Mathematical Models and Methods in Applied Sciences

Singularly perturbed elliptic system modelling the competitive interactions for a single resource

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In this work we consider an elliptic system of equations modelling the spatial heterogeneous interactions of species competing for a single resource. Coexistence of species is studied under the small diffusion approximation. Lyapunov type arguments, based on the construction of appropriated sub-harmonic maps, are proposed to determine the small diffusion asymptotic profile of the solutions. These profiles are then coupled together with topological degree arguments to prove various coexistence results.

Keywords: Elliptic systems; singular limit asymptotic; coexistence solutions; heterogeneous environment; prey-predator interactions.

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1. Introduction

This work deals with the following elliptic system of equations:

$$\begin{cases} \varepsilon\Delta r(x) + I(x) - \sum_{i=1}^{N} f_i(r(x),x)u_i(x) - m_0 r(x) = 0, \ x \in \Omega \\ \varepsilon d_i \Delta u_i(x) + f_i(r(x),x)u_i(x) - m_i u_i(x) = 0, \ i = 1, \cdots, N, \ \ x \in \Omega \\ \partial_\nu u_1(x) = ... = \partial_\nu u_N(x) = \partial_\nu r(x) = 0 \text{ for } x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter. Here Ω is a sufficiently smooth bounded domain in $\mathbb{R}^n$, while ∂_ν denotes the usual derivative along the outward normal vector to $\partial\Omega$. The above system of equations arises when looking at steady state solutions of the following reaction-diffusion system with spatially varying environment modelling the competition of N species for a single resource posed for time $t > 0$ and $x \in \Omega$

$$\begin{cases} (\partial_t - \varepsilon\Delta)\, r(t,x) = I(x) - \sum_{i=1}^{N} f_i\left(r(t,x),x\right) u_i(t,x) - m_0 r(t,x), \\ (\partial_t - \varepsilon d_i \Delta)\, u_i(t,x) = u_i(t,x) \left[f_i(r(t,x),x) - m_i\right], \, i = 1, \cdots, N, \end{cases} \tag{1.2}$$

supplemented together with the no flux boundary conditions:

$$\partial_\nu u_1(t,x) = ... = \partial_\nu u_N(t,x) = \partial_\nu r(t,x) = 0 \text{ for } t > 0, \ x \in \partial\Omega. \tag{1.3}$$

Coming back to (1.1) or to (1.2)-(1.3), r represents the density of the single resource, while u_1, .., u_N denote the density of the different competing species. In the above system, $I(x) \geq 0$ corresponds to the supply of resource while $m_0 > 0$ denotes its natural decay. Parameter $m_i > 0$ denotes the natural death rate of the $i^{th}-$species, while function $f_i(r,x) \geq 0$ corresponds to the consumption rate of the resource at each point $x \in \Omega$ of the $i^{th}-$species. Note that the interactions between species are of competition type through the consumption of the resource, so that System (1.1) (or (1.2)-(1.3)) exhibits a prey-predator like structure in some spatially heterogeneous environment.

In an homogeneous environment, namely $I(x) \equiv I > 0$ and $f_i(r,x) \equiv f_i(r)$, the above system reduces to the well known homogeneous chemostat system. The homogeneous chemostat system is widely used in theoretical biology to study population of micro-organisms such as plankton or bacteria as well as in bioscience to model industrial cultures of micro-organisms. Such a system is known to exhibit the so-called *exclusion principle*. Such a property holds true for a large class of growth (or consumption) functions f_i. Roughly speaking, this exclusion principle means that only one species (the strongest) will survive and coexistence cannot occur. There is a wide literature on this topic and we refer for instance to Hsu et al [31], Hsu [29], Sari and Mazenc [40] or to the monograph of Smith and Waltman [41](see also the references cited therein).

However, in various contexts, theoretical studies have highlighted the importance of spatial heterogeneities [9,10,36] and of the diffusion rates [6,2,22] on the coexistence of species (see also Ref. 1 and the references therein). In chemostat like models, both empirical [11,19,26] and numerical [24,37] results indicate that an intermediate diffusion rate maximize the number of species which can coexist. However, from a mathematical point of view, these questions are not yet fully elucidated.

The coexistence problem for a system similar to (1.1) with two species ($N = 2$) and for any value of ε has been investigated for the so-called *unstirred chemostat* model. We refer for instance to Hsu and Waltman in Ref. 32, Wu [44] and Nie and Wu [39]. In these works, the authors assume equi-diffusivity as well as equi-mortality rates to reduce the problem to a scalar elliptic equation. Bifurcation techniques are then used to construct coexistence branches of solutions. These results have been extended by Dung, Smith and Waltman in Ref. 21 where the authors used perturbation methods to obtain results close to the equi-mortality and equi-diffusivity case. Note that all the aforementioned works are devoted to the case of two species

($N = 2$). This has been extended to very general systems with an arbitrary number of competing species by Baxley and Robinson in Ref. 3 who focused on some properties of the solutions close to the bifurcation points.

In the case of fast diffusion ($\varepsilon \to +\infty$), more results can be obtained. Indeed, using aggregation methods, Castella and Madec have shown in Ref. 8 that system (1.1) behaves similarly to a suitable associated spatially averaged chemostat system. It turns out that, for large diffusion rates, there is generically no coexistence solution for (1.1), and such a result holds true for an arbitrary number of competing species.

In this work, we shall focus on System (1.1) for an arbitrary number of competing species in the framework of the small diffusion approximation, namely $\varepsilon \to 0$. Note that such a limiting case is well adapted in the modelling of the interactions of micro-organisms in large environment such as lake or large plug-flow. Let us emphasize that the existence of positive solutions is related to the instability of the semi-trivial solutions (i.e. with at least one zero component). The asymptotic profile of the solutions as $\varepsilon \to 0$ will allow us to obtain such information and therefore to compute a suitable topological degree ensuring the existence of at least one positive solution. We would like to mention that the instability of semi-trivial equilibria can also be used to study the permanence properties as well as global attractor for the parabolic system (1.2)-(1.3). We refer to the monograph of Cantrell and Cosner [7]. Hence such tools coupled together with Hale-Lopes like fixed point argument (see Ref. 25) can also be used instead of the topological degree arguments presented in this work. This will be investigated in detail in a forthcoming work.

Note that the small diffusion asymptotic is widely used in the study of reaction-diffusion system arising in population dynamics. Such studies are mainly focused on cooperative and competitive interactions for which monotonicity arguments are crucial to obtain convergence toward free boundary problems. We refer for instance to Bothe and Hilhorst [5], Dancer et al [18], Hilhorst et al [27] (and the references cited therein). We also refer to Hutson et al. in Ref. 34, 33 (see also the references cited therein) who studied parabolic equation as well as elliptic system of competitive type posed in heterogeneous environment with fast reaction and small diffusion. Here again the approaches are based on monotonicity properties, that, as noticed by the above mentioned authors, seem difficult to extend to prey-predator like interactions and/or to more than two competing species. Let us finally mention the work of Hilhorst et al in Ref. 28 dealing with the singular limit for a non-competitive two components reaction-diffusion system arising in theoretical chemistry. In this paper, the arguments are no longer based on monotonicity, which is not available for such interactions, and the authors crucially use the specific form of the reaction term.

As mentioned in a conjecture proposed by Huston et al. in Ref. 34, the asymptotic profile of the solutions of (1.1) as $\varepsilon \to 0$ seems to be related to the long time behaviour of the parametrised ODE system for $x \in \Omega$,

$$\begin{cases} \frac{dr(t,x)}{dt} = I(x) - \sum_{i=1}^{N} f_i(r(t,x),x)u_i(t,x) - m_0 r(t,x) \\ \frac{du_i(t,x)}{dt} = \left[f_i(r(t,x),x) - m_i\right] u_i(t,x),\ i = 1, \cdots, N. \end{cases} \tag{1.4}$$

We provide a methodology to study the relationship between the asymptotic shape of the solutions of (1.1) as $\varepsilon \to 0$ and the interior attractor of (1.4). Note that the long time behaviour of (1.4) can be derived by using a suitable Lyapunov functional. As formally explained below, the specific shape of this functional will allow us to construct appropriated sub-harmonic maps that will be used in the study of (1.1).

Let us now sketch this idea by performing some formal computations that will be rigorously justified throughout this work to study the singular limit profiles for (1.1). Consider an elliptic system of the form

$$\begin{cases} \varepsilon D \Delta U^\varepsilon(x) + F\left(x, U^\varepsilon(x)\right) = 0,\ x \in \Omega, \\ \partial_\nu U^\varepsilon(x) = 0\ \ \forall x \in \partial\Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^n$, $F : \overline{\Omega} \times \mathbb{R}^p \to \mathbb{R}^p$ is a smooth map, $D = \text{diag}\ (d_1, .., d_p)$ is a diagonal matrix with positive coefficients while $U = (u_1, .., u_p)$ is a vector valued function. In order to understand the behaviour of the solutions as $\varepsilon \to 0$, we assume that the solutions are uniformly bounded with respect to ε small enough. Let $x_0 \in \Omega$ be given and consider the rescaled vector valued function $V^\varepsilon(y) = U^\varepsilon\left(x_0 + y\sqrt{\varepsilon}\right)$. The uniform bound on U^ε together with elliptic regularity allows us to assume that $V^\varepsilon(y) \to V(y)$ locally uniformly with respect to $y \in \mathbb{R}^n$ as $\varepsilon \to 0$ and where $V \equiv V(y)$ becomes a bounded solution of the following homogeneous elliptic system of equation

$$D\Delta V(y) + F\left(x_0, V(y)\right) = 0,\ \ y \in \mathbb{R}^n. \tag{1.5}$$

The asymptotic profile of $U^\varepsilon(x_0)$ is therefore related to the solutions of the above elliptic equation. The study of the later elliptic equation will be performed by constructing suitable sub-harmonic maps. To explain this idea, we assume that there exist p sufficiently smooth, non-negative and convex maps $\mathcal{V}_i : \mathbb{R} \to \mathbb{R}$ such that

$$\sum_{i=1}^{p} \mathcal{V}_i'(u_i) F_i\left(x_0, u_1, .., u_p\right) \leq 0, \tag{1.6}$$

for each $U = (u_1, .., u_p)$ in some suitable domain of $\mathbb{R}^p$. Note that this assumption is related to the existence of a separable Lyapunov functional for the ODE system of equations:

$$\frac{dU(t)}{dt} = F\left(x_0, U(t)\right). \tag{1.7}$$

Indeed, if $t \mapsto U(t)$ is a suitable trajectory, then the map $t \mapsto \sum_{i=1}^{p} \mathcal{V}_i\left(u_i(t)\right)$ is decreasing in time. Note that such an assumption holds true for a large class of systems arising in population dynamics and more generally in mathematical biology. We refer to the survey paper of Hsu [30] and the references cited therein.

Coming back to (1.5), let $V \equiv (v_1, .., v_p)(y)$ be a given suitable solution of (1.5) and consider the function $W(y) := \sum_{i=1}^{p} d_i \mathcal{V}_i(v_i(y))$. Then one obtains that

$$\Delta W(y) = \sum_{i=1}^{p} d_i \mathcal{V}_i''(v_i(y)) |\nabla v_i(y)|^2 - \sum_{i=1}^{p} \mathcal{V}_i'(v_i(y)) F_i(x_0, V(y)) . \tag{1.8}$$

Note that (1.6) together with the convexity of functions $\mathcal{V}_i$ imply that function W is a sub-harmonic map on $\mathbb{R}^n$. Sub-harmonicity as well as the shape of functions $\mathcal{V}_i$ will impose strong constraints on the solutions of (1.5), that is on the asymptotic behaviour of $U^\varepsilon(x_0)$.

The goal of this manuscript is to develop this methodology in order to study of (1.1). This work is organized as follows: Section 2 is devoted to listing our main assumptions and to stating our main results. Section 3 deals with preliminary results that will be extensively used in the sequel of this work. It is more precisely concerned with elliptic eigenvalue estimates and a priori estimates of the solutions. Section 4 is concerned with the study of System (1.1) with $N = 1$. The construction of a positive solution is presented. Such a construction as well as the asymptotic analysis are then generalized in Section 5 where induction arguments are used to derive sufficient conditions ensuring the existence of coexistence solutions of (1.1). Finally this work is ended by an Appendix presenting technical results used throughout this work, topological degree on cones, some elliptic lemma, and rescaling techniques at a boundary point.

2. Main results

In this section we will state our main results that will be discussed and proved in this work.

We will assume that the following properties hold true:

Assumption 2.1. *We assume that $\Omega \subset \mathbb{R}^n$ is a regular and bounded domain. We assume that the external supply function $I \in C^{0,1}\left(\overline{\Omega}\right)$ is Lipschitz continuous on $\overline{\Omega}$, $I \geq 0$ and I is not identically zero*[a]*. Parameters $d_1 > 0$,..., $d_N > 0$ and $m_0 > 0$, $m_1 > 0$,.., $m_N > 0$ are fixed given constants.*

We furthermore assume specific assumptions on the consumption functions f_i.

Assumption 2.2. *We assume that for each $i = 1, \cdots, N$ function f_i satisfies:*

(i) for all $x \in \overline{\Omega}$, the function $r \mapsto f_i(r, x)$ is increasing from $\mathbb{R}^+$ into itself and satisfies $f_i(0, x) = 0$,
(ii) for all $r \in \mathbb{R}^+$, the function $x \mapsto f_i(r, x)$ belongs to $C^1(\overline{\Omega})$.

[a] If $I \equiv 0$ then $(0, \cdots, 0)$ is the only non-negative solution of (1.1).

Let us notice that such an assumption holds true for a large class of consumption functions including linear consumption functions as well as Michaelis-Mentens type functional response (see for instance Sari and Mazenc [40]).

The aim of this work is to provide sufficient conditions ensuring the existence of positive solutions of System (1.1). The construction procedure will follow an induction process and will make use of the important quantities (well defined using Assumption 2.2) $R_i(x)$ defined, for each $i = 1, .., N$ and $x \in \overline{\Omega}$, by

$$R_i(x) = \begin{cases} r_i(x) \text{ if } \lim_{r\to\infty} f_i(r,x) > m_i \\ +\infty \text{ else,} \end{cases} \tag{2.1}$$

where r_i is uniquely defined (if $\lim_{r\to\infty} f_i(r,x) > m_i$) by the resolution of the equation

$$f_i\left(r_i(x), x\right) = m_i.$$

These spatially dependent quantities will allow us to define a spatial ordering of the components of System (1.1) in the small diffusion asymptotic $\varepsilon \to 0$. Similarly to the ODE case, these quantities describe the strength of each competing species at a given spatial location. To be more precise, at each point $x \in \Omega$, the smaller $R_i(x)$, the stronger competitor is the i^{th} species at x. In what follows they will be used to provide a spatial comparison between the different species.

