \left(\left(\tilde{Y}_1, \dots, \tilde{Y}_{k-1}, \tilde{Y}_k \right), \left(\tilde{Y}_1, \dots, \tilde{Y}_{k-1}, Z_k \right) \right).$$

Since $\mathbf{E}e^{itV_k}$ is supported in the interval $[-\varepsilon_0, \varepsilon_0]$, by Lemma 9.5, for any $T > 0$ and $k \leq [n^\alpha]$,

$$(8.12) \quad \begin{aligned} & \pi \left(\left(\tilde{Y}_1, \dots, \tilde{Y}_{k-1}, \tilde{Y}_k \right), \left(\tilde{Y}_1, \dots, \tilde{Y}_{k-1}, Z_k \right) \right) \\ & \leq \frac{T}{\pi} \left(\int_{[-\varepsilon_0, \varepsilon_0]^k} |\phi(t, u) - \psi_1(t) \psi_2(u)|^2 dt du \right)^{1/2} \\ & + \mathbb{P} \left(\left\| \left(\tilde{Y}_1, \dots, \tilde{Y}_{k-1}, \tilde{Y}_k \right) \right\|_\infty > T \right), \end{aligned}$$

where ϕ is the characteristic function of $\left(\tilde{Y}_1, \dots, \tilde{Y}_{k-1}, \tilde{Y}_k \right)$ and ψ_1 and ψ_2 are the characteristic functions of the r.v.'s $\left(\tilde{Y}_1, \dots, \tilde{Y}_{k-1} \right)$ and \tilde{Y}_k . Condition **C1** implies that

$$(8.13) \quad \begin{aligned} |\phi(t, u) - \psi_1(t) \psi_2(u)| & \leq \lambda_0 (1 + a)^k \exp(-\lambda_1 b) \\ & \leq c_{\lambda_1} \lambda_0 (1 + n^{1-\alpha})^{n^\alpha} \exp(-\lambda_1 n^{\alpha+\rho}). \end{aligned}$$

Let $T = e^{\frac{\lambda_1}{2} n^{\alpha+\rho}}$. By Chebychev's inequality, taking into account that for $k \leq [n^\alpha]$, we have

$$(8.14) \quad \begin{aligned} \mathbb{P} \left(\left\| \left(\tilde{Y}_1, \dots, \tilde{Y}_{k-1}, \tilde{Y}_k \right) \right\|_\infty > T \right) & \leq T^{-1} \sum_{i=1}^k \|Y_i + V_i\|_{L^1} \\ & \leq T^{-1} \sum_{i=1}^k (\|Y_i\|_{L^{2+2\delta}} + c) \\ & \leq e^{-\frac{\lambda_1}{2} n^{\alpha+\rho}} [n^\alpha] ([n^{1-\alpha}] \mu_\delta + c) \\ & \leq c n e^{-\frac{\lambda_1}{2} n^{\alpha+\rho}} (1 + \mu_\delta). \end{aligned}$$

From (8.12), (8.13) and (8.14) we obtain

$$\begin{aligned} & \pi \left(\left(\tilde{Y}_1, \dots, \tilde{Y}_{k-1}, \tilde{Y}_k \right), \left(\tilde{Y}_1, \dots, \tilde{Y}_{k-1}, Z_k \right) \right) \\ & \leq \frac{T}{\pi} \lambda_0 \varepsilon_0^n (1 + n^{1-\alpha})^{n^\alpha} e^{-\lambda_1 n^{\alpha+\rho}} + c_\alpha n e^{-\frac{\lambda_1}{2} n^{\alpha+\rho}} (1 + \mu_\delta) \\ & \leq c \varepsilon_0^n n (1 + n^{1-\alpha})^{n^\alpha} e^{-\frac{\lambda_1}{2} n^{\alpha+\rho}} (1 + \lambda_0 + \mu_\delta). \end{aligned}$$

Implementing this bound in (8.11) we get

$$\begin{aligned} \pi \left(\left(\tilde{Y}_1, \dots, \tilde{Y}_{[n^\alpha]} \right), \left(Z_1, \dots, Z_{[n^\alpha]} \right) \right) & \leq c \varepsilon_0^n n^{1+\alpha} (1 + n^{1-\alpha})^{n^\alpha} e^{-\frac{\lambda_1}{2} n^{\alpha+\rho}} (1 + \lambda_0 + \mu_\delta) \\ & \leq \Delta = c_{\alpha, \lambda_1} e^{-\frac{\lambda_1}{4} n^{\alpha+\rho}} (1 + \lambda_0 + \mu_\delta). \end{aligned}$$

