Some probabilistic aspects of representation theory

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1 Introduction

This note is concerned with interactions between representation theory of simple Lie algebras over $\mathbb{C}$ and certain random walks defined on lattices in Euclidean spaces. We will restrict ourself to the so called "ballot random walk". Let $B = (e_1, \ldots, e_n)$ be the standard basis of $\mathbb{R}^n$. The ballot random walk can be defined as the Markov chain $(W_\ell = X_1 + \cdots + X_\ell)_{\ell \geq 1}$ where $(X_\ell)_{\ell \geq 1}$ is a sequence of independent and identically distributed random variables taking values in the base B. We will review some results due to O'Connell [12],[13] giving the law of the random walk $W = (W_\ell)_{\ell \geq 1}$ conditioned to never exit the cone

$$\overline{C} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geq \cdots \geq x_n \geq 0\}$$

and its probability to never exit $\overline{C}$. This can be regarded as the Weyl chamber associated to the root system of type A_{n-1}. This is achieved in [12] by considering first a natural transformation $\mathcal{P}$ which associates to any path with steps in B a path in the Weyl chamber $\overline{C}$, next by checking that the image of the random walk W by this transformation is a Markov chain and finally by establishing that this Markov chain has the same law as W conditioned to never exit $\overline{C}$. The transformation $\mathcal{P}$ is based on the Robinson-Schensted correspondence which maps the words on the ordered alphabet $\mathcal{A}_n = \{1 < \cdots < n\}$ (regarded as finite paths in $\mathbb{R}^n$) on pairs of semistandard tableaux. Here we will expose simpler proofs of O'Connell's results based on the reflection principle of Gessel and Zeilberger [4].

The results of O'Connell can be extended to a large class of random walk defined from representations of Lie algebras (or their generalisations as Kac-Moody algebras). The corresponding transformation $\mathcal{P}$ is then defined in terms of the Littelmann path model. In the opposite direction, the probabilistic approach can be used to obtain asymptotic behavior of tensor product multiplicities. We refer the interested reader to [1], [8], [9] and [10] for a complete exposition.

The results exposed in the sequel are also related to the asymptotic representation theory of the symmetric group. Here, we refer the reader to the works of Kerov and Vershik [7] for a complete exposition.

The note is self containing, in particular all the probabilistic results we need are recalled. We begin in Section 2 by the study of the simple random walk in dimension 1 and obtain the probability this random walk remains nonnegative. In Section 3, we generalise the results obtained in Section 2 to higher dimension. We define the transformation $\mathcal{P}$ in Section 4 and prove that $\mathcal{P}(W)$ has the same law as W conditioned to stay in $\overline{C}$.

2 Random walk on the integers

2.1 An elementary non trivial problem

To define a random walk on $\mathbb{Z}$ with steps $+1$ or -1, we need first to choose two positive reals p_1 and p_{-1} such that $p_1+p_{-1}=1$ and then consider the Bernouilli random variable X with values in $\{\pm 1\}$ such that

$$\mathbb{P}(X=1)=p_1 \text{ and } \mathbb{P}(X=-1)=p_{-1}=1-p_1.$$

Next, we consider a sequence $(X_\ell)_{\ell\geq 1}$ of independent random variables with the same law as X. Such a sequence is said i.i.d. (for independent and identically distributed). The random walk starting at $\gamma\in\mathbb{Z}$ defined from $(X_\ell)_{\ell\geq 1}$ is the sequence $W=(W_\ell)_{\ell\geq 1}$ of random variables such that

$$W_\ell=\gamma+X_1+\cdots+X_\ell \text{ for any integer } \ell\geq 1.$$

The common mean

$$m=E(X_i)=p_1-p_{-1}$$

is called the drift of the random walk W.

Our random walk W is defined on the set of infinite trajectories $\Omega=\{(x_\ell)_{\ell\geq 1}\mid x_\ell\in\{\pm 1\}\}$ where x_ℓ is the elementary step at time ℓ. The set Ω is obtained as the direct limit of the finite sets $\Omega_\ell=\{w=(x_1,\ldots,x_\ell)\mid x_k\in\{\pm 1\}\}$, $\ell\geq 1$ which is a finite probability space.

Lemma 2.1.1 *For any $w=(x_1,\ldots,x_\ell)\in\Omega_\ell$ such that $\mu=x_1+\cdots+x_\ell$, we have*

$$p_w=p_{x_1}\times\cdots\times p_{x_\ell}=p_1^{\frac{\ell+\mu}{2}}p_{-1}^{\frac{\ell-\mu}{2}}:=\mathbf{p}_\ell^\mu$$

where $\mu=x_1+\cdots+x_\ell$. In particular, two trajectories with the same length ℓ and the same ends have the same probability.

Remark: By definition, $\mathbf{p}_\ell^\mu$ is thus the probability of any trajectory of length ℓ from 0 to μ. We will also set $\mathbf{p}_0^\mu:=p_1^{\frac{\mu}{2}}p_{-1}^{-\frac{\mu}{2}}$. Our notation is then multiplicative:

$$\mathbf{p}_\ell^\mu\times\mathbf{p}_{\ell'}^{\mu'}=\mathbf{p}_{\ell+\ell'}^{\mu+\mu'}$$

for any nonnegative integer ℓ and ℓ'.

Theorem 2.1.2 (Strong law of large numbers) *Let $(X_\ell)_{\ell\geq 1}$ be a i.i.d. sequence of random variables and $E(X_1)=m$. Then*

$$\lim_{\ell\to+\infty}\frac{W_\ell}{\ell}=m \text{ almost surely}$$

that is

$$\mathbb{P}\left(\lim_{\ell\to+\infty}\frac{W_\ell}{\ell}=m\right)=1.$$

In particular, when $m>0$, $\lim_{\ell\to+\infty}\mathbb{P}(W_\ell>0)=1$.

