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PDEs FOR ISING CORRELATION FUNCTIONS ON THE CYLINDER

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In this note, I review some recent results concerning spin-spin Ising correlators on a cylinder: lattice form factor expansions, a scaling limit and nonlinear differential equations. In particular, the pair correlation function is shown to satisfy the same three PDEs of the fourth order in the paramagnetic and ferromagnetic phase. I derive determinant representations of these equations and their multilinear Hirota form. I also conjecture an overdetermined system of PDEs, related to the isomonodromy problem for the Dirac equation on the cylinder, for which n-point functions of the Ising model should give a solution.

1. Form factor expansions and scaling limit

The two-dimensional isotropic Ising model on a $M \times N$ square lattice with the periodic boundary conditions is characterized by the Hamiltonian

$$H[\sigma] = -K \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \sigma(i,j) \left[\sigma(i+1,j) + \sigma(i,j+1) \right],$$

where K denotes the coupling constant and the spins $\sigma(i,j)$ take the values ± 1. The main subject of the present paper is the corresponding two-point correlation function

$$\langle \sigma(0,0)\sigma(r_x,r_y) \rangle = Z^{-1} \sum_{[\sigma]} \sigma(0,0)\sigma(r_x,r_y)\, e^{-H[\sigma]}.$$

More precisely, we are interested in two limiting cases – the Ising model on the plane ($M \to \infty$, $N \to \infty$) and on the cylinder ($M \to \infty$, $N = \text{const}$).

The first situation was extensively studied by different authors for more than twenty years, starting from the pioneering work of Montroll, Potts and Ward. In their paper,[8] correlation functions were expressed via determinants of Toeplitz matrices with the dimension depending on the distance between correlating spins. Such representations, however, can be used in practice only for small separation of spins. To extract from them the long-distance behaviour of correlators, further considerable theoretical efforts were needed – the final answer was obtained only in the early eighties.[9,12] The explicit formulae for the correlation functions are different in the ferromagnetic and paramagnetic phase and have the form of the lattice form factor expansions:

$$\langle\sigma(0,0)\sigma(r_x,r_y)\rangle = \begin{cases} \xi \sum\limits_{n=0}^{\infty} g_{2n}, & s>1\,, \\ \xi \sum\limits_{n=0}^{\infty} g_{2n+1}, & s<1\,, \end{cases}$$

$$g_n = \frac{1}{n!(2\pi)^n} \int\limits_{-\pi}^{\pi} \ldots \int\limits_{-\pi}^{\pi} dq_1 \ldots dq_n \, F_n^2[q] \prod_{j=1}^{n} \frac{\mathrm{e}^{-|r_x|\gamma(q_j)+ir_y q_j}}{\sinh\gamma(q_j)}, \quad g_0 = 1.$$

Here $\gamma(q)$ is the positive root of the lattice dispersion relation

$$\cosh\gamma(q) = s + s^{-1} - \cos q, \qquad s = \sinh 2K\,.$$

The form factor $F_n[q]$ is given by

$$F_n[q] = \prod_{1\leq i<j\leq n} \frac{\sin\frac{q_i-q_j}{2}}{\sinh\frac{\gamma(q_i)+\gamma(q_j)}{2}}, \quad F_1 = 1, \tag{1.1}$$

and $\xi = \left|1 - s^{-4}\right|^{\frac{1}{4}}$ denotes the squared spontaneous magnetization.

A corresponding problem on the cylinder has a number of applications in mesoscopic physics and quantum field theory at nonzero temperature (or in a finite volume). However, technically it is even more complicated. Because of this it had remained unsolved[a] for a long time, until the recent ingenious calculation of Bugrij.[1] His results clearly demonstrate the finite-size effects and are very similar to the form factor expansions on the plane:

$$\langle\sigma(0,0)\sigma(r_x,r_y)\rangle = \begin{cases} (\xi\cdot\xi_T)e^{-|r_x|/\Lambda} \sum\limits_{n=0}^{[N/2]} g_{2n}, & s>1\,, \\ \\ (\xi\cdot\xi_T)e^{-|r_x|/\Lambda} \sum\limits_{n=0}^{[N/2]} g_{2n+1}, & s<1\,, \end{cases} \tag{1.2}$$

$$g_n = \frac{\mathrm{e}^{-n/\Lambda}}{n!N^n} \sum_{[q]}^{(b)} F_n^2[q] \prod_{j=1}^{n} \frac{\mathrm{e}^{-|r_x|\gamma(q_j)+ir_y q_j-\nu(q_j)}}{\sinh\gamma(q_j)}, \quad g_0 = 1. \tag{1.3}$$