According to Strassen-Dudley's theorem (see Lemma 9.1) there is a coupling of $(\tilde{Y}_1, \dots, \tilde{Y}_b)$ and (Z_1, \dots, Z_b) such that

$$P \left(\left\| \left(\tilde{Y}_1, \dots, \tilde{Y}_{[n^\alpha]} \right) - \left(Z_1, \dots, Z_{[n^\alpha]} \right) \right\|_\infty \geq \Delta \right) \leq \Delta.$$

Let $S = \tilde{Y}_1 + \dots + \tilde{Y}_{[n^\alpha]} - (Z_1 + \dots + Z_{[n^\alpha]})$. Taking into account that $\|V_i\|_{L^{2+2\delta}} \leq c$, we have

$$\begin{aligned} \|S\|_{L^{2+2\delta}} & = \left\| \tilde{Y}_1 + \dots + \tilde{Y}_{[n^\alpha]} - (Z_1 + \dots + Z_{[n^\alpha]}) \right\|_{L^{2+2\delta}} \\ (8.15) \quad & \leq cn^\alpha a \left(1 + \max_{l \geq 1} \|X_l\|_{L^{2+2\delta}} \right) \leq c'n (1 + \mu_\delta). \end{aligned}$$

Let $\eta \in (0, \delta - \delta')$, $p = 2 + 2\delta'$, $p' = p + 2\eta \leq 2 + 2\delta$ and $\gamma = \gamma(\eta) = \frac{2\eta}{p(p+2\eta)}$. By Hölder's inequality,

$$\left\| S^{2+2\delta'} \mathbf{1}(|S| \geq n^\alpha \Delta) \right\|_{L^{2+2\delta'}} \leq \|S\|_{L^{p'}} (\mathbb{P}(|S| \geq n^\alpha \Delta))^\gamma \leq \|S\|_{L^{2+2\delta}} (\mathbb{P}(|S| \geq n^\alpha \Delta))^\gamma.$$

Using the bound $|S| \leq n^\alpha \left\| \left(\tilde{Y}_1, \dots, \tilde{Y}_{[n^\alpha]} \right) - \left(Z_1, \dots, Z_{[n^\alpha]} \right) \right\|_\infty$ and we have

$$\begin{aligned} \|S\|_{L^{2+2\delta'}} & \leq n^\alpha \Delta + \left\| S^{2+2\delta'} \mathbf{1}(|S| \geq n^\alpha \Delta) \right\|_{L^{2+2\delta'}} \\ & \leq n^\alpha \Delta + \|S\|_{L^{2+2\delta}} (\mathbb{P}(|S| \geq n^\alpha \Delta))^\gamma \\ & \leq n^\alpha \Delta + \|S\|_{L^{2+2\delta}} \left(\mathbb{P} \left(\left\| \left(\tilde{Y}_1, \dots, \tilde{Y}_{[n^\alpha]} \right) - \left(Z_1, \dots, Z_{[n^\alpha]} \right) \right\|_\infty \geq \Delta \right) \right)^\gamma \\ & \leq n^\alpha \Delta + cn (1 + \mu_\delta) \Delta^\gamma. \end{aligned}$$

Taking into account the definition of Δ ,

$$\begin{aligned} \|S\|_{L^{2+2\delta'}} & \leq n^\alpha c_{\alpha, \lambda_1} e^{-\frac{\lambda_1}{4} n^{\alpha+\rho}} (1 + \lambda_0 + \mu_\delta) \\ & \quad + cn (1 + \mu_\delta) \left(c_{\alpha, \lambda_1} e^{-\frac{\lambda_1}{4} n^{\alpha+\rho}} (1 + \lambda_0 + \mu_\delta) \right)^\gamma \\ (8.16) \quad & \leq A' = c'_{\lambda_1, \lambda_2, \gamma, \alpha, \rho} (1 + \lambda_0 + \mu_\delta)^{1+\gamma}. \end{aligned}$$