Proof. We give below for completion a simple proof of the law of large numbers extract from [11]. This proof uses the additional assumption that the random variables X_1 is such that $E(X_1^4)$ is finite. This assumption is obviously satisfied for the random variables we consider in this note (for they only take a finite number of values). We can and do assume $E(X_\ell) = 0$ for any $\ell \geq 1$ by replacing X_ℓ by $X_\ell - m$. Now

$$E\left(\left(\frac{X_1 + \cdots + X_\ell}{\ell}\right)^4\right) = \frac{1}{\ell^4} \sum_{i_1,i_2,i_3,i_4 \in \{1,\dots,\ell\}} E(X_{i_1} X_{i_2} X_{i_3} X_{i_4}).$$

Recall that $E(YZ) = E(Y)E(Z)$ for any two independent random variables Y and Z. Since $E(X) = E(X_k) = 0$ for $k = 1, \dots, \ell$, we have $E(X_{i_1} X_{i_2} X_{i_3} X_{i_4}) \neq 0$ if and only if $i_1 = i_2 = i_3 = i_4$ or i_1, i_2, i_3, i_4 takes only two distinct values. We get

$$E\left(\left(\frac{X_1 + \cdots + X_\ell}{\ell}\right)^4\right) = \frac{1}{\ell^4}\left(\ell E(X_1^4) + 6\frac{\ell(\ell-1)}{2} E(X_1^2 X_2^2)\right) \leq \frac{C}{\ell^2}$$

where C is a constant depending only on X_1. We thus have

$$\sum_{\ell=1}^{+\infty} E\left(\left(\frac{X_1 + \cdots + X_\ell}{\ell}\right)^4\right) = E\left(\sum_{\ell=1}^{+\infty}\left(\frac{X_1 + \cdots + X_\ell}{\ell}\right)^4\right) < +\infty.$$

In particular the random variable $\sum_{\ell=1}^{+\infty}\left(\frac{X_1+\cdots+X_\ell}{\ell}\right)^4$ is almost surely finite (otherwise its mean would be infinite). This implies that $\frac{X_1+\cdots X_\ell}{\ell}$ tends to $m = 0$ almost surely and we are done. ■

Define the map

$$\psi : \begin{cases} \mathbb{N} \to [0,1[\\ \mu \mapsto \psi(\mu) := \mathbb{P}_\mu(W_\ell \geq 0 \text{ for any } \ell \geq 0) \end{cases}$$

where $\mathbb{P}_\mu(W_\ell \geq 0 \text{ for any } \ell \geq 0)$ is the probability that the random walk W starting at μ remains nonnegative.

Proposition 2.1.3 *The function ψ is harmonic on $\mathbb{N}$ in the sense that for any $\mu > 0$*

$$\psi(\mu) = \psi(\mu + 1)p_1 + \psi(\mu - 1)p_{-1}.$$

Problem 2.1.4 *How can we make explicit the harmonic function ψ ?*

2.2 Reflection principle

The reflection principle permits to compute easily the number of trajectories of length ℓ starting at $\mu \in \mathbb{N}$ and remaining in $\mathbb{N}$.

Consider β, γ in $\mathbb{Z}$ and λ, μ in $\mathbb{N}$. Write

- $N_\ell(\beta, \gamma)$ for the number of trajectories of length ℓ from β to γ with steps in $\{\pm 1\}$.
- $N_\ell^{\mathbb{N}}(\mu, \lambda)$ for the number of trajectories of length ℓ from μ to λ with steps in $\{\pm 1\}$ which remain in $\mathbb{N}$.

Proposition 2.2.1 *We have*

$$N_\ell^{\mathbb{N}}(\mu,\lambda) = N_\ell(\mu+1,\lambda+1) - N_\ell(-\mu-1,\lambda+1). \tag{1}$$

Proof. First observe that $N_\ell^{\mathbb{N}}(\mu,\lambda) = N_\ell^{\mathbb{N}^*}(\mu+1,\lambda+1)$. Let U be the set of paths with length ℓ starting at $\mu+1$ or $-\mu-1$ and ending at $\lambda+1$ which do not remain in $\mathbb{N}^*$. Consider $w = (\varepsilon(\mu+1), x_1, \ldots, x_\ell)$ with $\varepsilon \in \{\pm 1\}$a path in U. Then π must attain 0. Let $L \in \{1, \ldots, \ell\}$ be the the maximal integer such that $\varepsilon(\mu+1) + x_1 + \cdots + x_L = 0$. Define $\overline{\pi}$ as the path $\overline{\pi} = (-\varepsilon(\mu+1), -x_1, \ldots, -x_L, x_{L+1}, \ldots, x_\ell)$. Clearly $\overline{\pi}$ belongs to U. Now the map $\theta : U \to U$ such that $\pi \longmapsto \overline{\pi}$ is an involution. Moreover, in (1), the paths π and $\overline{\pi}$ are counted with an opposite sign. This proves the proposition. ■

For β, γ in $\mathbb{Z}$, we have

$$\mathbb{P}_\beta(W_\ell = \gamma) = N_\ell(\beta,\gamma)\mathbf{p}_\ell^{\gamma-\beta}$$

since each trajectory from β to γ has the same probability $\mathbf{p}_\ell^{\gamma-\beta}$. This observation combined to the reflection principle permits to prove the

Theorem 2.2.2 *Assume $m = p_1 - p_{-1} > 0$. Then, for any $\mu \in \mathbb{N}$, we have*

$$\psi(\mu) = 1 - \mathbf{p}_0^{-2(\mu+1)} = 1 - \left(\frac{p_{-1}}{p_1}\right)^{\mu+1}.$$

In particular, $\psi > 0$.[1]

Proof. For any integer $\ell \geq 1$ define the function ψ_ℓ on $\mathbb{N}$ by

$$\psi_\ell(\mu) := \mathbb{P}_\mu(W_k \geq 0, \text{ for any } k = 1, \ldots, \ell).$$

We have by Proposition 2.2.1

$$\begin{aligned}
\psi_\ell(\mu) = \sum_{\lambda\in\mathbb{N}} N_\ell^{\mathbb{N}}(\mu,\lambda)\mathbf{p}_\ell^{\lambda-\mu} = \\
\sum_{\lambda\in\mathbb{N}} \left(N_\ell(\mu+1,\lambda+1) - N_\ell(-\mu-1,\lambda+1)\right)\mathbf{p}_\ell^{\lambda-\mu} = \\
\sum_{\lambda\in\mathbb{N}} N_\ell(\mu+1,\lambda+1)\mathbf{p}_\ell^{\lambda-\mu} - \left(\sum_{\lambda\in\mathbb{N}} N_\ell(-\mu-1,\lambda+1)\mathbf{p}_\ell^{\lambda+\mu+2}\right)\mathbf{p}_0^{-2\mu-2} = \\
\mathbb{P}_{\mu+1}(W_\ell > 0) - \mathbb{P}_{-\mu-1}(W_\ell > 0)\mathbf{p}_0^{-2\mu-2}.
\end{aligned}$$

