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A multi structured epidemic problem with direct and indirect transmission in heterogeneous environments

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In this work we analyse a deterministic epidemic mathematical model motivated by the propagation of a hantavirus (Puumala hantavirus) within a bank vole population (*Clethrionomys glareolus*). The host population is split into juvenile and adult individuals. A heterogeneous spatial chronological age and infection age structure is considered, and also indirect transmission via the environment. Maturation rates for juvenile individuals are adult density-dependent. For the reaction-diffusion systems with age structures derived, we give global existence, uniqueness and global boundedness results. A model with transmission to humans is also studied here.

Keywords: Multi structured, epidemic problem, heterogeneous environment

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1. Introduction

We are mainly interested in the mathematical analysis of a deterministic mathematical model describing the propagation of a macroparasite within a single structured host population. This study is supplemented by a related epidemiological model wherein a macroparasite is transmitted from a reservoir host population to a second host population. This work is motivated by the specific Puumala hantavirus (PUU) - bank vole (*Clethrionomys glareolus*) system in Europe. In that particular system the macroparasite is benign in the reservoir host population and can be transmitted to humans, an epidemiological dead end, with a mild lethal impact. See Wolf et al. [32], Sauvage et al. [23] [24], Sauvage [22] and Wolf [31] for details.

First, in addition to age-dependence that is commonly used in population dynamics (see Anita [2], Gurtin [14], Ianelli [16] or [29] for examples), we want to take into account a stage structuration, due to the fact that sexual maturation of juveniles depends on the density of adults: the higher the density of adult, the slower the maturation (see Sauvage et al. [23]). We have included that hantavirus seems not to affect the demography of bank vole population (but may be lethal for humans) and that there is no vertical transmission of the disease (offspring of infected individuals are healthy at birth). Two modes of horizontal propagation are considered: by direct contacts from infected to healthy individuals, and by contacts

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of healthy individuals with the environment that can be contaminated by infected individuals (see Sauvage et al. [23]). In addition to time and chronological age, we consider a third structure variable which is the age of the disease in a given animal. This first model was constructed and studied in Wolf [30]. Numerical simulations based on this model (see Wolf [31]) leads to dynamics close to those which were observed on fields

Moreover, bank voles also move in space but the previous model does not take into account this fact. A spatial-dependent model was constructed and studied in Wolf et al. [32]. This model highlighted the importance of the spatial structuration in infection evolution, at local and global scales. But the model studied here was simply structured for age and disease status (noninfected and infected juveniles or adults), which is not well adapted to the evolution of the disease (see Wolf [30]).

The purpose of this work is to suggest and analyse a model combining the whole of these important phenomena into a single system. This will lead to a strongly structured system which is too complex for qualitatives studies of the dynamic. Nevertheless we prove that the solution of this system is unique and bounded and thus this model is a good one in order to understand the epidemic spreading and dynamic.

In Section 2 we construct a disease-free model for a closed population ; we derive here a global existence, uniqueness result and then we give a global uniform bound on solutions.
Next in Section 3 we construct the epidemic model and supply global existence, uniqueness and boundedness results.
Then in Section 4, we look at a simplify model which includes transmission of the parasite to a second host population.
Last in Section 5 we give the proof of the results stated in the previous sections.

2. The JA disease-free model

In this section we analyse a disease-free demographic model described in the Introduction.

2.1. *Modeling*

The construction of the model is first based on a disease-free one for the host population. Because of intraspecific competition and different behaviors between juvenile and adult individuals (Bujalska [5], Kostova et al. [17] or Sauvage [22] and references therein), the host population is split into juvenile (**J**) and adult (**A**) subpopulations. Let $J(t, x, a)$ and $A(t, x, a)$ be their respective densities at time t, position $x \in \Omega$, $\Omega \subset \mathbb{R}^n$, $n \geq 1$, and chronological age $a \in (0, a_\dagger)$ for juveniles and $a \in (a_1, a_\dagger)$ for adults, with $a_1 > 0$ (see Webb [29], Iannelli [16], Thieme [26] for example). The host population reads $P(t, a, x) = J(t, a, x) + A(t, a, x)$. We assume maturation of juveniles depends on the total density of adults and cannot occur prior age a_1. Let $\tau(t, a, x, \mathbb{A}(t, x)))$ be the maturation rate at time t of juveniles having age a and position x for a spatial density of adults given by $\mathbb{A}(t, x) = \int_{a_1}^{a_\dagger} A(t, a, x)da$; we assume τ is non increasing with respect to the last variable, $\mathbb{A}$. Let $\beta(t, a, x, \mathbb{P}(t, x))$, be the adult fertility rate depending on the spatial population density given by $\mathbb{P}(t, x) = \int_0^{a_\dagger} J(t, a, x)da + \int_{a_1}^{a_\dagger} A(t, a, x)da$. Let $\mu_J(t, a, x, \mathbb{P}(t, x))$ and $\mu_A(t, a, x, \mathbb{P}(t, x))$ be the respective mortality rates for juveniles and adults.

The resulting compartmental model is depicted in Figure 1 ; see Wolf et al. [32].

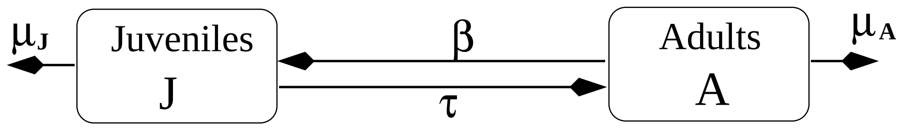

Figure 1. The Juvenile-Adult host population system

Populations disperse via Fickian law with diffusion rates $d_J(t,a,x)$ and $d_A(t,a,x)$. The resulting mathematical model is the following :

$$\begin{cases} \partial_t J + \partial_a J - \operatorname{div}(d_J(t,a,x)\cdot\nabla J) + \mu_J(t,a,x,\mathbb{P}(t,x))\cdot J \\ \qquad\qquad = -\tau(t,a,x,\mathbb{A}(t,x))\cdot J & \text{in } Q_J, \\ J(t,0,x) = \displaystyle\int_{a_1}^{a_\dagger} \beta(t,a,x,\mathbb{P}(t,x))\cdot A(t,a,x)\, da & \text{in } Q_{J,t}, \\ J(0,a,x) = J_0(a,x) & \text{in } Q_{J,a}, \\ (d_J(t,a,x)\cdot\nabla J(t,a,x))\cdot\eta(x) = 0 & \text{in } Q_{J,\partial}, \end{cases} \tag{2.1}$$

$$\begin{cases} \partial_t A + \partial_a A - \operatorname{div}(d_A(t,a,x)\cdot\nabla A) + \mu_A(t,a,x,\mathbb{P}(t,x))\cdot A \\ \qquad\qquad = \tau(t,a,x,\mathbb{A}(t,x))\cdot J & \text{in } Q_A, \\ A(t,a_1,x) = 0 & \text{in } Q_{A,t}, \\ A(0,a,x) = A_0(a,x) & \text{in } Q_{A,a}, \\ (d_A(t,a,x)\cdot\nabla A(t,a,x))\cdot\eta(x) = 0 & \text{in } Q_{A,\partial}, \end{cases} \tag{2.2}$$

with:

$$\begin{aligned} Q_J &= \mathbb{R}_+\times(0,a_\dagger)\times\Omega, & Q_A &= \mathbb{R}_+\times(a_1,a_\dagger)\times\Omega, \\ Q_{J,\partial} &= \mathbb{R}_+\times(0,a_\dagger)\times\partial\Omega, & Q_{A,\partial} &= \mathbb{R}_+\times(a_1,a_\dagger)\times\partial\Omega, \\ Q_{J,a} &= (0,a_\dagger)\times\Omega, & Q_{A,a} &= (a_1,a_\dagger)\times\Omega, \\ Q_{J,t} &= \mathbb{R}_+\times\Omega, & Q_{A,t} &= \mathbb{R}_+\times\Omega, \end{aligned}$$

and:

$$\begin{aligned} \mathbb{A}(t,x) &= \int_{a_1}^{a_\dagger} A(t,a,x)da, \quad \mathbb{J}(t,x) = \int_0^{a_\dagger} J(t,a,x)\, da, \\ \mathbb{P}(t,x) &= \mathbb{J}(t,x) + \mathbb{A}(t,x). \end{aligned}$$

2.2. *Assumptions*

We introduce a set of conditions used through out this work.

Hyp 2.1 Suppose:

- $0 < a_1 < a_\dagger \leq +\infty$,
- $\beta \in L^\infty(Q_A \times \mathbb{R}^+)$ is nonnegative,
- $\tau \in L^\infty(Q_J \times \mathbb{R}^+)$ is nonnegative,
- $\mu_J \in L^\infty(Q_J \times [0,R])$, $\forall R > 0$ is nonnegative,
- $\mu_A \in L^\infty(Q_A \times [0,R])$, $\forall R > 0$ is nonnegative.

Let

$$\beta_\infty = ||\beta||_{\infty, Q_A \times \mathbb{R}^+}, \quad \tau_\infty = ||\tau||_{\infty, Q_J \times \mathbb{R}^+}$$
$$\mu_\infty(R) = max\{||\mu_J||_{\infty, Q_J \times [0,R]}, ||\mu_A||_{\infty, Q_A \times [0,R]}\}.$$

Hyp 2.2 For all $R > 0$,

- There exists $K_\beta(R) > 0$ such that for $0 \leq |\xi|, |\widetilde{\xi}| \leq R$,

$$\forall (t,a,x) \in Q_A, \quad |\beta(t,a,x,\xi)) - \beta(t,a,x,\widetilde{\xi})| \leq K_\beta(R)|\xi - \widetilde{\xi}|$$

- There exists $K_\tau(R) > 0$ such that for $0 \leq |\xi|, |\widetilde{\xi}| \leq R$,

$$\forall (t,a,x) \in Q_J, \quad |\tau(t,a,x,\xi)) - \tau(t,a,x,\widetilde{\xi})| \leq K_\tau(R)|\xi - \widetilde{\xi}|$$

- For $Z = J, A$, there exists $K_Z(R) > 0$ such that for $0 \leq |\xi|, |\widetilde{\xi}| \leq R$,

$$\forall (t,a,x) \in Q_Z, \quad |\mu_Z(t,a,x,\xi)) - \mu_Z(t,a,x,\widetilde{\xi})| \leq K_Z(R)|\xi - \widetilde{\xi}|$$

Let Ω be an open bounded domain in $\mathbb{R}^n$ with a smooth bondary $\partial\Omega$, such that locally Ω lies on one side of its boundary. Let $\eta(x)$ be a unit normal vector to Ω along $\partial\Omega$. In order to take into account spatial heterogeneities we introduce open subsets $\theta_i \quad 1 \leq i \leq n_\theta$ with $\overline{\theta_i} \subset \Omega$, $\theta_i \cap \theta_j = \emptyset \quad \forall i,j$ having the same regularity properties as Ω ; see Figure 2.

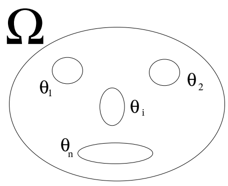

Figure 2. The spatial domain

Let:

$$\Theta = \bigcup_{1 \leq i \leq n_\theta} \theta_i \qquad \text{and} \quad \theta_0 = \Omega \setminus \overline{\Theta}$$

and assume diffusion rates satisfy:

Hyp 2.3 For $Z = J, A$, we suppose that:

- $0 < \underline{d} \leq d_Z(t,a,x) \leq \overline{d} < +\infty, \quad \forall (t,a,x) \in Q_Z$,
- $d_Z \in C(\overline{Q}_{Z,i})$ for $0 \leq i \leq n_\theta$, where $\overline{Q}_{Z,i} = \mathbb{R}_+ \times (0, a_\dagger) \times \overline{\theta_i}$.

Remark 1 : Discontinuity in diffusion rates implies that we cannot expect the spatial regularity afforded by classical diffusion processes. Systems with such diffusion rates are for example studied in Fitzgibbon et al. [8] [9] [10] [12].

2.3. *Main results*

We are now interested in the study of system (2.1)-(2.2). We first establish the existence of weak solutions, for which a definition is given belows (see also Garroni et Langlais [13], Langlais [19] and Naulin [21]).

Definition 2.4: For $a_\dagger < +\infty$, (J, A) is a weak solution of (2.1)-(2.2) in $\Big((0,T)\times(0,a_\dagger)\times\Omega\Big)\times\Big((0,T)\times(a_1,a_\dagger)\times\Omega\Big)$ if

$$\begin{gathered}
J \in L^\infty((0,T)\times(0,a_\dagger)\times\Omega)\cap L^2((0,T)\times(0,a_\dagger);H^1(\Omega)),\\
(\partial_t+\partial_a)J \in L^2((0,T)\times(0,a_\dagger);(H^1(\Omega))'),\\
A \in L^\infty((0,T)\times(a_1,a_\dagger)\times\Omega)\cap L^2((0,T)\times(a_1,a_\dagger);H^1(\Omega)),\\
(\partial_t+\partial_a)A \in L^2((0,T)\times(a_1,a_\dagger);(H^1(\Omega))'),\\
\mathbb{P}\in L^\infty((0,T)\times\Omega) \ \text{ with } \ \mathbb{P} \ \text{ defined in (2.1)},\\
\mu_J(\mathbb{P})\cdot J \in L^1((0,T)\times(0,a_\dagger)\times\Omega),\\
\mu_A(\mathbb{P})\cdot A \in L^1((0,T)\times(a_1,a_\dagger)\times\Omega),
\end{gathered}$$

solution in the weak form of (2.1)-(2.2), this is:

$$\begin{aligned}
\int_{(0,T)\times(0,a_\dagger)} < (\partial_t+\partial_a)J, u > + \int_\Omega \big(d_J\nabla J\cdot\nabla u + (\mu_J+\tau)Ju\big)dx\ dadt = 0\\
\int_{(0,T)\times(a_1,a_\dagger)} < (\partial_t+\partial_a)A, u > + \int_\Omega \big(d_A\nabla A\cdot\nabla v + (\mu_A A-\tau J)v\big)dx\ dadt = 0
\end{aligned}$$

for all $u \in L^\infty((0,T)\times(0,a_\dagger)\times\Omega)\cap L^2((0,T)\times(0,a_\dagger);H^1(\Omega))$ and all $v \in L^\infty((0,T)\times(a_1,a_\dagger)\times\Omega)\cap L^2((0,T)\times(0,a_\dagger);H^1(\Omega))$; and satisfying initial conditions of (2.1) et (2.2).

and a similar definition for $a_\dagger = +\infty$:

Definition 2.5: For $a_\dagger = +\infty$, (J, A) is a weak solution of (2.1)-(2.2) in $\Big((0,T)\times(0,+\infty)\times\Omega\Big)\times\Big((0,T)\times(a_1,\infty)\times\Omega\Big)$ if for all $0<\overline{a}<+\infty$, (J,A) is a weak solution of (2.1)-(2.2) in $\Big((0,T)\times(0,\overline{a})\times\Omega\Big)\times\Big((0,T)\times(a_1,\overline{a})\times\Omega\Big)$.

We have the following Theorem:

Theorem 2.6: *Suppose that assumption Hyp 2.1 to Hyp 2.3 are satisfied and that initial conditions* (J_0, A_0) *are continuous, nonnegative and* L^∞ *in* $Q_{J,a}$ *and* $Q_{A,a}$*. Then for all* $T > 0$ *problem (2.1)-(2.2) has a unique global weak solution* (J, A) *with nonnegative components defined in* $\Big((0,T)\times(0,a_\dagger)\times\Omega\Big)\times\Big((0,T)\times(a_1,a_\dagger)\times\Omega\Big)$.

Remark 2: We could also consider continuous μ_J and μ_A going to infinity when a is going to $a_\dagger$ when $a_\dagger < +\infty$. It follows that densities go to 0 in $a_\dagger$; see Naulin [21]. An additional truncation step is then required in the following proofs.