Here $\sum_{[q]}^{(b)}$ stands for the summation over the discrete bosonic quasimomenta ($q_j = 2\pi k/N$, $k = 0, 1, \ldots, N-1$), while ξ_T, Λ and $\nu(q)$ are determined by

$$\ln\xi_T = \frac{N^2}{2\pi^2} \int\limits_0^{\pi} \int\limits_0^{\pi} \frac{dp\; dq\, \gamma'(p)\gamma'(q)}{\sinh\left(N\gamma(p)\right)\sinh\left(N\gamma(q)\right)} \ln\left|\frac{\sin\left((p+q)/2\right)}{\sin\left((p-q)/2\right)}\right|, \tag{1.4}$$

[a]Actually, there exists one wrong conjecture in the scaling limit. Though it was rewritten by many authors, the origins lie most likely in Ref. 5.

$$\Lambda^{-1} = \frac{1}{\pi}\int_0^\pi dp\, \ln\coth\left(N\gamma(p)/2\right), \tag{1.5}$$

$$\nu(q) = \frac{1}{\pi}\int_0^\pi \frac{dp\,\left(\cos p - \mathrm{e}^{-\gamma(q)}\right)}{\cosh\gamma(q) - \cos p}\ln\coth\left(N\gamma(p)/2\right). \tag{1.6}$$

In the thermodynamic limit, when $N \to \infty$ and $s \neq 1$, the summation in (1.3) transforms into the integration, the integrals (1.4)-(1.6) vanish and formulae for the infinite lattice are easily recovered. When N is kept finite, the sums (1.2) and (1.3) contain a finite number of terms, and for small N the result can be even compared with, say, the transfer matrix calculations for the N-rows Ising chain.

In what follows, we will consider the scale-invariant limit of (1.2)-(1.6). It implies

$$\gamma(0) \to 0, \quad r_x \to \infty, \quad r_y \to \infty, \quad N \to \infty,$$

$$\gamma(0)r_x \to mx = \text{const}, \quad \gamma(0)r_y \to my = \text{const}, \quad \gamma(0)N \to m\beta = \text{const},$$

with m, x, y, β being the dimensional mass, coordinates and the length of the base of the cylinder. Then (1.2)-(1.6) are reduced to

$$\langle\sigma(0,0)\sigma(r_x,r_y)\rangle = \begin{cases} \xi\cdot\tilde{\xi}_T(\beta)\, e^{-m|x|/\tilde{\Lambda}(\beta)}\tau_-(x,y,\beta), & s>1\,, \\ \xi\cdot\tilde{\xi}_T(\beta)\, e^{-m|x|/\tilde{\Lambda}(\beta)}\tau_+(x,y,\beta), & s<1\,, \end{cases} \tag{1.7}$$

$$\tau_-(x,y,\beta) = \sum_{n=0}^{\infty}\tilde{g}_{2n}, \quad \tau_+(x,y,\beta) = \sum_{n=0}^{\infty}\tilde{g}_{2n+1},$$

$$\tilde{g}_n = \frac{1}{n!}\sum_{[l]}\prod_{j=1}^{n}\frac{\exp\left(-m|x|\cosh\theta_j + imy\sinh\theta_j - \tilde{\nu}(\theta_j,\beta)\right)}{m\beta\cosh\theta_j}\prod_{1\le i<k\le n}\tanh^2\left(\frac{\theta_i-\theta_k}{2}\right),$$

$$\tilde{g}_0 = 1, \tag{1.8}$$

where the relativistic notation

$$\sinh\theta_j = \frac{2\pi}{m\beta}\,l_j, \qquad l_j = -\infty,\ldots,0,\ldots,\infty$$

is used. The scaled cylindrical parameters $\tilde{\xi}_T(\beta)$, $\tilde{\Lambda}(\beta)$ and $\tilde{\nu}(\theta_j,\beta)$ are given by

$$\ln\tilde{\xi}_T(\beta) = \left(\frac{m\beta}{2\pi}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sinh\theta_1\sinh\theta_2\; d\theta_1 d\theta_2}{\sinh\left(m\beta\cosh\theta_1\right)\sinh\left(m\beta\cosh\theta_2\right)}$$
$$\times\;\ln\left|\coth\frac{\theta_1-\theta_2}{2}\right|, \tag{1.9}$$

$$\tilde{\Lambda}^{-1}(\beta) = \frac{m\beta}{2\pi} \int_{-\infty}^{\infty} \frac{\sinh^2 \theta \, d\theta}{\sinh (m\beta \cosh \theta)} , \tag{1.10}$$

$$\tilde{\nu}(\theta, \beta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\theta'}{\cosh(\theta' - \theta)} \ \ln \coth \frac{m\beta \cosh \theta'}{2}. \tag{1.11}$$

