

Bistability induced by generalist natural enemies can reverse pest invasions without the need for an Allee effect in prey dynamics

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Abstract. Reaction-diffusion analytical modeling of predator-prey systems has shown that specialist natural enemies can slow, stop and even reverse pest invasions, assuming that the prey population displays a strong Allee effect in its growth. Few additional analytical results have been obtained for other spatially distributed predator-prey systems, as traveling waves of non-monotonous systems are notoriously difficult to obtain. Traveling waves have indeed recently been shown to exist in predator-prey systems, but the direction of the wave, an essential item of information in the context of the control of biological invasions, is generally unknown. Preliminary numerical explorations have hinted that control by generalist predators might be possible for prey populations displaying logistic growth. We aimed to formalize the conditions in which spatial biological control can be achieved by generalists, through an analytical approach based on reaction-diffusion equations.

The population of the focal prey — the invader — is assumed to grow according to a logistic function. The predator has a type II functional response and is present everywhere in the domain, at its carrying capacity, on alternative hosts. Control, defined as the invader becoming extinct in the domain, may result from spatially independent demographic dynamics or from a spatial extinction wave. Using comparison principles, we obtain sufficient conditions for control and for invasion, based on scalar bistable partial differential equations (PDEs). The searching efficiency and functional response plateau of the predator are identified as the main parameters defining the parameter space for prey extinction and invasion. Numerical explorations are carried out in the region of those control parameters space between the super- and sub-solutions, in which no conclusion about controllability can be drawn on the basis of analytical solutions.

The ability of generalist predators to control prey populations with logistic growth lies in the bistable dynamics of the coupled system, rather than in the bistability of prey-only dynamics as observed for specialist predators attacking prey populations displaying Allee effects. The consideration of space in predator-prey systems involving generalist predators with a parabolic functional response is crucial. Analysis of the ordinary differential equations (ODEs) system identifies parameter regions with monostable (extinction) and bistable (extinction or invasion) dynamics. By contrast, analysis of the associated PDE system distinguishes different and additional regions of invasion and extinction. Depending on the relative positions of these different zones, four patterns of spatial dynamics can be identified : traveling waves of extinction and invasion, pulse waves of extinction and heterogeneous stationary positive solutions of the Turing type. As a consequence, prey control is predicted to be possible when space is considered in additional situations other than those identified without considering space. The reverse situation is also possible. None of these considerations apply to spatial predator-prey systems with specialist natural enemies.

1 Introduction

1.1 Modeling the biological control of invasive pests

Biological invasions are a major contemporary problem ([Pimentel (2011), Garnier et al. (2012), Mistro et al. (2012), Potapov & Rajakaryne (2013), Wang et al. (2013), Savage & Renton (2013)]) for which few solutions are available, all of which are very costly. The use of natural enemies for the biological control of invading insects is one of the most promising possibilities ([Moffat et al. (2013), Li et al. (2014), Ye et al. (2014), Basnet & Mukhopadhyay (2014)]). As invasion is essentially a spatial process, the potential of natural enemies to stop or even reverse an invasion is of particular interest. The fundamental analytical work of [Owen & Lewis (2001)] showed that *specialist* predators could potentially slow, stop or reverse the spread of invasive pests. The reversal of pest spread by specialist predation requires a strong Allee effect for the pest-only dynamics, defined as a negative growth rate for the prey population at low density. In the presence of a weak Allee effect, the predator can stop, but not reverse the wave of invasion. These conclusions have been confirmed in several other studies ([Cai et al. (2014), Boukal et al. (2007), Morozov & Petrovskii (2009)]).

Generalist predators can also control prey effectively ([Erbach et al. (2014), Chakraborty (2015)]). Their use could be promoted through conservation biological control programs without the need for exogenous specialist natural enemies. Unfortunately, the role of generalist predators in the spatial control of their prey has been much less studied than that of specialist predators, due to the intrinsic difficulties of having to work with a *system* of equations rather than with a *single* scalar equation. However, two important studies have been carried out in this area : the analytical and comprehensive study of [Du & Shi (2007)], and the preliminary simulation study of [Fagan et al. (2002)]. Both used the same model structure as we do here, with logistic growth for both prey and predator populations, and a type II functional response for predators. The convergence of these models was strengthened further by the in-depth analysis of [Magal et al. (2008)] in which space was not considered. It is difficult to use these models in a spatial context : the work of [Du & Shi (2007)] cannot deal with invasion and traveling waves, because it deals with a bounded space. The numerical simulations of [Fagan et al. (2002)] are restricted to a few parameter values. They are, however, valuable, because they suggest conditions in which a generalist predator might be able to stop, and even reverse the invasion wave of a pest population displaying logistic growth. Fagan *et al.* also reported the results of field studies indicating that predators with diffusion coefficients higher than those of their prey are poor control organisms. The authors provided an explanation for this finding founded on logical arguments, but without a firm mathematical foundation. This result has been confirmed by a few numerical simulations including space, as reported by [Magal et al. (2008)], revealing a strong dependence of system dynamics on the relative rates of diffusion of the prey and the predator. It is thus important to take space into account, by considering diffusion coefficients of both predator and prey. This conclusion accounts for the interest of scientists in questions of this type ([Lewis et al. (2013), Hastings (2000), De Roos et al. (1991), De Roos et al. (1998)]).

There are therefore hopes that it might be possible to extend the conditions for the control of invasive prey organisms to (i) generalist predators and (ii) prey populations displaying growth patterns not dependent on the restrictive assumption of Allee effects. Such control approaches would have a major impact in the field, given the high degree of generalism obtained. The aim of this study was, therefore, to formalize the conditions in which spatial biological control can be achieved by generalists, through an analytical approach based on traveling waves solutions of reaction-diffusion equations.

Travelling wave solution thus describes a constant profile U moving through space at a speed c . Such waves are often observed in nonlinear reaction-diffusion systems modeling various phenomena. They are particularly suitable for describing the propagation of invasive fronts. In systems modeling a single species, described by a *scalar* equation, this type of solution is very well

understood ([Fischer (1937), Kolmogorov et al. (1937)] and [Volpert et al. (1994)] for a complete theory). Two particular classes of equation can be distinguished : monostable equations (like the Fisher-KPP equations) and bistable equations (often modeling the Allee effect). In monostable equations, there is a minimal wave speed c^* such that, for any $c \geq c^*$, a wave solution with speed c exists. In bistable equations traveling waves exist for *a unique* speed c^* . The sign of this speed c^* distinguishes between invasion or extinction of prey, which is a key property for our purposes.

For interactions of several species (described by a *multidimensional* system), the situation is much more complex. However, for some type of interaction, cooperation for instance, the system possesses a strong structural property, namely monotonicity. Essentially, this monotony makes it possible to use the comparison principle, which is always possible for one-dimensional systems, and the theory is then complete (see [Volpert et al. (1994)]). Unfortunately, our system, and prey-predator systems in general, do not have such a monotonous structure. This method is then unsuitable for monotonous systems and only a few results have been published. One of the key reasons for this is as follows : when we search for traveling wave solutions for a system with N equations, we obtain a system of N second order ordinary equations that can be reformulated as a system of $2N$ first order ordinary equations. In the scalar case ($N = 1$), it is therefore necessary to study trajectories in a plane, available using classical tools for two-dimensional dynamical systems. For several species ($N > 1$), it is necessary to study trajectories in a $2N$ -dimensional space, which may be very difficult.

Hence, the first rigorous results demonstrating the existence of traveling waves in prey-predator systems were based on a generalization, to the fourth dimension, of the classical shooting method in the phase plane ([Dunbar (1984a), Dunbar (1984b)]). This approach has since been generalised ([Huang et al. (2003), Xu & Weng (2012)]). However, all these studies simply investigate the mere *existence* of traveling waves. They do not determine the direction of the wave or the global dynamics for general initial conditions. Other methods have recently been developed in similar models ([Huang & Weng (2013), Ducrot & Langlais (2012)]), but they are subject to the same limitations. We therefore required another method.

The analysis provided in [Magal et al. (2008)] gives conditions for preys' control by predators, but this analysis was carried out largely without reference to space. Thus, we have extended the system of [Magal et al. (2008)] by adding spatial diffusion. We find that the conditions for control are very different from those for the system in which space is not considered. The conditions for prey extinction and invasion are discussed in terms of two essential parameters : the encounter rate E and the handling time h . Increasing E clearly increases predator pressure. Conversely, increasing h decreases predator pressure.

The paper is organized as follows. In section 2 we present the mathematical model and the main result of this work : theorem 2.1 describes invasion conditions for the ODE system and the theorems 2.4 and 2.6 the invasion conditions for the PDE system. The mathematical results are completed by numerical simulations in section 3. The results are discussed in section 4. The final section 5 is devoted to the mathematical proofs.

2 Model and main results

2.1 Mathematical model

We analyze a system of partial differential equations for a prey population with logistic growth, and a generalist predator population with logistic growth on alternative prey in the absence of the invading host. The functional response is of Holling type II. The prey-predator interactions

are modeled by the following partial differential equation system :

$$\begin{cases} u_t = D_u \Delta_x u + r_1 u \left(1 - \frac{u}{K_1}\right) - \frac{Euv}{1+Ehu}, \\ v_t = D_v \Delta_x v + r_2 v \left(1 - \frac{v}{K_2}\right) + \gamma \frac{Euv}{1+Ehu}, \\ u(x, 0) = u_0(x), \quad v(x, 0) = 1 \end{cases} \quad x \in \mathbb{R}, t \in \mathbb{R}^+ \quad (1)$$

with :

$u(t, x)$ = prey density at time t and at point x . $v(t, x)$ = predator density at time t and at point x .
 D_u = diffusion rate of prey D_v = diffusion rate of predators
 r_1 = growth rate of prey r_2 = growth rate of predators
 K_1 = carrying capacity of prey K_2 = carrying capacity of predators in absence of focal prey
 E = encounter rate h = handling time
 γ = conversion efficiency $u_0 \geq 0$ the initial concentration of prey,
 $D_u, D_v, r_1, r_2, K_1, K_2, E, h$ and γ are positive constant parameters.

We carried out the following adimensionalization :

$$\begin{aligned} t' &= r_1 t; \quad x' = x \sqrt{\frac{r_1}{D_u}}; \quad u'(x', t') = u(t, x)/K_1; \quad v'(x', t') = v(t, x)/K_2 \\ d' &= D_v/D_u; \quad r' = r_2/r_1; \quad E' = EK_2/r_1; \quad h' = r_1 h K_1/K_2; \quad \gamma' = \gamma K_1/K_2; \quad \alpha = \frac{\gamma'}{r'}. \end{aligned}$$

Removing the sign ' to simplify the notation, the system reads

$$\begin{cases} u_t = \Delta_x u + u(1 - u) - \frac{Euv}{1+Ehu}, \\ v_t = d \Delta_x v + r \left(v(1 - v) + \alpha \frac{Euv}{1+Ehu} \right) \end{cases} \quad x \in \mathbb{R}, t \in \mathbb{R}^+ \quad (2)$$

with the initial conditions¹

$$\begin{cases} u(0, x) = u_0(x) \in [0, 1]; \quad \lim_{x \rightarrow -\infty} u_0(x) = 1; \quad \lim_{x \rightarrow +\infty} u_0(x) = 0 \\ v(0, x) = 1. \end{cases} \quad (3)$$

2.2 Main results

We distinguish two ways in which a predator can control the prey, one taking space into account and the other not considering this factor (mathematical definitions are provided in definition 2.2.1).

- The spatially *uniform extinction* results exclusively from *local* demographic processes and is independent of space.
- The *extinction wave* is due to both demographic and diffusive processes and may take various forms, from a traveling front to a pulse.

Conversely, invasion is defined as prey survival and we distinguish two ways in which the prey can invade.

- The spatially *uniform invasion*, which is independent of space.
- The *non-uniform invasion*, described by various *spatial* dynamics, from Turing phenomena to invasion waves.

Definition 2.2.1. Let $(u_0(x), v_0(x))$ be an initial condition verifying (3) and $(u(t, x), v(t, x))$ be the corresponding solution of (2).

- *Extinction of prey occurs if*

$$\forall x \in \mathbb{R}, \quad \lim_{t \rightarrow +\infty} u(t, x) = 0.$$

1. All our results remain true for various different initial conditions. The essential being that solution of the scalar systems we consider converge to traveling wave solutions. In particular, compact support may be allowed for u_0 . See [Fife (1979)] for a detailed discussion.