The proof goes through several steps: first we solve two auxiliary problems then we derive a fixed point method. The proof can be found in Section 5.1.

Under additional assumptions, we establish a global bound L^∞ for solutions of system (2.1)-(2.2). More precisely, we prove that we can estimate quantities $||\mathbb{J}(t,\cdot)||_{\infty,\Omega}$ and $||\mathbb{A}(t,\cdot)||_{\infty,\Omega}$ independently on t.

The additional assumption are:

Hyp 2.7 Diffusion rates d_J and d_A are not dependent on chronological age a.

and, in order to consider death rates of logistic types:

Hyp 2.8 For $Z = J, A$, we have an under bound of the form: there exists $\mu_0 > 0,\ \mu_1 > 0$ such that

$$\mu_0 + \mu_1\ \xi \leq \mu_Z(\cdot,\cdot,\cdot,\xi).$$

Thus one has:

Theorem 2.9 : *Suppose assumption Hyp 2.1 to Hyp 2.8 satisfied and* $(J_0, A_0) \in C(\overline{\Omega})$ *with nonnegative components. If* (J, A) *is solution of system (2.1)-(2.2), then there exists a positive constant* $M_0 = M_0(J_0,\ A_0)$ *independent on* t *such that:*

$$\max_{t>0}\{||\mathbb{J}(t,\cdot)||_{\infty,\Omega}, ||\mathbb{A}(t,\cdot)||_{\infty,\Omega}\} \leq M_0.$$

The proof is based on many estimates derive by iterations each depending on the others and is given in Section 5.2.

3. Epidemic model

We are now interested in the analysis of an epidemic model.

3.1. *Modeling*

Concerning the epidemic model for a single host population we shall consider a basic SI model with susceptible (**S**) and infective (**I**) classes. Newly infected individuals highly excrete the virus, and are very infectious, but chronically infected individuals excrete very few viruses and are less infectious; see Sauvage et al. [24]. Thus we will consider a continuous age structure with the age of infection $b \leq b_\dagger$ where the age of infection is the duration of the desease. Then we have 4 classes of population

- $J_s(t, a, x)$ represents susceptible (i.e. not yet infected) juveniles,
- $J_i(t, a, b, x)$ represents infected juveniles witch is infected since a time b
- $A_s(t, a, x)$ represents susceptible adults,
- and $A_i(t, a, b, x)$ represents adults infected since a time b

We assume the microparasite is benign in the host population: this means there is no additional mortality due to the parasite, fertility and maturation rates as well as diffusivities of infectives are identical to those of susceptibles. We use different incidence functions for direct transmission of the parasite from infected individual since differentes times b to susceptibles: a frequency dependent rate for the former and a density dependent one for the latter ; see Busenberg and Cooke [6], Diekmann and Heesterbeek [7] or Brauer and Castillo Chavez [4].

In our model we also consider that indirect transmission of the parasite through the environment is possible. We shall also need an equation to handle the evolution of the proportion (**G**) of the contaminated environment. The resulting compartmental model is depicted in Figure 3 ; see Wolf [31].

For direct propagation, newly infected individuals are more infective than chronically infected ones. Then the type of incidence change with the age of the infection (mass action type incidence is dominant for small values of b and proportionate mixing type incidence is dominant for high values of b). The incidence functions are given below (3.5).

Indirect transmission occurs by via the release of the virus from the feces, vomit, urine and other bodily fluids. Hence, infective individuals will contaminate the environment at a rate $\alpha(t,a,b,x)$ (α_J or α_A depending on the infected class) ; while susceptible individuals are infected by the contaminated environment at a rate $\gamma_J(t,a,x)$ for juveniles and $\gamma_A(t,a,x)$ for adults. $G(t,x)$ represents the proportion of contaminated environment. We consider that the environment eliminates viruses with time at a rate $\delta(t,x) > 0$. Let $\mathbb{I}(t)$ be the density of infected individuals, and $G(t) \geq 0$ be the percentage of contaminated environment ; for unstructured population, an equation for $G(t)$ has the form : $\partial_t G(t) = \alpha(t)\,\mathbb{I}(t)\cdot(1-G(t)) - \delta(t)\,G(t)$; see Berthier et al. [3].
The resulting compartmental model is depicted in Figure 3.

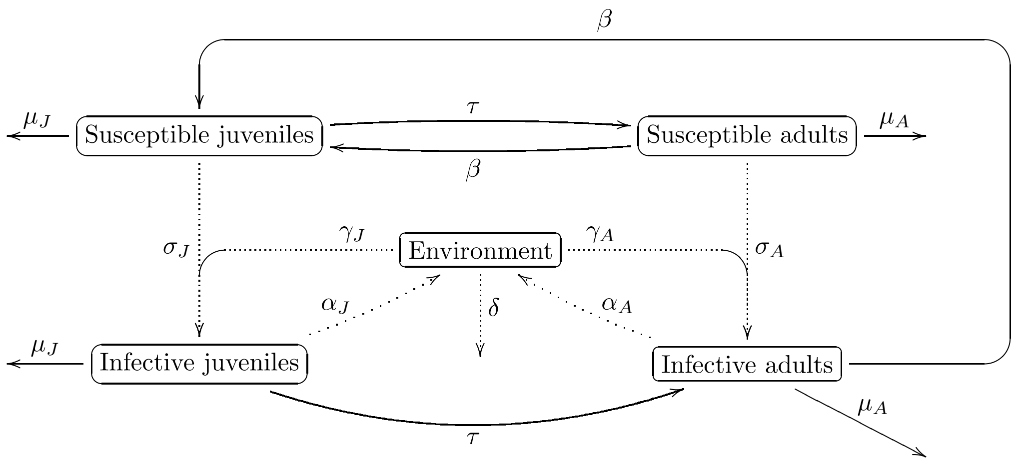

Figure 3. Epidemic model with continuous age of infection structure

The resulting mathematical model couples partial differential equations to an ordinary differential equations.

Let $U(t,a,b,x) = ^t \Big(J_s(t,a,x), J_i(t,a,b,x), A_s(t,a,x), A_i(t,a,b,x)\Big)$, the system corresponding to the epidemic model is

$$\begin{cases} \forall t > 0, \forall a \in (0,a_\dagger), \forall b \in (0,b_\dagger), \forall x \in \Omega, \\ (\partial_t + \partial_a + \partial_b)U - \operatorname{div}(D(t,a,x)\cdot\nabla U) = (\Phi(U) + \Psi(U)), \end{cases} \tag{3.1}$$

with the ordinary differential equation

$$\partial_t G(t,x) = \Big(\int_0^{a_\dagger}\int_0^{b_\dagger} \Upsilon(t,a,b,x)\cdot U(t,a,b,x)dadb\Big)\cdot(1-G(t,x)) - \delta(t,x)\cdot G(t,x), \tag{3.2}$$

and considering the initial conditions:

$$\begin{cases} J_s(t,0,x) = \displaystyle\int_{a_1}^{a_\dagger} \beta(t,a,x,\mathbb{P}(t,x)) \cdot A(t,a,x)da, \\ J_i(t,0,b,x) = 0, \\ A_s(t,a,x) = A_i(t,a,b,x) = 0 \quad \text{for} \quad a \leq a_1, \\ J_i(t,a,0,x) = \sigma_J(t,a,x)J_s(t,a,x) + \gamma_J(t,a,x) \cdot G(t,x) \cdot J_s(t,a,x) \\ A_i(t,a,0,x) = \sigma_A(t,a,x)A_s(t,a,x) + \gamma_A(t,a,x) \cdot G(t,x) \cdot A_s(t,a,x) \\ Z(0,a,x) = Z_0(a,x) \quad \text{for} \quad Z = J_s, A_s, \\ Z(0,a,b,x) = Z_0(a,b,x) \quad \text{for} \quad Z = J_i, A_i, \\ G(0,x) = G_0(x), \end{cases} \tag{3.3}$$

and Neumann boundary conditions: for $Z = J, A$,

$$\begin{cases} \forall t > 0, \forall a \in (0, a_\dagger), \forall b \in (0, b_\dagger) \forall x \in \partial\Omega, \\ d_Z(t,a,x) \; \nabla Z_s(t,a,x) \cdot \eta(x) = 0, \\ d_Z(t,a,x) \; \nabla Z_i(t,a,b,x) \cdot \eta(x) = 0, \end{cases} \tag{3.4}$$

The matrix of diffusion rates D and vectors $\Phi(U)$ for demography, $\Psi(U)$ for transmission rates in the host population (direct and indirect), and Υ representing the environment contamination by infected individuals are:

$$D(t,a,x) \;=\; \begin{pmatrix} d_J(t,a,x) & 0 & 0 & 0 \\ 0 & d_J(t,a,x) & 0 & 0 \\ 0 & 0 & d_A(t,a,x) & 0 \\ 0 & 0 & 0 & d_A(t,a,x) \end{pmatrix},$$

$$\Upsilon(t,a,x) \;=\; \begin{pmatrix} 0 \\ \alpha_J^i(t,a,x) \\ 0 \\ \alpha_A^i(t,a,x) \end{pmatrix},$$

$$\Phi(U)(t,a,x) \;=\; \begin{pmatrix} -\mu_J(t,a,x,\mathbb{P}(t,x)) \cdot J_s(t,a,x) - \tau(t,a,x,\mathbb{A}(t,x)) \cdot J_s(t,a,x) \\ -\mu_J(t,a,x,\mathbb{P}(t,x)) \cdot J_i(t,a,b,x) - \tau(t,a,b,x,\mathbb{A}(t,x)) \cdot J_i(t,a,b,x) \\ \tau(t,a,x,\mathbb{A}(t,x)) \cdot J_s(t,a,x) - \mu_A(t,a,x,\mathbb{P}(t,x)) \cdot A_s(t,a,x) \\ \tau(t,a,x,\mathbb{A}(t,x)) \cdot J_i(t,a,x) - \mu_A(t,a,x,\mathbb{P}(t,x)) \cdot A_i(t,a,b,x) \end{pmatrix},$$

$$\Psi(U)(t,a,x) \;=\; \begin{pmatrix} -\sigma_J(t,a,x) \cdot J_s(t,a,x) - \gamma_J(t,a,x) \cdot G(t,x) \cdot J_s(t,a,x) \\ 0 \\ -\sigma_A(t,a,x) \cdot A_s(t,a,x) - \gamma_A(t,a,x) \cdot G(t,x) \cdot A_s(t,a,x) \\ 0 \end{pmatrix},$$

and, for $Z = J, A$:

$$\sigma_Z(t,a,x) = \int_0^{b_\dagger}\int_0^{a_\dagger} \sigma^{ma}_{j,Z}(t,a,a',b,x)\cdot J_i(t,a',b,x) + \sigma^{ma}_{a,Z}(t,a,a',b,x)\cdot A_i(t,a',b,x)$$
$$+ \frac{\sigma^{pm}_{j,Z}(t,a,a',b,x)\cdot J_i(t,a',b,x)}{J(t,a',x)} + \frac{\sigma^{pm}_{a,Z}(t,a,a',b,x)\cdot A_i(t,a',b,x)}{A(t,a',x)} da' db. \quad (3.5)$$

Finally, we set for:

$$\begin{aligned} Z(t,a,x) &= Z_s(t,a,x) + \int_0^{b_\dagger} Z_i(t,a,b,x)db \quad \text{for } Z = J, A \\ \mathbb{A}(t,x) &= \int_{a_1}^{a_\dagger} A(t,a,x)\ da, \quad \mathbb{J}(t,x) = \int_0^{a_\dagger} J(t,a,x)\ da, \\ \mathbb{P}(t,x) &= \mathbb{J}(t,x) + \mathbb{A}(t,x). \end{aligned}$$

Remark 1 : Integrating in b the equation for J_i and adding the result with the equation for J_s in one hand and those for A_i and A_s, in the other hand, one gets system (2.1)-(2.2).

3.2. *Assumptions*

We suppose that initial conditions in $t = 0$ J_s^0, J_i^0, A_s^0 and A_i^0 are continuous, nonnegative and L^∞. In addition to assumptions Hyp 2.1 to Hyp 2.3, concerning demographic and diffusion rates, we make the following 2 assymptions concerning transmission rates.

Hyp 3.1

- for $Z = J, A$, let $\gamma_Z \in L^\infty(Q_Z)$ be nonnegative with:

$$0 \le \gamma_Z(t,a,x) \le \gamma_\infty \quad \forall t,\ a,\ x$$

- for $Z = J, A$ and $z = n, c$, let $\alpha^z_Z \in L^\infty(Q_Z)$ be nonnegative with:

$$0 \le \alpha^z_Z(t,a,x) \le \alpha_\infty \quad \forall t,\ a,\ x$$

- and for $Z = J, A$, $Z' = j, a$, and $z = pm, am$ let $\sigma^z_{Z,Z'} \in L^\infty(Q_Z)$ be nonnegative with:

$$0 \le \sigma^z_{Z,Z'}(t,a,a',b,x) \le \sigma_\infty \quad \forall t,\ a,\ a',\ b,\ x$$

Hyp 3.2 Let $J_s^0(a,x) > 0$ in $(0,\underline{a}) \times \Omega$ and $A_s^0(a,x) > 0$ in $(a_1, a_1 + \underline{a}) \times \Omega$ with $\underline{a} > 0$.

This last assumption is useful to prove Lemma 5.17 which is used to treat the proportionate mixing part in the following.
In order to simplify the notations, we set:

$$\mathcal{H}_{J_s} = L^2\big((0,a_\dagger)\times\Omega\big) \quad \mathcal{H}_{J_i} = L^2\big((0,a_\dagger)\times(0,b_\dagger)\times\Omega\big)$$
$$\mathcal{H}_{A_s} = L^2\big((a_1,a_\dagger)\times\Omega\big) \quad \mathcal{H}_{A_i} = L^2\big((a_1,a_\dagger)\times(0,b_\dagger)\times\Omega\big)$$

$$\mathcal{H}^2 = \mathcal{H}_{J_s} \times \mathcal{H}_{J_i} \times \mathcal{H}_{A_S} \times \mathcal{H}_{A_i}$$

and:

$$\begin{aligned} ||U(t)||_{\mathcal{H}^2} &= ||J_s(t,\cdot,\cdot)||_{\mathcal{H}_{J_s}} + ||J_i(t,\cdot,\cdot,\cdot)||_{\mathcal{H}_{J_i}} \\ &\qquad + ||A_s(t,\cdot,\cdot)||_{\mathcal{H}_{A_s}} + ||A_i(t,\cdot,\cdot,\cdot)||_{\mathcal{H}_{A_i}} \end{aligned}$$

And for $T > 0$:

$$\begin{aligned} \mathcal{H}^2(T) =& L^2\big((0,T)\times(0,a_\dagger)\times\Omega\big) \times L^2\big((0,T)\times(0,a_\dagger)\times(0,b_\dagger)\times\Omega\big) \\ & \times L^2\big((0,T)\times(a_1,a_\dagger)\times\Omega\big) \times L^2\big((0,T)\times(a_1,a_\dagger)\times(0,b_\dagger)\times\Omega\big) \end{aligned}$$

3.3. *Mains results*

As in the previous section, we prove existence and uniqueness of a global weak solutions for system (3.1)-(3.4). The notion of weak solution is defined in 2.4 and 2.5.

We have the Theorem:

Theorem 3.3: *Suppose assumptions Hyp 2.1 to Hyp 2.3, Hyp 3.1 and Hyp 3.2 are satisfied, and that initial conditions* $(J_s^0, J_i^0, A_s^0, A_i^0, G(0))$ *are continuous, nonnegative and* L^∞ *in* $Q_{J,a}$ *and* $Q_{A,a}$. *Then for all* $T > 0$ *problem (3.1)-(3.4) has a unique global weak solution* (J_s, J_i, A_s, A_i, G) *with nonnegative components, with* $0 \le G(t,x) \le 1$ *and defined in* $\Big((0,T)\times(0,a_\dagger)\times\Omega\Big) \times \Big((0,T)\times(0,a_\dagger)\times(0,b_\dagger)\times\Omega\Big) \times \Big((0,T)\times(a_1,a_\dagger)\times\Omega\Big) \times \Big((0,T)\times(a_1,a_\dagger)\times(0,b_\dagger)\times\Omega\Big) \times \Big((0,T)\times\Omega\Big)$.