We first investigate System (1.1) without any species and we will prove the following asymptotic result:

Proposition 2.1 (trivial solution). *Let Assumption 2.1 be satisfied. Let $\varepsilon > 0$ be given. Then the elliptic equation*

$$\begin{cases} \varepsilon\Delta r(x) + I(x) - m_0 r(x) = 0 \text{ on } \Omega, \\ \partial_\nu r = 0 \text{ on } \partial\Omega, \end{cases}$$

has a unique solution denoted by $s^\varepsilon > 0 \in C^2(\overline{\Omega})$. It furthermore satisfies the following asymptotic

$$\lim_{\varepsilon\to 0} s^\varepsilon(x) = S(x) := \frac{I(x)}{m_0}, \text{ uniformly for } x \in \overline{\Omega}. \tag{2.2}$$

We now consider System (1.1) with $N = 1$. Recalling (2.1), define

$$\Theta_0 = \left\{x \in \overline{\Omega},\ S(x) \le R_1(x)\right\}, \quad \Theta_1 = \left\{x \in \overline{\Omega},\ R_1(x) < S(x)\right\}. \tag{2.3}$$

Then the following result holds true:

Theorem 2.3 (Single species survival). *Let Assumption 2.1 and 2.2 be satisfied. Assume that $\Theta_1 \neq \emptyset$. Then there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, System (1.1) with $N = 1$ admits at least one positive solution $(r^\varepsilon, u_1^\varepsilon) \in \left(C^2(\overline{\Omega})\right)^2$.*

We will then derive the asymptotic behaviour $\varepsilon \to 0$ of such a survival solution.

Theorem 2.4. *Let $\varepsilon_0 > 0$ and let $\{(r^\varepsilon, u_1^\varepsilon)\}_{\varepsilon\in(0,\varepsilon_0)}$ be a family of positive solutions of (1.1) with $N = 1$. Then one has*

$$\lim_{\varepsilon\to 0} \|r^\varepsilon - R^*\|_\infty = 0 \text{ and } \lim_{\varepsilon\to 0} \|u_1^\varepsilon - U_1^*\|_\infty = 0,$$

wherein we have set

$$R^*(x) = \min\left(S(x), R_1(x)\right), \quad U_1^*(x) = \frac{m_0}{m_1}\left(S(x) - R_1(x)\right)^+ .$$

This result allows us to derive a criteria for non-existence of coexistence solution. Our result reads as follows:

Corollary 2.1. *Let $N = 1$ and assume that $S(x) < R_1(x)$ for all $x \in \overline{\Omega}$. Then for each $\varepsilon > 0$ small enough, function $(s^\varepsilon, 0)$ (see Proposition 2.1) is the only non-negative solution of (1.1).*

At this step, it is natural to investigate the stability of the solutions provided by Theorem 2.3. One can expect that this coexistence solution is stable when it exists. We are only able to prove this result as well as the uniqueness when $\Theta_1 = \overline{\Omega}$ (see Proposition 4.1).

We will generalize the above results and prove the existence of a positive solution for (1.1) if each species is the best competitors (expressed in term of $R_i(x)$) at least at some point $x \in \Omega$. To be more precise, let us introduce the following partition of $\overline{\Omega}$:

$$\Theta_0 = \{x \in \overline{\Omega},\ S(x) \le R_i(x),\ i = 1, \cdots, N\},$$
$$\Theta_j = \{x \in \overline{\Omega},\ R_j(x) < S(x),\ R_j(x) < R_i(x),\ \forall i \neq j\},\ j = 1, \cdots, N,$$

as well as the interface set defined by

$$\Gamma = \{x \in \overline{\Omega},\ R_i(x) = R_j(x) < S(x), \text{ for some } i \neq j\}.$$

Our analysis provides information on the asymptotic profile of the solutions, as $\varepsilon \to 0$, outside the interface Γ. We will assume the following geometrical assumption to ensure that each species is the strongest competitor at, at least, one location outside the interface:

Assumption 2.5. *For all $k \in \{1, \cdots, N\}$, $\Theta_k \neq \emptyset$ and $\Theta_k \not\subset \Gamma$.*

As formally described in the introduction, our asymptotic analysis relies on the existence of a separable Lyapunov function for the corresponding parametrized ODE system (see (1.4)). Such a property is ensured by assuming the following:

Assumption 2.6. *Let $j = 1, \cdots, N$ and $x \in \Theta_j$ be given. For $r \neq R_i(x)$, define*

$$G_i(r, x) = \frac{f_i(r, x)\left[f_j(r, x) - m_j\right]}{f_j(r, x)\left[f_i(r, x) - m_i\right]}.$$

We assume that for each $i \neq j$, then there exists a number $\alpha_i(x) > 0$ such that

$$\max_{0\le r\le R_j(x)} G_i(r, x) \le \alpha_i(x),$$

and if $R_i(x) < S(x)$*, then*

$$\alpha_i(x) \leq \min_{R_i(x) < r \leq S(x)} G_i(r, x).$$

Similarly to Assumption 2.2, the above structural assumption also holds true for a large class of functions f_i including linear function or Michaelis-Mentens function (also referred to as Holling I and Holling II functional responses). We refer to Ref. 40 for such a study (see also the references cited therein). We can now state our first result.

Theorem 2.7 (Asymptotic exclusion principle outside of the interface). *Let Assumption 2.1, 2.2, 2.5 and 2.6 be satisfied. Consider a family* $(r^\varepsilon, u_1^\varepsilon, \cdots, u_N^\varepsilon)$ *with* $\varepsilon \in (0, \varepsilon_0)$ *of positive solutions of (1.1). Then the following convergence holds true:*

$$\lim_{\varepsilon \to 0} (r^\varepsilon, u_1^\varepsilon, \cdots, u_N^\varepsilon) = (R^*, U_1^*, \cdots, U_N^*),$$

uniformly on each compact subset of $\overline{\Omega} \setminus \Gamma$ *and wherein we have set* $R^*(x) = \min(S(x), R_1(x), \cdots, R_N(x))$ *and for* $i = 1, \cdots, N$*,*

$$U_i^*(x) = \begin{cases} \frac{m_0}{m_i}(S(x) - R^*(x)) & \text{if } x \in \Theta_i, \\ 0 & \text{if } x \notin \Theta_i. \end{cases} \tag{2.4}$$

The above result shows that in the small diffusion asymptotic, outside the interface, the coexistence solutions converge to a segregative solution. This means that at a given spatial location, only the strongest species can survive. As a consequence, exclusion principle holds at any given spatial location far from the interface. Let us also mention that the asymptotic profile of the solutions at the interface remains an open problem. Moreover coming back to the parametrized ODE (1.4) let us recall that it corresponds for each given $x \in \Omega$ to the usual chemostat system and that it is formally obtained from (1.2)-(1.3) by setting $\varepsilon = 0$. One can notice that the asymptotic profile R^* described in Theorem 2.7 corresponds, for each value of $x \in \Omega$, to the so-called break-even concentration of nutriment associated to (1.4), that is well known in the homogeneous chemostat problem (we refer to the monograph of Smith and Waltman [41] and the recent paper of Sari and Mazenc [40] for a survey on this topic).

The above result furthermore provides a necessary condition for coexistence in the case $\Gamma = \emptyset$.

Corollary 2.2. *Let Assumption 2.1, 2.2, 2.5 and 2.6 be satisfied. Assume furthermore that* $\Gamma = \emptyset$*. If there exists* $i = 1, \cdots, N$ *such that* $R_i(x) > R^*(x)$ *for all* $x \in \overline{\Omega}$*, then there exists* $\varepsilon_0 > 0$ *such that for each* $\varepsilon \in (0, \varepsilon_0)$*, System (1.1) does not have any coexistence solution.*

Remark 2.1. Note that in the case $N = 2$, if $R_2(x) > \min(R_1(x), S(x))$ for each $x \in \overline{\Omega}$ then $\Gamma = \emptyset$ and Corollary 2.2 applies. For $N \geq 3$ species, we expect that the assumption $\Gamma = \emptyset$ can be weakened. However such a result would make use of a precise profile of the solutions on the interface, that remains an open question for the moment.

We end this section by stating the following theorem that ensures that coexistence holds for small diffusion rates when each species is the strongest competitor at, at least, one location $x \in \Omega$ outside the interface.

Theorem 2.8 (Coexistence state). *Let Assumption 2.1, 2.2, 2.5 and 2.6 be satisfied. Then there exists* $\varepsilon_0 > 0$ *such that for each* $\varepsilon \in (0, \varepsilon_0)$*, System (1.1) has a positive solution* $(r^\varepsilon, u_1^\varepsilon, \cdots, u_N^\varepsilon) \in \left(C^2(\overline{\Omega})\right)^{N+1}$.

Remark 2.2. The proof of Theorem 2.8 for $N = 2$ is based on the asymptotic profile as $\varepsilon \to 0$ of the solutions of the 1-species problems. Such profile is provided by Theorem 2.4 and it does not depend on Assumption 2.5. As consequence for $N = 2$, the above result does not require Assumption 2.5.

Remark 2.3. We finally would like to mention that all the results presented in this section also hold true when the decay rates arising in System (1.1) are spatially varying functions, namely $m_j \equiv m_j(x)$ for $j = 0, .., N$. The proofs associated to this situation are similar as soon as functions m_j are sufficiently smooth and $m_j(x) > 0$ for any $x \in \overline{\Omega}$.

3. Preliminary

3.1. *Uniform bound*

This aim of this section is to prove first a priori estimates (independent of ε) of the solutions of (1.1) and to complete the proof of Proposition 2.1. We start this section by proving Proposition 2.1.

Proof of Proposition 2.1. Let us first notice that the existence and positivity of s^ε is classical. Since $I \in C^{0,1}\left(\overline{\Omega}\right)$, due to elliptic regularity, one obtains that $s^\varepsilon \in C^2(\overline{\Omega})$. Introducing the linear operator $A : D(A) \subset C\left(\overline{\Omega}\right) \to C\left(\overline{\Omega}\right)$ defined by

$$A = \Delta, \quad D(A) = \left\{\varphi \in C\left(\overline{\Omega}\right) \cap H^2(\Omega) : \ \partial_\nu \varphi = 0 \text{ on } \partial\Omega, \ \Delta\varphi \in C\left(\overline{\Omega}\right)\right\},$$

one know that

$$s^\varepsilon = \frac{m_0}{\varepsilon}\left(\frac{m_0}{\varepsilon} - A\right)^{-1}\frac{I}{m_0}.$$

Recalling that Ω has a sufficiently smooth boundary, so that A is a densely defined operator satisfying the Hile-Yosida property (see for instance Ref. 4 and the references cited therein), the result follows.

∎

We now derive a uniform bound independent of $\varepsilon > 0$ of the solutions of (1.1).

Lemma 3.1. *Let Assumption 2.1 be satisfied. There exists a constant $M > 0$ (independent of $\varepsilon > 0$ and of function f_i) such that any non-negative solution $(r^\varepsilon, u_1^\varepsilon, .;, u_N^\varepsilon)$ of (1.1) satisfies*

$$\|r^\varepsilon\|_\infty + \sum_{i=1}^N \|u_i^\varepsilon\|_\infty \leq M.$$

Furthermore recalling the definition of s^ε in Proposition 2.1, function r^ε satisfies

$$r^\varepsilon(x) \leq s^\varepsilon(x), \ \ \forall x \in \overline{\Omega}.$$

Proof. Adding up the $N+1$ equations of (1.1), yields

$$\varepsilon\Delta\left(r + \sum_{i=1}^N (d_i u_i)\right) - \left(m_0 r + \sum_{i=1}^N m_i u_i\right) + I(x) = 0.$$

Setting $P = r^\varepsilon + \sum_{i=1}^N d_i u_i^\varepsilon$ and $\alpha = \min\left\{m_0, \frac{m_1}{d_1}, \cdots, \frac{m_N}{d_N}\right\}$, then one gets

$$(\alpha - \varepsilon\Delta)P(x) = (\alpha - m_0)r + \sum_{i=1}^N \left(\alpha - \frac{m_i}{d_i}\right) d_i u_i(x) + I(x) \leq I(x), \ \forall x \in \Omega.$$

The elliptic comparison principle therefore yields

$$\|P\|_\infty \leq \frac{1}{\alpha}\|I\|_\infty.$$

This completes the proof of the uniform bound. The proof of the upper estimates of r^ε directly follows from the elliptic maximum principle. ∎

3.2. *Eigenvalue lemma*

The aim of this section is to provide qualitative information on the principal eigenvalue of an elliptic operator as the diffusion rate tends to 0.

In order to state our results, for each $d > 0$ and each function $q \in L^\infty(\Omega)$ let us introduce the quantity $\Lambda(d,q) \in \mathbb{R}$ defined as the principal eigenvalue of the elliptic operator $d\Delta + q(x)$ on Ω supplemented together with homogeneous Neumann boundary condition on $\partial\Omega$, that is

$$\begin{cases} (d\Delta + q(x))\,\phi(x) = \Lambda(d,q)\phi(x) \text{ in } \Omega, \\ \partial_\nu \phi = 0 \text{ on } \partial\Omega, \\ \phi(x) > 0 \ \ \forall x \in \overline{\Omega}. \end{cases}$$

Recall that such a principle eigenvalue can also be characterized by the so-called Rayleigh quotient (see for instance Ref. 42)

$$\Lambda(d,q) = \max_{\phi \in H^1(\Omega)\setminus\{0\}} \frac{-d\int_\Omega \nabla\phi^2 dx + \int_\Omega q(x)\phi^2 dx}{\|\phi\|_2^2}, \tag{3.1}$$

and it is continuous with respect to q.
The first lemma is well known (we refer for instance to Ref. 33, 34, 35 see also Ref. 16 for general cooperative systems).

Lemma 3.2. *Consider a family* $\{q_\varepsilon\}_{\varepsilon>0} \subset C\left(\overline{\Omega}\right)$ *and assume furthermore that* $q_\varepsilon \to q$ *as* $\varepsilon \to 0$ *for the topology of* $C(\overline{\Omega})$. *Then one has:*

$$\lim_{\varepsilon\to 0} \Lambda(\varepsilon, q_\varepsilon) = \max_{\overline{\Omega}}(q).$$

In our applications, the strong convergence $q_\varepsilon \to q$ in $C(\overline{\Omega})$ will not be satisfied and we will need a weaker version of such a result to ensure $\Lambda(\varepsilon, q_\varepsilon) > 0$ for all sufficiently small ε. This previous result is adapted into the following lemma that will be used in Section 5.