Since $m > 0$, we have by using the law of large numbers

$$\lim_{\ell\to+\infty} \mathbb{P}_{\mu+1}(W_\ell > 0) = \mathbb{P}_{-\mu-1}(W_\ell > 0) = 1.$$

We thus get

$$\psi(\mu) = \lim_{\ell\to+\infty} \psi_\ell(\mu) = 1 - \mathbf{p}_0^{-2\mu-2} = 1 - \left(\frac{p_{-1}}{p_1}\right)^{\mu+1}.$$

[1] The positivity of ψ can also be deduced from the law of large numbers by purely probabilistics arguments

■

There is also a simple expression of ψ in terms of the Schur polynomials in two variables. For any $\mu \in \mathbb{N}$ define the symmetric polynomial

$$s_\mu(X_1, X_2) := \sum_{k=0}^{\mu} X_1^{\mu-k} X_2^k.$$

Then

$$\psi(\mu) = \left(1 - \frac{p_{-1}}{p_1}\right) p_1^{-\mu} s_\mu\left(p_1, p_{-1}\right).$$

3 Generalisation in higher dimension

3.1 Random walk on $\mathbb{Z}^n$

Let $B = \{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$ and let $\overline{C}$ be the cone

$$\overline{C} = \{x \in \mathbb{R}^n \mid x_1 \geq \cdots \geq x_n \geq 0\} \subset \mathbb{R}^n.$$

The elements of $P_+ = \overline{C} \cap \mathbb{Z}^n$ are the partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ of length at most n.

Set

$$|\lambda| = \lambda_1 + \cdots + \lambda_n$$

Let $(X_\ell)_{\ell \geq 1}$ be a sequence of random variables in B (i.i.d.)

$$\mathbb{P}(X_\ell = e_i) = p_{e_i} \in]0, 1[\text{ for } i = 1, \dots, n$$
$$p_{e_1} + \cdots + p_{e_n} = 1$$

$$m := E(X_\ell) = \sum_{i=1}^{n} p_{e_i} e_i.$$

The sequence of random variables $W = (W_\ell)_{\ell \geq 1}$ such that $W_\ell = X_1 + \cdots + X_\ell$ defines a random walk on $\mathbb{Z}^n$ with steps in B. This random walk is defined on the probabilistic space $\Omega = \{w = (x_\ell)_{\ell \geq 1} \mid x_\ell \in B\}$ of trajectories with steps in B which is the direct limit of the probabilistic spaces Ω_ℓ of trajectories with length ℓ. For such a trajectory $w = (x_1, \dots, x_\ell)$ we will have

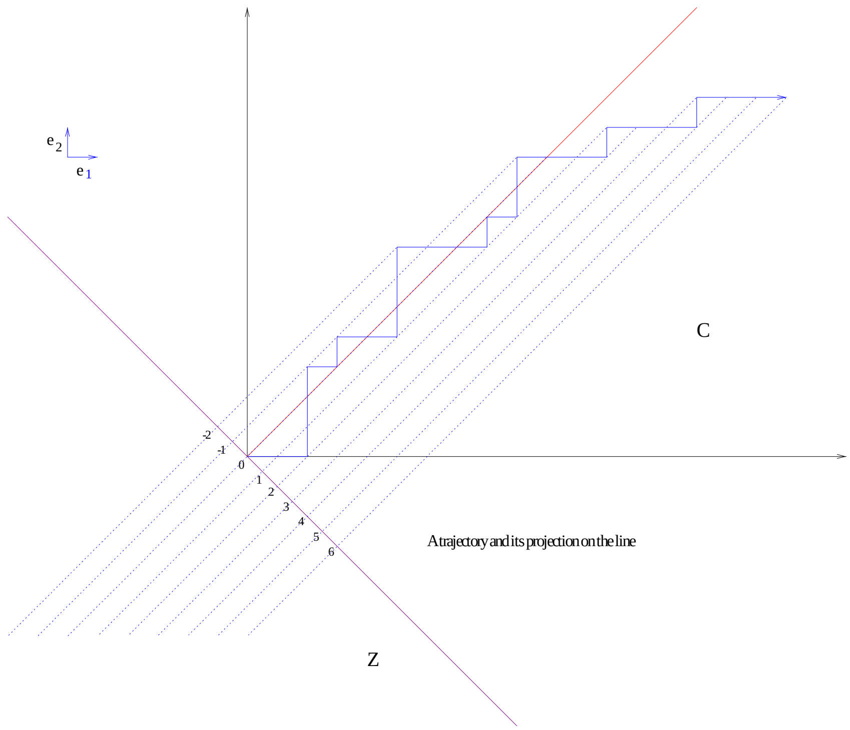
$$p_w = p_{x_1} \times \cdots \times p_{x_\ell} = p_1^{\gamma_1} \times \cdots \times p_n^{\gamma_n}$$

where for any $i = 1, \dots, n$, γ_i is the number of steps e_i in the trajectory w. We will call $\gamma = (\gamma_1, \dots, \gamma_n)$ the weight of w and set $\mathrm{wt}(w) = \gamma$. Also we write for short

$$p_w = p^{\mathrm{wt}(w)}.$$

The notation is clearly multiplicative in the sense that $p_{w*w'} = p_w \times p_{w'}$ where $w * w'$ is the trajectory of length $\ell(w) + \ell(w')$ obtained by concatenation of w and w'.

Observe that the case of the random walk on the line considered in § 2 can be recover from the random walk we consider here in dimension 2 by projection on the line $y = -x$ as illustrated by the figure below.



Assume $E(X_\ell) = m = (p_{e_1}, \dots, p_{e_n}) \in C$.
We define the function ψ on $\overline{C}$ by

$$\psi(\mu) = \mathbb{P}_\mu(W_\ell \in \overline{C}, \forall \ell \geq 1) \tag{2}$$

For any partition μ.