- Prey extinction is uniform if it is uniform with respect to $x \in \mathbb{R}$, that is, if there exists a map $\phi(t)$ verifying

$$\forall x \in \mathbb{R}, \forall t > t_0, 0 \leq u(t, x) \leq \phi(t) \text{ and } \lim_{t \rightarrow +\infty} \phi(t) = 0.$$

- Prey extinction is non uniform if there is extinction but no uniform extinction.
- Invasion of prey occurs if there is no extinction, that is if

$$\exists x \in \mathbb{R}, \limsup_{t \rightarrow +\infty} u(t, x) > 0$$

2.2.1 Analysis of the associated ODE system

If space is not taken into account, system (2) may be rewritten as follows.

$$\begin{cases} \frac{d}{dt} u = u(1 - u) - \frac{Euv}{1+Ehu}, & t \in \mathbb{R}^+ \\ \frac{d}{dt} v = r \left(v(1 - v) + \alpha \frac{Euv}{1+Ehu} \right) \\ 0 < u(0) = u_0 \leq 1 \leq v(0) = v_0 \end{cases} \quad (4)$$

System (4) is well understood ([Magal et al. (2008)]). Indeed, it is clear that there are always three trivial stationary states : $(0, 0)$ and $(1, 0)$, which are unstable and $(0, 1)$, which is asymptotically stable if, and only if, $E > 1$. Moreover, there are no more than three non-trivial positive steady states. We are interested principally in the case $E > 1$. In this case, there are either no or two stationary positive steady states. If the two steady states exist, denoted (u_1, v_1) and (u_2, v_2) with $u_1 < u_2$ and $v_1 < v_2$, then (u_1, v_1) is unstable and (u_2, v_2) is stable. In this case, there are two stable nonnegative solutions, $(0, 1)$ and (u_2, v_2) and the system is bistable.

We are interested principally in the conditions for prey extinction. If $E < 1$, then $(0, 1)$ is unstable and no extinction occurs. We are therefore interested only in the case $E > 1$. Now, if $E > 1$, there are two possibilities. In the non bistable case, $(0, 1)$ is globally stable and there is extinction. In the bistable case, provided that u is initially small enough, say $u_0 < \mu_1$ for some $0 < \mu_1 < 1$, then $u(t) \rightarrow 0$ as $t \rightarrow +\infty$. Conversely, if u is initially large enough, say $0 < \mu_2 < u_0$, then $u(t) \rightarrow u_2$ as $t \rightarrow +\infty$. Thus, in this case, the outcome — extinction or invasion — depends on the initial conditions. The following result provides an explicit statement of the above in the parameter space $(E ; h)$ and is proved in section 5.1.

Theorem 2.1. *Let $E > 1$ and $\alpha \geq 0$ be fixed.*

- (i) **Existence of positive solutions.** *There exists a unique $h^* = h^*(E, \alpha)$ such that*
 - *If $h < h^*$ there is no positive stationary solution and there is extinction of prey for the ODE system.*
 - *If $h > h^*(E, \alpha)$ then there exist two positive solutions for the ODE system $(\hat{u}, \hat{v})$ and (u^*, v^*) with $\hat{u} < u^*$.*
- (ii) **Stability of the solutions.** *Let $h > h^*(E, \alpha)$. The solution $(\hat{u}, \hat{v})$ is always unstable. Moreover, there exists a unique $h^{**}(E, \alpha) > h^*(E, \alpha)$ such that*
 - *If $h > h^{**}(E, \alpha)$ then (u^*, v^*) is stable.*
 - *If $h^*(E, \alpha) < h < h^{**}(E, \alpha)$ the stability of (u^*, v^*) depends on r . It is unstable if r is small enough and stable otherwise.*

Remark 1. *Our calculations show that the gap between h^* and h^{**} is very small, so, roughly speaking, (u^*, v^*) is stable whenever it exists. However, if h belongs to the conditional stability zone and r is very small, stability is lost and the system becomes excitable. This explains, in particular, the presence of pulses for small values of r when dealing with spatial interactions (see section 3).*

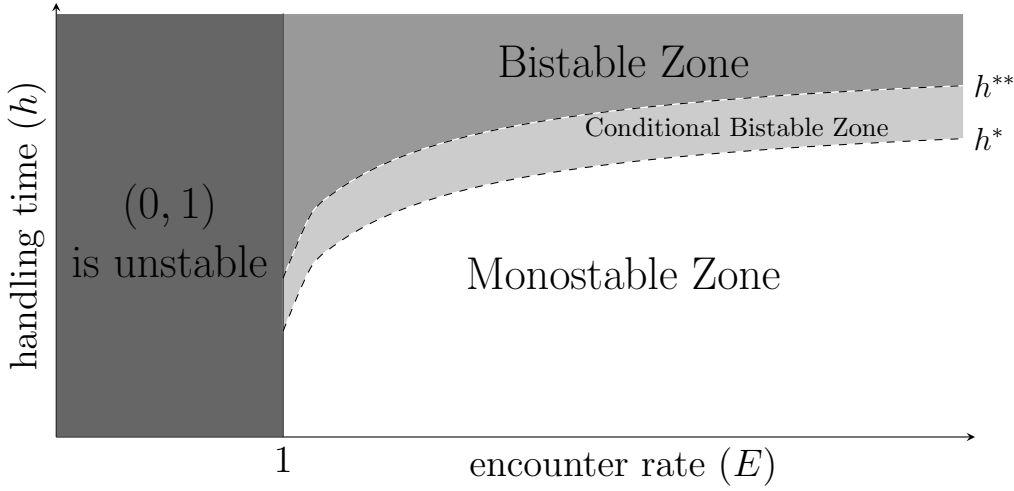


FIGURE 1: Description of the dynamics of the ODE system (4) in the $E - h$ plane. If $E < 1$, the control solution $(0, 1)$ is unstable. In this zone there exists at least one positive steady state and the prey never disappears entirely. If $E > 1$, then the control solution $(0, 1)$ is always (locally) stable. Moreover, the $E > 1$ zone is the union of three subzones. Below the h^* curve, $(0, 1)$ is the only non-negative steady state and is a global attractor : this is a monostable zone. Above the h^* curve, there are two additional positive steady states, one of which is always unstable while the second, denoted (u^*, v^*) , may be stable or unstable. Above the h^{**} curve, (u^*, v^*) is always stable. In this subzone, the asymptotic behaviors depends only on the initial conditions : this is a bistable zone. Between the h^* and the h^{**} curves, the stability of (u^*, v^*) depends on other parameters : this is a conditional bistable zone. For illustration purpose, the size of this last subzone has been considerably increased on the figure.

The map $(E, \alpha) \mapsto h^*(E, \alpha)$ has the following properties, as proved in section 5.1.

Properties 2.2. *Let $\alpha \geq 0$ be fixed. The map $E \mapsto h^*(E, \alpha)$ is increasing and one has the explicit limits :*

$$\lim_{E \rightarrow 1} h^*(E, \alpha) = \begin{cases} 1 + \alpha & \text{if } \alpha < 1 \\ 2\sqrt{\alpha} & \text{if } \alpha \geq 1 \end{cases} \quad \lim_{E \rightarrow +\infty} h^*(E, \alpha) = 2 + 2\sqrt{1 + \alpha}.$$

Let $E \geq 1$ be fixed. The map $\alpha \mapsto h^(E, \alpha)$ is increasing and one has the explicit limits :*

$$h^*(E, 0) = \frac{1}{E} \left(2E - 1 + 2\sqrt{E(E-1)} \right) := h_1(E) \quad \text{and} \quad \lim_{\alpha \rightarrow +\infty} h^*(E, \alpha) = +\infty.$$

Figure 1 illustrates the maps h^* and h^{**} and the possible situations for system (4).

2.2.2 Analysis of the PDE system

We wish to identify the parameter conditions required to obtain prey extinction in the PDE system (2). A simple stability analysis shows that, if $E < 1$, then invasion occurs in the PDE system. If $E > 1$, then the situation for the PDE system is more complex. In this situation, the spatial structure and diffusion processes result in additional conditions for extinction. The rationale is explained in detail below.

Let us assume that there is a positive stable stationary solution of (2) denoted by (u^*, v^*) and that the initial condition $u(x, 0)$ is close to u^* at some places x and close to 0 at other places. Since both u^* and 0 are stable, the demographic phenomena lead to an aggregation near u^* and an aggregation near 0. However, diffusion allows individuals to move around in space, so one of 0 or u^* may be the final global attractor. In other words, there may be a (stable) traveling wave joining u^* to 0. The direction of this wave, given by the sign of the speed of the wave, indicates whether extinction or invasion occurs. However, there are difficulties associated with this argument.

- There can be no homogeneous stationary solution of (2), only stable heterogeneous positive stationary solutions. In other words, it is possible that $h < h^*(E, \alpha)$ without control occurring.
- Even in the case of bistability ($h > h^*(E, \alpha)$), the bistable system (2) is neither competitive nor cooperative. Little theoretical knowledge is available concerning the occurrence of traveling waves in such systems, with even less known about the stability and direction of the wave.

Using super- and subsolutions, we show here how to obtain the conditions sufficient (but not necessary) for extinction and for invasion, based on well known *scalar* bistable PDEs. Roughly speaking, let $(u(t, x), v(t, x))$ be the solution of (2). If we find a *positive constant* $\underline{v}$ such that, for any² $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, $v(t, x) \geq \underline{v}$ then it comes

$$\partial_t u(t, x) - \Delta_x u(t, x) \leq u(t, x)(1 - u(t, x)) - \frac{Eu(t, x)}{1 + Ehu(t, x)}\underline{v}, \quad t > 0, x \in \mathbb{R}.$$

Let $\bar{u}$ be the solution of

$$\begin{cases} \partial_t \bar{u}(t, x) - \Delta_x \bar{u}(t, x) = \bar{u}(t, x)(1 - \bar{u}(t, x)) - \frac{E\bar{u}(t, x)}{1 + Ehu(t, x)}\underline{v}, \\ \bar{u}(0, x) \geq u(0, x). \end{cases}$$

The comparison principle implies that $\bar{u}(t, x) \geq u(t, x)$. Now, if $\bar{u}(x, t) \rightarrow 0$ when $t \rightarrow +\infty$ then $u(t, x) \rightarrow 0$ when $t \rightarrow +\infty$ and extinction of prey occurs (see figure 2-(a)). Moreover, if $\bar{u}(x, t) = \phi(t)$ does not depend on x , then there is aspatial control (see figure 2-(b)).

Conversely, if we can identify a *positive constant* $\bar{v}$ such that $v(x, t) \leq \bar{v}$, then it comes

$$\partial_t u(t, x) - \Delta_x u(t, x) \geq u(t, x)(1 - u(t, x)) - \frac{Eu(t, x)}{1 + Ehu(t, x)}\bar{v}, \quad t > 0, x \in \mathbb{R}.$$

Now, define $\underline{u}(t, x)$ as the solution of

$$\begin{cases} \partial_t \underline{u}(t, x) - \Delta_x \underline{u}(t, x) = \underline{u}(t, x)(1 - \underline{u}(t, x)) - \frac{E\underline{u}(t, x)}{1 + Ehu(t, x)}\bar{v}, \\ \underline{u}(0, x) \leq u(0, x). \end{cases}$$

The comparison principle implies that $\underline{u}(t, x) \leq u(t, x)$. It follows that if for some $x \in \mathbb{R}$, $\limsup_{t \rightarrow +\infty} \underline{u}(x, t) > 0$ then $\limsup_{t \rightarrow +\infty} u(t, x) > 0$ and there is (non uniform) invasion (see figure 3).

These arguments give rise to the following theorems yielding sufficient conditions, in terms of the parameters E , α and h , for extinction or invasion to occur. All theorems are proved in section 5. We begin with a sufficient condition for uniform extinction.

Theorem 2.3. *Let $E > 1$ and define*

$$h_1(E) = \frac{1}{E} \left(2E - 1 + 2\sqrt{E(E-1)} \right).$$

If $h < h_1(E)$ then there is uniform extinction. In other word, for any initial condition verifying (3), there exists $\phi(t) \geq 0$ such that any solution $(u(t, x), v(t, x))$ of (2) verifies

$$\forall x \in \mathbb{R}, 0 \leq u(t, x) \leq \phi(t) \text{ and } \lim_{t \rightarrow +\infty} \phi(t) = 0.$$

2. It suffices that this condition occurs for $t > t_0$ for some $t_0 > 0$.

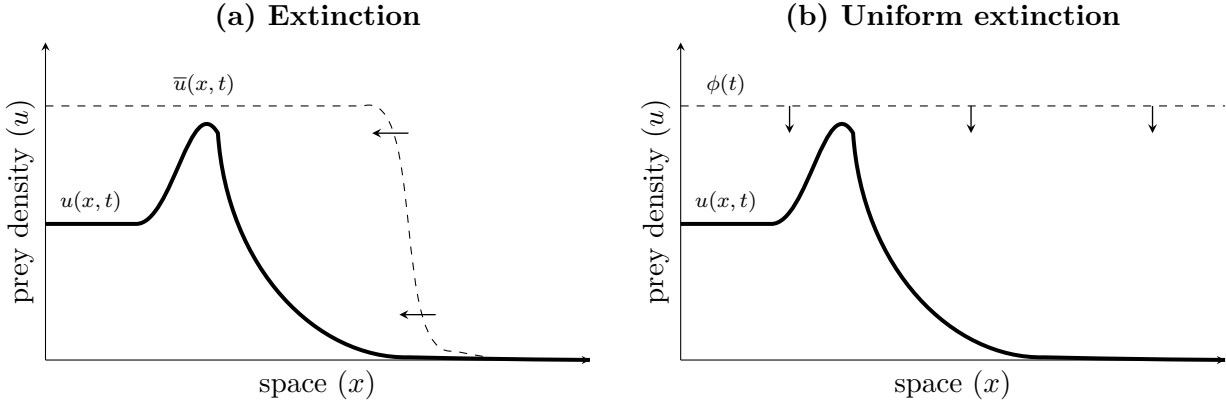


FIGURE 2: (a) Sufficient condition for extinction. The solution $u(t, x)$ is majored by a supersolution $\bar{u}(t, x)$. If $\lim_{t \rightarrow +\infty} \bar{u}(t, x) = 0$ then $\lim_{t \rightarrow +\infty} u(t, x) = 0$ and there is extinction.
(b) Sufficient condition for uniform extinction. The solution $u(t, x)$ is majored by a supersolution $\phi(t)$ which does not depend on x . If $\lim_{t \rightarrow +\infty} \phi(t) = 0$ then $\lim_{t \rightarrow +\infty} u(t, x) = 0$ uniformly in x and there is uniform extinction.