Lemma 3.3. *Let* $x \in \Omega$ *and* $\eta > 0$ *be given such that* $K := \overline{B}(x,\eta) \subset \Omega$. *Assume that* $q_\varepsilon \to q$ *in* $C(K)$ *with function* q *such that* $q(x) > 0$. *Then there exists* $\varepsilon_0 > 0$ *such that*

$$\Lambda(\varepsilon, q_\varepsilon) > 0, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Proof. On the one hand, using the Rayleigh quotient representation of $\Lambda(\varepsilon, q_\varepsilon)$ (see (3.1)), one has

$$\Lambda(\varepsilon, q_\varepsilon) = \max_{\phi\in H^1(\Omega)\setminus\{0\}} \frac{-\varepsilon\int_\Omega \nabla\phi^2 + \int_\Omega q_\varepsilon \phi^2}{\|\phi\|_2^2}. \tag{3.2}$$

Let $x \in \Omega$ be given and let us consider ε small enough such that $\overline{B}\left(x, \varepsilon^{1/4}\right) \subset \Omega$. Next fix a positive test function φ supported in $B(0,1)$, such that $\int_{\mathbb{R}^n} \varphi^2 = 1$, and set

$$\phi_\varepsilon^x(y) = \varepsilon^{-n/8}\varphi\left(\frac{x-y}{\varepsilon^{1/4}}\right).$$

Then ϕ_ε^x satisfies for each ε small enough:

(a) $\|\phi_\varepsilon^x\|_2 = 1$,
(b) $\varepsilon\int_\Omega |\nabla\phi_\varepsilon^x|^2 \to 0$ as $\varepsilon \to 0$,
(c) $\mathrm{supp}\,(\phi_\varepsilon^x) = \overline{B}\left(x, \varepsilon^{1/4}\right)$.

Using (3.2) and ϕ_ε^x as test function, one obtains:

$$\Lambda(\varepsilon, q_\varepsilon) \geq -\varepsilon\int_\Omega |\nabla\phi_\varepsilon^x(y)|^2 dy + \int_\Omega q_\varepsilon(y)(\phi_\varepsilon^x(y))^2 dy.$$

It thus follows that

$$\liminf_{\varepsilon\to 0} \Lambda(\varepsilon, q_\varepsilon) \geq \lim_{\varepsilon\to 0}\int_\Omega q_\varepsilon(y)(\phi_\varepsilon^x(y))^2 dy.$$

On the other hand, one has

$$\int_\Omega q_\varepsilon(y)(\phi_\varepsilon^x(y))^2 dy - q(x) = \int_\Omega (q_\varepsilon(y)-q(y))(\phi_\varepsilon^x(y))^2 dy + \int_\Omega (q(y)-q(x))(\phi_\varepsilon^x(y))^2 dy.$$

The choice of the test function, namely ϕ_ε^x, yields to

$$\int_\Omega (q(y)-q(x))|\phi_\varepsilon^x(y)|^2 dy \to 0$$

while the first term can be estimated as follows for each ε small enough:

$$\begin{aligned}\left|\int_\Omega (q_\varepsilon(y)-q(y))(\phi_\varepsilon^x(y))^2 dy\right| &\leq \max_{y\in \text{supp }(\phi_\varepsilon)}(|q_\varepsilon(y)-q(y)|)\\ &\leq \max_{y\in \overline{B}(x,\eta)}(|q_\varepsilon(y)-q(y)|),\end{aligned}$$

as soon as supp $(\phi_\varepsilon)\subset \overline{B}(x,\eta)$. Then $\max_{y\in \text{supp }(\phi_\varepsilon)}(|q_\varepsilon(y)-q(y)|)\to 0$ which implies that

$$\liminf_{\varepsilon\to 0}\Lambda(\varepsilon,q_\varepsilon)\geq q(x),$$

and the result follows. ∎

4. Single species problem

The aim of this section is to deal with System (1.1) with $N=1$. The system under consideration thus reads as

$$\begin{cases}\varepsilon\Delta r(x)+I(x)-f_1(r(x),x)u(x)-m_0 r(x)=0,\ x\in\Omega,\\ \varepsilon d_1\Delta u(x)+f_1(r(x),x)u(x)-m_1u(x)=0,\\ \partial_\nu u=\partial_\nu r=0 \text{ on } \partial\Omega.\end{cases} \tag{4.1}$$

In this section we will focus on the proofs of Theorem 2.3, Theorem 2.4 as well as Corollary 2.1.

4.1. *Existence, proof of Theorem 2.3*

To prove the existence of solution, we will use the degree theory in a positive cone (we refer for instance to Ref. 20, 13, 14, see also Appendix A). Let us consider the Banach space E defined by

$$E=C^0(\overline{\Omega})\times C^0(\overline{\Omega}),$$

endowed with the usual product norm, as well as its positive cone $C=P\times P$ wherein P is defined by

$$P=\left\{v\in C^0(\overline{\Omega})\,:\,v(x)\geq 0\quad \forall x\in\overline{\Omega}\right\}. \tag{4.2}$$

Now recall that Lemma 3.1 provides the existence of some constant $M>0$ independent on ε such that each non-negative solution of (4.1) satisfies $0\leq u\leq M$ and $0\leq r\leq M$.
Define

$$\mathcal{K}=\{(r,u)\in C, r<2M, u<2M\},$$

and let $\beta > 0$ be a positive constant such that

$$\beta > 2M \sup_{\substack{r\in(0,2M] \\ x\in\overline{\Omega}}} \frac{f_1(r,x)}{r}.$$

Let $B : C \to C$ be the compact operator defined by

$$B = \begin{pmatrix} (m_0+\beta-\varepsilon\Delta)^{-1} & 0 \\ 0 & (m_1-\varepsilon d_1\Delta)^{-1} \end{pmatrix},$$

and let us set, for each $t \in [0,1]$, the operator A_t defined by

$$A_t(r,u) = B\begin{pmatrix} I+\beta r - tf_1(r,.)u \\ tf_1(r,.)u \end{pmatrix}.$$

Lemma 3.1 as well as the above choice of β show that operator A_t is well defined and acts from $\mathcal{K} \to C$. Moreover, standard elliptic regularity ensures that A_t is a completely continuous operator and one can notice that (r,u) is a non-negative solution of (4.1) with f_1 replaced by tf_1 if and only if (r,u) is a fixed point of A_t in $\mathcal{K}$.
Note that the (unique) trivial solution $(s^\varepsilon, 0)$ is a non-negative fixed point of A_t for each $t \in [0,1]$ and, we furthermore have for each $t \in [0,1]$:

$$(r,0) \in C \text{ and } A_t(r,0) = (r,0) \ \Leftrightarrow \ r = s^\varepsilon.$$

We now aim to apply Proposition Appendix A.1 (see Appendix A) . To do so, let us set

$$\mathcal{U} = \{r \in P,\ r < 2M\},$$

and for each $\delta > 0$:

$$P_\delta = \{u \in P,\ \|u\| \le \delta\}.$$

Then the following lemma holds true:

Lemma 4.1. *The following assertions hold true:*

(i) $\forall t \in [0,1], deg_C(I - A_t, \mathcal{K}) = 1$,
(ii) *Assume that* $\Lambda\left(\varepsilon d_1, f_1(s^\varepsilon(.),.) - m_1\right) > 0$. *Then, for all small enough* $\delta > 0$, *one has*

$$deg_C(I - A_1, \mathcal{U} \times P_\delta) = 0.$$

Proof. In order to prove this lemma, let us introduce the following operators

$$A_t = (A_t^1, A_t^2),\ L_t^1 = D_r A_t^1(s^\varepsilon, 0) \text{ and } L_t^2 = D_u A_t^2(s^\varepsilon, 0).$$

Then one has $A_0(r,u) = (A_0^1(r,u), 0)$ with $A_0^1(r,u) = (m_0+\beta-\varepsilon\Delta)^{-1}(I+\beta r)$. Since $(s^\varepsilon, 0)$ is the only fixed point of A_0 in $\mathcal{K}$, one obtains for each $\delta > 0$,

$$deg_C(I - A_0, \mathcal{K}) = deg_C(I - A_0, \mathcal{U} \times P_\delta). \quad (4.3)$$

Simple computations show that $L_0^1 = \beta(m_0 + \beta - \varepsilon\Delta)^{-1}$ while $D_u A_0^2(s^\varepsilon, 0) = L_0^2 = 0$, so that $r_{spec}(L_0^2) < 1$ (where r_{spec} denotes the spectral radius). Let us denote by σ, the number of eigenvalues of L_0^1 greater than 1 counted with their algebraic multiplicities. Then Proposition Appendix A.1-(ii) applies and provides that

$$deg_C(I - A_0, \mathcal{U} \times P_\delta) = deg_P(I - A_0^1, \mathcal{U}) = (-1)^\sigma. \tag{4.4}$$

Note that the last equality arises since s^ε belongs to the interior of $\mathcal{U}$.

Now note that if μ is an eigenvalue of L_0^1 then there exists some function $\rho \neq 0$ such that

$$(m_0 - \varepsilon\Delta)\rho = \frac{\beta}{\mu}(1 - \mu)\rho.$$

It follows that $\frac{\beta}{\mu}(1-\mu)$ is an eigenvalue of operator $(m_0 - \varepsilon\Delta)$ and therefore $\mu < 1$. As a consequence, $\sigma = 0$ and (4.3)-(4.4) yield to

$$deg_C(I - A_0, \mathcal{K}) = 1.$$

Due to Lemma 3.1, for each $t \in [0, 1]$, operator A_t has no fixed point on $\partial\mathcal{K}$ (the boundary being relative to C). It follows from homotopy invariance of the fixed point degree that $deg_C(I - A_t, \mathcal{K}) = 1$ for all $t \in [0, 1]$. This completes the proof of (i).

In order to prove (ii), let us notice that for all ε sufficiently small,

$$\lim_{\sigma\to\infty} \Lambda\left(\varepsilon d_1, \frac{1}{\sigma} f_1(s^\varepsilon, .) - m_1\right) = \Lambda(\varepsilon d_1, -m_1) \leq -m_1 < 0.$$

Since $\Lambda(\varepsilon d_1, f_1(s^\varepsilon, .) - m_1) > 0$, there exists $\sigma_0 > 1$ such that

$$\Lambda\left(\varepsilon d_1, \frac{1}{\sigma_0} f_1(s^\varepsilon, .) - m_1\right) = 0.$$

Then this leads us to $r_{spec}(L_1^2) > \sigma_0 > 1$ and for all $\phi \in P \setminus \{0\}$, $L_1^2\phi > 0$. Thus Proposition Appendix A.1- (i) applies and it follows that $deg_C(I - A_1, \mathcal{U} \times P_\delta)$ is well defined for all small enough δ and that this last quantity equals 0. This completes the proof of (ii). ∎

Next, the following lemma holds true:

Lemma 4.2. *Let $\varepsilon > 0$ be given. Assume that $\Lambda(\varepsilon d_1, f_1(s^\varepsilon(.), .) - m_1) > 0$. Then there exists $\alpha_1 > 0$ such that for each solution $(r^\varepsilon, u^\varepsilon)$ of (4.1):*

$$u^\varepsilon > 0 \;\Rightarrow\; u^\varepsilon(x) > \alpha_1 \text{ for all } x \in \overline{\Omega}.$$

Proof. To prove this lemma, let us argue by contradiction by assuming that there exist a sequence of positive solutions $\{(r_k, u_k)\}_{k\geq 0}$ of (4.1) and a sequence $\{x_k\}_{k\geq 0} \subset \overline{\Omega}$, such that $u_k(x_k) \to 0$ as $k \to +\infty$. Then one may assume, possibly up to a subsequence, that $x_k \to x^* \in \overline{\Omega}$. By Lemma 3.1, standard elliptic regularity and Sobolev embedding theorem, one may assume that $(r_k, u_k) \to (r_\infty, u_\infty)$ in

$C^1(\overline{\Omega})$ where (r_∞, u_∞) is a non-negative solution of (4.1) such that $u_\infty(x^*) = 0$. Then the maximum principle and Hopf lemma yields to $(r_\infty, u_\infty) \equiv (s^\varepsilon, 0)$. Now, let us define for each $k \geq 0$, the function $U_k = \frac{u_k}{\|u_k\|_\infty}$. Then it satisfies for each $k \geq 0$:

$$\begin{cases} \varepsilon d_1 \Delta U_k + f_1(r_k(x), x)U_k - m_1 U_k = 0 \text{ in } \Omega, \\ \partial_\nu U_k = 0 \text{ on } \partial\Omega. \end{cases}$$

Since $\|U_k\|_\infty = 1$ for each $k \geq 0$, due to elliptic estimates, one may assume that, possibly along a subsequence, $U_k \to U_\infty$ for the topology of $C^1\left(\overline{\Omega}\right)$ and where U_∞ satisfies

$$\begin{cases} \varepsilon d_1 \Delta U_\infty + f_1(s^\varepsilon(x), x)U_\infty - m_1 U_\infty = 0, \text{ in } \Omega, \\ \partial_\nu U_\infty = 0 \text{ on } \partial\Omega, \\ U_\infty \geq 0 \text{ and } \|U_\infty\|_\infty = 1. \end{cases}$$

Elliptic maximum principle and Hopf lemma implies that $U_\infty > 0$ and therefore

$$\Lambda(\varepsilon d_1, f_1(s^\varepsilon(.), .) - m_1) = 0,$$

a contradiction together with the assumption in Lemma 4.2. This completes the proof of the result. ∎

Let $\alpha_1 > 0$ be given by Lemma 4.2. Define the subset $\mathcal{O} \subset \mathcal{K}$ by

$$\mathcal{O} = \{(r, u) \in \mathcal{K}, u(x) > \alpha_1, \quad \forall x \in \overline{\Omega}\}.$$

Then the following lemma holds true:

Lemma 4.3. *Assume that* $\Lambda(\varepsilon d_1, f_1(s^\varepsilon(.), .) - m_1) > 0$ *for some* $\varepsilon > 0$. *Then System (4.1) has at least one positive solution.*

Proof. Let us first notice that since $\Lambda(\varepsilon d_1, f_1(s^\varepsilon(.), .) - m_1) > 0$, Lemma 4.2 implies that a positive function pair (r, u) is a solution of (4.1) if and only if $(r, u) \in \mathcal{O}$ and (r, u) is a fixed point of operator A_1. Then Lemma 4.1 yields

$$1 = deg(I - A_1, \mathcal{K}) = deg_C(I - A_1, \mathcal{K} \setminus \mathcal{O}) + deg_C(I - A_1, \mathcal{O}).$$

Next we infer from Lemma 4.2 and the definition of $\mathcal{O}$ that for any small enough $\delta > 0$

$$deg_C(I - A_1, \mathcal{K} \setminus \mathcal{O}) = deg_C(I - A_1, \mathcal{U} \times P_\delta) = 0$$

where the last equality follows from Lemma 4.1. Thus, one gets $deg_C(I - A_1, \mathcal{O}) = 1 \neq 0$ that completes the proof of the lemma. ∎

We are now able to complete the proof of Theorem 2.3.