Proposition 3.1.1 *The function is harmonic on $\overline{C} \cap \mathbb{Z}^n$*

$$\psi(\mu) = \sum_{\mu \rightsquigarrow \lambda} \psi(\lambda) p_{\lambda - \mu}$$

where $\mu \rightsquigarrow \lambda$ means that the partition λ is obtained by adding a box to the partition μ.[2]

Proof. This follows from the decomposition

$$\mathbb{P}(W_0 = \mu, W_\ell \in \overline{C}, \forall \ell \geq 1) = \sum_{\mu \rightsquigarrow \lambda} \mathbb{P}(W_1 = \lambda \mid W_0 = \mu) \times \mathbb{P}(W_1 = \lambda, W_\ell \in \overline{C}, \forall \ell \geq 2).$$

■

[2] In fact, one can prove using the law of large numbers that $\psi > 0$ under our assumption $m \in C$. This will also folllow from Theorem 3.4.1

3.2 Markov chains and conditioning

Consider a probability space $(\Omega, \mathcal{T}, \mathbb{P})$ and a set $\overline{C}$. Let $Y = (Y_\ell)_{\ell\geq 0}$ be a sequence of random variables defined on Ω with values in S. The sequence Y is a Markov chain when

$$\mathbb{P}(Y_{\ell+1} = y_{\ell+1} \mid Y_\ell = y_\ell, \ldots, Y_1 = y_0) = \mathbb{P}(Y_{\ell+1} = y_{\ell+1} \mid Y_\ell = y_\ell)$$

for any $\ell \geq 0$ and any $y_0, \ldots, y_\ell, y_{\ell+1} \in S$. The Markov chains considered here will also be assumed time homogeneous, that is

$$\mathbb{P}(Y_{\ell+1} = y_{\ell+1} \mid Y_\ell = y_\ell) = \mathbb{P}(Y_\ell = y_{\ell+1} \mid Y_{\ell-1} = y_\ell) \text{ for any } \ell \geq 1.$$

For all x, y in S, the transition probability from x to y is then defined by

$$\Pi(x, y) = \mathbb{P}(Y_{\ell+1} = y \mid Y_\ell = x)$$

and we refer to Π as the transition matrix of the Markov chain Y. The distribution of Y_0 is called the initial distribution of the chain Y (in the situation we consider, Y_0 will be a constant). The initial distribution and the transition probability determine the law of the Markov chain. A class of examples of Markov chains is given by random walks. Notably the random walk W is a Markov chain on $S = \mathbb{Z}^n$ with transition matrix

$$\Pi(\alpha, \beta) = \begin{cases} p_{e_i} \text{ if } \beta - \alpha = e_i \in B, \\ 0 \text{ otherwise.} \end{cases}$$

Let $\overline{C}$ be a nonempty subset of S and consider the event $S = (Y_\ell \in \overline{C} \text{ for any } \ell \geq 0)$. Assume that $\mathbb{P}(S \mid Y_0 = \lambda) > 0$ for all $\lambda \in \overline{C}$. This implies that $\mathbb{P}[S] > 0$, and we can consider the conditional probability $\mathbb{Q}$ relative to this event: $\mathbb{Q}[\cdot] = \mathbb{P}[\cdot|S]$.

Proposition 3.2.1 *Under this new probability $\mathbb{Q}$, the sequence (Y_ℓ) is still a Markov chain, with values in $\overline{C}$, and with transitions probabilities given by*

$$\mathbb{Q}[Y_{\ell+1} = \lambda \mid Y_\ell = \mu] = \mathbb{P}[Y_{\ell+1} = \lambda \mid Y_\ell = \mu]\frac{\mathbb{P}[S \mid Y_0 = \lambda]}{\mathbb{P}[S \mid Y_0 = \mu]}. \tag{3}$$

We will denote by $Y^{\overline{C}}$ this Markov chain and by $\Pi^{\overline{C}}$ the restriction of the transition matrix Π to the entries which belong to $\overline{C}$ (in other words $\Pi^{\overline{C}} = (\Pi(\mu, \lambda))_{\lambda,\mu\in\overline{C}}$).

Proof. By definition of the conditional probability $\mathbb{Q}$, for any $\ell \geq 1$ and any $\mu_0, \cdots, \mu_\ell, \lambda \in \overline{C}$, we have

$$\begin{aligned}\mathbb{Q}(Y_{\ell+1} = \lambda \mid Y_\ell = \mu_\ell, \cdots, Y_0 = \mu_0) &= \frac{\mathbb{Q}(Y_{\ell+1} = \lambda, Y_\ell = \mu_\ell, \ldots, Y_0 = \mu_0)}{\mathbb{Q}(Y_\ell = \mu_\ell, \ldots, Y_0 = \mu_0)} \\ &= \frac{\mathbb{P}(S, Y_{\ell+1} = \lambda, Y_\ell = \mu_\ell, \ldots, Y_0 = \mu_0)}{\mathbb{P}(S, Y_\ell = \mu_\ell, \ldots, Y_0 = \mu_0)} =: \frac{N_\ell}{D_\ell}.\end{aligned}$$

We get

$$
\begin{aligned}
N_\ell = \mathbb{P}(Y_k \in \overline{C} \text{ for } k \geq 0, Y_{\ell+1} = \lambda, Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0)& \\
&= \mathbb{P}(Y_k \in \overline{C} \text{ for } k \geq \ell+1, Y_{\ell+1} = \lambda, Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0) \\
&= \mathbb{P}(Y_k \in \overline{C} \text{ for } k \geq \ell+1 \mid Y_{\ell+1} = \lambda, Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0) \\
&\quad \times \mathbb{P}(Y_{\ell+1} = \lambda, Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0) \\
&\overset{\text{Markov Property}}{=} \mathbb{P}(Y_k \in \overline{C} \text{ for } k \geq \ell+1 \mid Y_{\ell+1} = \lambda) \\
&\quad \times \mathbb{P}(Y_{\ell+1} = \lambda, Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0) \\
&\overset{\text{time homogeneous}}{=} \mathbb{P}(Y_k \in \overline{C} \text{ for } k \geq 0 \mid Y_0 = \lambda) \\
&\qquad \times \mathbb{P}(Y_{\ell+1} = \lambda, Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0)
\end{aligned}
$$

with

$$
\begin{aligned}
\mathbb{P}(Y_{\ell+1} = \lambda, Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0) =& \\
&= \mathbb{P}(Y_{\ell+1} = \lambda \mid Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0) \\
&\quad \times \mathbb{P}(Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0) \\
&\overset{\text{Markov Property}}{=} \mathbb{P}(Y_{\ell+1} = \lambda \mid Y_\ell = \mu_\ell) \times \mathbb{P}(Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0).
\end{aligned}
$$