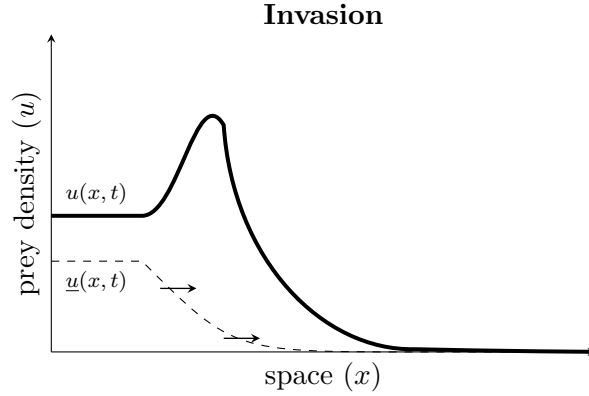


FIGURE 3: Sufficient condition for invasion. The solution $u(t, x)$ is minored by a subsolution $\underline{u}(t, x)$. If $\limsup_{t \rightarrow +\infty} \underline{u}(t, x) > 0$ then $\limsup_{t \rightarrow +\infty} u(t, x) > 0$ and there is invasion.

If $h > h_1(E)$ then there can be invasion or extinction. The following theorem give a sufficient condition for extinction to occur.

Theorem 2.4. *Assume that $E > 1$. There exists $h^- = h^-(E) > h_1(E)$, such that if $h < h^-(E)$ then there is extinction of prey. In other words, for any initial condition verifying (3) and for any $x \in \mathbb{R}$, $\lim_{t \rightarrow +\infty} u(t, x) = 0$.*

The map $E \mapsto h^-(E)$ verifies the following properties proved in section 5.3.

Properties 2.5. *The map $E \mapsto h^-(E)$ is increasing and admits the following explicit limits :*

$$\lim_{E \rightarrow 1} h^-(E) = 1 \quad ; \quad \lim_{E \rightarrow +\infty} h^-(E) = \frac{16}{3}.$$

Our last result gives a sufficient condition for invasion to occur.

Theorem 2.6. *For any $E > 1$ and $\alpha \geq 0$, there exists $h^+ = h^+(E, \alpha)$ verifying $h^+(E, \alpha) > h^-(E)$ and $h^+(E, \alpha) > h^*(E, \alpha)$ such that, if $h > h^+(E, \alpha)$ then prey invasion occurs. More precisely, for any x , $\limsup_{t \rightarrow +\infty} u(t, x) > 0$.*

Finally, the following result specifies the behavior of the map h^+ .

Properties 2.7. *The maps $E \mapsto h^+(E, \alpha)$ and $\alpha \mapsto h^+(E, \alpha)$ are increasing. For any $E > 1$, $h^+(E, 0) = h^-(E)$ and $\lim_{\alpha \rightarrow +\infty} h^+(E, \alpha) = +\infty$. Finally, $\alpha \geq 0$ being fixed, one has the explicit limit*

$$\lim_{E \rightarrow +\infty} h^+(E, \alpha) = \frac{8}{3} \left(1 + \sqrt{1 + \frac{3}{4}\alpha} \right).$$

The results above are summarized in figure 4. In the domain $\{(E, h), E > 1, h^-(E) < h < h^+(E, \alpha)\}$, which we will refer to as the ‘transition zone’, it is not possible to draw any conclusions concerning whether prey invasion or extinction is likely to occur. Indeed, this zone can be separated into two subzones, according to the parameter values :

$$\text{Zone I} = \{(E, h), E > 1, \max(h^*(E, \alpha), h^-(E)) < h < h^+(E, \alpha)\},$$

$$\text{Zone II} = \{(E, h), E > 1, h^-(E) < h < h^*(E, \alpha)\}.$$

In the Zone I, our numerical simulations show non-monotonous traveling waves. In the zone II, simulations show various types of behavior including pulse and even heterogeneous positive stationary solutions. This phenomena are discussed in the section 3.

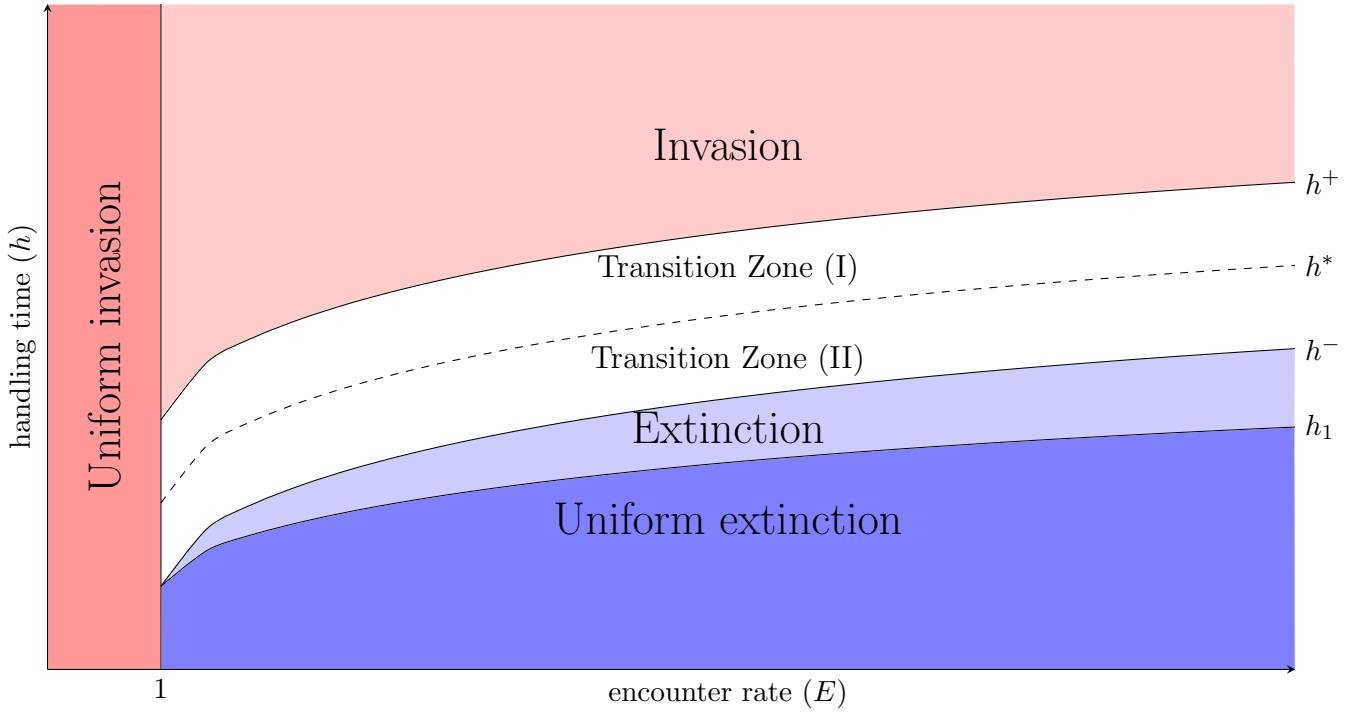


FIGURE 4: Description of the dynamic of the PDE system (2) in the $E - H$ plan. If $E < 1$ there is always invasion. If $E > 1$ there is a uniform extinction for $h < h_1(E)$, extinction for $h_1(E) < h < h^-(E)$ and invasion for $h^+(E, \alpha) < h$. The zone between h^- and h^+ is called the transition zone. This transition zone is splitted into two subzones : Zone I and Zone II, separated by the h^* curve. In these two zones, both extinction or invasion of prey may occur due to various spatial phenomena.

3 Numerical study of the transition zone

3.1 Influence of α .

The mathematical results above demonstrate the influence of the parameters E and h , and, indirectly, that of the conversion rate α , on the long-term behavior of the system.

In summary, we have demonstrated the following phenomena :

- An increase in the rate of encounter E increases predator efficacy and, thus, the chances of prey extinction.
- An increase in the handling rate h decreases predator efficacy and, thus, increases the risk of prey invasion.

More precisely, when $E > 1$, prey extinction or invasion may occur, depending on the value of h . Indeed, we can define two values $h^- = h^-(E) < h^+ = h^+(E, \alpha)$ (see theorems 2.4 and 2.6). Extinction occurs if $h < h^-$ and invasion occurs if $h > h^+$. When $h \in (h^-, h^+)$ we observe richer dynamics, which may depend on other factors. We refer to this zone as the **transition zone**. Note that, as h^- is not dependent on α and h^+ is an increasing function of α (proposition 2.7), the size of this transition zone increases with increasing α .

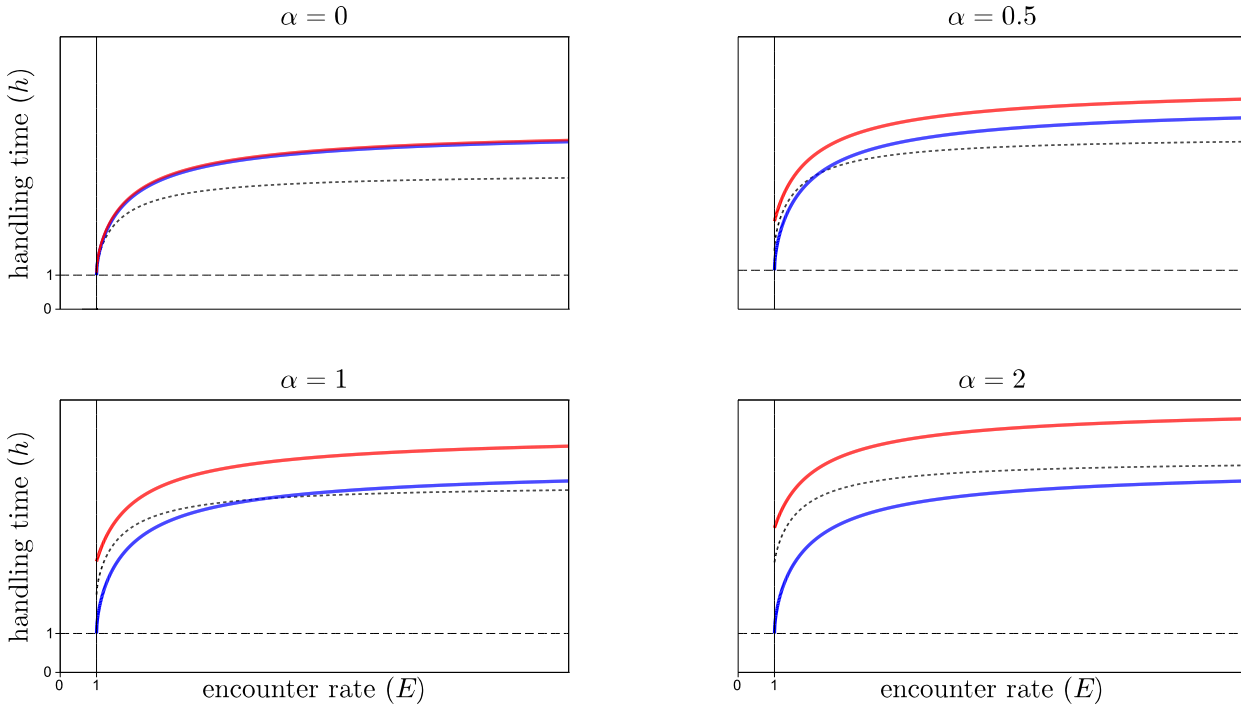


FIGURE 5: Computation of $h^*(E, \alpha)$, $h^-(E)$ and $h^+(E, \alpha)$ for four values of α . The so-called transition zone lies between the red curve h^+ and the blue curve h^- . The transition zone increases within increasing α . This transition zone can be split into two subzones separated by h^* (black line).

A first clue to the possible dynamics in the transition zone is provided by an understanding of the dynamics of the ODE system (4) described in the theorem 2.1. The dynamic of (4) is essentially dependent³ on the position of h relative to $h^* = h^*(E, \alpha)$. When $h < h^*$, there is no positive stationary solution, whereas, for $h > h^*$ there are two positive stationary solutions, one of which, the larger of the two, is (nearly always) stable.