Proof of Theorem 2.3. Let us first notice that we infer from Proposition 2.1 and Lemma 3.2 that

$$\lim_{\varepsilon \to 0} \Lambda(\varepsilon, f_1(s^\varepsilon(.), .) - m_1) = \max_{x \in \overline{\Omega}} (f_1(S(x), x) - m_1),$$

wherein we have set $S(x) \equiv \frac{I(x)}{m_0}$. Recalling (2.3) and (2.1), Assumption $\Theta_1 \neq \emptyset$ re-writes as

$$\max_{x\in\overline{\Omega}} \left(f_1(S(x), x) - m_1\right) > 0,$$

and therefore there exists $\varepsilon_0 > 0$ such that

$$\Lambda\left(\varepsilon, f_1\left(s^\varepsilon(.), .\right) - m_1\right) > 0, \ \ \forall \varepsilon \in (0, \varepsilon_0).$$

Hence Lemma 4.3 applies and provides the existence of a positive solution (r, u) of (4.1) for each $\varepsilon \in (0, \varepsilon_0)$. Finally due to elliptic regularity, such a solution belongs to $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ and this completes the proof of Theorem 2.3. ■

4.2. *Asymptotic behaviour, proof of Theorem 2.4*

In this section we investigate the behaviour of the above constructed coexistence solution when $\varepsilon \to 0$. Throughout this section, we assume that for any small enough $\varepsilon > 0$ there exists a positive solution denoted by $(r^\varepsilon, u^\varepsilon)$ of System (4.1). Before proving Theorem 2.4 we need to derive some preliminary lemmas.

Our first result reads as follows:

Lemma 4.4. *Let $x \in \overline{\Omega}$ be given. Assume that there exists a sequence $\{\varepsilon_k\}_{k\geq 0}$ tending to zero (as $k \to \infty$) such that*

$$\lim_{k\to\infty} u^{\varepsilon_k}(x) = 0, \tag{4.5}$$

then, up to subsequence, one has

$$\lim_{k\to\infty} r^{\varepsilon_k}(x + y\sqrt{\varepsilon_k}) = S(x) = \frac{I(x)}{m_0},$$

locally uniformly with respect to $y \in \mathbb{R}^n$.

Proof. We assume that $x \in \Omega$. The case $x \in \partial\Omega$ is more delicate and we refer to Appendix C to consider such a case. Consider the sequence of rescaled functions

$$R_k(y) = r^{\varepsilon_k}(x + y\sqrt{\varepsilon_k}), \ U_k(y) = u^{\varepsilon_k}(x + y\sqrt{\varepsilon_k}), \ k \geq 0,$$

defined when $x + y\sqrt{\varepsilon_k} \in \Omega$. Note that (4.5) re-writes as

$$\lim_{k\to\infty} U_k(0) = 0. \tag{4.6}$$

These functions satisfy on the above set:

$$\begin{cases} \Delta_y R_k(y) + I\left(x + y\sqrt{\varepsilon_k}\right) - f_1\left(R_k(y), x + y\sqrt{\varepsilon_k}\right) U_k(y) - m_0 R_k(y) = 0, \\ d_1 \Delta_y U_k(y) + f_1\left(R_k(y), x + y\sqrt{\varepsilon_k}\right) U_k(y) - m_1 U_k(y) = 0. \end{cases}$$

Let $\{M_k\}_{k\geq 0}$ be an increasing sequence tending to $+\infty$ as $k \to \infty$ and such that $\{x + y\sqrt{\varepsilon_k}, \ |y| < M_k\} \subset \Omega$. Denote by $B\left(M\right) = \{y \in \mathbb{R}^n, \ \|y\| < M\}$. Since R_k and U_k are uniformly bounded, L^p-elliptic estimates apply and provide that the sequences $\{R_k\}$ and $\{U_k\}$ are uniformly bounded in $W^{2,p}\left(B\left(\frac{M_k}{\sqrt{\varepsilon_k}}\right)\right)$ for each $p > 1$.

Due to Sobolev embeddings, the sequences $\{R_k\}$ and $\{U_k\}$ are uniformly bounded in $C^{1,\gamma}\left(\overline{B\left(\frac{M_k}{\sqrt{\varepsilon_k}}\right)}\right)$ for any $\gamma \in (0,1)$. Using a standard diagonal process and a compactness argument, possibly along a subsequence, one may assume that $U_k \to U^*$ and $R_k \to R^*$ as $k \to \infty$ where the convergence is uniform on each compact subset of $\mathbb{R}^n$. We furthermore obtain that functions R^* and U^* are bounded, both belong to $W^{2,p}_{loc}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ and they satisfy the following system of equations for each $y \in \mathbb{R}^n$:

$$\begin{cases} \Delta R^*(y) + I(x) - f_1(R^*(y), x)U^* - m_0 R^*(y) = 0, \\ d_1 \Delta U^*(y) + f_1(R^*(y), x)U^*(y) - m_1 U^*(y) = 0. \end{cases}$$

Let us furthermore notice that (4.6) implies that $U^*(0) = 0$ so that $U^*(y) \equiv 0$. Then function $R^* \equiv R^*(y)$ becomes a bounded solution of the scalar elliptic equation

$$\Delta R^*(y) + I(x) - m_0 R^*(y) = 0, \ \ \forall y \in \mathbb{R}^n.$$

The classification derived by Caffarelli and Littman in Ref. 12 ensures that $R^*(y) \equiv S(x)$ and the result follows. ∎

Lemma 4.5. *Recalling (2.3), let $x \in \Theta_1$ be given. Then one has*

$$\liminf_{\varepsilon \to 0} u^\varepsilon(x) > 0.$$

Proof. Let us assume that $x \in \Omega$ (see Appendix C for the case $x \in \partial\Omega$). In order to prove the above result let us argue by contradiction by assuming that there exists a sequence $\{\varepsilon_k\}_{k\geq 0}$ tending to zero such that

$$\lim_{k\to\infty} u^{\varepsilon_k}(x) = 0.$$

According to Lemma 4.4 one has

$$\lim_{k\to\infty} r^{\varepsilon_k}(x + y\sqrt{\varepsilon_k}) = S(x),$$

locally uniformly for $y \in \mathbb{R}^n$. Next consider the sequence of maps $\{w_k\}_{k\geq 0}$ with w_k defined by $w_k(y) = \frac{u^{\varepsilon_k}(x+y\sqrt{\varepsilon_k})}{u^{\varepsilon_k}(x)}$ and that satisfies the equation

$$\begin{aligned} & d_1 \Delta w_k(y) + f_1\left(r^{\varepsilon_k}(x + y\sqrt{\varepsilon_k}), x + y\sqrt{\varepsilon_k}\right) w_k(y) - m_1 w_k(y) = 0, \\ & w_k(0) = 1 \text{ and } w_k > 0. \end{aligned}$$

Due to Harnack inequality, the sequence $\{w_k\}_{k\geq 0}$ is locally bounded and, up to a subsequence, one may assume that

$$w_k \to w^* \text{ locally uniformly for } y \in \mathbb{R}^N.$$

Furthermore w^* satisfies

$$\begin{aligned} & d_1 \Delta w^*(y) + \left[f_1(S(x), x) - m_1\right] w^*(y) = 0, \ y \in \mathbb{R}^n \\ & w(0) = 1 \text{ and } w \geq 0. \end{aligned}$$

Hence $w > 0$ and Lemma Appendix B.1 provides that

$$f_1(S(x), x) - m_1 \le 0,$$

a contradiction together with $x \in \Theta_1$. ■

Lemma 4.6. *Let $x \in \Theta_1$ be given. Consider the following system posed for $y \in \mathbb{R}^n$*

$$\begin{cases} \Delta R(y) - m_0 R(y) - f_1\left(R(y), x\right) U(y) + I\left(x\right) = 0 \\ d_1 \Delta U(y) + f_1\left(R(y), x\right) U(y) - m_1 U(y) = 0, \end{cases} \tag{4.7}$$

This system has a unique bounded positive solution $(R, U)(y) \equiv (R_1(x), U_1^*(x))$ (Here recall that R_1 and U_1^* are defined in (2.1) and Theorem 2.4).

Proof. Let (R, U) be a bounded positive solution of (4.7). Then we claim that

$$\inf_{y \in \mathbb{R}^n} R(y) > 0 \text{ and } \inf_{y \in \mathbb{R}^n} U(y) > 0. \tag{4.8}$$

Before proving this claim, let us complete the proof of the lemma. To do so, consider the function $W \equiv W(y)$ defined by

$$W(y) = \int_{R_1(x)}^{R(y)} \left(1 - \frac{m_1}{f_1(\sigma, x)}\right) d\sigma + d_1 \int_{U_1^*(x)}^{U(y)} \left(1 - \frac{U_1^*(x)}{\xi}\right) d\xi. \tag{4.9}$$

Since $\sigma \mapsto f_1(\sigma, x)$ is increasing, W is non-negative. Besides, due to (4.8), W is bounded on $\mathbb{R}^N$. Moreover it satisfies (for notational simplicity, we do not explicitly write down the dependence with respect to the given point $x \in \Theta_1$ in the sequel of the proof)

$$\begin{aligned} \Delta W(y) &= \Delta R(y)\left(1 - \frac{m_1}{f_1(R(y))}\right) + d_1 \Delta U(y)\left(1 - \frac{U_1^*}{U(y)}\right) \\ &\quad + \frac{m_1 f_1'(R(y))}{f_1(R(y))^2}|\nabla R(y)|^2 + \frac{d_1}{U(y)^2}|\nabla U(y)|^2 \\ &= \frac{m_1 f_1'(R(y))}{f_1(R(y))^2}|\nabla R(y)|^2 + \frac{d_1}{U(y)^2}|\nabla U(y)|^2 \\ &+ \frac{m_0}{f_1(R(y)) m_1}\left[S(f_1(R(y)) - m_1) + m_1 R(y) - f_1(R(y)) R_1\right](f_1(R(y)) - m_1). \end{aligned} \tag{4.10}$$

On the other hand one has

$$\begin{aligned} &\left[S(f_1(R(y)) - m_1) + m_1 R(y) - f_1(R(y)) R_1\right](f_1(R(y)) - m_1) \\ &= (S - R_1)(f_1(R(y)) - m_1)^2 + m_1(R(y) - R_1)(f_1(R(y)) - m_1). \end{aligned} \tag{4.11}$$

Note that the first term in (4.11) is non-negative since $x \in \Theta_1$, that is $S > R_1$. Since $m_1 = f_1(R_1)$ and f_1 is increasing (see Assumption 2.2), one obtains

$$m_1(R(y) - R_1)(f_1(R(y)) - m_1) = m_1(R(y) - R_1)(f_1(R(y)) - f_1(R_1)) \ge 0.$$

As a consequence of the above computations one obtains that function W satisfies

$$\begin{cases} \Delta W(y) \geq 0 \text{ for each } y \in \mathbb{R}^n \\ \text{and } W \text{ is non-negative and bounded on } \mathbb{R}^n. \end{cases}$$

It follows that there exists a sequence $\{y_k\}_{k\geq 0} \in \mathbb{R}^n$ such that $W(y_k) \to \sup(W)$. Up to a subsequence, one may assume that $R(y+y_k)$ and $U(y+y_k)$ converges locally uniformly to $\widehat{R}(y)$ and $\widehat{U}(y)$ satisfying (4.7). Then $W(y+y_k)$ converges locally uniformly to $\widehat{W}(y)$ defined by

$$\widehat{W}(y) = \int_{R_1(x)}^{\widehat{R}(y)} \left(1 - \frac{m_1}{f_1(\sigma,x)}\right) d\sigma + d_1 \int_{U_1^*(x)}^{\widehat{U}(y)} \left(1 - \frac{U_1^*(x)}{\xi}\right) d\xi,$$

which satisfies $\widehat{W}(0) = \sup(W) = \sup\left(\widehat{W}\right)$ and $\Delta\widehat{W}(y) \geq 0$. It follows that $\widehat{W}(y) \equiv \sup(W)$ so that $\Delta\widehat{W}(y) = 0$. On the other hand, $\Delta\widehat{W}(y)$ is given by (4.10) where R and U are replaced by $\widehat{R}$ and $\widehat{U}$ respectively. This yields to

$$\nabla\widehat{R}(y) = \nabla\widehat{U}(y) \equiv 0, \quad \widehat{R}(y) \equiv R_1(x).$$

Plugging this into (4.7) yields $\widehat{U}(y) \equiv U_1^*(x)$. Furthermore one obtains that $\widehat{W}(y) \equiv 0$ so that $\sup(W) = 0$ and, since W is non-negative, $W \equiv 0$. Finally (4.10) implies that

$$\nabla R(y) = \nabla U(y) \equiv 0, \quad R(y) \equiv R_1(x).$$

Plugging this into (4.7) completes the proof of the result.

It remains to prove Claim (4.8). Let us focus on proving

$$\inf_{y\in\mathbb{R}^n} U(y) > 0.$$

To do so, let us argue by contradiction by assuming that there exists a sequence $\{y_k\}_{k\geq 0} \subset \mathbb{R}^n$ such that

$$\lim_{k\to\infty} U(y_k) = 0.$$

Next, consider the sequence of maps $U_k(y) = U(y_k+y)$ and $R_k(y) = R(y_k+y)$. Due to elliptic estimates one may assume that (U_k, R_k) converges to (U_∞, R_∞) locally uniformly with $U_\infty(0) = 0$. The strong comparison principle implies that $U_\infty \equiv 0$ and therefore $R_\infty \equiv S(x)$. Consider now the map $w_k(y) = \frac{U_k(y)}{U(y_k)}$. It satisfies

$$d_1 \Delta w_k(y) + f_1(R_k(y), x) w_k(y) - m_1 w_k(y) = 0.$$

Due to Harnack inequality the sequence $\{w_k\}_{k\geq 0}$ is locally bounded and due to elliptic estimates, one may assume that it converges to some function w_∞ locally uniformly while w_∞ satisfies

$$\begin{aligned} &d_1 \Delta w_\infty(y) + (f_1(S(x),x) - m_1)\, w_\infty(y) = 0, \ y \in \mathbb{R}^n, \\ &w_\infty(0) = 1, \ \ w_\infty(y) \geq 0. \end{aligned}$$

This implies that $w_\infty(y) > 0$ and that $f_1(S(x), x) - m_1 \leq 0$, a contradiction with the choice of $x \in \Theta_1$. The proof of the property for the $R-$component, namely

$$\inf_{y \in \mathbb{R}^n} R(y) > 0$$

can be proved similarly by using that $I(x) > 0$ since $x \in \Theta_1$. This completes the proof of the lemma. ∎

Our next result reads as:

Lemma 4.7. *Let $x \in \Theta_0$ be given. Then the elliptic system (4.7) posed for $y \in \mathbb{R}^n$ has a unique bounded non-negative solution (R, U) and one has*

$$(R, U)(y) \equiv (S(x), 0).$$

Proof. Let us first notice that due to the elliptic comparison principle one has

$$R(y) \leq S(x), \quad \forall y \in \mathbb{R}^n. \tag{4.12}$$

Consider the map $W \equiv W(y)$ defined by

$$W(y) = d_1 U(y).$$

Then W is bounded and satisfies

$$\Delta W(y) = U(y)\left[m_1 - f_1(R(y), x)\right] \geq 0, \ \forall y \in \mathbb{R}^n,$$

so that $\Delta W(y) \geq 0$ for all $y \in \mathbb{R}^n$. It follows that W is a constant function so that $\nabla W(y) = d_1 \nabla U(y) \equiv 0$. In order to conclude the proof of this result, let us show that $U(y) = U \equiv 0$. Assume by contradiction that $U > 0$. Then plugging this into the first equation of (4.7) yields $R(y) < S(x)$, for each $y \in \mathbb{R}^n$. This last inequality together with the second equation in (4.7) yields $U \equiv 0$, a contradiction. Therefore one has $U \equiv 0$ and $R \equiv S(x)$, that completes the proof of the lemma. ∎

We are now able to prove Theorem 2.4.