We therefore obtain

$$N_\ell = \mathbb{P}(S \mid Y_0 = \lambda) \times \mathbb{P}(Y_{\ell+1} = \lambda \mid Y_\ell = \mu_\ell) \times \mathbb{P}(Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0).$$

A similar computation yields

$$D_\ell = \mathbb{P}(S \mid Y_0 = \mu_\ell) \times \mathbb{P}(Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0).$$

Finally, we get

$$\mathbb{Q}(Y_{\ell+1} = \lambda \mid Y_\ell = \mu_\ell, \cdots, Y_0 = \mu_0) = \mathbb{P}(Y_{\ell+1} = \lambda \mid Y_\ell = \mu_\ell) \times \frac{\mathbb{P}(S \mid Y_0 = \lambda)}{\mathbb{P}(S \mid Y_0 = \mu_\ell)}$$

which proves the proposition. ■

A *substochastic matrix* on the countable set S is a map $\Pi : S \times S \to [0,1]$ such that $\sum_{y \in S} \Pi(x,y) \leq 1$ for any $x \in S$. If Π, Π' are substochastic matrices on S, we define their product $\Pi \times \Pi'$ as the substochastic matrix given by the ordinary product of matrices:

$$\Pi \times \Pi'(x,y) = \sum_{z \in S} \Pi(x,z)\Pi'(z,y).$$

The matrix $\Pi^{\overline{C}}$ defined in the previous proposition is an example of substochastic matrix.

A function $h : S \to \mathbb{R}$ is *harmonic* for the substochastic transition matrix Π when it satisfies $\sum_{y \in S} \Pi(x,y)h(y) = h(x)$ for any $x \in S$. Consider a strictly positive harmonic function h for Π. The Doob transform of Π by h (also called the h-transform of Π) is defined by

$$\Pi_h(x,y) = \frac{h(y)}{h(x)}\Pi(x,y).$$

We have $\sum_{y\in S}\Pi_h(x,y)=1$ for any $x\in S$. Thus Π_h can be interpreted as the transition matrix for a certain Markov chain.

Proposition 3.2.2 *The function ψ defined in (2) is harmonic for the substochastic matrix $\Pi^{\overline{C}}=(\Pi(\mu,\lambda))_{\lambda,\mu\in\overline{C}})$ such that*

$$\Pi(\mu,\lambda)=p_{\lambda-\mu}1_B(\lambda-\mu).$$

Assume $\psi>0$. Then, the matrix Π_ψ such that

$$\Pi_\psi(\mu,\lambda)=\frac{\psi(\lambda)}{\psi(\mu)}p_{\lambda-\mu}1_B(\lambda-\mu)$$

is stochastic and the associated Markov chain is called the conditioning of W to stay in the cone $\overline{C}$.

Proof. This follows directly from Proposition 3.1.1. ■

Determining the law of the conditioning of W to stay in $\overline{C}$ is equivalent to compute the transition matrix Π_ψ which is clearly obtained from the harmonic function ψ.

3.3 Reflection principle in $\mathbb{Z}^n$

Consider β,γ in $\mathbb{Z}^n$ and λ,μ in $P_+=\overline{C}\cap\mathbb{Z}^n$. Write

- $N_\ell(\beta,\gamma)$ for the number of paths of length ℓ from β to γ with steps in B,
- $N_\ell^{\overline{C}}(\mu,\lambda)$ the number of paths of length ℓ from μ to λ with steps in B remaining in $\overline{C}$.

Theorem 3.3.1 (Gessel-Zeilberger [4]) *We have*

$$N_\ell^{\overline{C}}(\mu,\lambda)=\sum_{\sigma\in S_n}\varepsilon(\sigma)N_\ell(\sigma(\mu+\rho),\lambda+\rho) \tag{4}$$

with $\rho=(n-1,n-2,\ldots,1,0)\in\mathbb{Z}^n$.

Proof. The proof follows the same line as for the random walk on $\mathbb{Z}$. First observe that $N_\ell^{\overline{C}}(\mu,\lambda)=N_\ell^{C}(\mu+\rho,\Lambda+\rho)$. Let U be the set of trajectories of length ℓ starting at one of the $\sigma(\mu+\rho)$ and ending at $\lambda+\rho$ which are not contained in $\overline{C}$. Such a trajectory must intersect $\overline{C}\setminus C$ since the steps we consider belong to B. Consider $w=(\sigma(\mu+\rho),e_{i_1},\ldots,e_{i_\ell})\in U$. Let L be maximal such that $w_L:=\sigma(\mu+\rho)+e_{i_1}+\cdots+e_{i_L}\in\overline{C}\setminus C$. Now $w_L\in\mathbb{Z}^\ell$ have at least two coordinates equal. Let $j\in\{1,\ldots,n-1\}$ be maximal such that the coordinates j and $j+1$ coincides in w_L. Let $s_j=(j,j+1)$ be the transposition of S_n which flips j and $j+1$. Define the path $\overline{w}$ by $\overline{w}=(s_j\sigma(\mu+\rho),s_j(e_{i_1}),\ldots,s_j(e_{i_L}),e_{i_{L+1}},\ldots e_{i_\ell})$. We have $\overline{w}\in U$ and the map $\theta:U\to U$ such that $\theta(w)=\overline{w}$ is an involution. Now the paths $\overline{w}$ and w are counted with an opposite sign in (4) because $\overline{w}$ starts at $s_jw(\mu+\rho)$. This shows that all the contributions of the paths in U cancel in (4) ■

3.4 Computing the probability to stay in $\overline{C}$

We can now compute the function ψ. The following result was obtained by O'Connell (see [12] and [13]). We give below a simpler proof based on the reflection principle.