The proof is similar to those of Theorem 2.6. Details for the points that make it more complicated are given in Section 5.3.

As for the JA demographic model, assuming assumption Hyp 2.7 is satisfied, it follows:

Theorem 3.4: *Suppose assumptions Hyp 2.1 to Hyp 2.8, Hyp 3.1 and Hyp 3.2 are satisfied. If* $(U,G) = (J_s, J_i, A_s, A_i, G)$ *is solution of system (3.1)-(3.4) with* $U_0 \in C(\overline{\Omega})$ *nonnegative and* $0 \le G_0 \le 1,\ G_0 \in C(\overline{\Omega})$, *then there exists a positive constant* $M_0 = M_0(U_0, G_0)$ *independent on* t *such that:*

$$\max_{t>0}\{||\mathbb{J}(t,\cdot)||_{\infty,\Omega}, ||\mathbb{A}(t,\cdot)||_{\infty,\Omega}\} \le M_0.$$

Proof: We have seen that $\forall x \in \Omega,\ \forall t > 0,\quad 0 \le G(t,x) \le 1$. Furthermore, it is easy to check that $\mathbb{R}_+^7$ is invariant by system (3.1)-(3.4).

Thus, integrating on b the equation for J_i and adding the equation of J_s, but also those for A_i and A_s, one gets equations (2.1)-(2.2). One can conclude using Theorem 2.9. □

4. Model with transmission to humans

Let us now consider the situation where the parasite is indirectly transmitted through the environment from the previous host population, a reservoir, to a second host population spatially distributed in a neighbooring spatial domain (Ω_H) with $\Omega \cap \Omega_H \neq \emptyset$. Assuming different times scales between these two host populations, neither age structure nor demography are considered in the second one. We use a basic spatially structured SIR epidemic model for the second population with an additional mortality rate.

4.1. *Modeling*

We extend the previous model by taking into account transmission to humans. We consider that this transmission is only due to the contamination of humans by the infected environment. Humans do not contaminate environment, and there is no transmission from human to human; see Sauvage [22]. The model used here is inspired by works of Sauvage [22] and Fitzgibbon et al. [12]. We consider three classes of human population : H_s represents susceptible individuals, H_i represents the infected (but not infectious) individuals and H_r represents recovered individuals, that we will consider as immune. Let γ_H be the contamination rate by the environment, λ be the rate at which Infected individuals become recovered and ε be the survival rate of the disease (it can be lethal for humans). Considering the smallness of times for bank voles demography, transmission and incubation of the virus, we do not take into account demography for humans.
It is also useless to introduce an age structure for humans ; only a space structure is considered in our model. Thus the system is composed by equations (3.1)-(3.4) with the additional equations for humans: for $t > 0$ and $x \in \Omega$,

$$\begin{cases} \partial_t H_s(t,x) - \operatorname{div}(d_{H,s}(t,x) \cdot \nabla H_s(t,x)) = -\gamma_H(t,x)\ G(t,x)\ H_s(t,x), \\ H_s(0,x) \ = \ H_s^0(x), \\ (d_{H,s}(t,x) \cdot \nabla H_s(t,x)) \cdot \eta(x) \ = \ 0 \quad \text{for} \quad t > 0, \quad x \in \partial\Omega, \end{cases} \tag{4.1}$$

$$\begin{cases} \partial_t H_i(t,x) - \operatorname{div}(d_{H,i}(t,x) \cdot \nabla H_i(t,x)) = \gamma_H(t,x) G(t,x) H_s(t,x) - \lambda H_i(t,x), \\ H_i(0,x) \ = \ H_i^0(x), \\ (d_{H,i}(t,x) \cdot \nabla H_i(t,x)) \cdot \eta(x) \ = \ 0 \quad \text{for} \quad t > 0, \quad x \in \partial\Omega, \end{cases} \tag{4.2}$$

$$\begin{cases} \partial_t H_r(t,x) - \operatorname{div}(d_{H,r}(t,x) \cdot \nabla H_r(t,x)) = \varepsilon\ \lambda\ H_i(t,x), \\ H_r(0,x) \ = \ H_r^0(x), \\ (d_{H,r}(t,x) \cdot \nabla H_r(t,x)) \cdot \eta(x) \ = \ 0 \quad \text{for} \quad t > 0, \quad x \in \partial\Omega. \end{cases} \tag{4.3}$$

We consider the following assumption:

HYP 4.1

- Let $\lambda > 0$ and $0 < \varepsilon < 1$,
- Let $\gamma_H \in L^\infty((0,+\infty)\times\Omega)$ be nonnegative,
- and for $x \in \Omega$, let $0 < \underline{d} \leq d_H(t,a,x) \leq \overline{d} < +\infty$.

Differences with the previous epidemic model comes only from the additional equations for humans. However, human population is only influenced by the equation for contaminated environment and does not influence equations for the host population neither those for contaminated environment. Thus results obtained in the previous section are still true and we can study system (4.1)-(4.3) with the minimal assumption that $G \in L^\infty((0,T)\times\Omega)$ for all T.

We have the following result; see Fitzgibbon et al. [12]:

Theorem 4.2 : *Suppose initial conditions (H_s^0, H_i^0, H_r^0) are nonnegative and continuous on Ω. Then there exists a unique global classical solution of system (4.1)-(4.3) with nonnegative components and uniformly bounded on $(0,+\infty)\times\Omega$. Furthermore one has:*

$$||H_s(t,\cdot)||_{\infty,\Omega} \leq ||H_s^0||_{\infty,\Omega},$$
$$||H_i(t,\cdot)||_{\infty,\Omega} + ||H_r(t,\cdot)||_{\infty,\Omega} \leq C(||H_s^0||_{\infty,\Omega}, ||H_i^0||_{\infty,\Omega}, ||H_r^0||_{\infty,\Omega}).$$

Proof: Local existence comes from Banach fixed point Theorem. Global existence is granted by a priori estimates and regularity results in Ladyzhenskaya et al. [18]. For the system considered here, this a priori estimates come from the maximum principle and the fact that $0 \leq G(t,x) \leq 1$ applied to the three equations for H_s, H_i and H_r ; it follows inequalities in (4.2); see Fitzgibbon et al. [10] [11] [12]. □

Remark 1 : Integrating the three equations (4.1)-(4.3) in space and adding them, one gets:

$$H'(t) = \int_\Omega (H_s' + H_i' + H_r')(t,x)\,dx \;=\; -(1-\varepsilon)\lambda \cdot \int_\Omega H_i(t,x)\,dx \leq 0,$$

thus the global Human population is logically nonincreasing. This come from the fact that there is no demographic supply in our model, but only mortality for infected individuals due to the virus.

5. Proofs

5.1. *Proof of Theorem 2.6*

First we will study two auxiliary problems, that will be useful in the general case.

5.1.1. *First auxiliary problem*

We consider the following system:

$$\begin{cases} \partial_t u + \partial_a u - \operatorname{div}(d(t,a,x)\cdot\nabla u) + \mu(t,a,x)\cdot u = f(t,a,x) & \text{in } Q_J, \\ u(t,0,x) \;=\; b(t,x) & \text{in } Q_{J,t}, \\ u(0,a,x) \;=\; u_0(a,x) & \text{in } Q_{J,a}, \\ (d(t,a,x)\cdot\nabla u(t,a,x))\cdot\eta(x) \;=\; 0 & \text{in } Q_{J,\partial}, \end{cases} \tag{5.1}$$

and we suppose that:

Hyp 5.1

- $u_0 \in L^2((0,a_\dagger)\times\Omega)$ and u_0 is nonnegative,
- $\mu \in L^\infty((0,T)\times(0,a_\dagger)\times\Omega)$ and μ is nonnegative,
- $b \in L^\infty((0,T)\times\Omega)$ and b is nonnegative,
- $f \in L^2((0,T)\times(0,a_\dagger)\times\Omega)\cap L^\infty((0,T)\times(0,a_\dagger)\times\Omega)$, with $f(t,a,x)\geq 0$,
- d satisfies the assumption Hyp 2.3.

Then we have the proposition:

Proposition 5.2: *Suppose assumption Hyp 5.1 is satisfied. Then problem (5.1) has a unique solution u in $(0,T)\times(0,a_\dagger)\times\Omega$ nonnegative and satisfying:*

$$u \in L^\infty((0,T)\times(0,a_\dagger)\times\Omega)\cap L^2((0,T)\times(0,a_\dagger);H^1(\Omega)),$$
$$(\partial_t+\partial_a)u \in L^2((0,T)\times(0,a_\dagger);(H^1(\Omega))'),$$

weak solution of (5.1), i.e. satisfying:

$$\int_{(0,T)\times(0,a_\dagger)} <(\partial_t+\partial_a)u,v> dadt + \int_{(0,T)\times(0,a_\dagger)\times\Omega}\big(d\nabla u\cdot\nabla v+\mu uv\big)dxdadt = \int_{(0,T)\times(0,a_\dagger)\times\Omega} f(t,a,x)v dxdadt$$

for all $v \in L^\infty((0,T)\times(0,a_\dagger)\times\Omega)\cap L^2((0,T)\times(0,a_\dagger);H^1(\Omega))$; and satisfying initial conditions of (5.1).

Proof: A proof is based on the Galerkin method using a convenient regular basis of $H^1(\Omega)$ and tools of Garroni and Langlais [13].
We can also use the characteristics method and classical results for hyperbolic problems (see for example Smoller [25]). We treat here the case $a_\dagger < +\infty$, but the case $a_\dagger = +\infty$ can be treated in similar ways.

We begin by considering $0 < t < a$. Let $0 < a_0 < a_\dagger$, $c \in (0, a_\dagger - a_0)$, and we set $t = c$, $a = a_0 + c$ and $w(c,x) = u(c, a_0 + c, x)$. Then w is solution of the following linear parabolic problem: for $c \in (0, a_\dagger - a_0)$ and $x \in \Omega$,

$$\begin{cases} \partial_c w - \operatorname{div}(d(c,a_0+c,x)\cdot\nabla w) + \mu(c,a_0+c,x)\cdot w = f(c,a_0+c,x), \\ w(0,x) \ = \ u_0(a_0,x), \\ (d(c,a_0+c,x)\cdot\nabla w(c,x))\cdot\eta(x) \ = \ 0. \end{cases}$$

Classical theory for linear parabolic problems gives existence, uniqueness and nonnegativeity of u under the characteristic $t = a$.

When $0 < a < t$, we consider $t_0 > 0$ and $c \in (0, a_\dagger)$ and we set $a = c$, $t = t_0 + c$ and $w(c,x) = u(t_0 + c, c, x)$. Then w is solution of the following linear parabolic problem: for $c \in (0, a_\dagger)$ and $x \in \Omega$:

$$\begin{cases} \partial_c w - \operatorname{div}(d(t_0+c,+c,x)\cdot\nabla w) + \mu(t_0+c,c,x)\cdot w = f(t_0+c,c,x), \\ w(0,x) \ = \ b(t_0,x), \\ (d(t_0+c,c,x)\cdot\nabla w(c,x))\cdot\eta(x) \ = \ 0. \end{cases}$$

Classical theory for linear parabolic problems gives existence, uniqueness and nonnegativeity of u over the characteristic $t = a$. □

There exists for parabolic equations a comparison theorem, from which we can get from the previous proof the following corollary:

Corollary 5.3: *If $f_1 \geq f_2 \geq 0$ in Q_J, $u_{01} \geq u_{02} \geq 0$ in $Q_{J,a}$, $b_1 \geq b_2 \geq 0$ in Q_J and $0 \leq \mu_1 \leq \mu_2$ in Q_J then corresponding solutions of system (5.1) satisfies $u_1 \geq u_2 \geq 0$ in Q_J.*

We now establish a boundedness result for solution of system (5.1):

Proposition 5.4: *Suppose that assumption Hyp 5.1 is satisfied for all $T > 0$. Then:*
If $a_\dagger < +\infty$, for all $T > 0$ there exists a constant $M_0(T) > 0$ depending on $\underline{d}$, $||u_0||_{\infty,(0,a_\dagger)\times\Omega}$, $||b||_{\infty,(0,T)\times\Omega}$, $||f||_{\infty,(0,T)\times(0,a_\dagger)\times\Omega}$ such that u solution of (5.1) satisfies:

$$||u(t,\cdot,\cdot)||_{\infty,(0,a_\dagger)\times\Omega} \leq M_0(T), \quad 0 < t < T.$$

If $a_\dagger = +\infty$, for all $T > 0$ and all $\overline{a} > 0$, there exists a constant $M_0(T,\overline{a}) > 0$ depending on $\underline{d}$, $||u_0||_{\infty,(0,+\infty)\times\Omega}$, $||b||_{\infty,(0,T)\times\Omega}$, $||f||_{\infty,(0,T)\times(0,+\infty)\times\Omega}$ such that u solution of (5.1) satisfies:

$$||u(t,\cdot,\cdot)||_{\infty,(0,\overline{a})\times\Omega} \leq M_0(T,\overline{a}), \quad 0 < t < T, \quad 0 < \overline{a} < +\infty.$$

Proof: We only deal with the case $a_\dagger < +\infty$. We yet know that $u \geq 0$ and using Corollary 5.3 it is sufficient to consider the case of $\mu(t,a,x) = 0$. We use the characteristics method and the results in Alikakos [1] and Ladyszenskaya et al. [18].

We first consider $0 < t < a$. We use the notations in the proof of Proposition 5.2 to get $w(c,x)$ solution for $c \in (0, a_\dagger - a_0)$ and $x \in \Omega$ of:

$$\begin{cases} \partial_c w - \mathrm{div}(d(c,a_0+c,x)\cdot\nabla w) = f(c,a_0+c,x), \\ w(0,x) \ = \ u_0(a_0,x), \\ (d(c,a_0+c,x)\cdot\nabla w(c,x))\cdot\eta(x) \ = \ 0. \end{cases}$$

If $f \equiv 0$, integrating on $(0,c)\times\Omega$ one gets:

$$||w(c,\cdot)||_{1,\Omega} \leq ||u_0(a_0,\cdot)||_{1,\Omega}.$$

A similar result than those in Alikakos [1] or the maximum principle gives the existence of $M_1(T) > 0$ depending on $\underline{d}$, $||u_0||_{\infty,(0,a_\dagger)\times\Omega}$ such that:

$$||w(c,\cdot)||_{\infty,\Omega} \leq M_1(T) < +\infty \quad 0 < c < a_\dagger - a_0, \quad 0 < a_0 < a_\dagger.$$

If $f \neq 0$, one has to use a result of Ladyzhenskaya et al. [18] to get the existence of M_1 depending this time also on $||f||_{\infty,(0,T)\times(0,a_\dagger)\times\Omega}$ such that $||w(c,\cdot)||_{\infty,\Omega} \leq M_1 < +\infty$.