Proof of Theorem 2.4. Let us denote by $R^*(x) = \min(R_1(x), S(x))$ and $U_1^*(x) = \frac{m_0}{m_1}(S(x) - R_1(x))^+$. We aim to prove

$$\lim_{\varepsilon \to 0} \|r^\varepsilon - R^*\|_\infty + \|u^\varepsilon - U_1^*\|_\infty = 0.$$

To do so, we will argue by contradiction by assuming that there exist $\alpha > 0$, a sequence $\{x_k\}_{k \geq 0} \subset \overline{\Omega}$ (that we may assume to be convergent toward some $x^* \in \overline{\Omega}$) and a sequence $\{\varepsilon_k\}_{k \geq 0}$ tending to zero as $k \to \infty$ such that

$$|r^{\varepsilon_k}(x_k) - R^*(x_k)| + |u^{\varepsilon_k}(x_k) - U_1^*(x_k)| > \alpha, \quad \forall k \geq 0. \tag{4.13}$$

Here again we assume that $x^* \in \Omega$. (The case where $x^* \in \partial\Omega$ can be handled similarly using technical arguments inspired by Ref. 38 and outlined in Appendix C). Since $x^* \in \Omega$, one may assume $x_k \in \Omega$ for all $k \geq 0$.

Let us first prove that

$$\lim_{k \to \infty} (r^{\varepsilon_k}(x_k), u^{\varepsilon_k}(x_k)) = (R^*(x^*), U_1^*(x^*)). \tag{4.14}$$

To prove this claim, define the sequence of rescaled maps

$$R_k(y) = r^{\varepsilon_k}(x_k + y\sqrt{\varepsilon_k}), \quad U_k(y) = u^{\varepsilon_k}(x_k + y\sqrt{\varepsilon_k}).$$

As in the above proofs, one may assume that $(R_k, U_k) \to (R_\infty, U_\infty)$ locally uniformly in $\mathbb{R}^n$ where $(R_\infty, U_\infty)(y)$ is a bounded solution of (4.7) with x replaced by x^*. Then Lemmas 4.6 and 4.7 yield $(R_\infty, U_\infty)(y) \equiv (R^*(x^*), U_1^*(x^*))$ and (4.14) follows. In addition one has for each $k \geq 0$:

$$\begin{aligned}|r^{\varepsilon_k}(x_k) - R^*(x_k)| + |u^{\varepsilon_k}(x_k) - U_1^*(x_k)| \leq |r^{\varepsilon_k}(x_k) - R^*(x^*)| + |u^{\varepsilon_k}(x_k) - U_1^*(x^*)|\\ + |R^*(x_k) - R^*(x^*)| + |U_1^*(x_k) - U_1^*(x^*)|.\end{aligned}$$

Since R^* and U_1^* are continuous on $\overline{\Omega}$, we infer from (4.14) that

$$\lim_{k\to\infty} \left(|r^{\varepsilon_k}(x_k) - R^*(x_k)| + |u^{\varepsilon_k}(x_k) - U_1^*(x_k)|\right) = 0,$$

a contradiction together with (4.13). This ends the proof of Theorem 2.4. ∎

We now complete the proof of Corollary 2.1.

Proof of Corollary 2.1. We argue by contradiction and we assume that there exists a sequence $\{\varepsilon_k\}_{k\geq 0}$ tending to zero as $k \to \infty$ and such that System (4.1) has a positive solution for each ε_k and $k \geq 0$. Such a solution is denoted by $(r^{\varepsilon_k}, u^{\varepsilon_k})$. Integrating the $u-$equation over Ω implies that for all ε, $\int_\Omega (f_1(r^\varepsilon) - m_1)u^{\varepsilon_k} = 0$. Since $S(x) < R_1(x)$ for each $x \in \overline{\Omega}$, it follows from Theorem 2.4 that r^{ε_k} converges uniformly to S. Therefore $r^{\varepsilon_k}(x) < R_1(x)$ for large enough k so that $\int_\Omega (f_1(r^{\varepsilon_k}) - m_1)u_k^\varepsilon < 0$ for large enough k, a contradiction. ∎

We finally complete this section by proving the stability as well as the uniqueness of positive solution when $\Theta_0 = \emptyset$ and for $\varepsilon > 0$ small enough. The precise statement of our result reads as:

Proposition 4.1. *Assume that $\Theta_1 = \overline{\Omega}$. Then there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, System (4.1) has a unique positive solution which is furthermore asymptotically stable (with respect to the parabolic system (1.2)-(1.3) with $N = 1$).*

Proof. The proof of this result is based on several steps. Let $\varepsilon_1 > 0$ be given and let $(r_\varepsilon, u_\varepsilon)$ be a positive solution of (4.1) for $\varepsilon \in (0, \varepsilon_1)$. Next consider the eigenvalue problem

$$\begin{cases} \varepsilon\Delta\psi - m_0\psi - a_\varepsilon(x)\phi - b_\varepsilon(x)\psi = \lambda\psi, \\ d_1\varepsilon\Delta\phi - m_1\phi + a_\varepsilon(x)\phi + b_\varepsilon(x)\psi = \lambda\phi, \\ \partial_\nu\psi = \partial_\nu\phi = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.15}$$

wherein we have set

$$a_\varepsilon(x) = f_1\left(r_\varepsilon(x), x\right), \quad b_\varepsilon(x) = \partial_r f_1(r_\varepsilon(x), x)u_\varepsilon(x),$$

and together with the normalisation condition

$$\|\psi\|^2_{L^2(\Omega)} + \|\phi\|^2_{L^2(\Omega)} = 1.$$

We claim that there exists some constant $C > 0$ such that for each $\varepsilon \in (0, \varepsilon_1)$, if $\lambda \in \mathbb{C}$ is an eigenvalue of (4.15) then

$$|\text{Im } \lambda| \leq C \text{ and } \text{Re } \lambda \leq C. \tag{4.16}$$

To prove this result, recall that from Lemma 3.1 there exists some constant $M > 0$ such that

$$0 \leq a_\varepsilon(x) \leq M, \;\; 0 \leq b_\varepsilon(x) \leq M, \;\; \forall \varepsilon \in (0, \varepsilon_1), \; \forall x \in \overline{\Omega}. \tag{4.17}$$

As a consequence multiplying the first equation of (4.15) by $\overline{\psi}$, the second equation by $\overline{\phi}$, integrating each of these expressions over Ω and adding up the two resulting equations leads us to

$$\begin{aligned}\lambda = -\varepsilon \int_\Omega \left(|\nabla \psi|^2 + d_1 |\nabla \phi|^2\right) dx + \int_\Omega \left[(a_\varepsilon(x) - m_1)|\phi|^2 - (m_0 + b_\varepsilon(x))|\psi|^2\right] dx \\ + \int_\Omega \left[b_\varepsilon(x)\psi\overline{\phi} - a_\varepsilon(x)\phi\overline{\psi}\right] dx.\end{aligned} \tag{4.18}$$

Therefore

$$|\text{Im } (\lambda)| \leq \int_\Omega \left[b_\varepsilon(x) + a_\varepsilon(x)\right] |\text{Im } (\psi\overline{\phi})| dx.$$

Due to the normalization condition of the eigenvectors as well as (4.17), the first part of (4.16) follows. To prove the second part, let us notice that

$$\text{Re } (\lambda) \leq \int_\Omega a_\varepsilon(x)|\phi|^2 + \left[b_\varepsilon(x) + a_\varepsilon(x)\right] |\text{Re } \left(\phi\overline{\psi}\right)| dx.$$

The result follows using the similar arguments as above, namely normalization and (4.17).

We are now able to prove the stability part of the result. To be more precise we show that there exists $\varepsilon_0 \in (0, \varepsilon_1)$ such that for each $\varepsilon \in (0, \varepsilon_0)$:

$$\lambda \text{ solution of (4.15)} \;\Rightarrow\; \text{Re } (\lambda) < 0. \tag{4.19}$$

To prove this claim, we will argue by contradiction by assuming that there exist a sequence $\{\varepsilon_k\}_{k \geq 0}$ tending to zero as $k \to \infty$ and $\{\lambda_k\}_{k \geq 0}$ a sequence of eigenvalue of (4.15) with $\varepsilon = \varepsilon_k$ for each $k \geq 0$ and such that

$$\text{Re } (\lambda_k) \geq 0, \;\; \forall k \geq 0. \tag{4.20}$$

Let us denote by (ψ_k, ϕ_k) an eigenvector of (4.15) with $\varepsilon = \varepsilon_k$ associated to λ_k. Assume the following normalization

$$\begin{aligned}&\max_{x \in \overline{\Omega}} (\max(|\phi_k(x)|, |\psi_k(x)|)) \leq 1, \\ &\exists x_k \in \overline{\Omega}, \;\; \max(|\phi_k(x_k)|, |\psi_k(x_k)|) = 1.\end{aligned}$$

Next set for each $k \geq 0$:

$$\begin{aligned}R_k(y) = r_{\varepsilon_k}(x_k + y\sqrt{\varepsilon_k}), \; U_k(y) = u_{\varepsilon_k}(x_k + y\sqrt{\varepsilon_k}) \\ \Psi_k(y) = \psi_k(x_k + y\sqrt{\varepsilon_k}), \; \Phi_k(y) = \phi_k(x_k + y\sqrt{\varepsilon_k}).\end{aligned}$$

Up to a subsequence, one may assume that $x_k \to x_0 \in \overline{\Omega}$ as $k \to \infty$. We will only deal with the case where $x_0 \in \Omega$, the case where $x_0 \in \partial\Omega$ can be handled similarly using Appendix C. Furthermore due to (4.16) one may assume that

$$\lambda_k \to \lambda_\infty \text{ with } \mathrm{Re}\ \lambda_\infty \geq 0.$$

Since $\Theta_1 = \overline{\Omega}$, the proof of Lemma 4.6 as well as Theorem 2.4 implies that

$$[R_k(y), U_k(y)] \to [R_1(x_0), U_1^*(x_0)] \text{ locally uniformly for } y \in \mathbb{R}^n.$$

Next note that Ψ_k and Φ_k satisfies

$$\begin{cases} \Delta\Psi_k(y) - a_k(y)\Phi_k(y) - [b_k(y) + m_0]\,\Psi_k(y) = \lambda_k\Psi_k(y), \\ d_1\Delta\Phi_k(y) + [a_k(y) - m_1]\,\Phi_k(y) + b_k(y)\Psi_k(y) = \lambda_k\Phi_k(y). \end{cases}$$

with

$$a_k(y) = f_1(R_k(y), x_k + y\sqrt{\varepsilon_k}), \quad b_k(y) = \partial_r f_1(R_k(y), x_k + y\sqrt{\varepsilon_k})U_k(y).$$

Due to elliptic estimates, one may assume, up to a subsequence, that (Ψ_k, Φ_k) converges locally uniformly to some function pair $(\Psi_\infty, \Phi_\infty)$ that satisfies for all $y \in \mathbb{R}^n$:

$$\begin{cases} \Delta\Psi_\infty(y) - \partial_r f_1(R_1(x_0), x_0)U_1^*(x_0)\Psi_\infty(y) - m_0\Psi_\infty(y) - m_1\Phi_\infty(y) = \lambda_\infty\Psi_\infty(y), \\ d_1\Delta\Phi_\infty(y) + \partial_r f_1(R_1(x_0), x_0)U_1^*(x_0)\Psi_\infty(y) = \lambda_\infty\Phi_\infty(y). \end{cases}$$

together with $\max(|\Psi_\infty|, |\Phi_\infty|) \leq 1$ and $\max(|\Psi_\infty(0)|, |\Phi_\infty(0)|) = 1$. Therefore, one obtains that

$$\lambda_\infty \in \{\sigma(A_\xi),\ \xi \in \mathbb{R}^n\} \tag{4.21}$$

wherein we have set

$$A_\xi = \begin{pmatrix} -|\xi|^2 - m_0 - \partial_r f_1(R_1(x_0), x_0)U_1^*(x_0) & -m_1 \\ \partial_r f_1(R_1(x_0), x_0)U_1^*(x_0) & -d_1|\xi|^2 \end{pmatrix}.$$

Indeed if one considers the elliptic operator $L : C^{2+\alpha}(\mathbb{R}^n, \mathbb{R}^2) \to C^\alpha(\mathbb{R}^n, \mathbb{R}^2)$ for some $\alpha \in (0,1)$ defined by

$$L = \begin{pmatrix} 1 & 0 \\ 0 & d_1 \end{pmatrix}\Delta + \begin{pmatrix} -\partial_r f_1(R_1(x_0), x_0)U_1^*(x_0) - m_0 & -m_1 \\ \partial_r f_1(R_1(x_0), x_0)U_1^*(x_0) & 0 \end{pmatrix},$$

then the equation $(\lambda_\infty I - L)\,u_\infty = 0$ has a bounded non-trivial solution. Using the results of Ref. 43 $(\lambda_\infty I - L)$ is a non-Fredholm elliptic operator. Hence λ_∞ belongs to the Fredholm spectrum of L and since L has constant coefficients, this spectrum is computed using Fourier transform and (4.21) follows (we also refer to the monograph of Volpert [43] for such computations).

Now a direct computation of the eigenvalues of A_ξ shows that $\Re(\lambda_\infty) < 0$, a contradiction together with the assumption. It follows that $\Re(\lambda_\varepsilon) < 0$ for small enough ε which ends the proof of (4.19).

It remains to prove the uniqueness of the solution for each $\varepsilon \in (0, \varepsilon_0)$. This result will follows from a topological degree argument. Recall that since each eigenvalue λ satisfies $\lambda \neq 0$ then $(r_\varepsilon, u_\varepsilon)$ is isolated in $\mathcal{O}$, so that $ind_C(A_1, (r, u))$ is well defined and equals $(-1)^\sigma$ where σ denotes the number of (real) eigenvalue λ greater than 0. Moreover, a usual compactness argument shows that there exists a finite number $m \geq 1$ of solutions denoted by $(r_i, u_i)_{1 \leq i \leq m}$ and one has

$$deg_C(I - A_1, \mathcal{O}) = \sum_{i=1}^{m} ind_C(A_1, (r_i, u_i)).$$

Finally, since each positive solution is asymptotically stable (see (4.19)) then $ind_C(A_1, (r_i, u_i)) = 1$. Now from the proof of Lemma 4.3 one obtains that

$$deg_C(I - A_1, \mathcal{O}) = 1 = m.$$

This completes the proof of the result. ■

5. System (1.1) with $N \geq 2$ species

The aim of this section is to investigate System (1.1) for an arbitrary number of species $N \geq 2$. Let us first give some definitions and notations that will be used in the sequel.