Theorem 3.4.1 (O'Connell [12]) *Assume $m = (p_{e_1} > \cdots > p_{e_n}) \in C$. For any partition μ,*

$$\psi(\mu) = \sum_{\sigma \in S_n} \varepsilon(\sigma) p^{\sigma(\mu+\rho)-\mu-\rho}.$$

Proof. Define the function ψ_ℓ on P_+ by

$$\psi_\ell(\mu) = \mathbb{P}_\mu(W_k \in \overline{C}, k = 1, \ldots, \ell).$$

We have

$$\psi_\ell(\mu) = \sum_{|\lambda|=\ell+|\mu|} N_\ell^{\overline{C}}(\mu,\lambda) p^{\lambda-\mu} =$$

$$\sum_{\sigma \in S_n} \varepsilon(\sigma) \left(\sum_{|\lambda|=\ell+|\mu|} N_\ell(\sigma(\mu+\rho), \lambda+\rho) p^{\lambda+\rho-\sigma(\mu+\rho)} \right) \times p^{\sigma(\mu+\rho)-\mu-\rho}$$

$$= \sum_{\sigma \in S_n} \varepsilon(\sigma) \mathbb{P}_{\sigma(\mu+\rho)}(W_\ell \in C) p^{\sigma(\mu+\rho)-\mu-\rho}$$

Since $m \in C$, we have by the law of large numbers

$$\lim_{\ell \to +\infty} \mathbb{P}_{\sigma(\mu+\rho)}(W_\ell \in C) = 1 \ \forall \sigma \in W.$$

We finally get

$$\psi(\mu) = \lim_{\ell \to +\infty} \psi_\ell(\mu) = \sum_{\sigma \in S_n} \varepsilon(\sigma) p^{\sigma(\mu+\rho)-\mu-\rho}.$$

■

3.5 Connection with the representation theory of $\mathfrak{gl}_n(\mathbb{C})$

Consider $\mathrm{Sym}[X_1, \ldots, X_n]$ the ring of symmetric polynomials in the indeterminates $X_1, \ldots, X_n$. For any partition $\lambda \in P_+$, the Schur polynomial s_λ is defined by

$$s_\lambda(X_1, \ldots, X_n) = \frac{\sum_{\sigma \in S_n} \varepsilon(\sigma) X^{\sigma(\lambda+\rho)}}{\sum_{\sigma \in S_n} \varepsilon(\sigma) X^{\sigma(\rho)}} \in \mathrm{Sym}[X_1, \ldots, X_n]$$

where $X^\beta = X_1^{\beta_1} \cdots X_n^{\beta_n}$ for any $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$.

To each partition $\lambda \in P_+$ corresponds a finite-dimensional irreducible $\mathfrak{gl}_n$-module of highest weight λ. Then s_λ is the character of the representation $V(\lambda)$ and is a polynomial with non-negative coefficients (see [3]). These characters permits to give simple expressions for both the harmonic function ψ and the transition matrix $\Pi^{\overline{C}}$.

Corollary 3.5.1 *We have*

$$\psi(\lambda) = \mathbb{P}_\lambda(\forall \ell \geq 0, W_\ell \in \overline{\mathcal{C}}) = \prod_{1 \leqslant i < j \leqslant n} \left(1 - \frac{p_{e_j}}{p_{e_i}}\right) p^{-\lambda} s_\lambda(p_{e_1}, \ldots, p_{e_n}) \text{ for any } \lambda \in P_+.$$

In particular $\psi > 0$ and

$$\Pi_\psi^{\overline{C}}(\mu, \lambda) = \frac{s_\lambda(p_{e_1}, \ldots, p_{e_n})}{s_\mu(p_{e_1}, \ldots, p_{e_n})} 1_B(\lambda - \mu) \text{ for any } \lambda, \mu \in P_+$$

gives the law of the random walk W conditioned to stay in $\overline{C}$.

Remark: The previous results can be generalised to a wide class of random walks defined from representations of semisimple Lie algebras $\mathfrak{g}$ (and their generalisations as Kac-Moody algebras). The corresponding harmonic functions and conditioned transition matrices have similar nice expressions in terms of the Weyl characters of $\mathfrak{g}$ (see [8], [9] and [10]).

4 Generalised Pitman transform

4.1 Robinson-Schensted correspondence

Consider $\lambda \in P_+$. A (semistandard) tableau of shape λ is a filling (let us call it T) of the Young diagram associated to λ by letters of the ordered alphabet $\mathcal{A}_n = \{1 < 2 < \cdots < n\}$ such that the rows of T weakly increase from left to right and its columns strictly increase from top to bottom (see Example 4.1.1). We denote by $T(\lambda)$ the set of all semistandard tableaux of shape λ. We define the reading of $T \in T(\lambda)$ as the word $\mathrm{w}(T)$ of $\mathcal{A}_n^*$ obtained by reading the rows of T from right to left and then top to bottom.

The weight of a word $w \in \mathcal{A}_n^*$ is the n-tuple $\mathrm{wt}(w) = (\mu_1, \ldots, \mu_n)$ where for any $i = 1, \ldots, n$ the nonnegative integer μ_i is the number of letters i in w. The weight $\mathrm{wt}(T)$ of $T \in T(\lambda)$ is then defined as the weight of its reading $\mathrm{w}(T)$. The Schur function s_λ (which is the character of $V(\lambda)$) can be expressed as a generating series over $T(\lambda)$, namely we have

$$s_\lambda(X) = \sum_{T \in T(\lambda)} X^{\mathrm{wt}(T)}.$$

Let T be a semistandard tableau of shape $\lambda \in \mathcal{P}$. We write $T = C_1 \cdots C_s$ as the juxtaposition of its columns. Consider $x \in \mathcal{A}_n$. We denote by $x \to T$ the tableau obtained by applying the following recursive procedure:

1. Assume $T = \emptyset$, then $x \to T$ is the tableau with one box filled by x.
2. Assume C_1 is nonempty.
   (a) If all the letters of C_1 are less than x, the tableau $x \to T$ is obtained from T by adding one box filled by x at the bottom of C_1.
   (b) Otherwise, let $y = \min\{t \in C_1 \mid t \geq x\}$. Write C_1' for the column obtained by replacing y by x in C_1. Then $x \to T = C_1'(y \to C_2 \cdots C_s)$ is defined as the juxtaposition of C_1' with the tableau obtained by inserting y in the remaining columns.