For $0 < a < t$, similar arguments gives the existence of $M_2(T) > 0$ or M_2 such that $||w(c,\cdot)||_{\infty,\Omega} \leq M_1 < +\infty$. □

5.1.2. Second auxiliary problem

We are interested in solutions (J^*, A^*) of the following problem:

$$\begin{cases} \partial_t J^*(t,a) + \partial_a J^*(t,a) = 0 & \text{in } (0,T)\times(0,a_\dagger), \\ J^*(t,0) = \displaystyle\int_{a_1}^{a_\dagger} \beta_\infty A^*(t,a)\, da & \text{for } t\in(0,T), \\ J^*(0,a) = ||J_0||_{\infty,\Omega} & \text{for } a\in(0,a_\dagger), \end{cases} \tag{5.2}$$

$$\begin{cases} \partial_t A^*(t,a) + \partial_a A^*(t,a) = \tau_\infty J^*(t,a) & \text{in } (0,T)\times(a_1,a_\dagger), \\ A^*(t,a_1) = 0 & \text{for } t\in(0,T), \\ A^*(0,a) = ||A_0||_{\infty,\Omega} & \text{for } a\in(a_1,a_\dagger). \end{cases} \tag{5.3}$$

Proposition 5.5: *For all $T>0$, for all $0<A<a_\dagger$, system (5.2)-(5.3) has a unique solution $(J^*,A^*)\in L^\infty((0,T)\times(0,A))\times L^\infty((0,T)\times(a_1,A))$ with nonnegative components. Furthermore, if $P^* = J^* + A^*$, the following estimate is satisfied:*

$$\int_0^{a_\dagger} P^*(t,a)da \le \Big(\int_0^{a_\dagger} P_0(a)da\Big)\cdot e^{(\beta_\infty+\tau_\infty)t}. \tag{5.4}$$

Proof: Existence and uniqueness in L^∞ is a consequence of a similar result in a more complicated case; see Wolf [30].
Furthermore, adding the two systems, one has:

$$\begin{cases} \partial_t P^*(t,a) + \partial_a P^*(t,a) \le \tau_\infty P^*(t,a), \\ P^*(t,0) \le \displaystyle\int_{a_1}^{a_\dagger} \beta_\infty P^*(t,a)\, da, \\ P^*(0,a) = ||P_0||_{\infty,\Omega}, \end{cases}$$

and integrating the first equation in age from 0 toward $a_\dagger$, and using Gronwall lemma, it follows estimate (5.4). □

5.1.3. End of proof of Theorem 2.6

We only deal with the case $a_\dagger < +\infty$, the case $a_\dagger = +\infty$ can be treated similarly by truncation.
Let (J^*, A^*) be the solution of (5.2)-(5.3).
Let also $\mathcal{K}$ be the closed convex set defined by:

$$\begin{aligned} \mathcal{K} = \{(J,A) \in L^2((0,T)\times(0,a_\dagger)\times\Omega)\times L^2((0,T)\times(a_1,a_\dagger)\times\Omega), \\ 0\le J(t,a,x)\le J^*(t,a) \text{ in } (0,T)\times(0,a_\dagger)\times\Omega, \\ \text{and } 0\le A(t,a,x)\le A^*(t,a) \text{ in } (0,T)\times(a_1,a_\dagger)\times\Omega\}. \end{aligned}$$

At least, let $\Phi:\mathcal{K}\to\mathcal{K}$ defined by $\Phi(\widetilde{J},\widetilde{A}) = (J,A)$ where J, A is solution of the linear problem:

$$\begin{cases} \partial_t J + \partial_a J - \text{div}(d_J(t,a,x)\cdot\nabla J) + \mu_J(t,a,x,\widetilde{\mathbb{P}}(t,x))\cdot J \\ \qquad\qquad\qquad\qquad +\tau(t,a,x,\widetilde{\mathbb{A}}(t,x))\cdot J = 0 \qquad \text{in } Q_J, \\ J(t,0,x) \;=\; \int_{a_1}^{a_\dagger} \beta(t,a,x,\widetilde{\mathbb{P}}(t,x))\cdot \widetilde{A}(t,a,x)\; da \qquad \text{in } Q_{J,t}, \\ J(0,a,x) \;=\; J_0(a,x) \qquad \text{in } Q_{J,a}, \\ (d_J(t,a,x)\cdot\nabla J(t,a,x))\cdot\eta(x) \;=\; 0 \qquad \text{in } Q_{J,\partial}, \end{cases} \tag{5.5}$$

$$\begin{cases} \partial_t A + \partial_a A - \text{div}(d_A(t,a,x)\cdot\nabla A) + \mu_A(t,a,x,\widetilde{\mathbb{P}}(t,x))\cdot A \\ \qquad\qquad\qquad\qquad -\tau(t,a,x,\widetilde{\mathbb{A}}(t,x))\cdot J = 0 \qquad \text{in } Q_A, \\ A(t,a_1,x) \;=\; 0 \qquad \text{in } Q_{A,t}, \\ A(0,a,x) \;=\; A_0(a,x) \qquad \text{in } Q_{A,a}, \\ (d_A(t,a,x)\cdot\nabla A(t,a,x))\cdot\eta(x) \;=\; 0 \qquad \text{in } Q_{A,\partial}. \end{cases} \tag{5.6}$$

Proposition 5.2 insures the existence of (J, A).
Comparing J to the solution of (5.1) with $\mu = 0$, $f = 0$ and $b(t,x) = \int_{a_1}^{a_\dagger} \beta_\infty A^*(t,a)\; da$ and using Corollary 5.3, one has

$$0 \le J(t,a,x) \le J^*(t,a) \quad \text{for } \; t\in(0,T),\; a\in(0,a_\dagger),\; x\in\Omega. \tag{5.7}$$

Similarly comparing A to the solution of (5.1) with $\mu = 0$, $f = \tau_\infty J^*(t,a)$ and $b(t,x) = \int_{a_1}^{a_\dagger} \beta_\infty A^*(t,a)\; da$ and using Corollary 5.3, one gets

$$0 \le A(t,a,x) \le A^*(t,a) \quad \text{for } \; t\in(0,T),\; a\in(0,a_\dagger),\; x\in\Omega. \tag{5.8}$$

This way $\Phi : \mathcal{K} \to \mathcal{K}$ is well defined.
It remains to prove that Φ is a strict contraction to have the existence of fixed point, and then to show that this fixed point is a weak solution.

Thus we consider $(J_1, A_1) = \Phi(\widetilde{J}_1, \widetilde{A}_1)$ and $(J_2, A_2) = \Phi(\widetilde{J}_2, \widetilde{A}_2)$. The following lemma holds:

Lemma 5.6: *There exist two constants k_1 and k_2 depending on $\underline{d}, \beta_\infty, a_\dagger, K_\beta, K_\tau, K_Z$ and $||Z^*||_{\infty,(0,T)\times(0,a_\dagger)}$, for $Z^* = J^*, A^*$ such that for $t \in (0,T)$, one has:*

$$\begin{aligned} \frac{d}{dt}\big(||(J_1-J_2)(t,\cdot,\cdot)||_{2,(0,a_\dagger)\times\Omega} + ||(A_1-A_2)(t,\cdot,\cdot)||_{2,(a_1,a_\dagger)\times\Omega}\big) \le \\ k_1\big(||(J_1-J_2)(t,\cdot,\cdot)||_{2,(0,a_\dagger)\times\Omega} + ||(A_1-A_2)(t,\cdot,\cdot)||_{2,(a_1,a_\dagger)\times\Omega}\big) \\ + k_2\big(||(\widetilde{J}_1-\widetilde{J}_2)(t,\cdot,\cdot)||_{2,(0,a_\dagger)\times\Omega} + ||(\widetilde{A}_1-\widetilde{A}_2)(t,\cdot,\cdot)||_{2,(a_1,a_\dagger)\times\Omega}\big). \end{aligned} \tag{5.9}$$

Proof: $k_i, i \ge 3$ will be constants with the same property as k_1 and k_2. We begin by estimate the equation corresponding to $J_1 - J_2$, that we multiplied by $J_1 - J_2$. one gets:

$$\frac{1}{2}(\partial_t+\partial_a\ )(J_1-J_2)^2-\mathrm{div}(d_J\nabla(J_1-J_2))(J_1-J_2)+(\mu_J(\widetilde{\mathbb{P}}_1)J_1-\mu_J(\widetilde{\mathbb{P}}_2)J_2)(J_1-J_2)$$
$$+(\tau(\widetilde{A}_1)J_1-\tau(\widetilde{A}_2)J_2)(J_1-J_2)=0$$

and

$$\frac{1}{2}(\partial_t+\partial_a\ )(J_1-J_2)^2-\mathrm{div}(d_J\nabla(J_1-J_2))(J_1-J_2)+(\mu_J(\widetilde{\mathbb{P}}_1)+\tau(\widetilde{A}_1))(J_1-J_2)^2$$
$$=-J_2(J_1-J_2)\big((\mu_J(\widetilde{\mathbb{P}}_1)-\mu_J(\widetilde{\mathbb{P}}_2))+(\tau(\widetilde{A}_1)-\tau(\widetilde{A}_2))\big).$$

By integrating on Ω, it follows:

$$\frac{1}{2}(\partial_t+\partial_a\ )\int_\Omega(J_1-J_2)^2\,dx+\int_\Omega d_J|\nabla(J_1-J_2)|^2\,dx+\int_\Omega(\mu_J(\widetilde{\mathbb{P}}_1)+\tau(\widetilde{A}_1))(J_1-J_2)^2\,dx$$
$$=-\int_\Omega J_2(J_1-J_2)\big((\mu_J(\widetilde{\mathbb{P}}_1)-\mu_J(\widetilde{\mathbb{P}}_2))+(\tau(\widetilde{A}_1)-\tau(\widetilde{A}_2))\big)\,dx,$$

thus as $0\le J_2\le J^*$, $J^*\in L^\infty((0,T)\times(0,a_\dagger))$ and using assumption Hyp 2.1- Hyp 2.3 and boundedness (5.7), one has:

$$\frac{1}{2}(\partial_t+\partial_a\ )\int_\Omega(J_1-J_2)^2\,dx+\underline{d}\int_\Omega|\nabla(J_1-J_2)|^2\,dx\le$$
$$k_3\int_\Omega|J_1-J_2|\cdot|\widetilde{\mathbb{P}}_1-\widetilde{\mathbb{P}}_2|dx+k_4\int_\Omega|J_1-J_2|\cdot|\widetilde{A}_1-\widetilde{A}_2|dx.$$

Integrating in a on $(0,a_\dagger)$, one gets:

$$\frac{1}{2}(\partial_t+\partial_a\ )||(J_1-J_2)(t,\cdot,\cdot)||_{2,(0,a_\dagger)\times\Omega}+\underline{d}||\nabla(J_1-J_2)(t,\cdot,\cdot)||_{2,(0,a_\dagger)\times\Omega}$$
$$\le I_1(t)+I_2(t)+I_3(t),\quad (5.10)$$

with:

$$\begin{aligned}I_1(t)&=k_3\int_{(0,a_\dagger)\times\Omega}|J_1-J_2|\cdot|\widetilde{\mathbb{P}}_1-\widetilde{\mathbb{P}}_2|dxda\\&=k_3\int_{(0,a_\dagger)\times\Omega}|J_1-J_2|\cdot\Big|\int_0^{a_\dagger}(\widetilde{J}_1-\widetilde{J}_2)da+\int_{a_1}^{a_\dagger}(\widetilde{A}_1-\widetilde{A}_2)da\Big|\,dxda\\&\le k_3\int_{(0,a_\dagger)\times\Omega}|J_1-J_2|\cdot\Big(\int_0^{a_\dagger}|\widetilde{J}_1-\widetilde{J}_2|da+\int_{a_1}^{a_\dagger}|\widetilde{A}_1-\widetilde{A}_2|da\Big)\,dxda\\&\le k_3\int_\Omega\Big(\int_0^{a_\dagger}|J_1-J_2|da\Big)\cdot\Big(\int_0^{a_\dagger}|\widetilde{J}_1-\widetilde{J}_2|da\Big)dx\\&\qquad+k_3\int_\Omega\Big(\int_0^{a_\dagger}|J_1-J_2|da\Big)\cdot\Big(\int_{a_1}^{a_\dagger}|\widetilde{A}_1-\widetilde{A}_2|da\Big)dx.\end{aligned}$$

Moreover, Holder's inequality for a function f implies that

$$\int_0^{a_\dagger} f da \leq \sqrt{a_\dagger}\Big(\int_0^{a_\dagger} f^2\ da\Big)^{1/2},$$

so using also Cauchy-Schwarz inequality, one has:

$$\begin{aligned} 0 \leq I_1(t) \leq k_3 a_\dagger \Big(&\int_{(0,a_\dagger)\times\Omega} (J_1 - J_2)^2\ dadx + \\ &\frac{1}{2}\int_{(0,a_\dagger)\times\Omega} (\widetilde{J}_1 - \widetilde{J}_2)^2\ dadx + \frac{1}{2}\int_{(a_1,a_\dagger)\times\Omega} (\widetilde{A}_1 - \widetilde{A}_2)^2\ dadx\Big). \end{aligned} \quad (5.11)$$

Also:

$$I_2(t) = k_4 \int_{(0,a_\dagger)\times\Omega} |J_1 - J_2| \cdot |\widetilde{A}_1 - \widetilde{A}_2| dxda.$$

It follows, as for I_1:

$$0 \leq I_2(t) \leq \frac{k_4 a_\dagger}{2}\Big(\int_{(0,a_\dagger)\times\Omega} (J_1 - J_2)^2\ dadx + \int_{(a_1,a_\dagger)\times\Omega} (\widetilde{A}_1 - \widetilde{A}_2)^2\ dadx\Big). \quad (5.12)$$

Finally:

$$\begin{aligned} I_3(t) &= \frac{1}{2}\int_\Omega \Big(\int_{(a_1,a_\dagger)\times\Omega} (\beta(\widetilde{\mathbb{P}}_1)\widetilde{A}_1 - \beta(\widetilde{\mathbb{P}}_2)\widetilde{A}_2\ da\Big)^2\ da \\ &= \frac{1}{2}\int_\Omega \Big(\int_{a_1}^{a_\dagger} \big(\beta(\widetilde{\mathbb{P}}_1)(\widetilde{A}_1 - \widetilde{A}_2) + \widetilde{A}_2(\beta(\widetilde{\mathbb{P}}_1) - \beta(\widetilde{\mathbb{P}}_2))\big) da\Big)^2\ dx, \end{aligned}$$

this way:

$$\begin{aligned} I_3(t) &\leq \int_\Omega \Big(\int_{a_1}^{a_\dagger} \beta(\widetilde{\mathbb{P}}_1)(\widetilde{A}_1 - \widetilde{A}_2)\ da\Big)^2 + \Big(\int_{a_1}^{a_\dagger} \widetilde{A}_2(\beta(\widetilde{\mathbb{P}}_1) - \beta(\widetilde{\mathbb{P}}_2)\ da\Big)^2\ dx \\ &\leq \beta_\infty^2 \int_\Omega \Big(\int_{a_1}^{a_\dagger} (\widetilde{A}_1 - \widetilde{A}_2)\ da\Big)^2\ dx + k_5 \int_\Omega \Big(\int_{a_1}^{a_\dagger} (\beta(\widetilde{\mathbb{P}}_1) - \beta(\widetilde{\mathbb{P}}_2))\ da\Big)^2\ dx \\ &\leq \beta_\infty^2 a_\dagger \int_{(a_1,a_\dagger)\times\Omega} (\widetilde{A}_1 - \widetilde{A}_2)^2\ dxda + k_5 K_\beta a_\dagger^2 \int_\Omega |\widetilde{\mathbb{P}}_1 - \widetilde{\mathbb{P}}_2| dx, \end{aligned}$$

and one gets:

$$0 \leq I_3(t) \leq k_6 \int_{(0,a_\dagger)\times\Omega} (\widetilde{J}_1 - \widetilde{J}_2)^2\ dadx + k_7 \int_{(a_1,a_\dagger)\times\Omega} (\widetilde{A}_1 - \widetilde{A}_2)^2\ dadx. \quad (5.13)$$