The above expressions for the scaled 2-point correlation function on the cylinder were obtained as a special case of the formulae (1.2)-(1.6). However, more recently Fonseca and Zamolodchikov have developed a very elegant indirect approach[2] that allows to get the results (1.7)-(1.11) without solving the initial discrete model. Using some field-theoretic arguments, these authors have found spin form factors (this is almost equivalent to a calculation of the 2-point function) in the scaling limit. Even more, they have conjectured (or, better to say, announced without a proof) the formulae for *all* spin matrix elements. Consequently, the computation of an arbitrary scaled n-point function becomes a simple exercise. It would be very interesting to obtain lattice versions of the results of Ref. 2. We plan to address this problem in the near future.

Finally, we remark that on the plane, when $\beta \to \infty$, the above formulae are reduced to the well-known result[9,11]

$$\langle \sigma(0,0)\sigma(r_x, r_y)\rangle = \begin{cases} \xi \sum\limits_{n=0}^{\infty} \tilde{g}_{2n} \quad = \xi \cdot \tau_-(r), & s > 1\,, \\ \\ \xi \sum\limits_{n=0}^{\infty} \tilde{g}_{2n+1} = \xi \cdot \tau_+(r), & s < 1\,, \end{cases} \tag{1.12}$$

$$\tilde{g}_n = \frac{1}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_n}{2\pi} \prod_{j=1}^{n} \mathrm{e}^{-m|x|\cosh\theta_j + imy \sinh\theta_j}$$

$$\times \prod_{1 \le i < k \le n} \tanh^2 \left(\frac{\theta_i - \theta_k}{2} \right). \tag{1.13}$$

It is easy to see that the functions $\tilde{g}_n$ depend only on the combination $r = \sqrt{x^2 + y^2}$. On the cylinder, however, such rotational invariance is obviously broken.

2. Differential equations for 2-point functions

Perhaps the most exciting feature of Ising correlators on the plane in the scale-invariant limit is their connection with integrable differential equations. For example, the combination

$$\eta(r) = \frac{\tau_-(2r) - \tau_+(2r)}{\tau_-(2r) + \tau_+(2r)}$$

of functions (1.12) is a solution of a particular case of the Painlevé III equation[11]

$$\eta'' = \frac{1}{\eta}\,(\eta')^2 - \frac{1}{r}\,\eta' + \eta^3 - \eta^{-1}.$$

Moreover, logarithmic derivatives of correlation functions

$$\zeta(t) = t\frac{d\,\ln\tau_\pm(t)}{dt}$$

can be shown to satisfy the same equation (closely related to Painlevé V)

$$(t\zeta'')^2 = 4\,(t\zeta'-\zeta)^2 - 4\,(\zeta')^2\,(t\zeta'-\zeta) + (\zeta')^2 \tag{2.1}$$

in both ferromagnetic and paramagnetic phase.[7,10]

As one might expect, similar results can also be obtained in the case of the cylinder. This time, however, instead of ordinary differential equations we obtain equations in partial derivatives. To write them, it is convenient to introduce the complex coordinates

$$z = -\,\frac{|x|-iy}{2}, \quad \bar{z} = -\,\frac{|x|+iy}{2}\,,$$

and the following combinations of functions (1.7):

$$\varphi = \ln\frac{\tau_- + \tau_+}{\tau_- - \tau_+}, \quad \psi = \ln\left(\tau_-^2 - \tau_+^2\right).$$

These new functions satisfy four relations[6]

$$\begin{gathered}\partial_{z\bar{z}}\varphi = \tfrac{m^2}{2}\sinh 2\varphi, \quad \partial_{z\bar{z}}\psi = \tfrac{m^2}{2}(1-\cosh 2\varphi),\\ \partial_{zz}\psi = -\left(\partial_z\varphi\right)^2, \quad \partial_{\bar{z}\bar{z}}\psi = -\left(\partial_{\bar{z}}\varphi\right)^2.\end{gathered} \tag{2.2}$$