One easily verifies that in any case $x \to T$ is a semistandard tableau. More generally, for any word $w = x_1 x_2 \cdots x_\ell \in \mathcal{A}_n^*$, we define the tableau $P(w)$ by setting

$$P(w) = x_\ell \to (\cdots (x_2 \to (x_1 \to \emptyset))). \tag{5}$$

Example 4.1.1 *With $n \geq 4$ and $w = 232143$, we obtain the following sequences of tableaux:*

$$\begin{array}{|c|}\hline 2\\\hline\end{array}\ ,\ \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\ ,\ \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 3\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}\ ,\ \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 3&3\\\cline{1-2} 4\\\cline{1-1}\end{array} = P(w).$$

Remark: Given a semistandard tableau T and a box b on its border strip (i.e. such that there is no box under and on the right of b), there is a unique letter x such that $T = x \to T'$ where the shape of T' is obtained by deleting b in the shape of T. This shows that our insertion procedure is reversible up to the choice of a box in the border strip of T.

Example 4.1.2 *Let b be the box with the bold letter in*

$$\begin{array}{|c|c|c|}\hline 1&2&\mathbf{3}\\\hline 2&3\\\cline{1-2} 4\\\cline{1-1}\end{array}$$

By reversing the steps of the insertion procedure, we obtain

$$\begin{array}{|c|c|c|}\hline 1&2&\mathbf{3}\\\hline 2&3\\\cline{1-2} 4\\\cline{1-1}\end{array} = 2 \to \begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline 4\\\cline{1-1}\end{array}.$$

For any word $w = x_1 \cdots x_\ell \in \mathcal{A}^\ell$ and any $k = 1, \ldots \ell$, let $\lambda^{(k)}$ be the shape of the tableau $P(x_1 \cdots x_k)$. The shape $\lambda^{(k)}$ is obtained from $\lambda^{(k-1)}$ by adding one box we denote by b_k. The recording tableau $Q(w)$ of shape $\lambda^{(\ell)}$ is obtained by filling each box b_k with the letter k. Observe that $Q(w)$ is a standard tableau: it contains exactly once all the integers $1, \ldots, \ell$, its rows strictly increase from left to right and its columns strictly increase from top to bottom. Note also that the datum of a standard tableau with ℓ boxes is equivalent to that of a sequence of shapes $(\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \in P_+^\ell$ such that $\lambda^{(1)} = (1)$ and for any $k = 1, \ldots, \ell$, the shape $\lambda^{(k)}$ is obtained by adding one box to $\lambda^{(k-1)}$.

Example 4.1.3 *From Example 4.1.1, we get*

$$Q(232143) = \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&6\\\cline{1-2} 5\\\cline{1-1}\end{array}$$

We can now state the Robinson-Schensted correspondence (see [3]). For any $\ell \in \mathbb{N}$ write $\mathcal{U}_\ell$ for the set of pairs (P, Q) where P is a semistandard tableau and Q a standard tableau *with the same shape* containing ℓ boxes.

Consider $\lambda \in \mathcal{P}$ and assume $|\lambda| = \ell$. Given τ a standard tableau of shape λ, we set

$$B(\tau) = \{w \in \mathcal{A}^\ell \mid Q(w) = \tau\}.$$

Theorem 4.1.4 (Robinson-Schensted correspondence)

1. *The map* $\begin{cases} \theta_\ell : & \mathcal{A}^\ell \to \mathcal{U}_\ell \\ & w \mapsto (P(w), Q(w)) \end{cases}$ *is a one-to-one correspondence.*

2. *In particular, the map P restricts to a weight preserving bijection $P : B(\tau) \longleftrightarrow T(\lambda)$.*

Proof. To construct recursively the map θ_ℓ^{-1}, it suffices to produce from any pair $(P, Q) \in \mathcal{U}_\ell$, a pair $(P', Q') \in \mathcal{U}_{\ell-1}$ and a letter x such that $x \to P' = P$ and Q is obtained by adding a box containing n in Q'. To do this, let b be the box of P corresponding to the letter n in Q. By the Remark just after Example 4.1.1, there exists a unique letter x and a unique tableau P' such that $P = x \to P'$. Let Q' be the standard tableau obtained by deleting n in Q. Then $(P', Q') \in \mathcal{U}_{\ell-1}$ and we are done. ■

4.2 The transformation $\mathcal{P}$ and the Markov chain H

Our aim is now to define a *transformation* $\mathcal{P}$ on the set of paths we have considered in § 3.1. Observe first, we can identify the set of infinite trajectories Ω with $\mathcal{A}^{\mathbb{N}}$ by associating to each trajectory $w = (x_\ell)_{\ell \geq 1}$ the infinite word $w = x_1 x_2 \cdots$. Set

$$\mathcal{P}(w) = (\lambda^{(\ell)})_{\ell \geq 1}$$

where for any $\ell \geq 1$, $\lambda^{(\ell)}$ is the shape of the tableau $P(x_1 \cdots x_\ell)$. OBserve we can also consider $\mathcal{P}(w)$ as a trajectory $(x_\ell)_{\ell \geq 1}$ which remains in $\overline{C}$ where $\lambda^{(\ell)} = x_1 + \cdots + x_\ell$ for any $\ell \geq 1$.

Example 4.2.1

1. *Consider $w = 232143$. From Example 4.1.1 we get*

ℓ	1	2	3	4	5	6
$\lambda^{(\ell)}$	(1)	(1, 1)	(2, 1)	(3, 1)	(3, 1, 1)	(3, 2, 1)

2. *Consider $w = 1121231212$. Observe that the path in $\mathbb{Z}^3$ associated to w remains in P_+. We obtain*

ℓ	1	2	3	4	5	6	7	8	9	10
$\lambda^{(\ell)}$	(1)	(2)	(2, 1)	(3, 1)	(3, 2)	(3, 2, 1)	(4, 2, 1)	(4, 3, 1)	(5, 3, 1)	(5, 4, 1)

   So by considering $\mathcal{P}(w)$ as a trajectory, we have $\mathcal{P}(w) = w$. The paths contained in P_+ are fixed by the transformation $\mathcal{P}$.

We then consider the sequence $H_\ell = (H_\ell)_{\ell \geq 0}$ of random variables $H_\ell := W_\ell \circ \mathcal{P}$ defined on the probability space $\left(\mathcal{A}^{\mathbb{N}}, \mathcal{P}\left(\mathcal{A}^{\mathbb{N}}\right), \mathbb{P}\right)$, with values in P_+. We can now state the main result of this section.