Working similarly on the equation in A, one gets:

$$\begin{aligned} \frac{1}{2}(\partial_t + \partial_a\ )||(A_1 - A_2)(t,\cdot,\cdot)||_{2,(a_1,a_\dagger)\times\Omega} + \underline{d}||\nabla(A_1 - A_2)(t,\cdot,\cdot)||_{2,(a_1,a_\dagger)\times\Omega} \\ \leq I_4(t) + I_5(t) + 0, \end{aligned} \quad (5.14)$$

with:

$$\begin{aligned}
I_4(t) = k_8 \int_{(a_1,a_\dagger)\times\Omega} |A_1 - A_2| \cdot |\widetilde{\mathbb{P}}_1 - \widetilde{\mathbb{P}}_2| dxda \\
\leq k_8\ a_\dagger \Big(\int_{(0,a_\dagger)\times\Omega} (A_1 - A_2)^2\ dadx + \\
\frac{1}{2} \int_{(0,a_\dagger)\times\Omega} (\widetilde{J}_1 - \widetilde{J}_2)^2\ dadx + \frac{1}{2} \int_{(a_1,a_\dagger)\times\Omega} (\widetilde{A}_1 - \widetilde{A}_2)^2\ dadx \Big), \quad (5.15)
\end{aligned}$$

and

$$\begin{aligned}
I_5(t) = k_9 \int_{(a_1,a_\dagger)\times\Omega} |J_1 - J_2| \cdot |\widetilde{A}_1 - \widetilde{A}_2| dxda \\
\leq \frac{k_9 a_\dagger}{2} \Big(\int_{(0,a_\dagger)\times\Omega} (J_1 - J_2)^2\ dadx + \int_{(a_1,a_\dagger)\times\Omega} (\widetilde{A}_1 - \widetilde{A}_2)^2\ dadx \Big). \quad (5.16)
\end{aligned}$$

Substituting inequalities (5.11), (5.12) and (5.13) in (5.10) and (5.15) and (5.16) in (5.14), one completes the proof of Lemma 5.6. □

We can deduce from lemma 5.6 the:

Lemma 5.7: *The mapping Φ is a strict contraction on $L^2((0,\tau^*)\times(0,a_\dagger)\times\Omega)\times L^2((0,\tau^*)\times(a_1,a_\dagger)\times\Omega)$ with τ^* small enough, i.e. there exists $\rho(\tau^*) < 1$ such that:*

$$\begin{aligned}
\big(||(J_1 - J_2)(t,\cdot,\cdot)||_{2,(0,a_\dagger)\times\Omega} + ||(A_1 - A_2)(t,\cdot,\cdot)||_{2,(a_1,a_\dagger)\times\Omega} \big) \leq \\
\rho(\tau^*) \big(||(\widetilde{J}_1 - \widetilde{J}_2)(t,\cdot,\cdot)||_{2,(0,a_\dagger)\times\Omega} + ||(\widetilde{A}_1 - \widetilde{A}_2)(t,\cdot,\cdot)||_{2,(a_1,a_\dagger)\times\Omega} \big). \quad (5.17)
\end{aligned}$$

Proof: First, note that if $y(t)$ is solution of system:

$$\begin{cases} y'(t) \leq k_1 y(t) + k_2 z(t), \\ y(0) = 0, \end{cases}$$

with $k_1, k_2 \geq 0$, then:

$$0 \leq y(t) \leq k_2 \int_0^t e^{k_1(t-s)} z(s) ds,$$

so when $t \to z(t)$ is nondecreasing:

$$0 \leq y(t) \leq k_2 \Big(\int_0^t e^{k_1(t-s)} ds \Big) z(t) \leq \frac{k_2}{k_1} (e^{k_1\ t} - 1) z(t).$$

Using (5.9) to use this with:

$$\begin{aligned}
y(t) &= ||(J_1 - J_2)||_{2,(0,a_\dagger)\times(0,t)\times\Omega} + ||(A_1 - A_2)||_{2,(a_1,a_\dagger)\times(0,t)\times\Omega}, \\
z(t) &= ||(\widetilde{J}_1 - \widetilde{J}_2)||_{2,(0,a_\dagger)\times(0,t)\times\Omega} + ||(\widetilde{A}_1 - \widetilde{A}_2)||_{2,(a_1,a_\dagger)\times(0,t)\times\Omega},
\end{aligned}$$

it follows (5.17) with $\rho(t) = \dfrac{k_2}{k_1}(e^{k_1\ t} - 1)$, smaller than 1 for t small enough. □

As Φ is a strict contraction on a Banach space, there exists a unique fixed point $(\widehat{J},\widehat{A}) \in L^2((0,a_\dagger)\times(0,\tau^*)\times\Omega)\times L^2((a_1,a_\dagger)\times(0,\tau^*)\times\Omega)$ such that $\Phi(\widehat{J},\widehat{A}) = (\widehat{J},\widehat{A})$. Furthermore, one has from (5.7)

$$\text{For} \quad Z = J, A, \quad 0 \le \widehat{Z}(t,a,x) \le Z^*(t,a).$$

Otherwise, by dominated convergence one checks that if (J_n, A_n) tends toward (J, A) in $\mathcal{K}$ then $(\mathbb{J}_n, \mathbb{A}_n)$ tends toward $(\mathbb{J}, \mathbb{A})$ in $(L^1((0,T)\times\Omega))^2$.
Thus, using dominated convergence, continuity of τ, β, μ_J and μ_A in the last variable and strong convergence in L^2 it follows that $(\widehat{J},\widehat{A})$ is a weak solution of (2.1)-(2.2) (see for example Naulin [21] for details in a similar case).
Then we can make again the same work to get the result on $(0, T)$. □

5.2. *Proof of Theorem 2.9*

First, in order to justify calculations below, we need the following Corollary of Proposition 5.2 for regularity of $\mathbb{J}$ and $\mathbb{A}$:

Corollary 5.8: *Suppose assumption Hyp 2.1 to Hyp 2.3 and Hyp 2.7 are satisfied. The the unique nonnegative weak solution u in $(0,T)\times(0,a_\dagger)\times\Omega$ of problem (5.1) satisfies:*

$$\int_0^{a_\dagger} u(t,a,x)da \ \in \ L^2(0,T;H^1(\Omega)).$$

Proof: We return to the proof of Proposition 5.2, if the diffusion rate d does not depend on the variable $a \in (0,a_\dagger)$ (Assumption 2.7) one can guarantee regularity on $\int_0^{a_\dagger} u(t,a,x)da$.
So as to get this, we work with the approximate solutions given by Galerkins method which have sufficient regularity: $\int_0^{a_\dagger} u_l(t,a,x)da \ \in \ L^2(0,T;H^1(\Omega))$.
Taking $v_l = \int_0^{a_\dagger} u_l(t,a,x)da$ in the weak formulation, one has:

$$\int_{(0,T)\times\Omega} d(t,x)|\nabla \int_0^{a_\dagger} u_l(t,a,x)da|^2 dtdx \le C,$$

where C depends on b, u_0 and f, and one gets the result by assumption Hyp 2.3.□

Integrating equation (2.1) in a from 0 toward $a_\dagger$, i.e. using $u = 1$ as a test function, one has:

$$\partial_t \mathbb{J} + J(t,a_\dagger,x) - \operatorname{div}(d_J(t,x)\cdot\nabla\mathbb{J}) + \int_0^{a_\dagger} \mu_J(t,a,x,\mathbb{P}(t,x))\cdot J\ da$$
$$+ \int_0^{a_\dagger} \tau(t,a,x,\mathbb{A}(t,x))\cdot J\ da = \int_0^{a_\dagger} \beta(t,a,x,\mathbb{P}(t,x))\cdot A(t,a,x)\ da,$$

thus one gets a first partial differential inequality for $\mathbb{J}$:

$$\partial_t \mathbb{J}(t,x) - \mathrm{div}(d_J(t,x)\cdot \nabla \mathbb{J}(t,x)) + (\mu_0 + \mu_1 \mathbb{P}(t,x))\cdot \mathbb{J}(t,x) \le \beta_\infty \mathbb{A}. \tag{5.18}$$

Similarly, one has a second partial differential inequality for $\mathbb{A}$:

$$\partial_t \mathbb{A}(t,x) - \mathrm{div}(d_A(t,x)\cdot \nabla \mathbb{A}(t,x)) + (\mu_0 + \mu_1 \mathbb{P}(t,x))\cdot \mathbb{A}(t,x) \le \tau_\infty \mathbb{J}.$$

Our goal is to prove the existence of a constant $M_0 > 0$, independent on t such that:

$$\max_{t>0}\{||\mathbb{J}(t,\cdot)||_{\infty,\Omega}, ||\mathbb{A}(t,\cdot)||_{\infty,\Omega}\} \le M_0.$$

In order to do this, we adapt a work of Fitzgibbon et al. [10]. First, we establish the following lemma:

Lemma 5.9: *If $\mathbb{J}(t,x)$, $\mathbb{A}(t,x)$ are classical nonnegative solutions of (5.18)-(5.2) in $[0,+\infty]\times\Omega$, then noting $b_\infty = \max(\beta_\infty,\tau_\infty)$:*

$$||\mathbb{P}(t,\cdot)||_{1,\Omega} \le \max(||\mathbb{P}_0||_{1,\Omega}, ((b_\infty - \mu_0)/\mu_1)|\Omega|) = C_1. \tag{5.19}$$

Furthermore

$$\lim_{t\to\infty} \sup ||\mathbb{P}(t,\cdot)||_{1,\Omega} \le (\beta_\infty/\mu_1)|\Omega|. \tag{5.20}$$

Moreover, for nonnegative l and l^* there exists a constant C_{l,l^*} depending on β_∞, μ_1 and $||\mathbb{P}_0||_{1,\Omega}$ such that if $Q(l,l+l^*) = (l,l+l^*)\times\Omega$,

$$||\mathbb{P}||_{2,Q(l,l+l^*)} \le C_{l,l^*}. \tag{5.21}$$

Also if l is large enough, then C_{l,l^*} can be taken independent on $||\mathbb{P}_0||_{1,\Omega}$ and l.

Proof: Integrating the inequality in $\mathbb{J}$ on Ω, one has:

$$\partial_t \int_\Omega \mathbb{J}(t,\cdot) - \int_\Omega \mathrm{div}(d_J(t,\cdot)\cdot\nabla\mathbb{J}(t,\cdot)) + \int_\Omega \left((\mu_0 + \mu_1\, \mathbb{P}(t,\cdot))\cdot \mathbb{J}(t,\cdot)\right) \le \beta_\infty \cdot \int_\Omega \mathbb{A}(t,\cdot)$$

But $\int_\Omega \mathrm{div}(d_J(t,\cdot)\cdot\nabla\mathbb{J}(t,\cdot)) = 0$ by the edge condition on $\partial\Omega$ so:

$$\frac{d}{dt}||\mathbb{J}(t,\cdot)||_{1,\Omega} \le \beta_\infty||\mathbb{A}(t,\cdot)||_{1,\Omega} - \mu_0||J_0(t,\cdot)||_{1,\Omega} - \mu_1\int_\Omega \mathbb{P}(t,x))\cdot \mathbb{J}(t,x)\,dx.$$

Similarly, one gets for the inequality in $\mathbb{A}$:

$$\frac{d}{dt}||\mathbb{A}(t,\cdot)||_{1,\Omega} \le \tau_\infty||\mathbb{J}(t,\cdot)||_{1,\Omega} - \mu_0||A_0(t,\cdot)||_{1,\Omega} - \mu_1\int_\Omega \mathbb{P}(t,x))\cdot \mathbb{A}(t,x)\,dx,$$

thus, adding the two inequalities:

$$\begin{aligned}\frac{d}{dt}||\mathbb{P}(t,\cdot)||_{1,\Omega} &\le (b_\infty - \mu_0)||\mathbb{P}(t,\cdot)||_{1,\Omega} - \mu_1 \int_\Omega \mathbb{P}^2(t,x))\ dx \\ &\le (b_\infty - \mu_0)||\mathbb{P}(t,\cdot)||_{1,\Omega} - \frac{\mu_1}{|\Omega|}||\mathbb{P}(t,\cdot)||^2_{1,\Omega}. \qquad (5.22)\end{aligned}$$

Then $||\mathbb{P}(t,\cdot)||_{1,\Omega}$ is bounded for $0 < t < \infty$ by the solution of problem:

$$y'(t) = (b_\infty - \mu_0)y - \frac{\mu_1}{|\Omega|}y^2, \qquad y(0) = ||\mathbb{P}_0||_{1,\Omega},$$

so (5.19) and (5.20) are proved.

In order to prove (5.21), remain that for $a, b \in \mathbb{R}$ and $\varepsilon > 0$, one has (Young inequality):

$$a \cdot b \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2.$$

Applying this to the right side of the first inequality of (5.22), it follows:

$$\frac{d}{dt}||\mathbb{P}(t,\cdot)||_{1,\Omega} \le c_1 - c_2||\mathbb{P}(t,\cdot)||^2_{2,\Omega},$$

thus integrating in time on $l, l + l^*$:

$$||\mathbb{P}||^2_{2,Q(\tau,\tau+\tau^*)} \le \frac{1}{c_2}\big(c_1\ \tau^* + ||\mathbb{P}(l,\cdot)||_{1,\Omega}\big) := C_{l,l^*},$$

and (5.20) achieved the proof. □

Now we give the following result, for regularity:

Lemma 5.10: *Suppose initial conditions (J_0, A_0) are nonnegative and continuous on $\overline{\Omega}$, and assumption Hyp 2.1 to Hyp 2.8 satisfied. Then there exists $C_7 \in C(\mathbb{R}_+)$ such that for $0 \le l < T$:*

$$\begin{aligned}\mathbb{J},\ \mathbb{A} \in L^6((0,T)\times\Omega) \quad &\textit{and} \quad ||\mathbb{J}||_{6,(0,T)\times\Omega},\ ||\mathbb{A}||_{6,(0,T)\times\Omega} \le C_7(T)\\ |\nabla\mathbb{J}|,\ |\nabla\mathbb{A}| \in L^5((0,T)\times\Omega) \quad &\textit{and} \quad ||\nabla\mathbb{J}||_{5,(l,T)\times\Omega},\ ||\nabla\mathbb{A}||_{5,(l,T)\times\Omega} \le C_7(T).\end{aligned}$$

Proof: From Lemma 5.9, ons has that $\mathbb{P}$ and so $\mathbb{J}$ and $\mathbb{A}$ are bounded in $L^2(Q(0,T))$. Multiplying inequality (5.18) by $\mathbb{J}$ and integrating on Ω one gets:

$$\begin{aligned}\frac{1}{2}\partial_t \int_\Omega \mathbb{J}^2(t,x)\ dx + \underline{d}\int_\Omega |\nabla\mathbb{J}|^2(t,x)\ dx + \mu_1 \int_\Omega \mathbb{J}^3(t,x)\ dx \\ \le \beta_\infty \int_\Omega \mathbb{J}(t,a)\mathbb{A}(t,x)\ dx \\ \le \frac{\beta_\infty}{2}\Big(\int_\Omega \mathbb{J}^2(t,x)\ dx + \int_\Omega \mathbb{A}^2(t,x)\ dx\Big).\end{aligned}$$

Similarly, one has:

$$\frac{1}{2}\partial_t\int_\Omega \mathbb{A}^2(t,x)\ dx + \underline{d}\int_\Omega |\nabla\mathbb{A}|^2(t,x)\ dx + \mu_1\int_\Omega \mathbb{A}^3(t,x)\ dx$$
$$\leq \frac{\tau_\infty}{2}\Big(\int_\Omega \mathbb{J}^2(t,x)\ dx + \int_\Omega \mathbb{A}^2(t,x)\ dx\Big).$$