In order to deal with System (1.1) for an arbitrary number of species, it is convenient to introduce the following definition:

Definition 5.1. Let $J \subset \{1, .., N\}$ be given. Let $\varepsilon > 0$ be given. A non-negative solution $(r, u_1, .., u_N)$ is said to be a $J-$**coexistence solution** of (1.1) if

$$u_i(x) = 0 \ \ \forall x \in \overline{\Omega}, \ i \in \{1, .., N\} \setminus J$$

Such a $J-$coexistence solution is said to be a **strict** $J-$**coexistence solution** if we furthermore impose that

$$u_j(x) > 0, \ \forall j \in J.$$

Now let $J \subset \{1, .., N\}$ be given. Let us introduce the following sets:

$$\Theta_0^J = \{x \in \overline{\Omega}, \ S(x) \leq R_k(x), \ \forall k \in J\}, \tag{5.1}$$

for each $k \in J$,

$$\Theta_k^J = \{x \in \overline{\Omega}, \ R_k(x) < S(x), \ R_k(x) < R_i(x), \ \forall i \in J \setminus \{k\}\}, \tag{5.2}$$

and for each $i, j \in \{1, .., N\}$,

$$\Gamma_{i,j} = \{x \in \overline{\Omega}, \ R_i(x) = R_j(x)\} \text{ and } \Gamma^J = \bigcup_{(i,j) \in J \times J} \Gamma_{i,j}. \tag{5.3}$$

We will now split this section into two parts, we first derive the asymptotic profile $(\varepsilon \to 0)$ of strict $J-$coexistence solutions of (1.1) for any subset $J \subset \{1, .., N\}$. We then prove using an induction argument that, under some suitable conditions, System (1.1) has a positive solution as soon as ε is small enough.

5.1. *Asymptotic profile* $\varepsilon \to 0$

In this section we study the asymptotic profile $\varepsilon \to 0$ of non-negative solutions of (1.1). This investigation will rely on several lemmas. Our first result reads as:

Lemma 5.1. *Let $J \subset \{1, \cdots, N\}$ be given such that $J \neq \emptyset$. Let $i \in J$ and $x \in \Theta_i^J$ be given. Then the elliptic system posed for $y \in \mathbb{R}^n$:*

$$\begin{cases} \Delta R(y) + I(x) - m_0 R(y) - \sum_{j \in J} f_j(R(y), x) U_j(y) = 0, \\ d_j \Delta U_j(y) - m_j U_j(y) + f_j(R(y), x) U_j(y) = 0, \ j \in J, \end{cases} \tag{5.4}$$

has a unique non-negative bounded solution $(R, U_j, j \in J)(y)$ such that $U_i(y) > 0$ and one has

$$R(y) \equiv R_i(x), \ U_i(y) \equiv U_i^*(x), \ U_j(y) \equiv 0, \ \forall j \in J \setminus \{i\}.$$

where in we have set $U_i^*(x) = \frac{m_i}{m_0}(S(x) - R_i(x))$.

Proof. Since $x \in \Theta_i^J$ it follows as in the proof of Lemma 4.6 that

$$\inf_{y \in \mathbb{R}^n} R(y) > 0 \text{ and } \inf_{y \in \mathbb{R}^n} U_i(y) > 0.$$

For each $j \in J \setminus \{i\}$, let $\alpha_j := \alpha_j(x)$ be defined as in Assumption 2.6. Consider the function $W : \mathbb{R}^n \to \mathbb{R}$ defined by

$$W(y) = \int_{R_i(x)}^{R(y)} \left(1 - \frac{m_i}{f_i(\sigma, x)}\right) d\sigma + d_i \int_{U_i^*(x)}^{U_i(y)} \left(1 - \frac{U_i^*(x)}{\xi}\right) d\xi + \sum_{j \in J \setminus \{i\}} d_j \alpha_j(x) U_j(y).$$

Note that since $\sigma \mapsto f_i(\sigma, x)$ is increasing, W is non-negative. In the sequel, for notational simplicity, we do not explicitly write down the dependence with respect to the given point $x \in \Theta_i^J$. Now note that W is bounded on $\mathbb{R}^n$ and satisfies for each $y \in \mathbb{R}^n$:

$$\Delta W(y) = \frac{m_i f_i'(R(y))}{f_i(R(y))^2} |\nabla R(y)|^2 + \frac{d_i}{U_i(y)^2} |\nabla U_i(y)|^2 + \sum_{j \in J} J_j(y),$$

wherein we have set

$$J_i(y) = \frac{m_0}{m_i f_i(R(y))} \left(f_i(R(y)) - m_i\right) \left[S\left(f_i(R(y)) - m_i\right) + m_i R(y) - f_i(R(y)) R_i\right],$$

and for $j \neq i$

$$J_j(y) = \frac{U_j(y)}{f_j(R(y))} \left[f_j(R(y)) \left(f_i(R(y)) - m_i\right) + \alpha_j f_i(R(y)) \left(m_j - f_j(R(y))\right)\right].$$

As in Lemma 4.6, one has $J_i(y) \geq 0$. Now, elliptic maximum principle implies that $R(y) < S(x)$ and it follows from Assumption 2.5 and from the choice of α_j that $J_j(y) \geq 0$ for any $j \neq i$. Hence it follows that the bounded non-negative function W satisfies $\Delta W \geq 0$ on $\mathbb{R}^n$. Let $\{y_k\}_{k \geq 0} \in \mathbb{R}^n$ be a sequence such that $W(y_k) \to sup(W)$. Up to a subsequence, one may suppose that $R(y + y_k)$ and

$U_j(y+y_k)$, $j \in J$ converges locally uniformly to $\widehat{R}(y)$, $\widehat{U_j}$ solution of (5.4). It follows that $W(y+y_k)$ converges locally uniformly to $\widehat{W}(y)$ satisfying $\widehat{W}(0) = \sup(W) = \sup\left(\widehat{W}\right)$ and $\Delta\widehat{W}(y) \geq 0$. Arguing similarly as in the proof of Lemma 4.6 leads us to $\nabla\widehat{R}(y) = \nabla\widehat{U_i}(y) \equiv 0$, $\widehat{R} \equiv R_i(x)$ and $\widehat{U_j} \equiv 0$ for $j \neq i$. Plugging this into (5.7) yields to $\widehat{U_i}(y) \equiv U_i^*(x)$ and then $\widehat{W}(y) \equiv 0$ so that $W \equiv 0$. As a consequence of $\Delta W(y) \equiv 0$, one obtains that $\nabla R(y) = \nabla U_i(y) \equiv 0$ while $J_j(y) \equiv 0$ for each $j \in J$. This leads us to $R \equiv R_i(x)$ and $U_j \equiv 0$ for $j \neq i$. Plugging this into (5.7), we conclude that $U_i(y) \equiv U_i^*(x)$, that completes the proof of the result. ■

Our next result is the following lemma:

Lemma 5.2. *Let $J \subset \{1, \cdots, N\}$ and $x \in \Theta_0^J$ be given. Then the elliptic system (5.4) posed for $y \in \mathbb{R}^n$ has a unique non-negative bounded solution $(R, U_j, j \in J)(y) \equiv (S(x), 0, \cdots, 0)$.*

Proof. The proof of the above lemma follows the same lines as the arguments of the proof of Lemma 5.1. Indeed, due to the elliptic comparison principle, one has $R(y) \leq S(x)$. Then one can check that the map $W : \mathbb{R}^n \to \mathbb{R}$ defined by

$$W(y) = \sum_{j \in J} d_j U_j(y),$$

is bounded on $\mathbb{R}^n$ and is a sub-harmonic function. The result follows by using the same computations as in Lemma 4.7. ■

Lemma 5.3. *Let $x \in \overline{\Omega} \setminus \Gamma$ be given and let $\{\varepsilon_k\}_{k \geq 0}$ be a sequence of positive numbers tending to zero. Then, up to a subsequence, one gets either*

$$\begin{aligned} &\lim_{k \to +\infty} r^{\varepsilon_k}(x + y\sqrt{\varepsilon_k}) = S(x), \\ &\lim_{k \to +\infty} u_j^{\varepsilon_k}(x + y\sqrt{\varepsilon_k}) = 0, \ \forall j \in \{1, \cdots, N\} \end{aligned} \tag{5.5}$$

or there exists $i \in \{1, \cdots, N\}$ such that $R_i(x) < S(x)$ and

$$\begin{aligned} &\lim_{k \to +\infty} r^{\varepsilon_k}(x + y\sqrt{\varepsilon_k}) = R_i(x), \\ &\lim_{k \to +\infty} u_i^{\varepsilon_k}(x + y\sqrt{\varepsilon_k}) = \tfrac{m_i}{m_0}(S(x) - R_i(x)), \\ &\lim_{k \to +\infty} u_j^{\varepsilon_k}(x + y\sqrt{\varepsilon_k}) = 0, \ \forall j \in \{1, \cdots, N\} \setminus \{i\}, \end{aligned} \tag{5.6}$$

where all the above convergences are locally uniform with respect to $y \in \mathbb{R}^n$.

Proof. Define for each $k \geq 0$, the functions $R_k(y) = r^{\varepsilon_k}(x + y\sqrt{\varepsilon_k})$ and $U_{j,k}(y) = u_j^{\varepsilon_k}(x + y\sqrt{\varepsilon_k})$. Then, due to elliptic estimates, possibly up to a subsequence, one may assume that $(R_k, U_{1,k}, \cdots, U_{N,k})$ converges locally uniformly to some bounded non-negative functions $(R, U_1, \cdots, U_N)(y)$, solution of the elliptic system posed for $y \in \mathbb{R}^n$

$$\begin{cases} \Delta R(y) + I(x) - m_0 R(y) - \sum_{j=1}^N f_j(R(y), x) U_j(y) = 0, \\ d_j \Delta U_j(y) - m_j U_j(y) + f_j(R(y), x) U_j(y) = 0, \ j = 1, \cdots, N. \end{cases} \tag{5.7}$$

Next, set $J = \{j \in \{1, \cdots, N\}, U_j > 0\}$. Recalling Definition (5.1)-(5.3), one has

$$\overline{\Omega} \setminus \Gamma \subset \overline{\Omega} \setminus \Gamma^J = \bigcup_{j \in J \cup \{0\}} \Theta_j^J.$$

Firstly, when $x \in \Theta_0^J$, due to Lemma 5.2 one obtains that $U_j \equiv 0$ for any $j \in J$ so that $J = \emptyset$. Hence $R(y) \equiv S(x)$ and Alternative (5.5) holds true.
Next assume that $x \in \Theta_i^J$ for some $i \in J$. Since $U_i > 0$, Lemma 5.1 applies and provides that $U_j(y) \equiv 0$ for any $j \in J \setminus \{i\}$ so that $J = \{i\}$. As a consequence of Lemma 5.1, Alternative (5.6) holds true. This completes the proof of the lemma. ∎

Lemma 5.4. *Let $i \in \{1, \cdots, N\}$ and $x \in \Theta_i \setminus \Gamma$ be given. Assume that for any small enough $\varepsilon > 0$, there exists a non-negative solution $(r^\varepsilon, u_1^\varepsilon, \cdots, u_N^\varepsilon)$ such that $u_i^\varepsilon > 0$. Then one has*

$$\liminf_{\varepsilon \to 0} u_i^\varepsilon(x) > 0.$$

Proof. To prove this lemma, let us argue by contradiction by assuming that there exists a sequence $\{\varepsilon_k\}_{k \geq 0}$ tending to zero such that $u_i^{\varepsilon_k}(x) = 0$. According to Lemma 5.3, up to a subsequence, one has $r^{\varepsilon_k}(x + y\sqrt{\varepsilon_k}) \to \widehat{R}(x)$ locally uniformly with either $\widehat{R}(x) = S(x)$ or $\widehat{R}(x) = R_j(x)$ for some $j \neq i$. Now since $x \in \Theta_i$, in any cases, one has

$$f_i\left(\widehat{R}(x), x\right) - m_i > 0. \tag{5.8}$$

Now consider the sequence of maps $W_k(y) = \frac{u_i^{\varepsilon_k}(x + y\sqrt{\varepsilon_k})}{u_i^{\varepsilon_k}(x)}$. Due to elliptic Harnack inequality, the sequence $\{W_k\}_{k \geq 0}$ is locally bounded and up to a subsequence one may assume that $W_k \to W$ locally uniformly in $\mathbb{R}^n$ where W satisfies

$$\begin{cases} d_i \Delta W(y) + f_i\left(\widehat{R}(x), x\right) W(y) - m_i W(y) = 0, \\ W \geq 0 \text{ and } W(0) = 1. \end{cases}$$

Then one obtains that $W > 0$ and Lemma Appendix B.1 yields

$$f_i\left(\widehat{R}(x), x\right) - m_i \leq 0,$$

a contradiction together with (5.8). This completes the proof of the lemma. ∎

We are now able to complete this section by proving the following theorem which contains Theorem 2.7 by taking $J = \{1, .., N\}$.

Theorem 5.1 (Asymptotic profile $\varepsilon \to 0$). *Let Assumptions 2.1, 2.2, 2.5 and 2.6 be satisfied. Let $J \subset \{1, \cdots, N\}$ be given and consider a family $(r^\varepsilon, u_1^\varepsilon, \cdots, u_N^\varepsilon)$ with $\varepsilon \in (0, \varepsilon_0)$ of strict J-coexistence solutions. Then the following convergence holds true:*

$$\lim_{\varepsilon \to 0} (r^\varepsilon, u_1^\varepsilon, \cdots, u_N^\varepsilon) = \left(R^{*,J}, U_1^{*,J}, \cdots, U_N^{*,J}\right),$$

where the convergence is uniform on each compact subset of $\overline{\Omega}\setminus\Gamma^J$ *and wherein we have set* $R^{*,J}(x)=\min\left(S(x),\min_{j\in J}R_j(x)\right)$ *and*

$$U_i^{*,J}(x)=\begin{cases}\frac{m_i}{m_0}\left(S(x)-R_i(x)\right), & \text{if } x\in\Theta_i^J,\\ 0 \text{ if } x\notin\Theta_i^J,\end{cases}\quad \text{if } i\in J,$$
$$\text{and } U_i^{*,J}(x)=0 \text{ if } i\in\{1,\cdots,N\}\setminus J.$$

Proof. The proof of this result is similar to the one of Theorem 2.4 using the Lemmas 5.4, 5.1 and 5.2. The details are left to the reader. ■

Let us also notice that the proof of Corollary 2.2 is similar to the one of Corollary 2.1 and thus omitted. Indeed since $\Gamma=\emptyset$, then the convergence $r^\varepsilon\to R^*$ holds uniformly on the whole domain $\overline{\Omega}$.