3. The dynamics generally also depends on a quantity h^{**} , defined in the theorem 2.1, slightly greater than h^* that is not taken into account here for the sake of simplicity.

The position of h^* relative to h^- and h^+ provides a first description of the transition zone. By virtue of proposition 2.2, one gets the following. We always have $h^* < h^+$. but the position of h^* relative to h^- is dependent on α . On the one hand, from the facts that $h^*(1, \alpha) > h^-(1)$ for $\alpha > 0$ and $h^*(E, 0) = h_1(E) < h^-(E)$ for $E > 1$, we deduce that $h^- > h^*$ for large enough values of E and small enough values of α . On the other hand, h^* is an increasing function of α tending to $+\infty$ and, thus, on have $h^*(E, \alpha) > h^-(E)$ for large values of α and any $E > 1$.

Remark 2. When $h^- > h^*$, which may occurs for sufficiently small values of α , we see that taking space into account automatically increases the potential of extinction of preys.

The transition zone can thus be separated into two subzones : one in which $h < h^*$ (Zone II) and one in which $h > h^*$ (Zone I). Figure 5 sums up this discussion on the transition zone. As we will see below, both extinction and invasion are possible in each of these zones, but the phenomena at work differ considerably, according to whether $h < h^*$ or $h > h^*$. These phenomena are studied in more detail below, using a numerical approach.

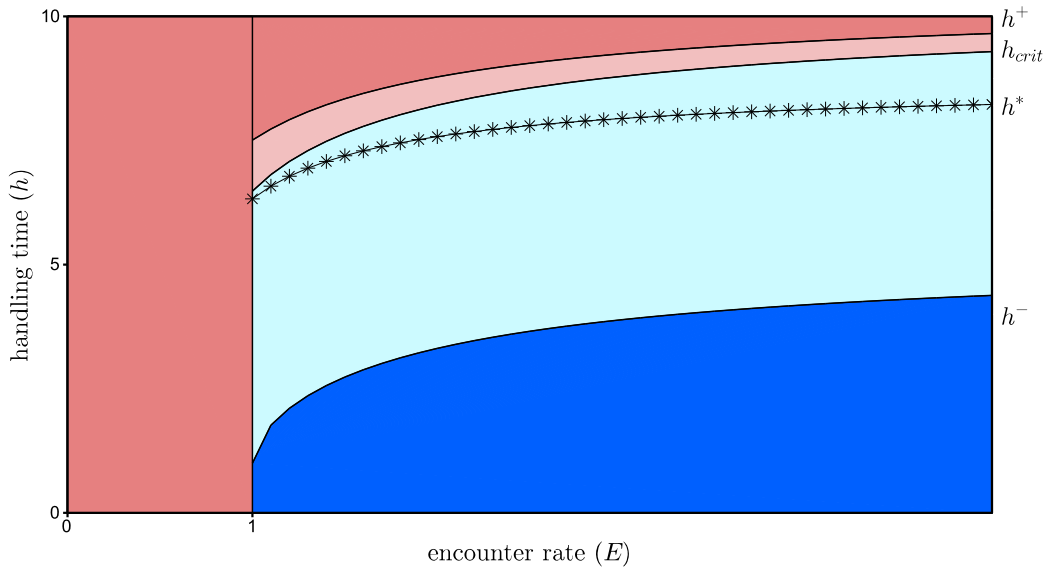


FIGURE 6: Numerical computation of h_{crit} in the $E - h$ plan for fixed values of α , d and r . The transition zone $h \in (h^-, h^+)$ is split into two subzones separated by h_{crit} . Extinction occurs below h_{crit} and invasion occurs above.

3.2 Extinction or invasion : influence of r and d .

Our numerical analysis shows that both invasion and extinction are possible in the transition zone. When space is taken into account we see that h^* does not separate the zone of invasion from that of extinction. These two zones are, indeed, separated by a new critical value of the handling time denoted $h_{crit} \in (h^-, h^+)$, which is dependent on E and α , of course, but also on the relative rates of growth (r) and diffusion (d) of the predator population :

$$h_{crit} = h_{crit}(E, \alpha, r, d).$$

As expected, when $h > h_{crit}$, prey invasion is observed, whereas extinction is observed when $h < h_{crit}$. Thus, higher values of h_{crit} are associated with more effective predation and thus with less effective invasion by the prey.

Figure 6 completes the theoretical scheme represented in figure 4, by presenting an example of the curve for a particular selection of values for the parameters α , r and d in the $E - h$ plan.

Like h^* and h^+ , h_{crit} increases with both E and α . This naturally translates the facts that, higher values of E increases the chance of meeting between predators and prey and that at higher α values, the predator is able to make greater use of the prey and can therefore eliminate it.

Given the multiple dependence of h_{crit} on different parameters, figure 6 can only represent a particular case, chosen for its simplicity. Figure 7 shows the relationships between h_{crit} and d for fixed values of E and α and for various values of r . We see that h_{crit} increases with r . This translates the fact that for small values of r , the predators growth rate is small and their effectiveness reduced. Conversely, h_{crit} (essentially) decreases with increasing d . This is due to the fact that for large d , predators spread into a zone in which prey are not present, resulting in a weakening predation.

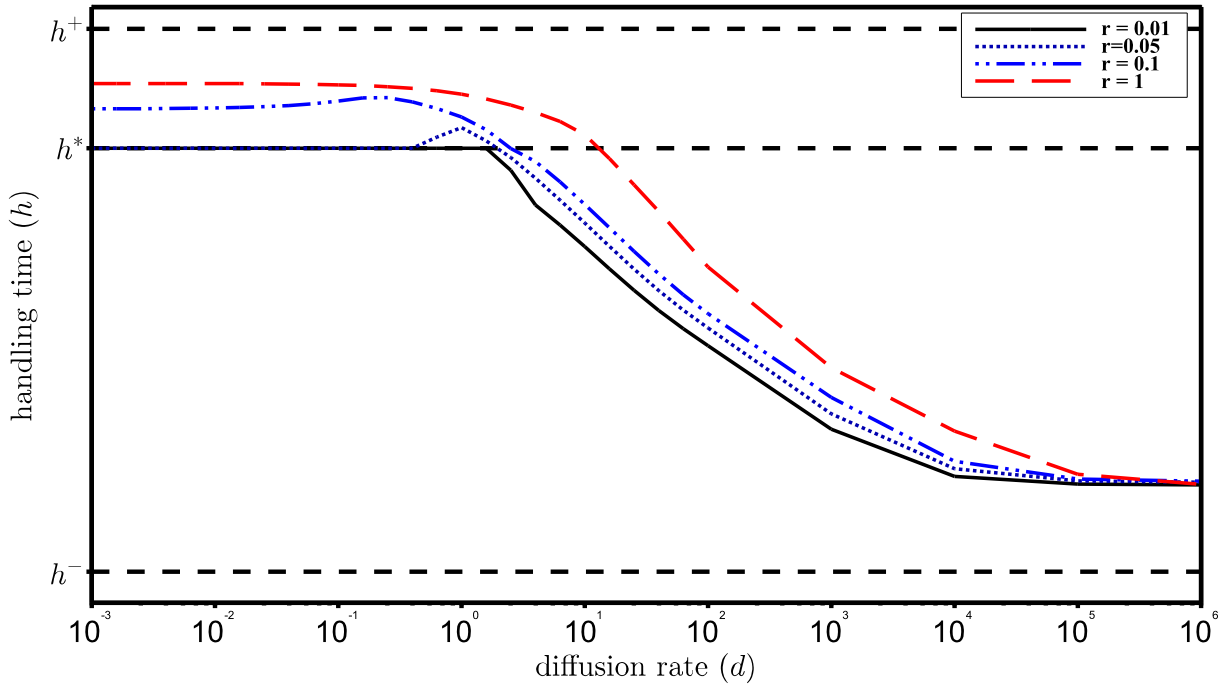


FIGURE 7: Numerical computation of h_{crit} with respect to d for four values of r with $E = 2$ and $\alpha = 4$. For a given value of the parameters, there is invasion of prey if $h > h_{crit}$ and extinction if $h < h_{crit}$. Therefore, the larger h_{crit} is, the greater is the potential for extinction of prey. We see that h_{crit} is increasing in r and (essentially) decreasing in d . Thus, an increase of d or a decrease of r decrease the impact of the predation on invasive prey.

Remark 3. For small values of r and intermediate values of d , predators may increase their effectiveness by increasing d . In that case, the predators growth rate being small, predators density remains high for a long time even if prey are absent. Now, if d is large enough but not large, predators may spread into a zone in which prey are not present and remain there at a high density and long enough to stop the prey to invade. This phenomenon may enable predators to form a barrier to prey's movement, preventing thereby prey propagation. For too large values of d , the loss in predators effectiveness due to movement is too strong and the above phenomenon does not happen any more. This explains why, for small values of r , h_{crit} first increase and then decreases with increasing d .

3.3 Dynamics of the system in the transition zone

We know that extinction occurs when $h < h_{crit}$, whereas invasion occurs when $h > h_{crit}$. It should be borne in mind that the existence or absence of non-trivial solutions that are homogeneous over space is dependent on the position of h with respect to h^* . Consequently, the processes at work during extinction or invasion are highly dependent on the position of h with respect to h^* .

We will now describe the different dynamics occurring in the transition zone, summarized in table 1. See section 2.2 for a precise definition of the various quantities described here.

A] Invasion ($h > h_{crit}$)

- **Turing instabilities (TI)** : $h < h^*$ and $d \gg 1$. As d increases, predators spread out, moving into areas from which the prey is absent, leading to a decrease in the size of the predator population. This phenomenon leads to a decrease in predator density throughout the space occupied by the predator, allowing the prey to survive in certain zones. We thus obtain a periodic distribution in space and a constant distribution over time of the densities of the prey and predator. Mathematically, this phenomenon is described by a Turing bifurcation.
- **Invasion traveling waves (ITW)** : $h > h^*$. The invasion is described simply by an invasion traveling wave : a wave of propagation linking the two stable solutions (u^*, v^*) and $(0, 1)$ in the direction of the positive solution (u^*, v^*) . When r is large this traveling wave is monotone. By contrast, when $r \ll 1$ it displays a rich dynamics. When $r \ll 1$ and d is not too large, we observe that h_{crit} is approximately equal to h^* (see figure 7). This indicates that there is an invasion traveling wave if a positive solution exists. In this case, ahead of the front, $v = 1$ and, since $r \ll 1$, the predator population increases very slowly. Besides, as $h > h^-$ the prey invades the space when the predator is at concentration 1. This leads to the front advancing. Behind the front, the predator has had sufficient time to increase the size of its population and, therefore, to decrease the size of the prey population. As $h > h^*$, the population of the prey decreases towards the positive solution u^* and we see a non-monotonous invasive traveling wave.

B] Extinction ($h < h_{crit}$)

- **Pulse** : $h \in [h^-, h^*]$ and $r \ll 1$ Since $r \ll 1$, The front of the wave is similar to the ITW described above. However, as $h < h^*$, here is no homogeneous positive solution u^* . The prey population therefore decreases to zero behind the front, whereas it continues to advance in front of the front. We thus obtain a pulse.
- **Extinction traveling wave (ETW)** : $h > h^*$ This corresponds to the simplest case described above. We observe a propagation wave linking the two stable solutions (u^*, v^*) and $(0, 1)$ in the direction of the control solution $(0, 1)$.

Finally, figure 8 presents the map in the $E - h$ plane for fixed values of α , d and r and specifies the possible dynamics in each zone.

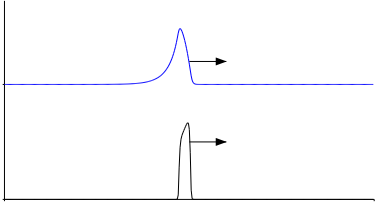
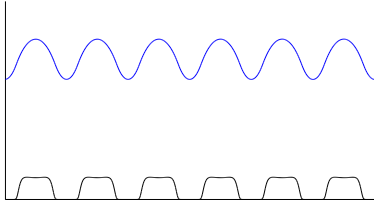
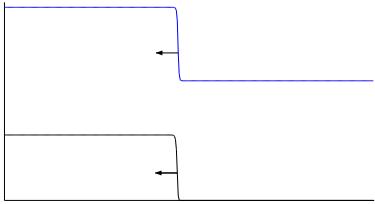
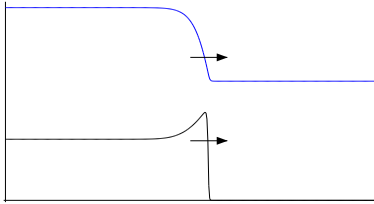
	$h < h_{crit} : \text{Extinction}$	$h > h_{crit} : \text{Invasion}$
$h < h^* : \text{Zone II}$	PULSE ($r \ll 1$) 	TURING ($d \gg 1$) 
$h > h^* : \text{Zone I}$	ETW 	ITW 

TABLE 1: Summary of the four different situations involving space in the transition zone. The X axis represent space and the Y axis the concentration of species. The blue curve (above) represent the concentration of predators in space while the black curve (below) is the concentration of prey. Extinction and Invasion Traveling Waves are abbreviated ETW and ITW respectively.

