16.338 Lab Report #2:
Kapitsa’s Stable Inverted Pendulum

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1 Introduction

The 1978 Nobel Laureate in Physics, Pyotr Leonidovich Kapitsa, discovered during the 1940’s that an inverted pendulum can be stabilized when forced with high frequency vertical oscillations. The present laboratory assignment investigates this phenomenon with a jigsaw-mounted inverted pendulum, as depicted in Figure 1.

![Figure 1: Experimental Setup](image)

2 Dynamic Model

We develop the governing equations from a moment balance about the pivot point.

\[ \sum M_{\text{pivot}} : I\ddot{\theta} = \frac{1}{2}Lm\lambda^2 \cos \omega t \sin \theta + \frac{1}{2}mgL \sin \theta \]  

(1)

Where the center of mass was taken at 1/2L. The right-hand-side terms account for forcing and gravity, respectively. The proper moment of inertia, \( I = 1/3mL^2 \), and the small angle approximation \( \sin \theta \approx \theta \) are introduced and the expression is simplified.

\[ \ddot{\theta} - \frac{3}{2} \left( \frac{g}{L} + \frac{\lambda}{L} \omega^2 \cos \omega t \right) \theta = 0 \]  

(2)

We have neglected rotational friction for the present derivation, but the damping effects will be discussed in a later section.
3 Analytical Investigation

We investigate the behavior of the dynamic model with two analytical treatments. First, with an approximate stability boundary derived from the broader theory of the Mathieu equation. Second, from an approximate solution form that is particular to this problem.

3.1 The Mathieu Equation

Equation 2 is a special form of Hill’s equation known as the Mathieu equation [1], for which there are no general closed-form solutions. The Mathieu equation can be written

\[ \frac{d^2y}{dz^2} + (a - 2q \cos 2z) y = 0 \]  

(3)

We can put Equation 2 into this form by making the arguments of the cosine functions identical.

\[ \frac{d^2y}{dz^2} + \left( \frac{2}{\omega} \right)^2 \left( -\frac{3g}{2L} - \frac{3\lambda}{2L} \cos 2z \right) y = 0 \]  

(4)

The coefficients of Equation 3 gives

\[ a = -\frac{6g}{\omega^2 L} \]  

(5)

\[ q = -\frac{3\lambda}{L} \]  

(6)

Approximate stability boundaries have been determined in the previously given reference by using the perturbation method. Figure 2 depicts the boundaries of interest, where the present system has values for \( a = f(\omega) \) around -0.01 and for \( q = \text{const.} \) at -0.015. The low frequency stability boundary for the Mathieu equation with parameters in this range has been found to be approximately

\[ a = -\frac{q^2}{2} \]  

(7)

which predicts the critical stabilization frequency to be

\[ \omega_{\text{crit}} = \frac{2}{\lambda} \sqrt{\frac{gL}{3}} \]  

(8)

3.2 Approximate Solution

Towards the end of finding an approximate solution for the slow oscillations, we conjecture that the solution is composed of a slow, large amplitude motion superimposed on a fast, small amplitude oscillation

\[ \theta = \theta_1 + \theta_2 \cos \omega t \]  

(9)
Substitution into Equation 2 yields

\[
\cos \omega t \left( -\omega^2 \theta_2 - \frac{3\lambda}{2L} \omega^2 \theta_1 - \frac{3g}{2L} \theta_2 \right) - \frac{3g}{2L} \theta_1 - \frac{3\lambda}{2L} \omega^2 \theta_2 \cos^2 \omega t = 0
\] (10)

Which suggests the relationship

\[
\theta_2 = -\frac{3\lambda}{2L} \theta_1
\] (11)

Giving an approximate form for the solution

\[
\theta = \theta_1 \left( 1 - \frac{3\lambda}{2L} \cos \omega t \right)
\] (12)

The approximate solution is now substituted back into Equation 2.

\[
\ddot{\theta}_1 \left( 1 - \frac{3\lambda}{2L} \cos \omega t \right) + 2\dot{\theta}_1 \frac{3\lambda}{2L} \sin \omega t - \frac{3g}{2L} \theta_1 + \left( \frac{3g}{2L} + \frac{3\lambda}{2L} \omega^2 \cos \omega t \right) \frac{3\lambda}{2L} \cos \omega t = 0
\] (13)

Averaging the above equation to extract slow motion behavior

\[
\ddot{\theta}_1 + \left( \frac{9\lambda^2 \omega^2}{8L^2} - \frac{3g}{2L} \right) \theta_1 = 0
\] (14)

Which tells us that above a certain critical forcing frequency, the inverted pendulum will be stable and have simple harmonic motion. The spring constant is determined by the forcing frequency, such that the slow motion behavior becomes stiffer at higher forcing frequencies. The critical stability forcing frequency is found to exactly match Equation 8 developed from the Mathieu equation framework discussed previously.