5.2. *Proof of Theorem 2.8*

In this section we investigate the existence of a positive solution of (1.1) in the general case of $N\geq 2$ species. The arguments will closely follow the ones developed for the one species problem using an induction argument on the number of species. Our precise result reads as:

Theorem 5.2. *Let Assumptions 2.1, 2.2, 2.5 and 2.6 be satisfied. Then there exists* $\varepsilon_0>0$ *such that for any* $J\subset\{1,\cdots,N\}$ *and any* $\varepsilon\in(0,\varepsilon_0)$*, System (1.1) has a strict* J*-coexistence solution belonging to* $\left(C^2(\overline{\Omega})\right)^{N+1}$.

The proof of this result will follow an induction argument on the cardinal of the subset J. Let us first notice that Proposition 2.1 as well as Theorem 2.3 proves the result in the case $J=\emptyset$ and $J=\{1\}$ respectively. Up to reordering the species, Theorem 5.2 holds true for each $J\subset\{1,\cdots,N\}$ with a cardinal in $\{0,1\}$.

Let $2\leq N'\leq N$ be given. Our induction assumption reads as for each $J\subset\{1,..,N\}$ with card $J\leq N'-1$ there exists $\varepsilon_1>0$ such that for any $\varepsilon\in(0,\varepsilon_1)$ System (1.1) has a strict J-coexistence solution in $\left(C^2(\overline{\Omega})\right)^{N+1}$.

Let $J'\subset\{1,..,N\}$ with card $J'=N'$ be given and we claim that

Claim 5.3. Up to reduce the value ε_1 if necessary, for each $\varepsilon\in(0,\varepsilon_1)$, then System (1.1) with ε has a strict J'-coexistence solution.

To prove this claim, up to reordering the species, one may assume that

$$J'=\{1,\cdots,N'\}.$$

The proof of the above mentioned claim will follows several lemmas. Our first lemma is the following:

Lemma 5.5. *Let Assumptions 2.1, 2.2, 2.5 and 2.6 be satisfied. Up to reduce* $\varepsilon_1>0$*, for each* $i\in J'$*, for each* $J\subset\{1,\cdots,N\}\setminus\{i\}$ *and for each strict* J*-coexistence solution* $(r^\varepsilon,u_1^\varepsilon,..,u_N^\varepsilon)$ *of (1.1) with* $\varepsilon\in(0,\varepsilon_1)$*, one has for any* $\varepsilon>0$ *small enough:*

$$\Lambda(d_i\varepsilon,f_i(r^\varepsilon(.),.)-m_i)>0.$$

Proof. Due to Assumption 2.5, one can choose a nonempty compact subset $K \subset \Theta_i \setminus \Gamma$. Due to the Definition of Γ^J (see (5.3)), one gets $K \subset \Theta_i \setminus \Gamma^J$. Then we infer from Theorem 5.1 that

$$\lim_{\varepsilon \to 0} r^\varepsilon = \min\left(S, \min_{j \in J} R_j\right) \text{ for the topology of } C^0(K). \tag{5.9}$$

Hence $r^\varepsilon(x) > R_i(x)$ for any $x \in K$ and any small enough $\varepsilon > 0$, so that for each $\varepsilon > 0$ small enough:

$$f_i(r^\varepsilon(x), x) - m_i > 0 \text{ for all } x \in K.$$

Finally Lemma 3.3 applies and completes the proof of the lemma. ∎

Due to the induction assumption, for each $J \subset \{1, .., N\}$ with card $J \le N' - 1$ and each $\varepsilon \in (0, \varepsilon_1)$ let us denote by $\left(r^{\varepsilon,J}, u_1^{\varepsilon,J}, \cdots, u_N^{\varepsilon,J}\right)$ a strict $J-$coexistence solution of (1.1) with ε. Now Lemma 5.5 applies and provides that for each $\varepsilon \in (0, \varepsilon_1)$, for each $i \in J'$ and each $J \subset \{1, .., N\} \setminus \{i\}$ with card $J \le N' - 1$:

$$\Lambda(d_i\varepsilon, f_i(r^{\varepsilon,J}(.), .) - m_i) > 0.$$

Coupling this result together with the same arguments as the ones of the proof of 4.2, one obtains the following a priori estimates for any J'-coexistence solution of (1.1) and ε small enough.

Lemma 5.6. *Let $\varepsilon \in (0, \varepsilon_1)$ be given. Then for each $j \in J'$ there exists $\alpha_j > 0$ such that for any J'-coexistence solution $(r^\varepsilon, u_1^\varepsilon, \cdots, u_N^\varepsilon)$ of (1.1) with ε satisfies for each $j \in J'$:*

$$u_j^\varepsilon > 0 \Rightarrow u_j^\varepsilon > \alpha_j.$$

This lemma allows us to define the following subsets for any $J \subset J'$

$$\mathcal{O}^J = \{(r, u_1, \cdots, u_{N'}),\ u_j > \alpha_j\ \forall j \in J,\ u_j < \alpha_j\ \forall j \in J' \setminus J\}.$$

Let $\varepsilon \in (0, \varepsilon_1)$ be given. Using Lemma 5.6 note that any strict J-coexistence solution of (1.1) belongs to $\mathcal{O}^J$. In particular a strict J'-coexistence solution belongs to $\mathcal{O}^{J'}$. Hence, it is sufficient to show that System (1.1) has a solution in $\mathcal{O}^{J'}$. This result will be obtained by using topological degree arguments. Let us first re-write Problem (1.1) (recalling that $\varepsilon \in (0, \varepsilon_1)$ is fixed) as a fixed point problem. Define $E = \left(C^0(\overline{\Omega})\right)^{N'+1}$, the positive cones $P = \{U \in C^0(\overline{\Omega}), U \ge 0\}$ and $C = P^{N'+1}$. Recall that, by Lemma 3.1, each non-negative solution $(r, u_1, \cdots, u_N)$ of (1.1) is uniformly bounded by some constant $M > 0$ independent of ε and f_i. Set

$$\mathcal{K} = \{(r, u_1, \cdots, u_{N'}) \in C,\ r < 2M,\ u_i < 2M\ \forall i \in J'\},$$

and consider $\beta > 0$ defined by

$$\beta = 2M \sup_{x \in \overline{\Omega}} \left(\sup_{r \in (0, 2M]} \sum_{i=1}^N \frac{f_i(r)}{r} \right).$$

Let us finally define the compact operator $B : C \to C$ by

$$B = \begin{pmatrix} (m_0 + \beta - \varepsilon d_0 \Delta)^{-1} & & & \\ & (m_1 - \varepsilon d_1 \Delta)^{-1} & & \\ & & \ddots & \\ & & & (m_{N'} - \varepsilon d_{N'} \Delta)^{-1} \end{pmatrix},$$

as well as for each $t \in [0, 1]$, the operator A_t by

$$A_t(r, u_1, \cdots, u_{N'}) = B \begin{bmatrix} I + \beta r - t \sum_{i \in J'} f_i(r) u_i \\ t f_1(r) u_1 \\ \vdots \\ t f_{N'}(r) u_{N'} \end{bmatrix}.$$

Note that $(r, u_1, \cdots, u_N)$ is a J'-coexistence solution of (1.1) with f_i replaced by tf_i if and only if it is a fixed point of A_t in $\mathcal{K}$. Let us also recall that due to standard elliptic estimates A_t is a completely continuous operator from $\mathcal{K}$ to C. Our first lemma is the following:

Lemma 5.7. *The following holds true.*

(i) $\forall t \in [0, 1]$, $deg_C(I - A_t, \mathcal{K}) = 1$.
(ii) *Let* $i \in J'$ *and* $J \subset J' \setminus \{i\}$ *be given. Then* $deg_C(I - A_1, \mathcal{O}^J) = 0$.

Proof. Since $(s^\varepsilon, 0, \cdots, 0)$ is the only fixed point of A_0 in $\mathcal{K}$, one can easily derive from Proposition Appendix A.1-(ii) that $deg_C(I - A_0, \mathcal{K}) = 1$. Since A_t does not have any fixed point on $\partial\mathcal{K}$, homotopy invariance of the topological degree completes the proof of (i).
In order to prove (ii), we only consider the case where $i = N'$ and $J = \{1, \cdots, N' - 1\}$. The other cases can be handled similarly. We aim to apply Proposition Appendix A.1-(i). Let us define

$$E_2 = C^0(\overline{\Omega}),\ E_1 = E_2^{N'-1},\ C_2 = P,$$
$$C_1 = P^{N'-1} \text{ and for } \delta > 0,\ P_\delta = \{u \in P,\ \|u\|_\infty < \delta\}.$$

Let us denote by $w = (r, u_1, \cdots, u_{N'-1})$, and $A = (A^1, A^2)$ with

$$A^2(w, u_{N'}) = (m_{N'} - \varepsilon d_{N'} \Delta)^{-1}(f_{N'}(r) u_{N'}).$$

Define the two following subsets of C_1 by

$$\mathcal{U} = \left\{(r, u_1, \cdots, u_{N'-1}) \in C_1,\ (r, u_1, \cdots, u_{N'}) \in \mathcal{O}^J \text{ for some } u_{N'}\right\},$$
$$\mathcal{T} = \left\{w \in \mathcal{U},\ A^1(w, 0) = w\right\}.$$

Using the definition of $\mathcal{O}^J$, the subset $\mathcal{U}$ is relatively[b] open and bounded in C_1. Then Lemma 5.6 implies that $A^1(w, 0) \neq w$ for any $w \in \partial\mathcal{U}$. Indeed if there exists

[b]In the case $J = \{1, \cdots, N' - 1\}$, $\mathcal{U}$ is open in C_1.

$w \in \partial\mathcal{U}$ such that $A^1(w,0) = w$ then $(w,0)$ is a $J-$coexistence solution of (1.1) such that $u_i \geq \alpha_i$ for all $i = 1,..,N'$. Lemma 5.6 applies and provides that $u_i > \alpha_i$ for all $i = 1,..,N'$, that is $w \in \mathcal{U}$. A contraction together with $w \in \partial\mathcal{U}$.

Note that due to the induction hypothesis and Lemma 5.6 $\mathcal{T} \neq \emptyset$. Using the same notations as in Proposition Appendix A.1-(i), one has for any $w \in \mathcal{T}$:

$$L^2(w) = (m_{N'} - \varepsilon d_{N'}\Delta)^{-1}(f_{N'}(r)\cdot).$$

Since for any $w \in \mathcal{T}$, $(w,0)$ is a strict $J-$coexistence solution, Lemma 5.5 ensures that $\Lambda(\varepsilon d_i, f_i(r^\varepsilon(\cdot),\cdot) - m_i) > 0$. Then one gets

$$\text{For any } w \in \mathcal{T},\ r_{spec}(L^2(w)) > 1,$$

and, for any $\phi \in C_2 \setminus \{0\}$, $L^2(w)\phi \neq \phi$ and $L^2(w)\phi > 0$. Proposition Appendix A.1-(i) applies and yields $deg_C(I - A_1, \mathcal{U} \times P_\delta) = 0$ for any small enough $\delta > 0$. Next from Lemma 5.6, one has $deg_C(I - A_1, \mathcal{O}^J) = deg_C(I - A_1, \mathcal{U} \times P_\delta) = 0$ which ends the proof of the lemma. ■

We are now able to complete the proof of Claim 5.3.

Proof of Claim 5.3. Let us first notice that Lemma 5.6 ensures that for any subset $J \subset J'$, operator A_1 does not have any fixed point on $\partial\mathcal{O}^J$. Using once again Lemma 5.6 one gets

$$deg_C(I - A_1, \mathcal{K}) = \sum_{J \subsetneq J'} deg_C(I - A_1, \mathcal{O}^J) + deg_C(I - A_1, \mathcal{O}^{J'}). \tag{5.10}$$

We infer from Lemma 5.7 that $deg(I - A_1, \mathcal{K}) = 1$ and $\deg_C(I - A_1, \mathcal{O}^J) = 0$ for any $J \subsetneq J'$. As a consequence we obtain that

$$deg_C(I - A_1, \mathcal{O}^{J'}) = 1.$$

Hence A_1 has at least one fixed point in $\mathcal{O}^{J'}$ that completes the proof of Claim 5.3. ■

Appendix A. Degree in a positive cone

We set up the fixed point index machinery used in this paper. In Ref. 13, the author describes various results to compute the topological degree of a completely continuous operator $A : C \to C$ defined in a cone C of a banach space E. This degree is denoted by $deg_C(I - A, \mathcal{U})$ where $\mathcal{U}$ is an open subset of C. Basically, if $\mathcal{U}$ lies in the interior of C then one get

$$deg_C(I - A, \mathcal{U}) = deg_E(I - A, \mathcal{U})$$

where $deg_E(I - A, \mathcal{U})$ denotes the usual Leray-Schauder degree. In particular, if $w^* \in int(C)$ is a solution of the equation $A(w) = w$ such that $I - D_w A(w^*)$ is invertible on E, then for any sufficiently small neighbourhood $\mathcal{U}$ of w^* in E, one gets

$$deg_C(I - A, \mathcal{U}) = index_E(A, w^*) = (-1)^\sigma$$

where σ is the sum of the (real) eigenvalues of $D_w A(w^*)$ smaller than 1 (counted with there algebraic multiplicities). If w^* does not belong to the interior of C, then $deg_C(I - A, \mathcal{U})$ can be computed if $I - D_w A(w^*)$ is invertible on E (see Theorem 1 in Ref. 13). Unfortunately, in our application, such an invertibility condition may fail and the aforementioned result cannot be used. However, in Ref. 14, the author states a result allowing to relax this condition. This result is proved by Dancer and Du (see Theorem 2.1 in Ref. 17).

Hence, following Ref. 14, for $i = 1, 2$, let E_i be a Banach space[c] and let C_i be a cone in E_i. Set $E = E_1 \times E_2$ and $C = C_1 \times C_2$.
Let $A^i : C \to C_i$ be two completely continuous operators and consider $A = (A^1, A^2)$. Denote I_i the identity map on E_i as well as $I = (I_1, I_2)$ the identity operator on E. For each $(w, v) \in E_1 \times E_2$, set $L^1(w, v) = D_w A^1(w, v)$ and $L^2(w, v) = D_v A^2(w, v)$ and for each $\varepsilon > 0$, consider $C_{2,\varepsilon} = \{v \in C_2, \|v\| \leq \varepsilon\}$. Together with these notations, the following proposition holds true (this is a stronger form of the result stated in Proposition 2 in Ref. 14, see Ref. 17 for a proof).