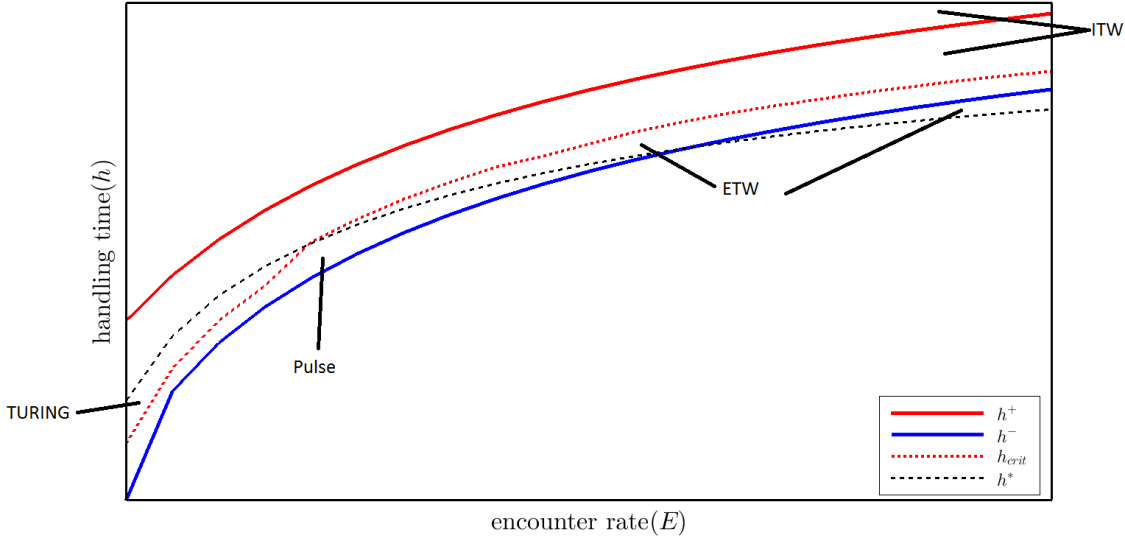


FIGURE 8: Different spatial dynamics in the transition zone. To ensure that all possible situations are represented, we chose $\alpha = 0.5$, $r = 0.01$ and $d = 10$. Note that traveling waves of extinction (ETW) and traveling waves of invasion (ITW) may be obtained outside the transition zone

4 Conclusion and discussion

4.1 Summary of the results

We have shown that invasion occurs if $E < 1$. If predators do not encounter their prey they cannot control them. If $E > 1$, then extinction or invasion can occur, depending on the parameters h , E and α . Uniform extinction occurs for $h < h_1(E)$ and extinction occurs for $h_1(E) < h < h^-(E)$. Invasion occurs for $h^+(E, \alpha) < h$. Thus, if h increases, we move from a zone of extinction without a consideration of space to a zone of extinction requiring a consideration of spatial aspects and then to a zone of invasion. For intermediate values of E , the zones of extinction increase with increasing E resulting in a higher potential of extinction, as h_1, h^*, h^- are increasing functions of E . When E is large, the zones of control do not depend on E any more because h_1, h^-, h^+ have finite limits when $E \rightarrow +\infty$. Thus, E can only play a role in prey control if it takes intermediate values and if h is not too large. In summary, for low values of E ($E < 1$) or high values of E or h , the outcome of the interaction (extinction or invasion) is independent of E .

There is furthermore a transition zone splitted in two subzones, Zone I and Zone II, with various spatio-temporal phenomena and wherein both extinction and invasion can occur. The size of this transition zone greatly increases when the conversion rate α increases. Depending on the relative positions of these two zones with regard to the zone of extinction and the zone of invasion, four spatial dynamics were identified : extinction and invasion traveling waves, extinction pulse waves and heterogeneous stationary positive solutions of the Turing type.

4.2 Biological interpretation of the main results

We have shown that an increase in E increases the potential of extinction while an increase in h increases the potential of invasion. This translates the fact that a highly effective predator does have a high encounter rate and a small handling time. Furthermore, since h^+ is an increasing function of α , an increase in α decreases the potential of invasion and increases the size of the transition zone, which in turn increases the potential for the system to have complex dynamics. Finally, an increase of the diffusion rate d and a decrease of the amplitude of the predators growth rate r both increase the potential of invasion of prey. Thus, a generalist predator loses its effectiveness to exterminate invasive prey if it diffuses too fast or if it has a too slow dynamics.

The above results are stated in term of adimensionalized parameters (see section 2.1). By choosing the appropriate spatio-temporal variables, we may define $D_u = r_1 = 1$. Thus, the biological interpretations of d and r are accurate. Conversely, the definitions of the searching efficiency, $E = EK_2$, the handling time $h = h \frac{K_1}{K_2}$ and the conversion rate $\alpha = \frac{\gamma}{r}$ complicate the biological interpretation of this three parameters. Thus, in addition to the above discussion about the influence of E and h , we now discuss our results in terms of the other biological variables : K_1 , K_2 , γ and r . The parameter h being increasing in the carrying capacity K_1 of prey, prey with high carrying capacity show a high risk of being invasive. Conversely, h and E are respectively decreasing and increasing in the carrying capacity of predators K_2 . Thus, predators with high carrying capacity have a high potential to control prey invasion. Otherwise $\gamma < 1$ means that predators growth is mostly due to alternative prey while $\gamma > 1$ implies that predators growth is due to consumption of the focal prey. Finally, an increase in γ and a decrease in the amplitude of the predators growth rate r yield an increase of α . Therefore, predators with a preference for focal prey or predator with a slow dynamics potentially lead to a high potential of complex dynamics. In particular, the likelihood of the system to exhibit a pulse wave is then important.

4.3 The consideration of space often, but not always, increases the potential for control of pest invasion

The model analyzed here was studied without taking space into account, except for a numerical exploration in the discussion, in an article by [Magal et al. (2008)]. As explained in the introduction, models of identical structure have been proposed independently by [Fagan et al. (2002)] and [Chakraborty (2015)]. We will now discuss our results in the context of these previous studies. Adding a spatial component to predator-prey systems makes any prediction about the controllability of the system difficult, as it then depends on the values of several parameters. The comparison between situations with and without the consideration of space is epitomized by the distinction between h^* , separating parameter regions of mono- and bistability in the ODE system, and h_{crit} , separating parameter regions of invasion and extinction in the PDE system. We will now focus on the case of $E > 1$, as values of $E < 1$ do not promote control, predators encountering prey too infrequently.

If space is not taken into account, control occurs if $h < h^*$, as 0 is a global attractor. This is still true in situations in which space is taken into account, if $h < h_1$ where h_1 is smaller than h^* . When h is between h_1 and h^* , 0 is only a local attractor, so it is not possible to state that control is always attained. In this respect, adding consideration of space decreases the potential for control. Furthermore, when space is not taken into account, there is either extinction or invasion when $h > h^*(E, \alpha)$, depending on initial conditions. Incorporating consideration of space changes the invasion region into $h > h^+(E, \alpha)$, with $h^+ > h^*$. Thus, the consideration of space reduces the size of the invasion zone and is detrimental to the invading prey. Finally, the relative levels of predator and prey diffusion also determine the potential for control. Our model shows that control is increased by predators being less mobile than prey. If predator mobility levels are too high, the predators become too thinly spread on the ground. For similar reasons, too high a level of prey mobility leaves the prey vulnerable to predators. This is entirely consistent with the experimental findings of [Fagan et al. (2002)]. In conclusion, taking space into account can lead to an increase or a decrease in the controllability of invading prey by predators; the addition of space to the model has no generic implication for considerations of predator-prey dynamics (see also [Lam & Ni (2012), Braverman et al. (2015)]).

4.4 How can generalist predators reverse invasion by pest?

The originality of this study lies in its consideration of a generalist predator in a spatial context. When studying generalist predators, it is common practice to assume that the functional response is of type III, due to switching between prey species ([Erbach et al. (2014), van Leuven et al. (2007), van Leuven et al. (2013), Morozov & Petrovskii (2013)]). However, this approach is not mandatory, and other works ([Basnet & Mukhopadhyay (2014), Krivan & Eisner (2006), Hoyle & Bowers (2007)]) have considered a type II functional response. Altering our model to include a type III functional response would be very costly in terms of understanding, because such responses lead to a loss of bistability. Its derivative would be null without prey, so some of our demonstration would fail and the analytical complexity would be greatly increased. However, traveling waves for *specialist* predator with type III functional response is known to exist ([Li & Wu (2008)]) which indicates that our result may be extended to this case.

The complexity of analytical studies of spatial predator-prey interactions lies in the reaction terms being of alternative signs in the equations, making the study of the *systems* of equations essential ([Dunbar (1984b), Huang et al. (2003), Huang & Weng (2013)]). Other interactions, such as competition of two species (all negative) and symbiosis (all positive), are simpler, as their studies are similar to the study of a single equation ([Volpert et al. (1994), Alzahrani et al. (2012)]). This accounts for the slow scientific progress in this otherwise highly relevant topic. However, several major results have been obtained in recent decades, including those of the fundamental work of

[Owen & Lewis (2001)]. The finding of Owen and Lewis that predators can slow, stop, and even reverse invasion by their prey was based on the bistability of the prey-only dynamics of systems consisting of specialist predators attacking prey populations displaying Allee effects. By contrast, our work shows that the ability of generalist predators to control prey populations with logistic growth lies in the bistable dynamics of the *coupled* system. We also observe pseudo-Allee effects in our system, but their physics is quite different. An analysis of the ODE system identified parameter regions of monostable (extinction) and bistable (extinction or invasion) dynamics, but analysis of the associated PDE was able to distinguish different and additional regions of invasion and extinction. As a consequence, prey control was predicted to be possible when space was considered in additional situations other than those identified without considering space. The reverse situation was also possible. None of these considerations apply to spatial predator-prey systems with specialist natural enemies.

5 Proofs

5.1 Proof of theorem (2.1)

Let $E > 1$ and $\alpha \geq 0$ be fixed. For any $h \geq 0$, the system (4) can be rewritten as

$$\begin{cases} \frac{d}{dt}u = \Theta_h(u)(f_h(u) - v) \\ \frac{d}{dt}v = rv(g_h(u) - v) \end{cases} \quad (5)$$

where $\Theta_h(u) = \frac{Eu}{1+Ehu}$, $f_h(u) = \frac{1}{E}(1-u)(1+Ehu)$ and $g_h(u) = 1 + \alpha \frac{Eu}{1+Ehu}$.

(i) *Proof of the existence* Define $H(h, u) = f_h(u) - g_h(u)$. For a given $h \geq 0$, a couple (u, v) is a positive stationary solution of (5) if and only if $v = f_h(u)$ and

$$u \in]0, 1[\text{ is a solution of } H(h, u) = 0. \quad (6)$$

Now, fix $u \in]0, 1[$. Since $\partial_h H(h, u) = u(1-u) + \alpha \left(\frac{Eu}{1+Ehu} \right)^2 > 0$, one sees that the map $h \mapsto H(h, u)$ is increasing. From $E > 1$, we get $H(0, u) < 0$ and from $u \in]0, 1[$ we get $\lim_{h \rightarrow +\infty} H(h, u) = +\infty$.