### 4 Experiment

The forcing frequency was measured with a strobe light. The frequency of the strobe was increased until registration marks on the pendulum were not only stationary, but doubled. Thus we were sure that the strobe was sampling at twice the forcing frequency. Slow motion oscillations were measure with a stopwatch. The experimental parameters are given in Table 4, and the measurements are given in Table 4.

| \(L\) | 9.75 in |
| \(\lambda\) | 0.5 in |

Table 1: Experimental Parameters

It is worth noting some observations made during the experiment. First, and most important, the slow frequency oscillations typically did not exhibit simple harmonic behavior for more than three or four periods. After which, the oscillations would occasionally cease somewhat suddenly, or the effects of a damped decay would become large enough to substantially reduce the amplitude.
Table 2: Experimental Measurements

of the oscillation. Second, the pivot point exerts some frictional damping on the system as well as backlash effects due to play between the pendulum and the bolt holding it to the jigsaw. The experiment was terminated after the pendulum sheared off the pivot bolt. The mounting hole in the pendulum had been elongated to failure in the direction parallel to the length of the pendulum. Third, there were small-scale flexing modes present in the pendulum.

While the effects of friction will be explored below, we will neglect backlash effects under the argument that small amounts of backlash introduce delay between the forcing mechanism and the system, but since there is no feedback or control process which will suffer from this phase shift, its effects can be safely neglected. To state it another way, for small amounts of play, the cosineoidal vertical acceleration will not change in magnitude, only in phase, which is immaterial to the present analysis.

5 Data Reduction

The goal is to measure the critical forcing frequency for stabilization, but because this is a point of neutral stability, we cannot measure it directly. We approach this difficulty by measuring the slow oscillation behavior at multiple forcing frequencies and using this information to deduce the stabilization frequency from Equation 14.

By analogy with the simple harmonic motion equation, we identify the slow oscillation frequency as

$$\omega_{\text{slow}}^2 = \frac{9\lambda}{8L^2} \omega_{\text{fast}}^2 - \frac{3g}{2L}$$

(15)

From which we recognize quadratic behavior of the form

$$\omega_{\text{slow}}^2 = \omega_{\text{crit}}^2 + \mu \omega_{\text{fast}}^2$$

(16)

Using the above relationship to extrapolate the critical stabilization forcing frequency from the measured data using least squares, we estimate the frequency given in Table 3. Figure 3 shows

<table>
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<tr>
<th>Run</th>
<th>$\omega_{\text{fast}}$ (RPM)</th>
<th>$T_{\theta_{\text{slow}}}$ (sec)</th>
<th>$\omega_{\theta_{\text{slow}}}$ (RPM)</th>
</tr>
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<td>1</td>
<td>2240</td>
<td>1.0</td>
<td>60</td>
</tr>
<tr>
<td>2</td>
<td>2640</td>
<td>0.75</td>
<td>80</td>
</tr>
<tr>
<td>3</td>
<td>2780</td>
<td>0.68</td>
<td>88</td>
</tr>
</tbody>
</table>

Table 3: Critical Stabilizing Forcing Frequency

the measured values and the extrapolation.
6 Frictional Effects

The system dynamics equation is modified by the addition of a viscous friction term in the following way:

$$\ddot{\theta} + \frac{3}{mL^2} \mu \dot{\theta} - \frac{3}{2} \left( \frac{g}{L} + \frac{\lambda}{L} \omega^2 \cos \omega t \right) \theta = 0$$  \hspace{1cm} (17)

Carrying out all the calculations as done in section 2, we obtain the following equation:

$$\ddot{\theta} + \frac{3\mu}{mL^2} \dot{\theta} - \frac{3}{2} \left( \frac{g}{L} + \frac{\lambda}{L} \omega^2 \cos \omega t \right) \theta = 0$$  \hspace{1cm} (18)

From the above equation we see that, within the limits of the approximations used, viscous friction does not affect the stability properties of the pendulum.