From (8.15) and (8.16), it follows that

$$\begin{aligned} \left\| \tilde{Y}_1 + \dots + \tilde{Y}_{[n^\alpha]} \right\|_{L^{2+2\delta'}} & \leq \|S\|_{L^{2+2\delta'}} + \|Z_1 + \dots + Z_{[n^\alpha]}\|_{L^{2+2\delta'}} \\ (8.17) \quad & \leq A' + \|Z_1 + \dots + Z_{[n^\alpha]}\|_{L^{2+2\delta'}}. \end{aligned}$$

Since the r.v.'s $Z_1, \dots, Z_{[n^\alpha]}$ are independent, by Rosenthal's inequality (Theorem 3 in [30]), there exists some constant $c_{\delta'}$ such that

$$(8.18) \quad \|Z_1 + \dots + Z_{[n^\alpha]}\|_{L^{2+2\delta'}} \leq c_{\delta'} \left(\sum_{i=1}^{[n^\alpha]} \mathbb{E} Z_i^2 \right)^{\frac{1}{2}} + c_{\delta'} \left(\sum_{i=1}^{[n^\alpha]} \mathbb{E} |Z_i|^{2+2\delta'} \right)^{\frac{1}{2+2\delta'}}.$$

Taking into account that $Y_i = \sum_{j \in I_i} X_j$ and that $|I_i| \leq a - b \leq n^{1-\alpha}$, by Proposition 8.3 we have

$$(8.19) \quad \mathbf{E} Z_i^2 = \left\| \tilde{Y}_i \right\|_{L^2}^2 \leq (c + \|Y_i\|_{L^2})^2 \leq c'_{\lambda_1, \lambda_2, \gamma} (1 + \lambda_0 + \mu_\delta)^{2+\gamma} n^{1-\alpha}.$$

Note also that $\|Z_i\|_{L^{2+2\delta'}} \leq v_{a-b} + c$, where $v_n = \sup_{m \geq 1} \left\| \sum_{i=m}^{m+n-1} X_i \right\|_{L^{2+2\delta'}}$. Therefore, from (8.18) and (8.19), it follows that

$$\begin{aligned} & \|Z_1 + \dots + Z_b\|_{L^{2+2\delta'}} \\ & \leq A' + c_{\lambda_1, \lambda_2, \gamma, \alpha, \rho, \delta'} (1 + \lambda_0 + \mu_\delta)^{1+\frac{\gamma}{2}} n^{\frac{1}{2}} + c_{\delta'} \left(\sum_{i=1}^{[n^\alpha]} (v_{a-b} + c)^{2+2\delta'} \right)^{\frac{1}{2+2\delta'}} \\ & \leq c_{\lambda_1, \lambda_2, \gamma, \alpha, \rho, \delta'} (1 + \lambda_0 + \mu_\delta)^{1+\gamma} n^{\frac{1}{2}} + c_{\delta'} v_{a-b} n^{\frac{\alpha}{2+2\delta'}}. \end{aligned}$$

Using (8.17), we get

$$\begin{aligned} \left\| \tilde{Y}_1 + \dots + \tilde{Y}_{[n^\alpha]} \right\|_{L^{2+2\delta'}} & \leq A' + \|Z_1 + \dots + Z_{[n^\alpha]}\|_{L^{2+2\delta'}} \\ & \leq c'_{\lambda_1, \lambda_2, \gamma, \alpha, \rho, \delta'} (1 + \lambda_0 + \mu_\delta)^{1+\gamma} n^{\frac{1}{2}} + c_{\delta'} v_{a-b} n^{\frac{\alpha}{2+2\delta'}}. \end{aligned}$$

Since $\tilde{Y}_k = Y_k + V_k$ and $\|V_k\|_{L^{2+2\delta}} \leq c$, it holds

$$\begin{aligned} \|Y_1 + \dots + Y_{[n^\alpha]}\|_{L^{2+2\delta'}} & \leq c [n^\alpha] + \left\| \tilde{Y}_1 + \dots + \tilde{Y}_{[n^\alpha]} \right\|_{L^{2+2\delta'}} \\ & \leq c n^\alpha + c_{\lambda_1, \lambda_2, \gamma, \alpha, \rho, \delta'} (1 + \lambda_0 + \mu_\delta)^{1+\gamma} n^{\frac{1}{2}} + c_{\delta'} v_{a-b} n^{\frac{\alpha}{2+2\delta'}} \\ & \leq c_{\lambda_1, \lambda_2, \gamma, \alpha, \rho, \delta'} (1 + \lambda_0 + \mu_\delta)^{1+\gamma} n^{\frac{1}{2}} + c_{\delta'} v_{a-b} n^{\frac{\alpha}{2+2\delta'}}, \end{aligned}$$