Proposition Appendix A.1. *Assume that $\mathcal{U} \subseteq C_1$ is a relatively open and bounded set such that $\forall w \in \partial\mathcal{U}$, $w \neq A^1(w, 0)$ (the boundary being relative to C_1). Assume that $A^1(\mathcal{U} \times \{0\}) \subset \mathcal{U}$ and let $\mathcal{T} = \{w \in \mathcal{U}, w = A^1(w, 0)\}$. Assume furthermore that $v \mapsto A^2(w, v)$ is linear and let us denote $L^2(w). = L^2(w, .)$. Then the following holds true:*

(i) Assume that for all $w \in \mathcal{T}$, $r_{spec}(L^2(w)) > 1$ and $L^2(w)\phi \neq \phi$ for any $\phi \in C_2 \setminus \{0\}$. Assume in addition that for each $w \in \mathcal{U}$, and $\phi \in C_2$, $L^2(w)\phi \geq 0$. Then $deg_C(I - A, \mathcal{U} \times C_{2,\varepsilon})$ is well defined for all sufficiently small positive ε and one has

$$deg_C(I - A, \mathcal{U} \times C_{2,\varepsilon}) = 0.$$

(ii) Assume that for each $w \in \mathcal{T}$, $r_{spec}(L^2(w)) < 1$. Then $deg_C(I - A, \mathcal{U} \times C_{2,\varepsilon})$ is well defined for all small enough positive ε and

$$deg_C(I - A, \mathcal{U} \times C_{2,\varepsilon}) = deg_{C_1}(I_1 - A^1_{|\mathcal{U} \times \{0\}}, \mathcal{U}).$$

Appendix B. A Lemma for an elliptic system on $\mathbb{R}^n$

This aim of this section is to prove the following lemma:

Lemma Appendix B.1. *Let $v \in C^2(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$ be given such that $v(x) > 0$ for each $x \in \mathbb{R}^n$ and*

$$\Delta v(x) + \lambda v(x) = 0, \quad x \in \mathbb{R}^n.$$

[c] In our applications, $E_2 = C^0(\overline{\Omega})$, C_2 is the natural positive cone in E_2, $E_1 = \underbrace{E_2 \times \cdots, E_2}_{N-1 \text{ copies}}$ and $C_1 = \underbrace{C_2 \times \cdots \times C_2}_{N-1 \text{ copies}}$.

Then $\lambda \leq 0$.

Note that such a result should be well known but we were not able to find any references for such a result. For the sake of completeness, we prefer to provide a proof of this result.

Proof. Let us first notice that due to elliptic gradient estimates as well as elliptic Harnack inequality, the function $\frac{\nabla v(x)}{v(x)}$ is uniformly bounded on $\mathbb{R}^n$. Indeed let $R > 0$ be given. Then there exists some constant $M = M(R)$ such that for each $y \in \mathbb{R}^n$,

$$\|\nabla v(x)\| \leq M \sup_{x \in B(y,2R)} v(x), \quad \forall x \in B(y, R).$$

One the other hand, due to Harnack inequality, there exists some constant $N = N(R)$ such that

$$\sup_{x \in B(y,2R)} v(x) \leq N v(y),$$

and the result follows.

Consider the real number $\Lambda \in \mathbb{R}$ defined by

$$\Lambda := \limsup_{\|y\| \to \infty,\ e \in S^{n-1}} \frac{e.\nabla v(y)}{v(y)}.$$

Then there exist two sequences $\{y_k\}_{k \geq 0} \subset \mathbb{R}^n$ and $\{e_k\}_{k \geq 0} \subset S^{n-1}$ such that $\|y_k\| \to \infty$ and

$$\lim_{k \to \infty} \frac{e_k.\nabla v(y_k)}{v(y_k)} = \Lambda.$$

Up to a subsequence and without loss of generality (up to rotation), one may assume that $e_k \to e_1 = (1, 0, .., 0)^T$. Next consider the sequence of map $w_k(y) = \frac{\partial_{x_1} v(y+y_k)}{v(y+y_k)}$ that satisfies

$$\Delta w_k(x) + 2 \frac{\nabla v(x + y_k)}{v(x + y_k)}.\nabla w_k(x) = 0, \quad x \in \mathbb{R}^n.$$

The sequence $\{w_k\}_{k \geq 0}$ is uniformly bounded on $\mathbb{R}^n$ and due to elliptic estimates, one may assume that $w_k \to w_\infty$, locally uniformly. Moreover it satisfies

$$\begin{aligned} &\Delta w_\infty(x) + \overrightarrow{a}(x) \nabla w_\infty(x) = 0, \\ &w_\infty(0) = \Lambda, \quad w_\infty(x) \leq \Lambda, \quad \forall x \in \mathbb{R}^n, \end{aligned}$$

for some some globally bounded function $\overrightarrow{a}(x)$. Then the strong maximum principle implies that $w_\infty(y) \equiv \Lambda$.

On the other hand, if one considers the sequence of maps $Z_k(x) = \frac{v(y_k+x)}{v(y_k)}$, then it satisfies

$$\Delta Z_k(x) + \lambda Z_k(x) = 0, \quad Z_k(0) = 1,$$

so that it is locally bounded due to Harnack inequality and, up to a subsequence, it converges locally uniformly to some positive function Z_∞ such that

$$\Delta Z_\infty(x) + \lambda Z_\infty(x) = 0, \quad Z(0) = 1. \tag{B.1}$$

Therefore $Z_\infty(x) > 0$. Furthermore one has $w_\infty = \frac{\partial_{x_1} Z_\infty}{Z_\infty} \equiv \Lambda$. This implies that

$$Z_\infty(x) = e^{\Lambda x_1},$$

and plugging this last equality into (B.1), leads us to

$$\Lambda^2 + \lambda = 0,$$

and the result follows. ∎

Appendix C. Rescalling when $x \in \partial\Omega$

The proof of many results in this text use a rescaling argument around a point $x \in \Omega$ by considering sequence of rescaled functions of the form $U^\varepsilon = u^\varepsilon(x + y\sqrt{\varepsilon})$. When $x \in \Omega$, this rescaling argument holds for $y \in B\left(x, \frac{d(x,\partial\Omega)}{\sqrt{\varepsilon}}\right)$ so that elliptic regularity allows us to get that (up to extraction) $U^\varepsilon \to V$ locally uniformly in $\mathbb{R}^n$. When $x \in \partial\Omega$ this methods cannot be directly applied. Here, we closely follow Ref. 34 and we also refer to Ref. 38 for more details. If $x \in \partial\Omega$, the idea is basically to "straighten the boundary at x" and to rescale it. Once the boundary is straighten, L^p elliptic estimates up to the boundary are used to pass to the limit $\varepsilon \to 0$ and then to obtain an elliptic system posed on the half space $\{y \in \mathbb{R}^n,\ y_n \geq 0\}$. Finally, thanks to the Neumann boundary condition, a reflection argument with respect to the hyperplane $y_n = 0$ allow to extend such an elliptic system on the whole space yielding to the same kind of system than the one obtained when $x \in \Omega$. Here, we describe this method only in the case of section 4. Hence, consider the system

$$\begin{cases} \varepsilon\Delta r_\varepsilon - m_0 r_\varepsilon - f_1(r_\varepsilon)u_\varepsilon + I = 0 \text{ on } \Omega, \\ d_1\varepsilon\Delta u_\varepsilon - m_1 u_\varepsilon + f_1(r_\varepsilon)u_\varepsilon = 0 \text{ on } \Omega, \\ \partial_\nu r_\varepsilon = \partial_\nu u_\varepsilon = 0, \text{ on } \partial\Omega. \end{cases} \tag{C.1}$$

The above arguments can be slightly modified in each case appearing in this text. Let $x_0 \in \partial\Omega$. Without lose of generality one may assume that $x_0 = 0 \in \mathbb{R}^n$. Due to the regularity of the boundary, one may assume that there exists a C^2 function $h(x')$ where $x' = (x_1, \cdots, x_{N-1})$ defined for $|x'| \ll 1$ such that $h(0) = 0$, $\partial_{x_i} h(0) = 0,\ 1 \leq i \leq N-1$ and for small neighbourhood V of the origin, $\Omega \cap V = \{(x', x_N) \in V,\ x_n > h(x')\}$, $\partial\Omega \cap V = \{(x', x_N) \in V,\ x_N = h(x')\}$.
For $z = (z', z_n) \in \mathbb{R}^n$ let us define $H(z) = (H_1(z), \cdots, H_n(z))$ by

$$H_j(z) = z_j - z_n \partial_{x_j} h(z'),\ j = 1, \cdots, N-1$$
$$H_n(z) = z_n + h(z').$$

Since the function H has an inverse function G defined nearby the origin, for $|x| \ll 1$ we introduce the new local coordinates $z = G(x) := (G_i(x))_{1 \leq i \leq n}$. The reason for this is that near x_0 one has $G(\partial\Omega) = \{|z| \ll 1,\ z_n = 0\}$, i.e. the boundary is flat in the new local coordinates. Now, define

$$\widetilde{a}_{ij}(z) = \sum_{k=1}^{n} \frac{\partial G_i}{\partial x_k}(H(z)) \frac{\partial G_j}{\partial x_k}(H(z)),\ 1 \le i, j \le n,$$
$$\widetilde{b}_j(z) = \Delta G_j(H(z)),\ 1 \le j \le n,$$

as well as the operator

$$\widetilde{\Xi} f(z) = \sum_{1 \le i,j \le n} \widetilde{a}_{ij}(z) \frac{\partial^2 f}{\partial z_i \partial z_j}(z) + \sum_{j=1}^{n} \widetilde{b}_j(z) \frac{\partial f}{\partial z_j}(z).$$

Let $(r_\varepsilon, u_\varepsilon)$ be a (sequence of) non-negative solutions of (C.1) and define $\widetilde{r}_\varepsilon(z) = r_\varepsilon(H(z))$, $\widetilde{u}_\varepsilon(z) = u_\varepsilon(H(z))$ that satisfy

$$\begin{cases} \varepsilon \widetilde{\Xi} \widetilde{r}_\varepsilon(z) - m_0 \widetilde{r}_\varepsilon(z) - f_1(\widetilde{r}_\varepsilon(z), H(z)) \widetilde{u}_\varepsilon(z) + I(H(z)) = 0, \\ \varepsilon d_1 \widetilde{\Xi} \widetilde{u}_\varepsilon(z) - m_1 \widetilde{u}_\varepsilon(z) + f_1(\widetilde{r}_\varepsilon(z), H(z)) \widetilde{u}_\varepsilon(z) = 0, \\ z \in B_\delta^+, \end{cases} \tag{C.2}$$

together with

$$\partial_{z_n} \widetilde{r}_\varepsilon = \partial_{z_n} \widetilde{u}_\varepsilon = 0, \text{ on } \{z_n = 0\} \cap B_\delta, \tag{C.3}$$

where $\delta > 0$ is small enough while $B_\delta = \{z \in \mathbb{R}^n,\ |z| < \delta\}$ and $B_\delta^+ = B_\delta \cap \mathbb{R}_+^n$. Now we consider the rescaled sequence of maps

$$R_\varepsilon(y) = \widetilde{r}_\varepsilon(y\sqrt{\varepsilon}),\ \ U_\varepsilon(y) = \widetilde{u}_\varepsilon(y\sqrt{\varepsilon}),$$

and for $y \in \mathbb{R}^n$, denote by

$$\Xi^\varepsilon f(y) = \sum_{1 \le i,j \le n} a_{ij}^\varepsilon(y) \frac{\partial^2 f}{\partial y_i \partial y_j}(y) + \sqrt{\varepsilon} \sum_{j=1}^{n} b_j^\varepsilon(y) \frac{\partial f}{\partial y_j}(y),$$

wherein we have set $a_{ij}^\varepsilon(y) = \widetilde{a}_{ij}(y\sqrt{\varepsilon})$ and $b_j^\varepsilon(y) = \widetilde{b}_j(y\sqrt{\varepsilon})$. The rescaled functions R_ε and U_ε satisfy

$$\begin{cases} \Xi^\varepsilon R_\varepsilon(y) - m_0 R_\varepsilon(y) - f_1(R_\varepsilon(y), H(y\sqrt{\varepsilon})) U_\varepsilon(y) + I(H(y\sqrt{\varepsilon})) = 0,\ y \in B^+_{\frac{\delta}{\sqrt{\varepsilon}}} \\ d_1 \Xi^\varepsilon U_\varepsilon(y) - m_1 U_\varepsilon(y) + f_1(R_\varepsilon(y), H(y\sqrt{\varepsilon})) U_\varepsilon(y) = 0,\ y \in B^+_{\frac{\delta}{\sqrt{\varepsilon}}} \\ \partial_{y_n} R_\varepsilon = \partial_{y_n} U_\varepsilon = 0, \text{ on } \{y_n = 0\} \cap B_{\frac{\delta}{\sqrt{\varepsilon}}}. \end{cases} \tag{C.4}$$

Choose a sequence M_ε tending to $+\infty$ as $\varepsilon \to 0$ such that $B^+_{4M_\varepsilon} \subset B^+_{\frac{\delta}{\sqrt{\varepsilon}}}$. Since a_{ij}^ε and b_j^ε are uniformly bounded with respect to ε in $C^2\left(\overline{B_{\frac{\delta}{\varepsilon}}}\right)$, one can apply elliptic L^p estimates up to the boundary (see Ref. 23 p238) to (C.4) in $\overline{B^+_{2M_\varepsilon}}$ as well as Sobolev embedding theorem, to obtain that R_ε and U_ε are uniformly bounded in $C^{1,\gamma}\left(\overline{B^+_{M_\varepsilon}}\right)$ for every $\gamma \in (0,1)$. Moreover $a_{ij}^\varepsilon \to \delta_{ij}$ as $\varepsilon \to 0$. Therefore, up to a subsequence, R_ε and U_ε can be assumed to converge uniformly on any compact subset of $\mathbb{R}^n_+$ to some functions R, $U \in W^{2,p}_{loc}(\mathbb{R}^n_+) \cap C^1(\mathbb{R}^n_+)$ solution of

$$\begin{cases} \Delta R(y) - m_0 R(y) - f_1(R(y),0)U(y) + I(0) = 0,\ y \in \mathbb{R}^n_+ \\ d_1 \Delta U(y) - m_1 U(y) + f_1(R(y),0)U_(y) = 0,\ y \in \mathbb{R}^n_+ \\ \partial_{y_n} R = \partial_{y_n} U = 0, \text{ on } \{y_n = 0\}. \end{cases} \tag{C.5}$$

Finally, using reflection with respect to the hyperplane $y_n = 0$, one can extend R and U to the whole space $\mathbb{R}^n$ so that (R,U) satisfies (4.7). Note that R and U are non-negative and bounded in $\mathbb{R}^n$.

The case involving more species follows exactly the same lines. In some case appearing in the text, as for instance in the proof of Proposition 4.1, one needs to consider a sequence $x_\varepsilon \to x_0 \in \partial\Omega$ rather than a given and fixed $x_0 \in \partial\Omega$. A slight modification of the above arguments provides the result (see Ref. 34 for more details).

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