A more physically meaningful modeling of the friction could be done by using Coulomb friction:

$$\ddot{\theta} + \frac{3}{mL^2} F(\dot{\theta}) - \frac{3}{2} \left( \frac{g}{L} + \frac{\lambda}{L} \omega^2 \cos \omega t \right) \theta = 0$$  \hspace{1cm} (19)

where

$$F(\dot{\theta}) := \begin{cases} -\phi & \text{for } \dot{\theta} < 0 \\ \phi & \text{for } \dot{\theta} > 0 \\ \tilde{\phi} \in [-\phi, \phi] & \text{for } \dot{\theta} = 0 \end{cases}$$  \hspace{1cm} (20)
The effect of the jigsaw excitation is like that of artificial dithering, transforming the Coulomb friction (which behaves like a switch) to a saturation, with gain:

\[
\nu = \frac{2\phi}{\pi|\theta_2|\omega} = \frac{4L\phi}{3\lambda\pi\omega|\theta_1|} \tag{21}
\]

for small amplitudes of the velocity (slow) oscillation:

\[
|\dot{\theta}_1| \ll |\theta_2|\omega = \frac{3\lambda}{2L}|\theta_1|\omega \tag{22}
\]

We obtained a “viscous” friction, with a coefficient depending on the value of \( \theta_1 \).

Since the friction term is a nonstatic nonlinearity, we cannot apply Popov’s criterion. However, since the effect of the friction is to reduce the energy of the system, we can say that the stabilizing frequency will be stabilizing also in the presence of friction: however, stability in this case must be seen as convergence to a small zone around the inverted position, whose size depends on the stiction coefficient and \( \omega_{\text{fast}} \) where the pendulum may get “stuck”.

7 Simulation

The equations of motion can be easily integrated numerically. Of course, care must be taken in the selection of the numerical integration scheme, since the differential system is stiff (we have two different time scales in the dynamics of the system).

Simulations were carried out for different values of the jigsaw frequency. In Figure 4 an example of the angular time history is shown, for \( \omega_{\text{fast}} = 2500 \) RPM, while in Figure 5 we have the power spectrum (obtained with an FFT) of the same signal, where we can clearly distinguish the slow and fast frequencies.

A number of simulations was carried out for values of \( \omega \) ranging from 1000 RPM to 3000. For each simulation the slow frequency was determined with a FFT: a plot of \( \omega_{\text{slow}} \) vs. \( \omega_{\text{fast}} \) is shown in Figure 6.

As we can see, we have a very good agreement with the theoretical relation between \( \omega_{\text{fast}} \) and \( \omega_{\text{slow}} \), including the value of \( \omega_{\text{crit}} \).

Unfortunately, the agreement is not very good when experimental data are concerned. This is most probably due to the limited capabilities in the available measuring equipment, which did not allow us to take very precise measurement.

An interesting note can be made on the effects of friction on the stability of the system: according to the numerical simulation, the nonlinear system oscillations are slowly diverging in the case of no friction. The addition of a small friction term corrects this divergence, giving the expected stable behavior.

Further investigation is needed to assess the origin of the divergence (i.e. to make sure it is not due to numerical errors in the propagation). However the theoretical results give only marginal stability for an approximate model: a slow divergence in the frictionless case is not to be excluded.
Figure 4: Time history of $\theta$

Figure 5: Power spectrum of $\theta$
8 Conclusion

The critical stabilizing forcing frequency has been determined by two approximate analytical methods and numerically. The analytical methods consisted of an approximate solution specific to the expected behavior of the inverted pendulum, and an approximate stability boundary taken from the broader theory of the Mathieu equation. The two analytical and the numerical predictions for the stabilizing frequency agreed very well with each other, but there was a larger than desired difference between the predictions and the experimental measurements.

Frictional effects and other unmodeled phenomena have been considered by qualitative argument and by simulation. While there were a number of unmodeled behaviors occurring during the experiment, it is our conclusion that the critical component is the pivot point. The interaction of the Coulomb friction, the backlash and the forcing created very complex behavior. Simple harmonic motion of the slow oscillation was not observed as our analytical models predicted and the crude measurement devices (strobe light and a stopwatch) were not adaptable take the necessary measurements to record this complexity. We suggest that the use of a bearing at the pivot point be investigated for future incarnations of the experiment.

The final word on the laboratory investigation is a good one. We were able to treat the difficult nonlinear system model by three different methods and we obtained excellent agreement and we were able to deduce the critical stabilizing frequency from a rather crude experiment, obtaining a value within 21% of the predictions.
References