The map $h \rightarrow H(h, u)$ being continuous, this implies that for any $u \in]0, 1[$, there exists a unique

$$h(u) > 0 \text{ such that } \begin{cases} H(h, u) < 0 \text{ if } h < h(u), \\ H(h, u) = 0 \text{ if } h = h(u), \\ H(h, u) > 0 \text{ if } h > h(u). \end{cases}$$

The smooth function $u \mapsto h(u)$ may be computed explicitly by noting that for any $h \geq 0$, the equation (6) is equivalent to the algebraic equation

$$u \in]0, 1[\text{ is a solution of } P_h(u) = 0 \quad (7)$$

wherein we have set

$$P_h(u) = (1-u)(1+Ehu)^2 - E(1+Ehu + E\alpha u).$$

This yields the explicit formula

$$h(u) = \frac{1}{Eu(1-u)} \left[\frac{E}{2} \left(1 + \sqrt{1 + 4\alpha u(1-u)} \right) + u - 1 \right]. \quad (8)$$

In particular

$$\lim_{u \rightarrow 0^+} h(u) = \lim_{u \rightarrow 1^-} h(u) = +\infty. \quad (9)$$

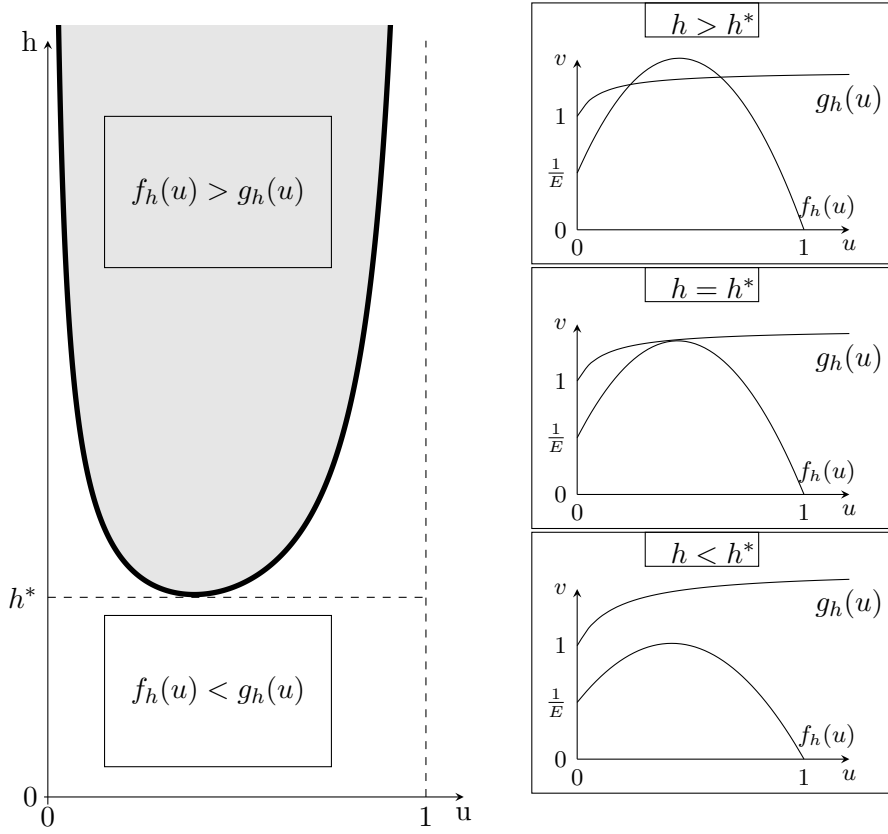


FIGURE 9: The four figures are computed for $E = \alpha = 2$, which gives $h^* \approx 4.36$. The figure on the left represents the curve $u \mapsto h(u)$ (in bold). Above the curve $f_h(u) > g_h(u)$ while below the curve $f_h(u) < g_h(u)$. For a given h , the ordinate u of a point of this curve verifies $f_h(u) = g_h(u)$ and corresponds to the positive stationary solution $(u, f_h(u))$ of (5). The figures on the right represent the isoclines $v = f_h(u)$ and $v = g_h(u)$ for the three fixed values $h = 3$, $h = 4.36$ and $h = 5$. A positive stationary solution of the system (5) corresponds to an intersection of these two isoclines. The system (5) admits two positive solutions for $h > h^*$, one (double) solution for the critical case $h = h^*$ and zero positive solution for $h < h^*$.

This implies that the minimum of $h(u)$ is obtained for some $u_{crit} \in]0, 1[$. We define

$$h^* = \inf_{u \in (0,1)} h(u) = \min_{u \in (0,1)} h(u) = h(u_{crit}). \quad (10)$$

The definition of h^* shows that if $h < h^*$, then (7) has zero solution. This also imply, together with the limits (9) and the continuity of $u \mapsto h(u)$, that for any fixed $h > h^*$ the equation (7) admits at least two solutions⁴ (see the figure 9). In addition to this, for any fixed $h > 0$, one has $\deg(P_h) = 3$ and P_h always admits a negative roots for $P(0) = 1 - E < 0$ and $\lim_{x \rightarrow -\infty} P_h(x) = +\infty$.

This implies that (7) admits at most two solutions⁵. In conclusion, (7) has exactly two positive solutions if $h > h^*$ and zero positive solution if $h < h^*$. This ends the proof of (i).

(ii) *Proof of the stability.* Let $h > h^*$ be fixed. Let (u, v) be a positive stationary solution of (5). Since (u, v) verifies $f_h(u) = g_h(u) = v$, the Jacobian matrix at (u, v) reads

4. Remark that $h^* > \frac{1}{E}$ because $h \leq \frac{1}{E}$, implies that $f'_h < 0$ on $(0, 1)$ and (6) has no solution.

5. These arguments also show that $u \mapsto h(u)$ is decreasing on $(0, u^*)$ and increasing on $(u^*, 1)$; for otherwise it is possible to choose $h > 0$ such that there is at least four different $u \in (0, 1)$ such that $h = h(u)$, which is equivalent to P_h having at least 4 roots. See the figure 9.

$$J(u, v) = \begin{bmatrix} \Theta_h(u)f'_h(u) & -\Theta_h(u) \\ rv g'_h(u) & -rv \end{bmatrix}$$

hence

$$\det(J(u, v)) = r\Theta_h(u)v(g'_h(u) - f'_h(u)).$$

From the proof of (i), we know that the system (5) admits exactly two positive solutions denoted respectively as $(\hat{u}, \hat{v})$ and (u^*, v^*) with $\hat{u} < u_{crit} < u^*$ and such that

$$h = h(\hat{u}) = h(u^*), \quad (11)$$

where $u \mapsto h(u)$ is given by (8). In particular, $\hat{u}$ and u^* are the solution of

$$H(h(u), u) = 0. \quad (12)$$

Differentiating the equation (12) with respect to u gives

$$\partial_u H(h(u), u) = -h'(u)\partial_h H(h(u), u).$$

Thus, using the known fact that $\partial_h H(h(u), u) > 0$, the identity (11) and the footnote 5, one gets $\partial_u H(h, \hat{u}) < 0$ and $\partial_u H(h, u^*) > 0$. Since $\partial_u H(h, u) = -(g'_h(u) - f'_h(u))$, this shows that $\det(J(\hat{u}, \hat{v})) < 0$ and the instability of $(\hat{u}, \hat{v})$ follows.

By contrast, one has $\det(J(u^*, v^*)) > 0$ and it appears that the stability of (u^*, v^*) is given by the sign of

$$\text{tr}(J(u^*, v^*)) = \Theta_h(u^*)f'_h(u^*) - rv^*. \quad (13)$$

In order to highlighted the dependence on h , for any $h > h^*$, we note $u^* = u^*(h)$ and we also define $\mu(h) = \frac{Eh-1}{2Eh}$. From $f'_h(u) = \frac{1}{2E^2h}(\mu(h) - u)$ and (13), we infer the following :

- If $\mu(h) \leq u^*(h)$ then (u^*, v^*) is asymptotically stable.
- If $\mu(h) > u^*(h)$ then the stability of (u^*, v^*) depends on r . More precisely, define $r_{crit} = \frac{\Theta_h(u^*)f'_h(u^*)}{v^*} > 0$. (It is easy to show that $r_{crit} \leq 1$).
- If $r > r_{crit}$, then (u^*, v^*) is asymptotically stable.
- If $r < r_{crit}$, then (u^*, v^*) is unstable.

The sign of $\mu(h) - u^*(h)$ with respect to the parameter h remains to be found.

On a first hand, the explicit expression of f'_h shows that f'_{h^*} is decreasing and that $f'_{h^*}(\mu(h^*)) = 0$. Moreover, the definition (10) of h^* yields $u^*(h^*) = u_{crit}$ and $f'_{h^*}(u_{crit}) = g'_{h^*}(u_{crit}) > 0$. Hence, $\mu(h^*) < u^*(h)$. It is also clear that $\frac{1}{2} = \lim_{h \rightarrow +\infty} \mu(h) < \lim_{h \rightarrow +\infty} u^*(h) = 1$. By continuity, we infer that the equation $\mu(h) = u^*(h)$ has at least one solution in $(h^*, +\infty)$.

On another hand, if $\mu(h) = u^*(h)$ then $H(h, \mu(h)) = 0$, which may be rewritten as

$$(Eh + 1)^3 = 4E((Eh)^2 + Eh(E\alpha + 1) - E\alpha).$$

A simple analysis shows that this equation has exactly one negative solution, one solution in $(0, \frac{1}{E})$ and one solution in $(\frac{1}{E}, +\infty)$. Since $h^* > \frac{1}{E}$ (see footnote 4), this implies that $\mu(h) = u^*(h)$ has exactly one solution in $(h^*, +\infty)$. We note this unique solution $h^{**} = h^{**}(E, \alpha)$.

Finally, it is clear from the above arguments that $\mu(h) < u^*(h)$ if $h \in (h^*, h^{**})$ and that $\mu(h) > u^*(h)$ if $h > h^{**}$. This ends the proof of the Theorem. $\square$

Proofs of properties 2.2 Similarly to the proof of the theorem 2.1, and to highlight the role of the parameters E and α , let us define

$$H(E, \alpha, h, u) = \frac{1}{E}(1 - u)(1 + Ehu) - \left(1 + \alpha \frac{Eu}{1 + Ehu}\right).$$

From the proof of the theorem 2.1, we know that the quantity $h^* = h^*(E, \alpha)$ and the corresponding $u_{crit} = u_{crit}(E, \alpha)$ are characterized by the two equations

$$H(E, \alpha, h^*, u_{crit}) = 0 \quad (14a)$$

$$\partial_u H(E, \alpha, h^*, u_{crit}) = 0. \quad (14b)$$

The implicit function theorem immediatly shows that the maps $(E, \alpha) \mapsto (h^*, u_{crit})$ belongs to $C^1((1, +\infty) \times [0, +\infty); \mathbb{R}_+^2)$.

– **Proofs of the growth of $E \mapsto h^*(E, \alpha)$ and of $\alpha \mapsto h^*(E, \alpha)$.**

Let $\alpha \geq 0$ be fixed. Differentiate (14a) with respect to E and use (14b) yields

$$\partial_E H(E, \alpha, h^*, u_{crit}) + \partial_E h^*(E, \alpha) \cdot \partial_h H(E, \alpha, h^*, u_{crit}).$$

We already know that $\partial_h H < 0$ and an explicit computation give

$$\partial_E H(E, \alpha, h^*, u_{crit}) = \frac{1}{E(1 + Eh^*u_{crit})} > 0.$$

It follows that $\partial_E h^*(E, \alpha) > 0$. Similar arguments show that $\partial_\alpha h^*(E, \alpha) > 0$.

– **Computation of $\lim_{E \rightarrow 1} h^*(E, \alpha)$.**

Let $\alpha \geq 0$ be fixed. Since $h^*(\cdot, \alpha)$ is increasing and positive on $(1, +\infty)$, there exists a nonnegative scalar $h^*(\alpha)$ such that $h^*(E, \alpha) \rightarrow h^*(\alpha)$ as $E \rightarrow 1$. To compute this limit, denote $P(E, \alpha, h, u) = (1 + Ehu)H(E, \alpha, h, u)$. From (14) and the definition of h^* , one see that h^* is the minimal value of h such that

$$\exists u \in [0, 1], P(E, \alpha, h, u) = \partial_u P(E, \alpha, h, u) = 0. \quad (15)$$

In other words, h^* is the minimal value of h such that $P(E, \alpha, h, \cdot)$ admits a multiple root in $[0, 1]$.

By passing to the limit $E \rightarrow 1$ in (15), we see that that $h^*(\alpha)$ is the minimal value of h such that

$$\exists u \in [0, 1], P(1, h, \alpha, u) = \partial_u P(1, h, \alpha, u) = 0. \quad (16)$$

Explicit computations give

$$P(1, \alpha, h, u) = u(h^2u^2 + h(2 - h)u + \alpha + 1 - h).$$

The multiplicity of the roots of $P(1, \alpha, h, \cdot)$ need now to be discussed.

- If $h < 2\sqrt{\alpha}$ then 0 is the only root of $P(1, \alpha, h, \cdot)$ and the multiplicity is one.
- If $h \geq 2\sqrt{\alpha}$ then $P(1, \alpha, h, \cdot)$ has two other real roots and may be explicitly written as

$$P(1, \alpha, h, u) = h^2u(u - u_-)(u - u_+)$$

where

$$u_{\pm} = \frac{1}{2h} \left(h - 2 \pm \sqrt{(2 - h)^2 - 4(\alpha + 1 - h)} \right).$$

- If $h = 2\sqrt{\alpha}$ then $u_- = u_+ = 1 - \frac{1}{\sqrt{\alpha}}$ and this multiple root belongs to $[0, 1]$ if and only if $\alpha \geq 1$.

- If $2\sqrt{\alpha} < h < \alpha + 1$ then the three roots 0 , u_- and u_+ are distinct and there is no multiple root.
- If $h = \alpha + 1$ then either $0 = u_-$ or $0 = u_+$, depending on the sign of $\alpha - 1$. In both cases 0 is a multiple root.
- Finally, if $h > \alpha + 1$ then $u_- < 0 < u_+$ and all the roots of $P(1, \alpha, h, \cdot)$ have multiplicity one.

The above discussion shows, using the characterization of $h^*(\alpha)$, that

$$h^*(\alpha) = \begin{cases} 1 + \alpha & \text{if } \alpha \leq 1 \\ 2\sqrt{\alpha} & \text{if } \alpha > 1. \end{cases} \quad (17)$$

Note that $\alpha \mapsto h^*(\alpha)$ belongs to $C^1([0, +\infty), \mathbb{R}^+)$. □

- **Computation of $\lim_{E \rightarrow +\infty} h^*(E, \alpha)$.**

Since $h^*(\cdot, \alpha)$ is positive and increasing, one has

$$\lim_{E \rightarrow +\infty} h^*(E, \alpha) = \frac{1}{\ell_\alpha}$$

for some nonnegative number ℓ_α (wherein we have denote $\frac{1}{0} = +\infty$).