Then, adding the two previous estimates, it follows:

$$\frac{1}{2}\partial_t\int_\Omega (\mathbb{J}^2+\mathbb{A}^2)(t,x)\ dx + \underline{d}\int_\Omega (|\nabla\mathbb{J}|^2+|\nabla\mathbb{A}|^2)(t,x)\ dx$$
$$+\mu_1\int_\Omega (\mathbb{J}^3+\mathbb{A}^3)(t,x)\ dx \leq k_1\int_\Omega (\mathbb{J}^2+\mathbb{A}^2)(t,x)\ dx. \quad (5.23)$$

thus there exists $C_2 \in C(\mathbb{R}^+)$ such that for $t < T$:

$$\mathbb{J}(t,\cdot),\ \mathbb{A}(t,\cdot) \in L^2(\Omega) \quad \text{et} \quad ||\mathbb{J}(t,\cdot)||_{2,\Omega},\ ||\mathbb{A}(t,\cdot)||_{2,\Omega} \leq C_2(t),$$

then, integrating (5.23) in time and using Lemma 5.9, one gets the existence of $C_3 \in C(\mathbb{R}^+)$ such that for $0 \leq l < T$:

$$\mathbb{J},\ \mathbb{A} \in L^3((0,T)\times\Omega) \quad \text{et} \quad ||\mathbb{J}||_{3,(0,T)\times\Omega},\ ||\mathbb{A}||_{3,(0,T)\times\Omega} \leq C_3(T)$$
$$|\nabla\mathbb{J}|,\ |\nabla\mathbb{A}| \in L^2((0,T)\times\Omega) \quad \text{et} \quad ||\nabla\mathbb{J}||_{2,(0,T)\times\Omega},\ ||\nabla\mathbb{A}||_{2,(0,T)\times\Omega} \leq C_3(T).$$

Moreover multiplying the inequality in $\mathbb{J}$ by $\mathbb{J}^2$ and integrating on Ω, one has:

$$\frac{1}{3}\partial_t\int_\Omega \mathbb{J}^3(t,x)\ dx + 2\underline{d}\int_\Omega |\nabla\mathbb{J}|^2(t,x)\ \mathbb{J}(t,x)\ dx + \mu_1\int_\Omega \mathbb{J}^4(t,x)\ dx$$
$$\leq \beta_\infty\int_\Omega \mathbb{J}^2(t,a)\mathbb{A}(t,x)\ dx$$
$$\leq \beta_\infty\Big(\frac{\varepsilon}{2}\int_\Omega \mathbb{J}^4(t,x)\ dx + \frac{1}{2\varepsilon}\int_\Omega \mathbb{A}^2(t,x)\ dx\Big),$$

thus for ε small enough the existence of $\widetilde{\mu_1} > 0$ and $\widetilde{\beta_\infty} > 0$ such that:

$$\frac{1}{3}\partial_t\int_\Omega \mathbb{J}^3(t,x)\ dx + \widetilde{\mu_1}\int_\Omega \mathbb{J}^4(t,x)\ dx \leq \widetilde{\beta_\infty}\int_\Omega \mathbb{A}^2(t,x)\ dx.$$

Similar work for the equation in $\mathbb{A}$ gives:

$$\frac{1}{3}\partial_t\int_\Omega \mathbb{A}^3(t,x)\ dx + \widetilde{\mu_1}\int_\Omega \mathbb{A}^4(t,x)\ dx \leq \widetilde{\tau_\infty}\int_\Omega \mathbb{J}^2(t,x)\ dx. \quad (5.24)$$

and, adding the two estimates:

$$\frac{1}{3}\partial_t\int_\Omega (\mathbb{J}^3+\mathbb{A}^3)(t,x)\ dx + \widetilde{\mu_1}\int_\Omega (\mathbb{J}^4+\mathbb{A}^4)(t,x)\ dx \leq k_2\int_\Omega (\mathbb{J}^2+\mathbb{A}^2)(t,x)\ dx.$$

So one has existence of $C_4 \in C(\mathbb{R}^+)$ such that for $t < T$:

$$\mathbb{J}(t,\cdot),\ \mathbb{A}(t,\cdot) \in L^3(\Omega) \quad \text{and} \quad ||\mathbb{J}(t,\cdot)||_{3,\Omega},\ ||\mathbb{A}(t,\cdot)||_{3,\Omega} \leq C_3(t),$$

and integrating (5.24) in time, one gets existence of $C_5 \in C(\mathbb{R}^+)$ such that for $0 \leq l < T$:

$$\begin{array}{rl} \mathbb{J},\ \mathbb{A} \in L^4((0,T)\times\Omega) & \text{et} \quad ||\mathbb{J}||_{4,(0,T)\times\Omega},\ ||\mathbb{A}||_{4,(0,T)\times\Omega} \leq C_5(T) \\ |\nabla\mathbb{J}|,\ |\nabla\mathbb{A}| \in L^3((0,T)\times\Omega) & \text{and} \quad ||\nabla\mathbb{J}||_{3,(0,T)\times\Omega},\ ||\nabla\mathbb{A}||_{3,(0,T)\times\Omega} \leq C_5(T). \end{array}$$

Similarly, multiplying inequalities in $\mathbb{J}$ and $\mathbb{A}$ by $\mathbb{J}^3$ and $\mathbb{A}^3$, it follows estimates of $\mathbb{J}(t,\cdot)$ and $\mathbb{A}(t,\cdot)$ in $L^4(\Omega)$, but also of $\mathbb{J}$ and $\mathbb{A}$ in $L^5((0,T)\times\Omega)$. Thus, with multiplication by $\mathbb{J}^4$ and $\mathbb{A}^4$, one gets existence of C_7 and estimates of $\mathbb{J}(t,\cdot)$ and $\mathbb{A}(t,\cdot)$ in $L^5(\Omega)$, and also of $\mathbb{J}$ and $\mathbb{A}$ in $L^6((0,T)\times\Omega)$. □

As $\mathbb{J}$ and $\mathbb{A}$ are bounded in $L^6(Q(0,T))$, each component of

$$F(t,x,\mathbb{J},\mathbb{A}) \;=\; \begin{pmatrix} -(\mu_0+\mu_1\ \mathbb{P}(t,x))\cdot\mathbb{J}(t,x)+\beta_\infty\cdot\mathbb{A}(t,x) \\ -(\mu_0+\mu_1\ \mathbb{P}(t,x))\cdot\mathbb{A}(t,x)+\tau_\infty\cdot\mathbb{J}(t,x) \end{pmatrix}$$

is bounded in $L^3(Q(0,T))$ thus one has existence of $M(t) \geq 0$ continuous on R^+ such that for $t > T$:

$$\max\{||\mathbb{J}(t,\cdot)||_{\infty,\Omega}, ||\mathbb{A}(t,\cdot)||_{\infty,\Omega}\} \leq M(t).$$

The following lemma complete the estimates given in Lemma 5.9:

Lemma 5.11: *If $\mathbb{J}(t,x)$, $\mathbb{A}(t,x)$ are nonnegative classical solutions of (5.18)-(5.2) on $[0,+\infty]\times\Omega$, then:*

$$||\mathbb{P}(t,\cdot)||_{2,\Omega} \leq C(||\mathbb{P}_0||_{2,\Omega}). \tag{5.25}$$

Furthermore

$$\lim_{t\to\infty}\sup ||\mathbb{P}(t,\cdot)||_{2,\Omega} \leq C, \tag{5.26}$$

where C is independent on initial conditions.

Proof: For $u \in L^2(\Omega)$, one has using Holder Inequality:

$$\int_\Omega u^2(x)dx \leq |\Omega|^{\frac{1}{3}} \left(\int_\Omega u^3(x)dx\right)^{\frac{2}{3}}.$$

In particular, for $u, v \in (L^2(\Omega))^2$ nonnegative one gets:

$$\int_\Omega u^2(x)dx \leq |\Omega|^{\frac{1}{3}} \left(\int_\Omega (u^3(x)+v^3(x))dx\right)^{\frac{2}{3}},$$

and

$$\int_\Omega (u^2(x)+v^2(x))dx \le 2|\Omega|^{\frac{1}{3}} \left(\int_\Omega u^3(x)+v^3(x)dx\right)^{\frac{2}{3}}$$
$$\left(\int_\Omega (u^2(x)+v^2(x))dx\right)^{\frac{3}{2}} \le 2^{\frac{3}{2}}|\Omega|^{\frac{1}{2}} \int_\Omega u^3(x)+v^3(x)dx. \tag{5.27}$$

Otherwise, from (5.23), it follows:

$$\frac{1}{2}\partial_t \int_\Omega (\mathbb{J}^2+\mathbb{A}^2)(t,x)\ dx \le k_1 \int_\Omega (\mathbb{J}^2+\mathbb{A}^2)(t,x)\ dx - \mu_1 \int_\Omega (\mathbb{J}^3+\mathbb{A}^3)(t,x)\ dx,$$

thus, using (5.27) with $u=\mathbb{J}$ and $v=\mathbb{A}$:

$$\frac{1}{2}\partial_t \int_\Omega (\mathbb{J}^2+\mathbb{A}^2)(t,x)\ dx \le k_1 \int_\Omega (\mathbb{J}^2+\mathbb{A}^2)(t,x)\ dx - \frac{\mu_1}{2^{\frac{3}{2}}|\Omega|^{\frac{1}{2}}} \left(\int_\Omega (\mathbb{J}^2+\mathbb{A}^2)(t,x)\ dx\right)^{\frac{3}{2}}.$$

Then $\int_\Omega (\mathbb{J}^2+\mathbb{A}^2)(t,x)\ dx \le y(t)$ where $y(t)$ is solution of a logistic equation:

$$y'(t) = c_1 y - c_2 y^{\frac{3}{2}}, \qquad y(0) = ||\mathbb{P}_0||_{2,\Omega},$$

and (5.25) and (5.26) follows. □

We now have the following result:

Proposition 5.12: *For fixed l large enough and $l^*>0$, there exists a constant $C(6,l^*)$ independent on initial conditions $||\mathbb{J}_0||_{1,\Omega}$ and $||\mathbb{A}_0||_{1,\Omega}$ such that for $Z = J, A$,*

$$||\mathbb{Z}||_{6,Q(l,l+l^*)} \le C(6,l^*).$$

Proof: Integrating estimate (5.23) in time from l toward $l+l^*$, one has:

$$\begin{aligned}\frac{1}{2}\int_\Omega (\mathbb{J}^2+\mathbb{A}^2)(l+l^*,x)\ dx + \underline{d}\int_{(l,l+l^*)\times\Omega} (|\nabla\mathbb{J}|^2+|\nabla\mathbb{A}|^2)(t,x)\ dxdt \\ + \mu_1 \int_{(l,l+l^*)\times\Omega} (\mathbb{J}^3+\mathbb{A}^3)(t,x)\ dxdt \\ \le k_1 \int_{(l,l+l^*)\times\Omega} (\mathbb{J}^2+\mathbb{A}^2)(t,x)\ dxdt + \frac{1}{2}\int_\Omega (\mathbb{J}^2+\mathbb{A}^2)(l,x)\ dx.\end{aligned} \tag{5.28}$$

But using Lemma 5.11, for l large enough the second term of the right side of (5.28) is controlled thus one gets existence of $C(3,l^*) \in C(\mathbb{R}^+)$ such that:

$$\begin{array}{c} \mathbb{J},\ \mathbb{A} \in L^3((l,l+l^*)\times\Omega) \quad \text{et} \quad ||\mathbb{J}||_{3,(l,l+l^*)\times\Omega},\ ||\mathbb{A}||_{3,(l,l+l^*)\times\Omega} \le C(3,l^*) \\ |\nabla\mathbb{J}|, |\nabla\mathbb{A}| \in L^2((l,l+l^*)\times\Omega) \text{ et } ||\nabla\mathbb{J}||_{2,(l,l+l^*)\times\Omega}, ||\nabla\mathbb{A}||_{2,(l,l+l^*)\times\Omega} \le C(3,l^*). \end{array}$$

Similarly to the proof of Lemma 5.9 and continuing estimates, Lemma 5.11 follows.
□

Finally, we can get the global $L^\infty(\Omega)$ estimates given in Theorem 2.9:
Proof: From Proposition 5.12, one has existence for l large enough ($l \ge l_0$) of $C(6,l^*)$ such that:

$$\max\{||\mathbb{J}||_{6,Q(l,l+l^*)}, ||\mathbb{A}||_{6,Q(l,l+l^*)}\} \le C(6,l^*).$$

Let $\overline{J}$ and $\overline{A}$ solutions in $L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^1(\Omega))$ of:

$$\begin{cases} \partial_t\overline{J}(t,x) - \text{div}(d_J(t,x)\cdot\nabla\overline{J}(t,x)) = \beta_\infty\mathbb{A}(t,x), \\ \overline{J}(0,x) \ = \ \mathbb{J}_0(x), \\ (d_J(t,x)\cdot\nabla\overline{J}(t,a,x))\cdot\eta(x) \ = \ 0, \end{cases}$$

$$\begin{cases} \partial_t\overline{A}(t,x) - \text{div}(d_A(t,x)\cdot\nabla\overline{A}(t,x)) = \tau_\infty\mathbb{J}(t,x), \\ \overline{A}(0,x) \ = \ \mathbb{A}_0(x), \\ (d_A(t,x)\cdot\nabla\overline{A}(t,a,x))\cdot\eta(x) \ = \ 0. \end{cases}$$

The maximum principle (see Smoller [25]) gives for $t \ge 0$ and $x \in \Omega$:

$$\begin{array}{c} 0 \le \mathbb{J}(t,x) \le \overline{J}(t,x), \\ 0 \le \mathbb{A}(t,x) \le \overline{A}(t,x). \end{array}$$

Using regularity results in Ladyzhenskaya et al. [18], it follows:

$$\max\{||\overline{J}||_{\infty,Q(0,l_0)}, ||\overline{A}||_{\infty,Q(0,l_0)}\} \le C(l_0, ||\mathbb{J}_0(x)||_\infty, ||\mathbb{A}_0(x)||_\infty),$$

and for $l \ge l_0$ et $l^* > 0$:

$$\max\{||\overline{J}||_{\infty,Q(l,l+l^*)}, ||\overline{A}||_{\infty,Q(l,l+l^*)}\} \le C(l^*, ||\overline{J}(l,x)||_\infty, ||\overline{A}(l,x)||_\infty.$$

But we want an estimate independent on $\overline{J}$ and $\overline{A}$ at time l.
Thus we set $l^* = 1$, $l \ge l_0 - 1$ and an auxiliary mapping $\phi(t)$ nonnegative, C^1 on $\mathbb{R}$ such that:

$$\begin{array}{rcll} \phi(s) & = & 0 & \text{for } \ s \le 0, \\ \phi(s) & = & 1 & \text{for } \ s > 1, \\ \phi'(s) & \ge & 0 & \text{for } \ s \in (0,1). \end{array}$$

Then we set the function $\rho(t,x) \ = \ \phi(t-l)\,\overline{J}(t,x)$. If $l \ge 0$, one has $\rho(t,x) \ = \ \overline{J}(t,x)$

for $t \in [l+1, l+2]$ and $\rho(l,x) = 0$. Deriving ρ toward time, one gets:

$$\begin{aligned}\partial_t \rho &= \phi'(t-l)\overline{J} + \mathrm{div}(d_J(t,x)\nabla\rho) + \phi(t-l)\beta_\infty \mathbb{A} \\ &= \mathrm{div}(d_J(t,x)\nabla\rho) + g(t,x),\end{aligned}$$

with:

$$\begin{aligned}\rho(l,x) &= 0 \\ (d_J(t,x)\cdot\nabla\rho(t,x))\cdot\eta(x) &= 0 \qquad x \in \partial\Omega,\ t \geq 0.\end{aligned}$$

Thus we have global estimates of $\mathbb{J}, \mathbb{A}$ in $L^3(Q(l,l+2)$, and using regularity results in Ladyzhenskaya et al. [18] one has a global estimate in $L^\infty(Q(l,l+2)$ for $\rho(t,x)$, and it follows a global estimate in $L^\infty(Q(l+1,l+2)$ for $\overline{J}(t,x)$.
Then for all $(l,l+1)$ with $l \geq l_0$ one has

$$||\overline{J}||_{\infty,Q(l,l+1)} \leq C$$

with C independent on $||\overline{J}(l,x)||_\infty$ and l. So, as:

$$||\overline{J}||_{\infty,Q(0,\infty)} \leq \max\{||\overline{J}||_{\infty,Q(0,l_0)}, ||\overline{J}||_{\infty,Q(l_0,\infty)}\},$$

one gets a global bound for $\overline{J}$, so for $\mathbb{J}$. Similar arguments give the same result for $\mathbb{A}$. □

Remark 1: A priori estimates allowing estimates in Lemma 5.9 and $M(t)$ in the proof of Proposition 5.10 can be obtained directly from equations of system (2.1)-(2.2) by integrating also equations in age a. However, we do not have results concerning the theory of parabolic equations from for example Smoller [25] or Ladyzhenskaya et al. [18] to conclude to global existence of solutions of the system.