where for the last line we use the fact that $\alpha < \frac{1-\rho}{2} < \frac{1}{2}$. Filling up the gaps in the final interval I_{fin} , we get

$$\begin{aligned} \left\| \sum_{i=1}^{m+n-1} X_i \right\|_{L^{2+2\delta'}} & \leq \|Y_1 + \dots + Y_{[n^\alpha]}\|_{L^{2+2\delta'}} \\ & \quad + \sum_{k=1}^{[n^\alpha]} \sum_{i \in J_k} \|X_i\|_{L^{2+2\delta'}} + \left\| \sum_{i=m+[n^\alpha]}^{m+n} X_i \right\|_{L^{2+2\delta'}} \\ & \leq c_{\lambda_1, \lambda_2, \gamma, \alpha, \rho, \delta'} (1 + \lambda_0 + \mu_\delta)^{1+\gamma} n^{\frac{1}{2}} + c_{\delta'} v_{a-b} n^{\frac{\alpha}{2+2\delta'}} \\ & \quad + n^{2\alpha+\rho} \mu_\delta + v_{n-[n^{1-\alpha}][n^\alpha]}. \end{aligned}$$

From this, we deduce the inequality

$$(8.20) \quad v_n \leq c_{\lambda_1, \lambda_2, \gamma, \alpha, \rho, \delta'} (1 + \lambda_0 + \mu_\delta)^{1+\gamma} n^{\frac{1}{2}} + n^{2\alpha+\rho} \mu_\delta + c_{\delta'} v_{[n^{1-\alpha}][n^\alpha]} n^{\frac{\alpha}{2+2\delta'}} + v_{n-[n^{1-\alpha}][n^\alpha]}.$$

Denote $\bar{v}_n = \frac{v_n}{(1+\lambda_0+\mu_\delta)^{1+\gamma}}$. Then from (8.20), it follows that

$$\bar{v}_n \leq c_{\lambda_1, \lambda_2, \gamma, \alpha, \rho, \delta'} n^{\frac{1}{2}} + n^{2\alpha+\rho} + c_{\delta'} \bar{v}_{[n^{1-\alpha}] - [n^{\alpha+\rho}]} n^{\frac{\alpha}{2+2\delta'}} + \bar{v}_{n - [n^{1-\alpha}][n^\alpha]}.$$

Fixing $\alpha = \frac{1}{6}$ and $\rho = \frac{1}{6}$, we get

$$(8.21) \quad \bar{v}_n \leq c_{\lambda_1, \lambda_2, \gamma, \delta'} n^{\frac{1}{2}} + c_{\delta'} \bar{v}_{[n^{\frac{5}{6}}] - [n^{\frac{1}{3}}]} n^{\frac{1}{6} \frac{1}{2+2\delta'}} + \bar{v}_{n - [n^{\frac{5}{6}}][n^{\frac{1}{6}}]}.$$

We start with the inequality $\bar{v}_n \leq n^{q_0}$, where $q_0 = 1$. Since $n - [n^{1-\alpha}][n^\alpha] \leq cn^{1-\alpha}$, we have $v_{n - [n^{1-\alpha}][n^\alpha]} \leq cn^{\frac{5}{6}q_0}$ and $v_{[n^{1-\alpha}] - [n^{\alpha+\rho}]} \leq cn^{\frac{5}{6}q_0}$. Implementing this in (8.21) gives,

$$\bar{v}_n \leq c_{\lambda_1, \lambda_2, \gamma, \delta'} n^{\frac{1}{2}} + c_{\delta'} n^{\frac{5}{6}q_0 + \frac{1}{6} \frac{1}{2+2\delta'}} \leq c_{1r_1, \gamma, \delta'} n^{\max\{\frac{1}{2}, q_1\}},$$

where $q_1 = \frac{5}{6}q_0 + \frac{1}{6} \frac{1}{2+2\delta'}$. Continuing in the same way, at the iteration $k+1$, we obtain

$$\bar{v}_n \leq c_{kr_1, \gamma, \delta'} n^{\max\{\frac{1}{2}, q_{k+1}\}},$$

where $q_{k+1} = \frac{5}{6}q_k + \frac{1}{6} \frac{1}{2+2\delta'}$. Since $\lim_{k \rightarrow \infty} q_k = \frac{1}{2+2\delta'}$, there exists a constant $k_0 < \infty$, such that $q_{k_0+1} \leq \frac{1}{2}$. With this k_0 , we get

$$\bar{v}_n \leq c_{k_0 \lambda_1, \lambda_2, \gamma, \delta'} n^{\frac{1}{2}}.$$