The proof of this statement is based on two simple observations:

- φ, ψ can be written in terms of the determinants of some infinite-dimensional matrices

  $$\varphi = \ln|1+\widehat{K}| - \ln|1-\widehat{K}|, \qquad \psi = \ln|1+\widehat{K}| + \ln|1-\widehat{K}|.$$

  Their elements have the form

  $$\begin{aligned}K_{ij} = \frac{2\,e^{-\tilde{\nu}(\theta_i,\beta)/2}}{\sqrt{1+e^{-2\theta_i}}}\cdot\frac{2\,e^{-\tilde{\nu}(\theta_j,\beta)/2}}{\sqrt{1+e^{-2\theta_j}}}&\\ \times\frac{\exp\left\{\frac{z}{2}(e^{\theta_i}+e^{\theta_j}) + \frac{\bar{z}}{2}(e^{-\theta_i}+e^{-\theta_j})\right\}}{e^{\theta_i}+e^{\theta_j}}&.\end{aligned} \tag{2.3}$$

  Here the functions $\tilde{\nu}(\theta,\beta)$ are the same as defined earlier, but for θ_j another definition is used: $\sinh\theta_j = \frac{2\pi j}{m\beta}$, $\;j = -\infty,\ldots,0,\ldots,\infty$.
- The N-soliton solution of the sinh-Gordon equation (first equation in (2.2)) depends on $2N$ parameters $\{\mu_1\ldots\mu_N, a_1\ldots a_N\}$ and is usually written as

  $$\varphi^{(N)} = \ln|1+K^{(N)}| - \ln|1-K^{(N)}|,$$

  $$K_{ij}^{(N)} = \frac{\sqrt{\mu_i\mu_j}}{\mu_i+\mu_j}\,\sqrt{X_iX_j}, \qquad X_i = a_i\,e^{\mu_i z+\mu_i^{-1}\bar{z}}, \qquad i,j = 1,\ldots,N.$$

Our function φ corresponds to an infinite number of solitons and a particular choice of their parameters:

$$\mu_i = e^{\theta_i}, \qquad a_i = 2\, e^{-\tilde{\nu}(\theta_i,\beta)} / \cosh\theta_i.$$

Three remaining relations can be proved by a straightforward differentiation, using the special form of the matrix elements (2.3).

2.1. *Determinant representations*

It is very desirable to derive differential equations for the functions $\tau_\pm$ separately. They can be proved to be more suitable for various generalizations. In particular, they allow to avoid some difficulties in the conformal limit, where the correlators $\tau_\pm$ are expected to be equal and the function φ should tend to infinity. An interesting feature of these equations is that they turn out to be the same in both phases.

One of the equations for $\tau_\pm$ on the cylinder has been already obtained in Ref. 6. More precisely, it was shown that the function

$$u = \ln \tau_\pm \tag{2.4}$$

in both cases satisfies

$$u_{z\bar{z}}(u_{zz\bar{z}\bar{z}} + 2u_{z\bar{z}}^2 - u_{z\bar{z}}) = m^2(u_{zz\bar{z}}u_{z\bar{z}\bar{z}} - u_{zz}u_{\bar{z}\bar{z}}). \tag{2.5}$$

In a completely similar manner, one can derive two more relations – the equation

$$\begin{aligned}(u_{zzz\bar{z}} + 2u_{zz}u_{z\bar{z}})(u_{z\bar{z}}^2 - u_{zz}u_{\bar{z}\bar{z}}) = \\ = (u_{zz\bar{z}}u_{z\bar{z}} - u_{z\bar{z}\bar{z}}u_{zz})u_{zz\bar{z}} - (u_{zz\bar{z}}u_{\bar{z}\bar{z}} - u_{z\bar{z}\bar{z}}u_{z\bar{z}})u_{zzz}\end{aligned} \tag{2.6}$$

and its complex conjugate.