5.3. *Proof of Theorem 3.3*

In order to prove Theorem 3.3, we will need this two result :

5.3.1. Environnement equation

This result is prooven by variation of the constant

Lemma 5.13: *The solution of equation*

$$\begin{cases} G'(t) = f(t) - g(t)\cdot G(t) \\ G(0) = G_0, \end{cases}$$

with $f,\ g \in L^\infty$ is given by:

$$G(t) \;=\; G_0\ e^{-\int_0^t g(s)\ ds} + \int_0^t f(l)\ e^{-\int_l^t g(s)\ ds}\ dl, \qquad t > 0.$$

We also need an other auxiliary problem :

5.3.2. Third auxiliary problem

We consider the following system :

$$\begin{cases} \partial_t u + \partial_a u + \partial_b u - div(d(t,a,x)\nabla u) + \mu(t,a,b,x)u = f(t,a,b,x) \\ u(0,a,b,x) = u_0(a,b,x) \\ u(t,a,b,x) = 0 \text{ if } a \leq a_1 \\ u(t,a,0,x) = X(t,a,x) \\ d(t,a,x)\nabla u(t,a,b,x) \cdot \eta(x) = 0 \text{ for } x \in \partial\Omega \end{cases} \tag{5.29}$$

and we suppose that :

Hyp 5.14

- $u_0 \in L^2((0,a_\dagger) \times \Omega)$ and u_0 is nonnegative,
- $\mu \in L^\infty((0,T) \times (0,a_\dagger) \times \Omega)$ and μ is nonnegative,
- $b \in L^\infty((0,T) \times \Omega)$ and b is nonnegative,
- $f \in L^2((0,T) \times (0,a_\dagger) \times \Omega) \cap L^\infty((0,T) \times (0,a_\dagger) \times \Omega)$, with $f(t,a,x) \geq 0$,
- d satisfies the assumption Hyp 2.3.

Then one gets :

Proposition 5.15: *Suppose assumption 5.14 is satisfied. Then the problem* (5.29) *has a unique weak solution u in* $(0,T) \times (0,a_\dagger) \times (0,b_\dagger) \times \Omega$ *nonnegative and satisfying :*

$$u \in L^\infty\left((0,T) \times (0,a_\dagger) \times (0,a_\dagger) \times \Omega\right) \cap L^2\left((0,T) \times (0,a_\dagger) \times (0,a_\dagger); H^1(\Omega)\right)$$
$$(\partial_t + \partial_a)u \in L^2\left((0,T) \times (0,a_\dagger) \times (0,a_\dagger); H^0(\Omega)\right)$$

Proof : As in the proposition 5.2 we proove this result by the characteristics method and classical results for hyperbolic problems. This time there are three differents cases : $0 < a < t, b$, $0 < b < t, a$ and $0 < t < a, b$.

We begin by considering $0 < a < t, b$. Let u be a solution of (5.29). For $c > 0$, we set $w(c,x) = u(t-a+c, c, b-a+c, x) = u(t_c, c, b_c, x)$. w is a solution of the following system :

$$\begin{cases} \partial_c w(c,x) - div(d(t_c,c,x).\nabla w) + \mu(t_c,c,x)w(c,x) = f(t_c,c,b_c,x) \\ w(a,x) = 0 \text{ if } a \leq a_1 \\ d(t_c,c,x)\nabla w(c,x) \cdot \eta(x) = 0 \text{ for } x \in \partial\Omega \end{cases}$$

The two other case lead to two similar parobolic problems by denoting respectively : $w(c,x) = u(t-b+c, a-b+c, c, x) = u(t_c, a_c, c, x)$ if $0 < b < t, a$ and $w(c,x) = u(c, a-t+c, b-t+c, x) = u(c, a_c, b_c, x)$ if $0 < t < a, b$.

In each cases, the parabolic equations theory implies the existence of a unique solution. □

As for the first auxiliary problem , comparison theorem for parabolic equations implies the following corollary:

Corollary 5.16: *If* $f_1 \geq f_2 \geq 0$,$b_1 \geq b_2 \geq 0$ *and* $\mu_2 \geq \mu_1 \geq 0$ *in* Q_J, $u_{01} \geq u_{02} \geq 0$ *in* $Q_{J,a}$ *then the solutions of* (5.29) *are non negatives.*

5.3.3. Proof of the Theorem 3.3

Proof: Let (J^*, A^*) be the solution of (5.2)-(5.3).
Let $\mathcal{K}$ be the closed convex subset of $\mathcal{H}^2(T) \times L^\infty$ defined by:

$$\mathcal{K} = \{(U,G) \in \mathcal{H}^2(T) \times L^2((0,T)\times\Omega),\ \ 0 \le G \le 1,\ \ 0 \le Z \le J^*,\ Z = J_s, J_i, \\ \text{and } 0 \le Z \le A^*,\ Z = A_s, A_i\}.$$

First note that we have the following lemma, useful in order to treat the incidence part corresponding to proportionate mixing term:

Lemma 5.17: *Let w_j (respectively w_a) the nonnegative solution of the linear problem (5.1) with $d = d_J$ (respectively $d = d_A$), $\mu = \mu_\infty + \tau_\infty + \gamma_\infty + \sigma_\infty(\mathbb{J}^* + \mathbb{A}^* + 2a_\dagger)$ (respectively $\mu = \mu_\infty + \gamma_\infty + \sigma_\infty(\mathbb{J}^* + \mathbb{A}^* + 2a_\dagger)$), $f = 0$, $\beta = 0$ and $w_{j,0} = \mathbb{J}_s(0)$ (respectively $w_{j,0} = \mathbb{A}_s(0)$). Then there exists a constant $m(T) > 0$ such that:*

$$0 < m(T) \le w_j(t,a,x) \le J_s(t,a,x), \quad 0 \le t \le T,\ 0 < a < \underline{a},\ x \in \Omega,$$
$$0 < m(T) \le w_a(t,a,x) \le A_s(t,a,x), \quad 0 \le t \le T,\ a_1 < a < a_1 + \underline{a},\ x \in \Omega.$$

Proof: By the assumption 3.1 :

$$\frac{1}{J(t,a',x)} \int_0^{b_\dagger} \sigma^{pm}_{j,Z}(t,a,a',b,x) J_i(t,a',b,x)db \le \sigma_\infty \frac{\int_0^{b_\dagger} J_i(t,a',b,x)db}{J(t,a',x)} \le \sigma_\infty$$

and in a similar way

$$\int_0^{b_\dagger} \frac{\sigma^{pm}_{a,Z}(t,a,a',b,x) A_i(t,a',b,x)}{A(t,a',x)} db \le \sigma_\infty$$

Thus, inequalities $w_j(t,a,x) \le J_s(t,a,x)$ and $w_a(t,a,x) \le J_a(t,a,x)$ come from the comparison result, Corollary 5.3). $m(T)$ results from integration along the characteristics of $(\partial_t + \partial_a)$, as detailed in [20] using assumption Hyp 3.2, as it is done in Wolf [30] for the proportionate mixing part. □

Let also defined the mapping $\Phi : \mathcal{K} \to \mathcal{K}$ by $\Phi(\widetilde{U}, \widetilde{G}) = (U, G)$ where (U, G) is solution of the linear problem:

$$\begin{cases} \forall t > 0, \forall a \in (0, a_\dagger), \forall x \in \Omega, \\ (\partial_t + \partial_a + \partial_b)U(t,a,b,x) - \operatorname{div}(D(t,a,x)\cdot \nabla U(t,a,b,x) \\ \qquad\qquad\qquad = (\widetilde{\Phi}(U,\widetilde{U}) + \widetilde{\Psi}(U,\widetilde{U},\widetilde{G}))(t,a,b,x), \end{cases}$$

$$G'(t,x) = \Big(\int_0^{a_\dagger}\int_0^{b_\dagger} \Upsilon(t,a,b,x)\cdot \widetilde{U}(t,a,b,x)dadb\Big)\cdot(1 - G(t,x)) - \delta(t,x)\cdot G(t,x),$$

$$\begin{cases} J_s(t,0,x) = \displaystyle\int_{a_1}^{a_\dagger} \beta(t,a,x,\widetilde{P}(t,x)) \cdot \widetilde{A}(t,a,x)da, \\ J_i(t,0,b,x) = 0, \\ A_s(t,a,x) = A_i(t,a,b,x) = 0 \quad \text{for} \quad a \le a_1, \\ Z(0,a,x) = Z_0(a,x) \quad \text{for} \quad Z = J_s, A_s, \\ Z(0,a,b,x) = Z_0(a,b,x) \quad \text{for} \quad Z = J_i, A_i, \\ G(0,x) = G_0(x), \end{cases}$$

$$\begin{cases} \forall t > 0, \forall a \in (0,a_\dagger), \forall b \in (0,b_\dagger), \forall x \in \partial\Omega, \\ d_J \ \nabla J_s(t,a,x) \cdot \eta(x) = 0, \\ d_J \ \nabla J_i(t,a,b,x) \cdot \eta(x) = 0, \\ d_A \nabla A_s(t,a,x) \cdot \eta(x) = 0, \\ d_A \nabla A_i(t,a,b,x) \cdot \eta(x) = 0, \end{cases}$$

with D and Υ as in (3.1) and (3.1), and:

$$\widetilde{\Phi}(U,\widetilde{U})(t,a,b,x) = \begin{pmatrix} -\mu_J(t,a,x,\widetilde{\mathbb{P}}(t,x)) \cdot J_s(t,a,x) - \tau(t,a,x,\widetilde{\mathbb{A}}(t,x)) \cdot J_s(t,a,x) \\ -\mu_J(t,a,x,\widetilde{\mathbb{P}}(t,x)) \cdot J_i(t,a,b,x) - \tau(t,a,x,\widetilde{\mathbb{A}}(t,x)) \cdot J_i(t,a,b,x) \\ \tau(t,a,x,\widetilde{\mathbb{A}}(t,x)) \cdot J_s(t,a,x) - \mu_A(t,a,x,\widetilde{\mathbb{P}}(t,x)) \cdot A_s(t,a,x) \\ \tau(t,a,x,\widetilde{\mathbb{A}}(t,x)) \cdot J_i(t,a,b,x) - \mu_A(t,a,x,\widetilde{\mathbb{P}}(t,x)) \cdot A_i(t,a,b,x) \end{pmatrix},$$

$$\widetilde{\Psi}(U,\widetilde{U},\widetilde{G})(t,a,b,x) = \begin{pmatrix} -\widetilde{\sigma_J}(t,a,x) \cdot J_s(t,a,x) - \gamma_J(t,a,x) \cdot \widetilde{G}(t,x) \cdot J_s(t,a,x) \\ 0 \\ -\widetilde{\sigma_A}(t,a,x) \cdot A_s(t,a,x) - \gamma_A(t,a,x) \cdot \widetilde{G}(t,x) \cdot A_s(t,a,x) \\ 0 \end{pmatrix},$$

with for $Z = J, A$:

$$\widetilde{\sigma_Z}(t,a,x) = \int_0^{b_\dagger}\int_0^{a_\dagger} \sigma_{j,Z}^{ma}(t,a,a',b,x) \cdot \widetilde{J}_i(t,a',b,x) + \sigma_{a,Z}^{ma}(t,a,a',b,x) \cdot \widetilde{A_i}(t,a',b,x) + \frac{\sigma_{j,Z}^{pm}(t,a,a',b,x) \cdot \widetilde{J}_i(t,a',b,x)}{\widetilde{J}(t,a',x)} + \frac{\sigma_{a,Z}^{pm}(t,a,a',b,x) \cdot \widetilde{A_i}(t,a',b,x)}{\widetilde{A}(t,a',x)} da'db.$$

On one side, integrating in b the equation in J_i and adding with the one in J_s and on the other side those in A_i and A_s one has equations (5.5)-(5.6). Thus one gets $\mathbb{J} \ge 0$ and $\mathbb{A} \ge 0$.
Equation for J_s is of the form (5.1) with $\mu = \mu_J + \tau + \widetilde{\sigma_J} + \gamma_J \widetilde{G}$ and $f = 0$, so that J_s is nonnegative.
Equation for J_i is of the form (5.29) with $\mu = \mu_J + \tau$ and $f = 0$, so that J_i is nonnegative.

Equations for A_s and A_i can be treated in the same way. Furthermore, one then gets $0 \leq G(t) \leq 1$ for all t, because all rates are nonnegative. Using results for linear equations in the previous section and lemma 5.13, $\Phi : \mathcal{K} \to \mathcal{K}$ is well defined.

We now have to check that Φ is a strict contraction. We consider $(U_1, G_1) = \Phi(\widetilde{U}_1, \widetilde{G}_1)$ and $(U_2, G_2) = \Phi(\widetilde{U}_2, \widetilde{G}_2)$.

Lemma 5.18: *There exists constants k_1 and k_2 depending on $\underline{d}, \beta_\infty, \alpha_\infty, \gamma_\infty, \sigma_\infty, a_\dagger,$, K_β, K_τ, K_Z and $||Z^*||_{\infty,(0,T)\times(0,a_\dagger)\times\Omega}$ for $Z = J, A$ such that for $t \in (0,T)$, one has:*

$$\frac{d}{dt}\big(||(U_1 - U_2)(t)||_{\mathcal{H}^2} + ||(G_1 - G_2)(t,\cdot)||_{2,\Omega}\big) \leq$$
$$k_1\big(||(U_1 - U_2)(t)||_{\mathcal{H}^2} + ||(G_1 - G_2)(t,\cdot)||_{2,\Omega}\big)$$
$$+ k_2\big(||(\widetilde{U}_1 - \widetilde{U}_2)(t)||_{\mathcal{H}^2} + ||(\widetilde{G}_1 - \widetilde{G}_2)(t,\cdot)||_{2,\Omega}\big).$$

Proof: Let k_i, $i \geq 3$ be constants with the same property as k_1. The main differences with the proof of Lemma 5.6 are that there is also a term for the transmission of infection, the equation of infected individuals are more structured, and there is the equation for G . Let us focus on the equations for J_s , J_i and G.