Since $\gamma = \gamma(\eta) = \frac{2\eta}{p(p+2\eta)} \leq \frac{2\eta}{p^2}$, we have, for any $m \geq 1$,

$$\left\| \sum_{i=m}^{m+n} X_i \right\|_{L^{2+2\delta'}} \leq c_{\lambda_1, \lambda_2, \eta, \delta'} (1 + \lambda_0 + \mu_\delta)^{1+\gamma} n^{\frac{1}{2}} \leq c_{\lambda_1, \lambda_2, \delta', \eta} (1 + \lambda_0 + \mu_\delta)^{1+\frac{2\eta}{p^2}} n^{\frac{1}{2}}.$$

Since η is arbitrary we obtain the assertion of Proposition 8.2.

9. APPENDIX

9.1. Some general bounds for the Prokhorov distance. Let (E, d) be a metric space endowed with the metric d and \mathcal{E} be the Borel σ -algebra on E . For any $B \in \mathcal{E}$ denote by B^ε its ε -extension: $B^\varepsilon = \{x \in E : d(x, B) \leq \varepsilon\}$. Let $\pi(\mathbf{P}, \mathbf{Q})$ be the Prokhorov distance between two probability measures \mathbf{P} and \mathbf{Q} defined by

$$\pi(\mathbf{P}, \mathbf{Q}) = \inf \left\{ \varepsilon : \sup_{B \in \mathcal{E}} |\mathbf{P}(B) - \mathbf{Q}(B^\varepsilon)| \leq \varepsilon \right\}.$$

The following assertion is known as Strassen-Dudley theorem and is a consequence of the results in Strassen [36] (see also Dudley [8]). Let $\mathcal{P}_E(\mathbf{P}, \mathbf{Q})$ be the set of probability measures on $E \times E$ with given marginals \mathbf{P} and \mathbf{Q} . Denote by $\mathcal{D}_{E,d}(\varepsilon)$ the ε -extension of the diagonal in $E \times E$, i.e. $\mathcal{D}_{E,d}(\varepsilon) = \{(s, s') \in E \times E : d(s, s') \leq \varepsilon\}$ and by $\overline{\mathcal{D}}_{E,d}(\varepsilon)$ its complement.

Lemma 9.1. (*Strassen-Dudley*) *If (E, d) is a complete separable metric space, then*

$$\pi(\mathbf{P}, \mathbf{Q}) = \min \left\{ \varepsilon : \exists \mathbb{P} \in \mathcal{P}_E(\mathbf{P}, \mathbf{Q}) \text{ such that } \mathbb{P}(\overline{\mathcal{D}}_{E,d}(\varepsilon)) \leq \varepsilon \right\}.$$

Let (E_1, d_1) and (E_2, d_2) be two complete separable metric spaces endowed with Borel σ -algebras \mathcal{E}_1 and \mathcal{E}_2 respectively. Consider the product space $E = E_1 \times E_2$ endowed with the metric $d(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$, where $x = (x_1, x_2), y = (y_1, y_2) \in E$. Denote by \mathcal{E} the Borel σ -algebra on E .