The change of variables (2.4) makes the derivation of the last two relations much simpler (cf Ref. 6). However, the final answer appears to be more beautiful in terms of the correlators $\tau_\pm$ themselves. Namely, the equation (2.5) can be rewritten as

$$\begin{vmatrix} \tau & \tau_z & \tau_{zz} \\ \tau_{\bar{z}} & \tau_{z\bar{z}} & \tau_{zz\bar{z}} \\ \tau_{\bar{z}\bar{z}} & \tau_{z\bar{z}\bar{z}} & \tau_{zz\bar{z}\bar{z}} \end{vmatrix} = m^2 \begin{vmatrix} \tau_{z\bar{z}} & \tau_z & \tau_{zz} \\ \tau_{\bar{z}} & \tau & \tau_z \\ \tau_{\bar{z}\bar{z}} & \tau_{\bar{z}} & \tau_{z\bar{z}} \end{vmatrix}, \tag{2.7}$$

whereas the equation (2.6) and its conjugate transform into

$$\begin{vmatrix} \tau & \tau_z & \tau_{\bar{z}} & \tau_{z\bar{z}} \\ \tau_z & \tau_{zz} & \tau_{z\bar{z}} & \tau_{zz\bar{z}} \\ \tau_{\bar{z}} & \tau_{z\bar{z}} & \tau_{\bar{z}\bar{z}} & \tau_{z\bar{z}\bar{z}} \\ \tau_{zz} & \tau_{zzz} & \tau_{zz\bar{z}} & \tau_{zzz\bar{z}} \end{vmatrix} = \begin{vmatrix} \tau & \tau_z & \tau_{\bar{z}} & \tau_{z\bar{z}} \\ \tau_z & \tau_{zz} & \tau_{z\bar{z}} & \tau_{zz\bar{z}} \\ \tau_{\bar{z}} & \tau_{z\bar{z}} & \tau_{\bar{z}\bar{z}} & \tau_{z\bar{z}\bar{z}} \\ \tau_{\bar{z}\bar{z}} & \tau_{z\bar{z}\bar{z}} & \tau_{\bar{z}\bar{z}\bar{z}} & \tau_{z\bar{z}\bar{z}\bar{z}} \end{vmatrix} = 0. \tag{2.8}$$

2.2. *Multilinear Hirota form*

Note that the starting point in the derivation of the equations (2.7) and (2.8) was actually their multisoliton solution[b]. Therefore, it is natural to suppose that they possess some integrable structure. In particular, their homogeneity suggests us to try to rewrite them in the Hirota form.

An usual Hirota operator is defined by its action on the "dot product" of functions

$$D_x f \cdot g = (\partial_{x_1} - \partial_{x_2}) f(x_1) g(x_2)\Big|_{x_1 = x_2 = x}.$$

However, since (2.7) is trilinear and (2.8) is quadrilinear, one should use Hietarinta's multilinear extension[3] of the Hirota formalism. To this end, in the n-linear case one introduces $\frac{n(n-1)}{2}$ operators

$$D_x^{ij} = \partial_{x_i} - \partial_{x_j}$$

acting on n-tuple products of functions. Of course, only $(n-1)$ of these operators are linearly independent. We may also consider the "symmetric" basis

$$T_x^m = \sum_{k=0}^{n-1} e^{\frac{2\pi i k m}{n}} \partial_{x_k}, \qquad m = 1, \ldots, n-1.$$

For example, in the trilinear case we have only two operators

$$T_x = \partial_{x_1} + j\partial_{x_2} + j^2 \partial_{x_3}, \quad T_x^* = \partial_{x_1} + j^2 \partial_{x_2} + j\partial_{x_3},$$

with $j = e^{2\pi i/3}$. A straightforward computation shows that the equation (2.7) can be rewritten in terms of these operators in the following way:

$$\left[(T_z^3 - T_z^{*3})(T_{\bar z}^3 - T_{\bar z}^{*3}) + 9(T_z T_{\bar z}^* - T_z^* T_{\bar z})^2 \right] \tau \cdot \tau \cdot \tau = 0.$$

Two quadrilinear equations (2.8) can also be reduced to the Hirota form

$$D_z^{41} D_z^{42} D_z^{43} D_{\bar z}^{41} \left(D_z^{13} D_{\bar z}^{12} - D_z^{12} D_{\bar z}^{13} \right)^2 \tau \cdot \tau \cdot \tau \cdot \tau = 0,$$

$$D_{\bar z}^{41} D_{\bar z}^{42} D_{\bar z}^{43} D_z^{41} \left(D_z^{13} D_{\bar z}^{12} - D_z^{12} D_{\bar z}^{13} \right)^2 \tau \cdot \tau \cdot \tau \cdot \tau = 0,$$

but this time the nonsymmetric basis is more suitable.