Theorem 4.2.2 *The stochastic process H is a Markov chain with transition matrix*

$$\Pi_H(\mu, \lambda) = \frac{s_\lambda(p)}{s_\mu(p)} 1_B(\lambda - \mu) \quad \lambda, \mu \in P_+. \tag{6}$$

Proof. Consider a sequence of dominant weights $\lambda^{(1)}, \dots, \lambda^{(\ell)}, \lambda^{(\ell+1)}$ in P_+ such that $\lambda^{(k)} \rightsquigarrow \lambda^{(k+1)}$ for any $k = 1, \dots, \ell$. We have seen in § 4.1 that this determines a unique standard tableau τ of shape $\lambda^{(\ell+1)}$ and we have

$$\mathbb{P}(H_{\ell+1} = \lambda^{(\ell+1)}, H_k = \lambda^{(k)} \text{ for any } k = 1, \dots, \ell) = \sum_{w \in B(\tau)} p_w = s_{\lambda^{(\ell+1)}}(p)$$

since there is a weight preserving bijection between $B(\tau)$ and $T(\lambda^{(\ell+1)})$ by Theorem 4.1.4. Similarly, we have

$$\mathbb{P}(H_k = \lambda^{(k)} \text{ for any } k = 1, \dots, \ell) = s_{\lambda^{(\ell)}}(p).$$

Hence

$$\mathbb{P}(H_{\ell+1} = \lambda^{(\ell+1)} \mid H_k = \lambda^{(k)} \text{ for any } k = 1, \dots, \ell) = \frac{s_{\lambda^{(\ell+1)}}(p)}{s_{\lambda^{(\ell)}}(p)}.$$

In particular, $\mathbb{P}(H_{\ell+1} = \lambda^{(\ell+1)} \mid H_k = \lambda^{(k)}$ for any $k = 1, \dots, \ell)$ depends only on $\lambda^{(\ell+1)}$ and $\lambda^{(\ell)}$, which is the Markov property. ■

Combining Corollary 3.5.1 and Theorem 4.2.2 we finally get :

Theorem 4.2.3 *Assume that $m \in C$ (the drift belongs to the interior of the cone $\overline{C}$). Then, the law of the Markov chain H is the same as the law of the random walk W conditioned to never exit the closed cone $\overline{C}$.*

4.3 The transformation $\mathcal{P}$ in dimension 2

Let us consider in more details the case of dimension 2. The relevant algebra is then $\mathfrak{g} = \mathfrak{gl}_2$ with root system A_1. Recall we can reduce our two dimensional processes problem to random processes on the line by the projection R on $y = -x$. For any $w \in \Omega$ (that is any infinite trajectory with steps e_1 or e_2), write $\overline{w}$ the trajectory on $\mathbb{Z}$ (which is an infinite trajectory with steps ± 1) obtained by applying R. Now observe that each trajectory on $\mathbb{Z}$ can be regarded as a function

$$f : \mathbb{N} \to \mathbb{Z}$$

where $f(\ell)$ is the integer attained at time ℓ. Here we assume that $f(0) = 0$. The original Pitman transform $\overline{\mathcal{P}}$ (see [14]) on the trajectory f is defined by

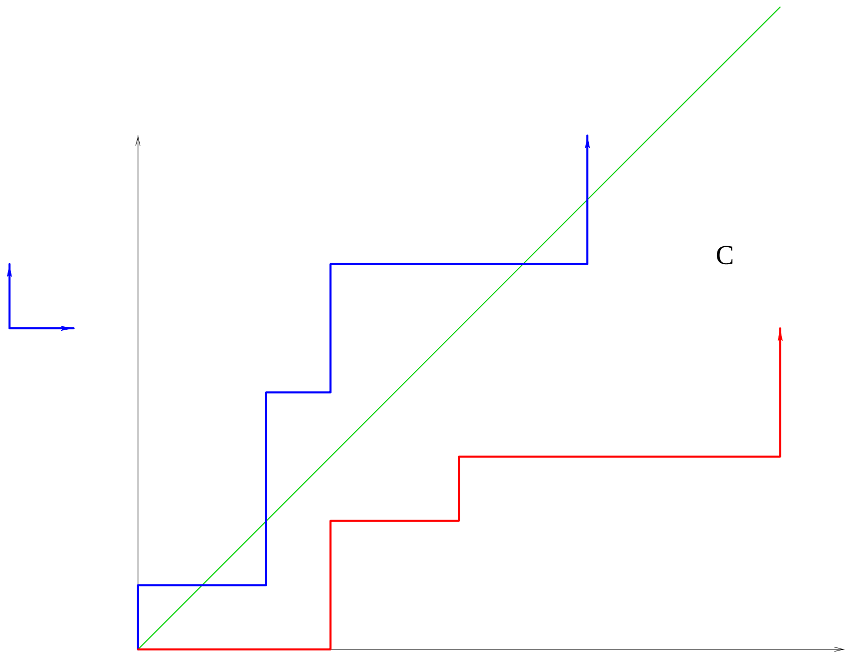
$$\overline{\mathcal{P}}(f)(\ell) = f(\ell) - 2 \min_{0 \le k \le \ell} f(k).$$

Clearly $\overline{\mathcal{P}}(f)$ is a trajectory on $\mathbb{N}$ since $\min_{0 \le k \le \ell} f(k) \le 0$. The following proposition can be proved by induction on ℓ.

Proposition 4.3.1 *For any $w \in \Omega$, we have*

$$\overline{\mathcal{P}(w)} = \overline{\mathcal{P}}(\overline{w})$$

i.e. the projection on the line intertwines the transformations $\mathcal{P}$ and $\overline{\mathcal{P}}$.



Un chemin (en bleu) et son image par P (en rouge) pour la representation vectorielle de sl(2,C)

$$\mathcal{P}(211222122111122) = 111221121111122$$

ℓ	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\overline{W}_\ell$	-1	0	1	0	-1	-2	-1	-2	-3	-2	-1	0	1	0	-1
$2\min_{0\leq k\leq \ell}\overline{W}_\ell$	-2	-2	-2	-2	-2	-4	-4	-4	-6	-6	-6	-6	-6	-6	-6
$\overline{H}_\ell$	1	2	3	2	1	2	3	2	3	4	5	6	7	6	5

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