Lemma 9.2. *Consider the r.v.'s $X, Y \in E_1$ and $Z \in E_2$ such that Z and (X, Y) are independent. Then*

$$\pi(\mathcal{L}_{X,Z}, \mathcal{L}_{Y,Z}) = \pi(\mathcal{L}_X, \mathcal{L}_Y).$$

Proof. Let $\mathbb{P}_1 \in \mathcal{P}_{E_1}(\mathcal{L}_X, \mathcal{L}_Y)$ and $\mathbb{P}_2 \in \mathcal{P}_{E_2}(\mathcal{L}_Z, \mathcal{L}_Z)$. If \mathbb{P}_2 is concentrated on the diagonal of $E_2 \times E_2$, then with $\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2$ we have $\mathbb{P}(\overline{\mathcal{D}}_{E,d}(\varepsilon)) = \mathbb{P}_1(\overline{\mathcal{D}}_{E_1,d_1}(\varepsilon))$. This means that

$$\begin{aligned} A &= \{\varepsilon : \exists \mathbb{P}_1 \in \mathcal{P}_E(\mathcal{L}_X, \mathcal{L}_Y) \text{ such that } \mathbb{P}_1(\overline{\mathcal{D}}_{E_1,d_1}(\varepsilon)) \leq \varepsilon\} \\ &= \{\varepsilon : \exists \mathbb{P} \in \mathcal{P}_E(\mathcal{L}_{X,Z}, \mathcal{L}_{Y,Z}) \text{ such that } \mathbb{P}(\overline{\mathcal{D}}_{E,d}(\varepsilon)) \leq \varepsilon\} = B. \end{aligned}$$

By Lemma 9.1, $\pi(\mathcal{L}_X, \mathcal{L}_Y) = \inf A = \inf B = \pi(\mathcal{L}_{X,Z}, \mathcal{L}_{Y,Z})$. \square

Let $(E_1, d_1), \dots, (E_n, d_n)$ be complete separable metric spaces. On the product space $E = E_1 \times \dots \times E_n$ consider the metric $d(x, y) = \max\{d_1(x_1, y_1), \dots, d_n(x_n, y_n)\}$, where $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in E$.

Lemma 9.3. *Consider the r.v.'s $X = (X_1, \dots, X_n) \in E$ and $Y = (Y_1, \dots, Y_n) \in E$. If X and Y are independent and Y_1, \dots, Y_n are independent, then*

$$\pi(\mathcal{L}_{X_1, \dots, X_n}, \mathcal{L}_{Y_1, \dots, Y_n}) \leq \sum_{k=1}^n \pi(\mathcal{L}_{X_1, \dots, X_{k-1}, X_k}, \mathcal{L}_{X_1, \dots, X_{k-1}, Y_k}).$$

Proof. The assertion of the lemma is obtained using the telescope rule and Lemma 9.2. \square

Lemma 9.4. *Consider the r.v.'s $X = (X_1, \dots, X_n) \in E$ and $Y = (Y_1, \dots, Y_n) \in E$. If $(X_1, Y_1), \dots, (X_n, Y_n)$ are independent, then*

$$\pi(\mathcal{L}_{X_1, \dots, X_n}, \mathcal{L}_{Y_1, \dots, Y_n}) \leq \sum_{k=1}^n \pi(\mathcal{L}_{X_k}, \mathcal{L}_{Y_k}).$$

Proof. The assertion of the lemma is obtained using Lemma 9.3 and Lemma 9.2. \square

The following lemma is taken from [15].

Lemma 9.5. *Let \mathbf{P} and \mathbf{Q} be two probability measures on $(\mathbb{R}^N, \mathcal{B}^N)$. Assume that the characteristic functions $\widehat{p}(t)$ and $\widehat{q}(t)$ pertaining to \mathbf{P} and \mathbf{Q} are square integrable with respect to the Lebesgue measure in \mathbb{R}^N . Then*

$$(9.1) \quad \pi(\mathbf{P}, \mathbf{Q}) \leq (T/\pi)^{N/2} \left(\int_{\mathbb{R}^N} |\widehat{p}(t) - \widehat{q}(t)|^2 dt \right)^{1/2} + \mathbf{P}(\|x\|_\infty > T).$$

Proof. Assume first that \mathbf{P} , \mathbf{Q} have square integrable densities p and q respectively. Denote $C_T = \{x \in \mathbb{R}^N : \|x\|_\infty \leq T\}$ and $B_T = \mathbb{R}^N \setminus C_T$. Assume that $A \in \mathcal{B}^N$ and let $\varepsilon > 0$. Then

$$\begin{aligned} & |\mathbf{P}(A) - \mathbf{Q}(A^\varepsilon)| \\ &= |\mathbf{P}(A^\varepsilon \cap C_T) + \mathbf{P}(A^\varepsilon \cap B_T) - \mathbf{Q}(A^\varepsilon)| \\ &\leq |\mathbf{P}(A^\varepsilon \cap C_T) - \mathbf{Q}(A^\varepsilon \cap C_T)| + \mathbf{P}(B_T) \\ &= \left| \int_{A^\varepsilon \cap C_T} (p(x) - q(x)) dx \right| + \mathbf{P}(B_T) \\ &\leq \int_{\mathbb{R}^N} |p(x) - q(x)| 1_{C_T}(x) dx + \mathbf{P}(\|x\|_\infty > T). \end{aligned}$$