Generally speaking, we were able to find the Hirota representation of the equations (2.7) and (2.8) due to their invariance under a gauge transformation $\tau \to e^{\alpha z + \gamma \bar z} \tau$. The requirement of such invariance was exactly a guiding principle in the construction of the multilinear Hirota formalism.[3]

[b]There is another interesting family of solutions: namely, every function satisfying the Helmholtz equation $\tau_{z\bar z} = m^2 \tau$ solves also (2.7) and (2.8).

3. Discussion

It should be mentioned that the PDEs for scaled correlation functions on the plane can be derived by two different approaches (this resembles the situation described in Section 1). The first one was developed by Wu, McCoy, Tracy and Barouch.[11] It uses the explicit formulae for correlators and, therefore, can be applied only to the 2-point case. The second approach does not need such explicitness and gives the equations for all n-point functions. It was constructed by Sato, Miwa and Jimbo[c] in their celebrated series of works on holonomic quantum fields.[10] These authors related Ising correlators in the scale-invariant limit to certain solutions of the Dirac equation with prescribed singular and asymptotic behaviour. The equations for correlation functions arise from monodromy preserving deformations of such solutions.

Our derivation is, in some sense, analogous to the WMTB computation. To construct an analog of the SMJ analysis, one must first establish the relation between the continuum version of the Ising model and the Dirac equation on the cylinder. It is most probably known and the same as in the plane case, though we have not found corresponding results in the literature. The second part of the method – a derivation of deformation equations – can also be carried out: the paper, devoted to the isomonodromy problem on the cylinder, is currently in preparation. We report here some results, relevant to the Ising model, under a conjecture that the above mentioned relation exists.

Let us consider a scaled n-point correlation function $\tau(q_1, \bar{q}_1, \ldots, q_n, \bar{q}_n)$ in the ferromagnetic phase. Here $q_\mu = m z_\mu$, $\bar{q}_\mu = m\bar{z}_\mu$, and $\{z_\mu\}$, $\{\bar{z}_\mu\}$ denote scaled complex coordinates of correlating spins on the cylinder ($0 \leq \operatorname{Im} z_\nu < \beta/2$, $\nu = 1, \ldots, n$). Then the following system of equations holds:

$$d\ln\tau = \frac{1}{2}\mathrm{tr}\left(\eta\, dQ + \bar{\eta}\, d\bar{Q}\right) - \frac{1}{2}\,\mathrm{tr}\left\{\frac{1-G}{1+G}\left(\Lambda\, dQ - \bar{\Lambda}\, d\bar{Q}\right)\right\}, \tag{3.1}$$

$$dG = G[\Lambda, dQ] - [\bar{\Lambda}, d\bar{Q}]G, \tag{3.2}$$

$$d\eta = d\bar{Q} - \mathrm{diag}\left\{[\Lambda, dQ]\,\Lambda + G^{-1} d\bar{Q}\, G\right\}, \tag{3.3}$$

$$d\Lambda = S dQ + \mathrm{diag}\left\{G^{-1} d\bar{Q}\, G\right\} - G^{-1} d\bar{Q}\, G. \tag{3.4}$$

Here matrices G, Λ, η obey the relations

$$G^{-1} = G^T = \bar{G}, \quad \Lambda = \Lambda^T, \quad \mathrm{diag}\,\Lambda = 0,$$

η is diagonal, dQ and $d\bar{Q}$ are defined by

$$dQ = \begin{pmatrix} dq_1 & 0 & \cdots & 0 \\ 0 & dq_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & dq_n \end{pmatrix}, \qquad d\bar{Q} = \begin{pmatrix} d\bar{q}_1 & 0 & \cdots & 0 \\ 0 & d\bar{q}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d\bar{q}_n \end{pmatrix}.$$

[c] A good pedagogical exposition can also be found in Ref. 4.

For example, if $n = 2$, the matrix G can be parameterized as follows

$$G = \begin{pmatrix} \cosh\varphi & i\sinh\varphi \\ -i\sinh\varphi & \cosh\varphi \end{pmatrix}.$$

Then, from (3.2) we obtain

$$\Lambda = \begin{pmatrix} 0 & -i\,\frac{\partial\varphi}{\partial z} \\ -i\,\frac{\partial\varphi}{\partial z} & 0 \end{pmatrix},$$

with $z = q_1 - q_2$. The substitution of the last relation into the equation (3.4) leads to the sinh-Gordon equation (2.2a).

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