First multiply the equation corresponding to $J_{s,1} - J_{s,2}$ by $J_{s,1} - J_{s,2}$, and integrating on Ω, it follows:

$$\frac{1}{2}(\partial_t + \partial_a) \int_\Omega (J_{s,1} - J_{s,2})^2 \, dx + \int_\Omega d_J |\nabla (J_{s,1} - J_{s,2})|^2 \, dx$$
$$+ \int_\Omega (\mu_J(\widetilde{\mathbb{P}}_1) + \tau(\widetilde{A}_1) + \sigma_{\widetilde{J}_1} + \gamma_J \widetilde{G}_1)(J_{s,1} - J_{s,2})^2 \, dx$$
$$= - \int_\Omega J_{s,2}(J_{s,1} - J_{s,2})\big((\mu_J(\widetilde{\mathbb{P}}_1) - \mu_J(\widetilde{\mathbb{P}}_2)) + (\tau(\widetilde{A}_1) - \tau(\widetilde{A}_2))$$
$$+ (\widetilde{\sigma_{J_1}} - \widetilde{\sigma_{J_2}}) + \gamma_J(\widetilde{G}_1 - \widetilde{G}_2)\big) \, dx.$$

Then, integrating in a on $(0, a_\dagger)$, one gets:

$$\frac{1}{2}(\partial_t)||(J_{s,1} - J_{s,2})(t,\cdot,\cdot)||^2_{\mathcal{H}_{J_s}} + \underline{d}||\nabla (J_{s,1} - J_{s,2})(t,\cdot,\cdot)||^2_{\mathcal{H}_{J_s}}$$
$$\leq I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t),$$

with I_1, I_2 and I_3 as in the proof of Lemma 5.6 and:

$$I_4(t) = k_5 \int_{(0,a_\dagger)\times\Omega} |J_{s,1} - J_{s,2}| \cdot |\widetilde{\sigma_{J_1}} - \widetilde{\sigma_{J_2}}| dxda,$$

thus using the positivity result Lemma 5.17:

$$I_4(t) \leq k_6 \int_{(0,a_\dagger)\times\Omega} |\mathbb{J}_{s,1} - \mathbb{J}_{s,2}|(|\widetilde{J}_{n,1} - \widetilde{J}_{n,2}| + |\widetilde{A}_{n,1} - \widetilde{A}_{n,2}| + |\widetilde{J}_{c,1} - \widetilde{J}_{c,2}| + |\widetilde{A}_{c,1} - \widetilde{A}_{c,2}|)dxda$$
$$\leq k_7(||(U_1 - U_2)(t)||_{\mathcal{H}^2} + ||(\widetilde{U}_1 - \widetilde{U}_2)(t)||_{\mathcal{H}^2}.$$

Moreover, one has:

$$\begin{aligned} I_5(t) &= k_8 \int_{(0,a_\dagger)\times\Omega} |J_{s,1} - J_{s,2}| \cdot |\widetilde{G}_1 - \widetilde{G}_2| dxda \\ &\leq k_9\big(||(U_1 - U_2)(t)||_{\mathcal{H}^2} + ||(G_1 - G_2)(t,\cdot)||_{2,\Omega} \\ &\qquad + ||(\widetilde{U}_1 - \widetilde{U}_2)(t)||_{\mathcal{H}^2} + ||(\widetilde{G}_1 - \widetilde{G}_2)(t,\cdot)||_{2,\Omega}\big). \end{aligned}$$

Now, focus on the equation for $J_{i,1} - J_{i,2}$. Multiply the equation for $J_{i,1} - J_{i,2}$ by $J_{i,1} - J_{i,2}$ and integrating on Ω, it follows :

$$\begin{aligned} &\frac{1}{2}(\partial_t + \partial_a + \partial_b) \int_\Omega (J_{i,1} - J_{i,2})^2 dx \\ &+ \int_\Omega d_J(t,x) \left(\nabla(J_{i,1} - J_{i,2})\right)^2 dx + \int_\Omega \Big(\mu_J(\widetilde{\mathbb{P}}_1) + \tau(\widetilde{\mathbb{A}}_1)\Big) (J_{i,1} - J_{i,2})^2 \, dx = \\ &\qquad - \int_\Omega J_{i,2} \, (J_{i,1} - J_{i,2}) \Big(\mu_J(\widetilde{\mathbb{P}}_1) - \mu_J(\widetilde{\mathbb{P}}_2) + \tau(\widetilde{\mathbb{A}}_1) - \tau(\widetilde{\mathbb{A}}_2)\Big) \, dx \end{aligned}$$

Integrating in a on $(0, a_\dagger)$ and in b on $(0, b_\dagger)$ and using initial condition, one gets :

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big(\|J_{i,1}(t,.,.) - J_{i,2}(t,.,.)\|^2_{\mathcal{H}_{J_i}} \\ &\qquad + \underline{d} \, \|\nabla(J_{i,1} - J_{i,2})(t,.,.)\|^2_{\mathcal{H}_{J_i}} \big) \\ &\qquad\qquad \leq I_6(t) + I_7(t) + I_8(t) + I_9(t) \end{aligned}$$

where

- $I_6(t) = k_{10} \int_{(0,a_\dagger)\times(0,b_\dagger)\times\Omega} |J_{i,1} - J_{i,2}| \left|\widetilde{\mathbb{P}}_1(t,x) - \widetilde{\mathbb{P}}_2(t,x)\right| dadbdx$
- $I_7(t) = k_{11} \int_{(0,a_\dagger)\times(0,b_\dagger)\times\Omega} |J_{i,1} - J_{i,2}| \left|\widetilde{\mathbb{A}}_1(t,x) - \widetilde{\mathbb{A}}_2(t,x)\right| dadbdx$
- $I_8(t) = k_{12} \int_{(0,a_\dagger)\times(0,b_\dagger)\times\Omega} |J_{i,1} - J_{i,2}| \left|\sigma_J(\widetilde{\mathbf{U}}_1) - \sigma_J(\widetilde{\mathbf{U}}_2)\right| dadbdx$
- $I_9(t) = k_{13} \int_{(0,a_\dagger)\times(0,b_\dagger)\times\Omega} |J_{i,1} - J_{i,2}| \left|\widetilde{G}_1(t,x) - \widetilde{G}_2(t,x)\right| dadbdx$

I_6 and I_7 are estimated as in the proof of Lemma 5.6. I_8 and I_9 are estimated as I_4 and I_5 in the inequation for $J_{s_1} - J_{s,2}$

Same work on $G_1 - G_2$ gives:

$$\frac{1}{2}\partial_t(G_1-G_2)^2 = \Big(\int_0^{a_\dagger}\Upsilon.(\widetilde{U}_1-\widetilde{U}_2)da\Big)(G_1-G_2) \\ -\Big(\int_0^{a_\dagger}\Upsilon.(\widetilde{U}_1G_1-\widetilde{U}_2G_2)da\Big)\cdot(G_1-G_2)-\delta(G_1-G_2)^2,$$

$$\frac{1}{2}\partial_t(G_1-G_2)^2 \le \Big(\int_0^{a_\dagger}\Upsilon.(\widetilde{U}_1-\widetilde{U}_2)da\Big)(G_1-G_2) \\ +k_3\Big(\int_0^{a_\dagger}|\widetilde{U}_1-\widetilde{U}_2|\cdot|G_1-G_2|\;da+|G_1-G_2|^2\Big),$$

so integrating in x and using Holder and Cauchy-Schwarz inequalities and assumption Hyp 3.1 :

$$\frac{1}{2}\partial_t(G_1-G_2)^2 \le k_4\big(||(\widetilde{U}_1-\widetilde{U}_2)(t)||_{\mathcal{H}^2}+||(G_1-G_2)(t,\cdot)||_{2,\Omega}\big).$$

Similar work on equations for A_s and A_i gives Lemma 5.18. □

Similarly to Lemma 5.7, there is:

Lemma 5.19: *The mapping Φ is a strict contraction on $\mathcal{H}^2(\tau^*)\times L^2((0,\tau^*)\times\Omega)$ with τ^* small enough, i.e. there exists $\rho(\tau^*)<1$ such that:*

$$\big(||(U_1-U_2)(t)||_{\mathcal{H}^2}+||(G_1-G_2)(t,\cdot)||_{2,\Omega}\big) \\ \le \rho(\tau^*)\big(||(\widetilde{U}_1-\widetilde{U}_2)(t)||_{\mathcal{H}^2}+||(\widetilde{G}_1-\widetilde{G}_2)(t,\cdot)||_{2,\Omega}\big).$$

The end of the proof of Theorem 3.3 is similar to those of Theorem 2.6. □

6. Conclusion

Our objective in this paper was to build and study a deterministic mathematical model describing the propagation of a virus within a structured host population.

In the existing litterature on propagation of deseases, various features have been identified, which govern the propagation of a given virus ([4],[5],[6],[7],[16],[17],[22],[24],[26], [29],[30],[32]). Motivated by these previous works, our model has been built so as to take into account three major features which are important in the specific case of the Puumala hantavirus - bank vole system in Europe.

(1) Maturation of juveniles depending on the density of adult individuals. This leads to a stage structure with juvenile and mature individuals, and a chronological age structure on each stage;
(2) transmission rates depend on the time elasped since infection. Hence, in addition to the usual stage structure between susceptible and infected, we use a third chronological variable: the age of the disease;
(3) a spatial structure is considered for the host population.

Our new model combines all the structures into a single strongly structured system. In this model, we also considered three other assumptions, based on the Puumala hantavirus - bank vole system: (1) the virus is benign in the host population, (2) virus propagation occurs through direct transmission from infective to susceptible individuals and through indirect contamination of susceptibles via the contaminated environment, and (3) the dispersion rates are discontinuous. Most of these features may be of some interest for lots of epidemiological systems.

We first analysed a demographic model for a closed population with chronological age and spatial structure and we derived here a mathematical analysis of this model. We get global existence, uniqueness and global boundedness results.
Then we studied an epidemic model with a continuous structure in age of infection and direct and indirect transmission. Global existence, uniqueness and global boundedness results was also performed in this case.
Lastly, we looked at a model including the transmission of the virus to Human populations with possible lethal consequences, we also had global existence, uniqueness and global boundedness results.
The next step of this work will be to take into account some others biological assumptions, such as density dependencies for mortality rates or maturation rates (that should be decreasing toward adults density because of adults pressure on maturation in the Puumala hantavirus - bank vole system). In the same way, density dependent diffusion rates may also be of some interest: diffusion is favoured by high population densities because of territorial reasons.

Hence, we obtained a well posed model taking into account many significant features. We believe this model can be very useful for diseases propagations studies. Unfortunately this system seems to be too complex to allow qualitative studies mathematically; but numerical simulations may give lots of information of biological interest and may be compared with data collected on the fields. A difficult point simulating this system is its very strong structured character that leads to a $1 + 1 + 1 + 2$ dimensions problem, but parameters t, a and b are basically the time, thus numerical simulations can be related to 3D ones. Some parameters are quantifiable with field data, but others will be more difficult to estimate. However, qualitative studies are possible and sensitivity studies can help to determine importance of parameters poorly known; we will focus on this.

Numerical simulations (C. Wesley et al. [27], W. Wang et al. [28] and references therein) have studied the hantavirus system, without spatial strucure, in the case when the parameters depend periodically with time. These works show that the demography and the propogation of desesases change dramatically when the coefficients differ from their average. This is especially true as far as propagation to a human population is concerned. In this direction, our model investigates the effect of the *spatial* variations of the coefficients (as opposed to temporal variations). The corresponding numerical study is works in progress.

References

[1] N.D. Alikakos, *An application of the invariance principle to reaction-diffusion equations*, J. Diff. Eq., **33** (1979), 201-225.

[2] S. Anita. *Analysis and Control of Age-Dependent Population Dynamics*, Kluwer Academic Publishers, Dordrecht, 2000.

[3] K. Berthier, M. Langlais, P. Auger and D. Pontier, *Dynamics of a feline virus with two transmission modes within exponentially growing host populations*, Proc. R. Soc. London, séries B, **267** (2000), 2049-2056.

[4] F. Brauer and C. Castillo Chavez, *Mathematical Models in Population Biology and Epidemiology*, Texts in Applied Mathematics, Vol. 40. Springer Verlag, 2001.

[5] G. Bujalska, *Life history consequences of territoriality in the bank vole*, in Evolution of Life Histories : Theories and Patterns from Mammals, Ed M.S. Boyce, pp. 75-90, Yale University Press, Yale (1988).

[6] S. Busenberg and K. C. Cooke, *Vertically transmitted diseases*, Biomathematics Volume 23, Springer Verlag, New York, 1993.

[7] O. Diekmann and J.A.P. Heesterbeek, *Mathematical Epidemiology of infectious diseases*, Mathematical and computational biology, J. Wiley & Sons, Chichester 2000.

[8] W.E. Fitzgibbon and J.J. Morgan, *Diffractive diffusion systems with locally defined reactions*, Evolution Equation, ed. G. Goldstein et al., M. Dekker, New York, 177-186 (1994).

[9] W.E. Fitzgibbon, S. Hollis and J.J. Morgan, *Steady-state solutions for balanced reaction diffusion systems on heterogeneous domains*, J. Diff. and Integral Equations, **12** (1999), 225-241.

[10] W.E. Fitzgibbon, M. Langlais and J.J. Morgan, *A mathematical model of the spread of Feline Leukemia Virus (FeLV) through a highly heterogeneous spatial domain*, SIAM J. Math. Analysis, **33** (2001), 570-588.

[11] W.E. Fitzgibbon, M. Langlais and J.J. Morgan, *A reaction-diffusion system modelling direct and indirect transmission of a disease*, DCDS-B, 893-910, 2004.

[12] W.E. Fitzgibbon, M. Langlais and J.J. Morgan, *A mathematical model for indirectly transmitted disease*, Manuscript, 2005.

[13] M.G. Garroni and M. Langlais, *Age dependent population diffusion with external constraints*, J. Math. Biol. **14** (1982), 77-94.

[14] M.E. Gurtin, *A system of equations for age dependent population diffusion*, J. Theoret. Biol., 40:389-392, 1973.

[15] M.E. Gurtin and R.C. Mac Camy. *Nonlinear age-dependent population dynamics.* Arch. Ration. Mech. Anal. 54, 1974, p. 281.

[16] M. Iannelli. 1994. *Mathematical Theory of Age-Structured Population Dynamics.* Applied Mathematics Monographs No 7, C.N.R. Pisa.

[17] Kostova, T., Li, J. and Friedman, M. (1999). Two models for competition between age classes. *Math. Biosciences* 157:65-89.

[18] O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Translations of mathematical Monographs, **23** (1968).

[19] M. Langlais, *A nonlinear problem in age dependent population diffusion*, SIAM J. Math Anal **16** (1985), 510-529.

[20] M. Langlais, *Large time behavior in a nonlinear age-dependant population dynamics problem with spatial diffusion*, J. Math. Biol., **26** (1988) :319-346.

[21] J.M. Naulin, *Simulations numriques et analyse mathmatique de modles de viroses dans des populations structures* (In French), PhD Thesis, Univ. Bordeaux I, 2001.

[22] F. Sauvage, *Synergy between the dynamics of the red bank vole populations and the shedding of the hantavirus Puumala: description of the emergence mechanism of* Nephropathia Epidemica (In French), PhD Thesis, Univ. C.Bernard Lyon I, 2004.

[23] F. Sauvage, M. Langlais, N.G. Yoccoz and D. Pontier, *Modelling Hantavirus in cyclic bank voles: the role of indirect transmission on virus persistence*, Journal of Animal Ecology, **72** (2003), 1-13.

[24] F. Sauvage, M. Langlais and D. Pontier, *Predicting the emergence of human hantavirus disease using a combination of viral dynamics and rodent demographic patterns*, Manuscript, 2005.

[25] J. Smoller, *Shock wawes and reaction diffusion equations.* Springer Verlag New York, 1983.

[26] H. R. Thieme. *Mathematics in population biology.* Princeton University Press, Princeton, 2003.

[27] C. Wesley, L.Allen and M. Langlais, *Models for the Spread and Persistence of Hantavirus In fection in Rodents with Direct and Indirect Transmission*, Mathematical Biosciences and Engineering, Volume 7, Number 1, January 2010 199215

[28] W. Wang and X-Q Zhao, Threshold dynamics for compartmental epidemic models in periodic environments, J. Dyn. Diff. Eqns., 20 (2008), 699717.

[29] G.F. Webb. 1985. *Theory of Nonlinear Age-Dependent Population Dynamics*, Marcel Dekker, New York.

[30] C. Wolf, *A mathematical model for the propagation of a hantavirus in structured populations*, DCDS-B, Vol.4, Number 4 (2004), 1065-1089.

[31] C. Wolf, *Modelling and mathematical analysis of the propagation of a microparasite in a structured population in heterogeneous environment* (in French), PhD Thesis, Univ. Bordeaux I, 2005.

[32] C. Wolf, F. Sauvage, D. Pontier and M. Langlais, *Multi-Patch Epidemic Model with Periodic Demography, Direct and Indirect Transmission and Variable Maturation Rate*, Mathematical Population studies, volume 13, Number 3 july-September 2006, 153-177.