Using Hölder's inequality, we get

$$\begin{aligned} \pi(\mathbf{P}, \mathbf{Q}) &\leq |\mathbf{P}(A) - \mathbf{Q}(A^\varepsilon)| \\ &\leq \left(\int_{\mathbb{R}^N} |p(x) - q(x)|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}^N} 1_{C_T}(x) dx \right)^{1/2} + \mathbf{P}(\|x\|_\infty > T). \end{aligned}$$

Since, by Plancherel's identity

$$\int_{\mathbb{R}^N} |p(x) - q(x)|^2 dx = (2\pi)^{-N} \int_{\mathbb{R}^N} |\widehat{p}(t) - \widehat{q}(t)|^2 dt,$$

we obtain (9.1) for \mathbf{P} and \mathbf{Q} having square integrable densities.

If \mathbf{P} and \mathbf{Q} do not have densities, denote by $\mathbf{P}_v = \mathbf{P} * \mathbf{G}_v$ and $\mathbf{Q}_v = \mathbf{Q} * \mathbf{G}_v$ the smoothed versions of \mathbf{P} and \mathbf{Q} , where \mathbf{G}_v is the normal distribution of mean 0 and variance v^2 . Using (9.1) and the obvious inequality $|\widehat{p}_v(t) - \widehat{q}_v(t)| \leq |\widehat{p}(t) - \widehat{q}(t)|$, we obtain

$$\pi(\mathbf{P}_v, \mathbf{Q}_v) \leq (2\pi)^{-N/2} (2T)^{N/2} \left(\int_{\mathbb{R}^N} |\widehat{p}(t) - \widehat{q}(t)|^2 dt \right)^{1/2} + \mathbf{P}_v(\|x\|_\infty > T).$$

Since $\pi(\mathbf{P}_v, \mathbf{P}) \rightarrow 0$ and $\pi(\mathbf{Q}_v, \mathbf{Q}) \rightarrow 0$ it follows that $\pi(\mathbf{P}_v, \mathbf{Q}_v) \rightarrow \pi(\mathbf{P}, \mathbf{Q})$ as $v \rightarrow 0$. Note also that $\limsup_{v \rightarrow 0} \mathbf{P}_v(\|x\|_\infty > T) \leq \mathbf{P}(\|x\|_\infty \geq T)$. The inequality (9.1) follows for arbitrary \mathbf{P}, \mathbf{Q} . \square

9.2. Coupling independent and Gaussian r.v.'s. The following result is proved in Theorem 5 of Sakhanenko [32] (see also [31], [33], [34] for related results). Let X_1, \dots, X_n be a sequence of independent r.v.'s satisfying $\mathbb{E}X_i = 0$ and $\mathbb{E}|X_i|^p < \infty$ for some $p \geq 2$ and all $1 \leq i \leq n$.

Theorem 9.6. *On some probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ there is a sequence of independent normal r.v.'s Y_1, \dots, Y_n satisfying $\mathbb{E}'Y_i = 0$ and $\mathbb{E}'Y_i^2 = \mathbb{E}X_i^2$, $1 \leq i \leq n$, and a sequence of independent r.v.'s X'_1, \dots, X'_n satisfying $X'_i \stackrel{d}{=} X_i$, $1 \leq i \leq n$, such that*

$$\mathbb{E}' \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k X'_i - \sum_{i=1}^k Y_i \right| \right)^p \leq c_p \sum_{i=1}^n \mathbb{E}|X_i|^p.$$

In particular, by Chebyshev's inequality, for the same construction and any $a > 0$ it holds

$$(9.2) \quad \mathbb{P}' \left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k X'_i - \sum_{i=1}^k Y_i \right| > a \right) \leq \frac{C_p}{a^p} \sum_{i=1}^n \mathbb{E} |X_i|^p.$$

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