On an other hand, since $u_{crit}(E, \alpha)$ is bounded, up to a subsequence again denoted by E , one has $u_{crit}(E, \alpha) \rightarrow \mu$ for some positive number μ (eventually depending on α). By passing to the limit in (14a), we deduce $\mu > 0$; for otherwise one obtains $0 = H(E, \alpha, h^*, u_{crit}) \rightarrow +\infty$. It follows that taking $E \rightarrow +\infty$ in (14b), one obtains $\mu = 1/2$. Thus, by passing to the limit in (14a), one gets

$$\frac{1}{\ell_\alpha^2} = \frac{2}{\ell_\alpha} + 2\alpha$$

and finally (ℓ_α being nonnegative)

$$\frac{1}{\ell_\alpha} = 2 + 2\sqrt{1 + \alpha};$$

□

- **Computation of $h^*(E, 0)$.** If $\alpha = 0$ one gets

$$P(E, h, 0, u) = \frac{1}{E}(1 + Ehu)(-Ehu^2 + u(Eh - 1) + 1 - E). \quad (18)$$

Standard computations show that $P(E, h, 0, \cdot)$ admits two nonnegative roots if and only if $h > \frac{1}{E}(2E - 1 + \sqrt{E(E - 1)}) := h_1(E)$ and one double nonnegative root if $h = h_1(E)$. This shows that $h^*(E, 0) = h_1(E)$. □

- **Computation of $\lim_{\alpha \rightarrow +\infty} h^*(E, \alpha)$.** Since $E \mapsto h^*(E, \alpha)$ is increasing one gets for any $E \geq 1$, $h^*(E, \alpha) \geq h^*(\alpha)$. The explicit expression (17) of $h^*(\alpha)$ implies $\lim_{\alpha \rightarrow +\infty} h^*(E, \alpha) = +\infty$. □

5.2 Proof of the Theorem 2.3.

Let $E > 1$ be fixed and $(u(t, x), v(t, x))$ be a solution of (2) with initial condition $(u_0(x), v_0(x))$ verifying (3).

Step 1. It is clear that $u(t, x) \geq 0$ for any $t > 0$ which implies

$$\partial_t v(t, x) - d\Delta v(t, x) \geq rv(t, x)(1 - v(t, x)), \quad t > 0, \quad x \in \mathbb{R}.$$

Thus, using the comparison principle and the hypothesis that $v(0, x) \geq 1$, we deduce that $v(t, x) \geq 1$ for any $t \geq 0$ and $x \in \mathbb{R}$. It follows that

$$\partial_t u(t, x) - \Delta u(t, x) \leq u(t, x)(1 - u(t, x)) - \frac{Eu(t, x)}{1 + Ehu(t, x)}, \quad t > 0, \quad x \in \mathbb{R}.$$

By the comparison principle, it follows that any solution $\bar{u}$ of

$$\partial_t \bar{u}(t, x) - \Delta \bar{u}(t, x) = \bar{u}(t, x)(1 - \bar{u}(t, x)) - \frac{E\bar{u}(t, x)}{1 + Eh\bar{u}(t, x)}, \quad t > 0, \quad x \in \mathbb{R} \quad (19)$$

such that $\bar{u}(0, x) \geq u_0(x)$ for any $x \in \mathbb{R}$ satisfies

$$\forall t > 0, \quad \forall x \in \mathbb{R}, \quad u(t, x) \leq \bar{u}(t, x).$$

In particular, let $\phi(t)$ be the solution of the ordinary differential equation

$$\frac{d}{dt}\phi(t) = \phi(t)(1 - \phi(t)) - \frac{E\phi(t)}{1 + Eh\phi(t)}, \quad t > 0 \quad (20)$$

together with the initial condition $\phi(0) = 1$.

ϕ is a homogeneous solution of (19) and from $u(0, x) \leq 1 = \phi(0)$ we deduce

$$\forall t > 0, \quad \forall x \in \mathbb{R}, \quad 0 \leq u(t, x) \leq \phi(t).$$

Step 2. We study now the behavior of $\phi(t)$ as $t \rightarrow +\infty$.

It is clear that 0 is always a steady state of (20) and is asymptotically stable for $E > 1$.

Now, let $\phi_0 > 0$ be a positive steady state of (20). ϕ_0 is a root of the polynomial $P(E, h, 0, \cdot)$ which is studied in the proof of the property 2.2 (see (18)). Hence, if $h \geq h_1(E)$ then (20) has two positive steady states that we denote as $u^-(E, h) \leq u^+(E, h) \leq 1$ explicitly given by

$$u^\pm(E, h) = \frac{1}{2} \left(1 - \frac{1}{Eh} \pm \sqrt{\left(1 - \frac{1}{Eh}\right)^2 - 4\frac{E-1}{Eh}} \right). \quad (21)$$

A linear analysis shows that for $h > h_1(E)$, $u^-(E, h)$ is unstable and $u^+(E, h)$ is (asymptotically) stable.

Finally, classical arguments show that $\phi(t) \rightarrow u^+(E, h)$ if $h > h_1(E)$ and $\phi(t) \rightarrow 0$ if $h < h_1(E)$. This ends the proof of the theorem. $\square$

5.3 Proof of the Theorem 2.4.

Let $E > 1$ be fixed and $(u(t, x), v(t, x))$ be a solution to (2) with initial condition $(u_0(x), v_0(x))$ verifying (3).

By the argument of the step 1 of the proof of 5.2, one already knows that :

$$u(t, x) \leq \bar{u}(t, x) \quad (22)$$

where $\bar{u}(t, x)$ is a solution of (19) with $\bar{u}(0, x) = u_0(x)$. Moreover, from the step 2 of the proof of 5.2, one knows that if $h > h_1(E)$ then the equation (19) is *bistable* for this equations admits two

stable nonnegative steady states : $u = 0$ and $u = u^+(E, h) < 1$.

We prove here that, in that case, there exists a traveling wave connecting 0 to u^+ at a negative speed if and only if $h < h^-(E)$ for some (implicit) number $h^-(E) > h_1(E)$. This implies that for any $x \in \mathbb{R}$, $\bar{u}(t, x) \rightarrow 0$ if $h < h^-(E)$ which proves the theorem.

Let $h > h_1(E)$ be fixed. It can be proven (see [Fife (1979)], [Volpert et al. (1994)] and the reference therein) that there exists a unique speed $c = c(E, h)$ such that (19) admits a traveling solution of speed c which connects u^+ to 0. More precisely, there exists a profile $U(\xi)$ verifying $U(-\infty) = u^+$, $U(+\infty) = 0$ and $U'(\pm\infty) = 0$, such that $U(x - ct) = u(x, t)$ is a solution of (19). Moreover (see [Fife (1979)]) this traveling wave describes the asymptotic behavior of all solutions provided the initial condition (3) are verified. The theorem is proved by showing that there exists $h^-(E)$ such that $c(E, h) < 0$ if $h < h^-(E)$ and $c(E, h) > 0$ if $h > h^-(E)$. This result is a direct consequence of the two following lemmas. The first lemma gives a characterization of the sign of c by an explicit function (see figure 10).

Lemma 5.1. *Define*

$$W(E, h, u) = \int_0^u \left(s(1-s) - \frac{Es}{1+Ehs} \right) ds.$$

Then $\text{sign}(c(E, h)) = \text{sign}(W(E, h, u^+))$ where $u^+ := u^+(E, h)$.

Démonstration. Denote $c = c(E, h)$ and let $Z = x - ct$ and $U(Z) = u(x, t)$ with $U(-\infty) = u^+$, $U(+\infty) = 0$ and $U'(\pm\infty) = 0$. We have

$$-cU' = U'' + U(1-U) - \frac{EU}{1+EhU} = U'' + \frac{\partial W}{\partial U}(E, h, U).$$

Multiplying by U' and integrating over $\mathbb{R}$ one gets

$$\begin{aligned} -c \int_{-\infty}^{+\infty} (U')^2 dZ &= \int_{-\infty}^{+\infty} \left(U(1-U) - \frac{EU}{1+EhU} \right) U' dZ \\ &= \int_{u^+}^0 \frac{\partial W}{\partial U}(E, h, U) dU = -W(E, h, u^+) \end{aligned}$$

and $\text{sign}(c) = \text{sign}(W(E, h, u^+(E, h)))$ follows. □ □

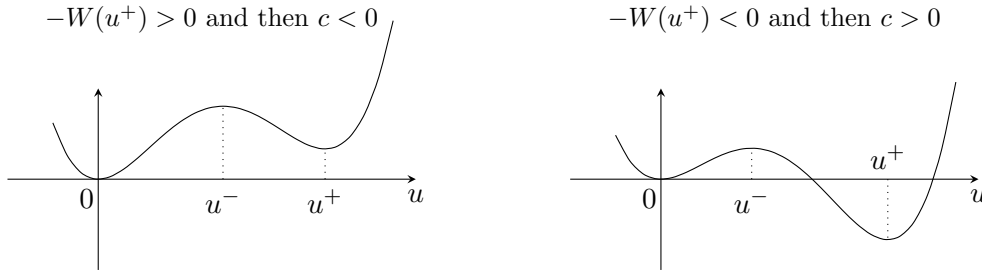


FIGURE 10: Graph of $u \mapsto -W(E, h, u)$. The steady states 0, u^- and u^+ correspond to critical points of the potential $-W$. A stable steady state is a local minima of W and an unstable steady state is a local maxima. One sees that 0 and u^+ are two stable steady states. The sign of c characterizes which of them is the final global attractor : if $c < 0$, then 0 is the global attractor. If $c > 0$, then u^+ is the global attractor.

The second lemma gives the sign of W with respect to h .

Lemma 5.2. *For any $E > 1$, there exists (a unique) $h^-(E) > h_1(E)$ such that*

$$\text{sign}(W(E, h, u^+(E, h))) = \text{sign}(h - h^-(E)).$$

Lemma 5.1 and 5.2 show that

- There is invasion of prey for (19) ($c(E, h) > 0$) if $h \in (h^-(E), +\infty)$. In that case, for any $x \in \mathbb{R}$, $\bar{u}(t, x) \rightarrow u^+(E, h)$ as $t \rightarrow +\infty$.
- There is extinction of prey for (19) ($c(E, h) < 0$) if $h \in [h_1(E), h^-(E))$. In that case, for any $x \in \mathbb{R}$, $\bar{u}(t, x) \rightarrow 0$ as $t \rightarrow +\infty$.

In particular, we infer from the inequality (22), that if $h \in [h_1(E), h^-(E))$, then for any $x \in \mathbb{R}$, $u(t, x) \rightarrow 0$ as $t \rightarrow +\infty$. This ends the proof of the theorem. $\square$

It remains to prove the lemma 5.2.

Proof of lemma 5.2. Define $\mathcal{W}(E, h) = W(E, h, u^+(E, h))$. From the lemma 5.1, we know that $\text{sign}(c) = \text{sign}(\mathcal{W}(E, h))$. We show here that there exists $h^-(E) > h_1(E)$ such that $\text{sign}(\mathcal{W}(E, h)) = \text{sign}(h - h^-(E))$.

Step 1. Differentiate $\mathcal{W}$ with respect to h gives

$$\partial_h \mathcal{W}(E, h) = \partial_h W(E, h, u^+(E, h)) + \partial_u W(E, h, u^+(E, h)) \partial_h u^+(E, h).$$

By the very definition of W and u^+ , one has $\partial_u W(E, h, u^+(E, h)) = 0$, which yields

$$\partial_h \mathcal{W}(E, h) = \partial_h W(E, h, u^+(E, h)).$$

Explicit calculations show that

$$W(E, h, u) = \frac{u^2}{2} - \frac{u^3}{3} - \frac{u}{h} + \frac{1}{Eh^2} \ln(1 + Ehu). \quad (23)$$

Differentiate this expression with respect to h and denoting $x = Ehu^+(E, h)$ provides

$$\partial_h W(E, h, u^+(E, h)) = \frac{1}{Eh^3} \left(x - 2 \ln(1 + x) + \frac{x}{1 + x} \right).$$

A standard analysis shows that the map $x \mapsto x - 2 \ln(1 + x) + \frac{x}{1 + x}$ takes positive values for $x > 0$. It follows that the map $h \mapsto \mathcal{W}(E, h)$ is increasing.

Step 2. Recalling that u^+ is the largest roots of (18), one verifies that $u^+(E, h) \rightarrow 1$ as $h \rightarrow +\infty$ and therefore $\mathcal{W}(E, +\infty) = 1/2 - 1/3 = 1/6 > 0$.

Step 3. This step consists in proving that for any $E > 1$, $\mathcal{W}(E, h_1(E)) = W(E, h_1(E), u^+(E, h_1(E))) := g(E)$ is negative.

From the explicit expression (21) of $u^+(E, h)$ and the definition of $h_1(E)$, we get

$$u^+(E, h_1(E)) = \frac{1}{2} \left(1 - \frac{1}{Eh_1(E)} \right) = 1 - E + \sqrt{E(E - 1)}.$$

On a first hand, we have

$$g'(E) = \partial_E W(E, h_1(E), u^+(E, h_1(E))) + \partial_h W(E, h_1(E), u^+(E, h_1(E))) \cdot h_1'(E).$$

Straightforward calculations give

$$g'(E) = \left(\frac{1}{Eh_1(E)} \right)^2 \left[2 - \ln(1 + \xi) \left(1 + \frac{2}{\xi} \right) \right].$$

wherein we have set

$$\xi = Eh_1(E)u^+(E, h_1(E)) = \sqrt{E-1} \left(\sqrt{E} + \sqrt{E-1} \right) = \frac{1}{2} (Eh_1(E) - 1).$$

A standard analysis shows that $2\frac{\xi}{\xi+2} < \ln(1 + \xi)$ for any $\xi > 0$. Thus $g'(E) < 0$ for any $E > 1$. On another hand, we have $h_1(1) = 1$ and $u^+(1, 1) = 0$, so that $g(1) = 0$. It follows that $g(E) < 0$ for any $E > 1$.

Conclusion For any $E > 1$, the map $h \mapsto \mathcal{W}(E, h)$ is increasing and verifies $\lim_{h \rightarrow +\infty} \mathcal{W}(E, h) = 1/6$ and $\mathcal{W}(E, h_1(E)) < 0$. By continuity, for any $E > 1$, there exists a unique $h^- = h^-(E) \in (h_1(E), +\infty)$ such that $\text{sign}(\mathcal{W}(E, h, u^+)) = \text{sign}(h - h^-(E))$. This ends the proof of lemma 5.2. $\square$

Proofs of properties 2.5. Let $E > 1$ be fixed.

– **Growth of $h^-(\cdot)$.**

Recall that $h^-(E)$ is characterized by

$$W(E, h^-(E), u^+(E, h^-(E))) = 0 \quad (24)$$

where the expressions of W and u^+ are respectively given in (23) and (21). By definition of W , one has

$$\partial_u W(E, h^-(E), u^+(E, h^-(E))) = 0.$$

Hence

$$\partial_E W(E, h^-(E), u^+(E, h^-(E))) + \frac{dh^-}{dE}(E) \cdot \partial_h W(E, h^-(E), u^+(E, h^-(E))) = 0.$$

The explicit computation of $\partial_E W$ and $\partial_h W$ are done in proof 5.3 and one gets

$$\partial_E W(E, h^-(E), u^+(E, h^-(E))) < 0 \text{ and } \partial_h W(E, h^-(E), u^+(E, h^-(E))) > 0$$

so that

$$\frac{dh^-}{dE}(E) = -\frac{\partial_E W(E, h^-(E), u^+(E, h^-(E)))}{\partial_h W(E, h^-(E), u^+(E, h^-(E)))} > 0$$

as needed. $\square$

– **Limit of $h^-(E)$ as $E \rightarrow 1$.**

Let $E > 1$. One knows that $h^-(\cdot)$ is increasing on $(1, +\infty)$ and minored by $h_1(E) > 0$. Hence $h^-(E)$ admits a limit ℓ^- as $E \rightarrow 1$, and from $h_1(1) = 1$, one obtains

$$\ell^- \geq 1.$$

The explicit expressions (21) of $u^\pm$ yield

$$u^-(1, \ell^-) = 0 \text{ and } u^+(1, \ell^-) = \frac{1 - \ell^-}{\ell^-}.$$

Assume by contradiction that $u^+(1, \ell^-) > 0$. Then, for any $h \geq 1$ and $0 < u < u^+(h, 1)$, je have

$$\partial_u W(1, h, u) = u(1 - u) - \frac{1}{1 + hu} > 0 \text{ and } W(1, h, 0) = 0.$$

This implies $W(1, \ell^-, u^+(1, \ell^-)) < 0$, which contradicts (24).

It follows that $u^+(1, \ell^-) = 0$. Thus $\ell^- = 1$, which reads $\lim_{E \rightarrow 1} h^-(E) = 1$. $\square$

– **Limit of $h^-(E)$ as $E \rightarrow +\infty$.**

One knows that $h^-(E)$ is increasing so that there is some $\ell \geq 0$ such that $\lim_{E \rightarrow +\infty} h^-(E) = \frac{1}{\ell}$ (wherein we have set $\frac{1}{\infty} = 0$). By taking the limit $E \rightarrow +\infty$ in (21), we obtain

$$\lim_{E \rightarrow +\infty} u^+(E, h^-(E)) = u_\ell := \frac{1}{2}(1 + \sqrt{1 - 4\ell}) > 0.$$

and then, by taking the limit in (24),

$$\frac{u_\ell^2}{2} - \frac{u_\ell^3}{3} - \frac{u_\ell}{\ell} = 0.$$

Thus, $\ell = \frac{3}{16}$ which reads, $\lim_{E \rightarrow +\infty} h^-(E) = \frac{16}{3}$.

5.4 Invasion conditions, proof of Theorem 2.6

As in the proof 5.3, we start here by showing that $0 \leq \underline{u}(t, x) \leq u(t, x)$ for some function $\underline{u}$ verifying a *scalar* reaction-diffusion equation (25) depending on E , h and α . Next, we show that there exists $h^+(E, \alpha)$ such that $\liminf_{t \rightarrow +\infty} \underline{u}(t, x) > 0$ when $h > h^+(E, \alpha)$. This step is done using the proof of Theorem 2.4 together with an appropriate change of variables.

Step 1. Let $E > 1$ be fixed. In general, using $0 \leq u \leq 1$, one has $1 \leq v \leq \bar{v}$ with

$$\bar{v} := \bar{v}(E, h, \alpha) = 1 + \alpha \frac{E}{1 + Eh}.$$

From the estimate $v \leq \bar{v}$, we get $\underline{u}(t, x) \leq u(t, x)$ where $\underline{u}$ verifies

$$\partial_t u = \Delta_x u + u(1 - u) - \frac{E\bar{v}u}{1 + Ehu} \quad t > 0, \quad x \in \mathbb{R}. \quad (25)$$

Denoting $\tilde{E} = E\bar{v}$ and $\tilde{h} = \frac{h}{\bar{v}}$, this equation reads simply

$$\partial_t u = \Delta_x u + u(1 - u) - \frac{\tilde{E}u}{1 + \tilde{E}hu} \quad t > 0, \quad x \in \mathbb{R}. \quad (26)$$

Note that, removing the $\tilde{\cdot}$, this equation is nothing but (19). As a consequence, the proof of the theorem 2.4 implies the following result on (26).

Lemma 5.3. *Let $\tilde{u}(t, x)$ be a solution of (26) verifying the initial condition (3). There exists $h^-(\tilde{E}) > h_1(\tilde{E})$ such that*

- *if $\tilde{h} < h^-(\tilde{E})$ then for any $x \in \mathbb{R}$, $\lim_{t \rightarrow +\infty} \tilde{u}(t, x) = 0$ (extinction),*
- *if $\tilde{h} > h^-(\tilde{E})$ then for any $x \in \mathbb{R}$, $\limsup_{t \rightarrow +\infty} \tilde{u}(t, x) > 0$ (invasion).*

Step 2. Recalling that $\tilde{E} = E\bar{v}(E, h, \alpha)$ and $\tilde{h} = \frac{h}{\bar{v}(E, h, \alpha)}$ do depend on h , the previous lemma gives the following implicit condition on h for invasion to occur :

There is invasion for (25) if

$$h > \bar{v}(E, h, \alpha) h^-(E\bar{v}(E, h, \alpha)). \quad (27)$$

The following lemma gives an equivalent condition for this implicit condition to occur. This ends the proof of theorem 2.4.

Lemma 5.4. *For any $E > 1$ and $\alpha \geq 0$, there exists (a unique) $h^+(E, \alpha) \geq h^-(E)$ such that (27) holds true if and only if $h > h^+(E, \alpha)$.*

Proof of lemma 5.4 : Let $E > 1$ and $\alpha \geq 0$ be fixed and define the function

$$\mathcal{F}_{E,\alpha}(h) = h - \bar{v}(E, h, \alpha)h^-(E\bar{v}(E, h, \alpha)). \quad (28)$$

By the construction of h^- via the implicit function theorem, one knows that $\mathcal{F}_{E,\alpha}(\cdot)$ is a C^1 function. Moreover, since h^- is bounded, we have $\mathcal{F}_{E,\alpha}(+\infty) = +\infty$ and

$$\mathcal{F}_{E,\alpha}(0) = -\bar{v}(E, 0, \alpha)h^-(E\bar{v}(E, 0, \alpha)) = -(1 + \alpha E)h^-(E(1 + \alpha E)) < 0.$$

therefore, it suffices to show that $\mathcal{F}_{E,\alpha}$ is increasing.

One has

$$\mathcal{F}'_{E,\alpha}(h) = 1 - \partial_h \bar{v}(E, h, \alpha) (h^-(E\bar{v}) + \bar{v}(h^-)'(E\bar{v})).$$

From the expression of $\bar{v}$, we infer $\partial_h \bar{v} < 0$ and from the proof of the properties 2.5, we know that $(h^-)' > 0$. It follows that $\mathcal{F}'_{E,\alpha}(h) > 0$. This ends the proof of the lemma. $\square$

Proofs of properties 2.7.

- **Proof of the growth of $h^+(E, \cdot)$ and $h^+(\cdot, \alpha)$.**
 $h^+(E, \alpha)$ is characterized by an equality in (27), that is,

$$\mathcal{F}_{E,\alpha}(h^+(E, \alpha)) = 0 \quad (29)$$

where $\mathcal{F}_{E,\alpha}$ is defined in (28). Differentiating (29) with respect to E gives

$$\partial_E h^+(E, \alpha) \cdot \mathcal{F}'_{E,\alpha}(h^+(E, \alpha)) + \partial_E \mathcal{F}_{E,\alpha}(h^+(E, \alpha)) = 0$$

and then

$$\partial_E h^+(E, \alpha) \cdot \mathcal{F}'_{E,\alpha}(h^+(E, \alpha)) = [(E\bar{v} + h^-)\partial_E \bar{v} + \bar{v}^2] \cdot (h^-)'(E\bar{v}).$$

where in we have set

$$h^- = h^-(E\bar{v}), \quad \bar{v} = \bar{v}(E, h^+(E, \alpha), \alpha) \text{ and } \partial_E \bar{v} = \partial_E \bar{v}(E, h^+(E, \alpha), \alpha).$$

One already knows that $\mathcal{F}'_{E,\alpha}(h^+(E, \alpha)) > 0$ and that $(h^-)' > 0$. A direct computation shows that $\partial_E \bar{v} > 0$, which leads to $\partial_E h^+(E, \alpha) > 0$ as needed.

Similarly, differentiating (29) with respect to α gives, with obvious notations,

$$\partial_\alpha h^+(E, \alpha) \cdot \mathcal{F}'_{E,\alpha}(h^+(E, \alpha)) = \partial_\alpha \bar{v} \cdot (h^- + E(h^-)'),$$

and since $\partial_\alpha \bar{v} > 0$, one obtains $\partial_\alpha h^+(E, \alpha) > 0$.

- **Limit of $h^+(E, \alpha)$ as $\alpha \rightarrow 0$ and $\alpha \rightarrow +\infty$.**

From $\bar{v}(E, h, 0) = 1$ we deduce $h^+(E, 0) = h^-(E)$. Now, it is clear from the construction of h^+ , that $h^+(E, \alpha) \geq h^*(E, \alpha)$. From the properties 2.2, we obtain $h^+(E, \alpha) \rightarrow +\infty$ as $\alpha \rightarrow +\infty$. $\square$

- **Limit of $h^+(E, \alpha)$ as $E \rightarrow +\infty$.**

First, recall that $h^+(E, \alpha)$ is characterized by (29), which reads

$$h^+(E, \alpha) = \left(1 + \alpha \frac{E}{1 + Eh^+(E, \alpha)}\right) \cdot h^- \left(E + \alpha \frac{E^2}{1 + Eh^+(E, \alpha)}\right). \quad (30)$$

Since $h^+(\cdot, \alpha)$ is increasing, there exists $\ell_\alpha \geq 0$ such that

$$\lim_{E \rightarrow +\infty} h^+(E, \alpha) = \frac{1}{\ell_\alpha}.$$

Taking the limit $E \rightarrow +\infty$ in (30) and that $h^-(E) \rightarrow \frac{16}{3}$, one obtains

$$\frac{1}{\ell_\alpha} = \frac{16}{3} \cdot (1 + \alpha \ell_\alpha)$$

and finally

$$\lim_{E \rightarrow +\infty} h^+(E, \alpha) = \frac{8}{3} \left(1 + \sqrt{1 + \frac{3\alpha}{4}} \right).$$

□

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