Kazhdan-Lusztig cells in affine Weyl groups (with unequal parameters)

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Let V be an euclidean space of dim r, with inner product $\langle .,. \rangle$. A root system Φ is a set of non-zero vectors that satisfy the following:

- Φ spans V.
- For $\alpha \in \Phi$, we have $\mathbb{R}\alpha \cap \Phi = \{\alpha, -\alpha\}$.
- For $\alpha \in \Phi$, let σ_α the orthogonal reflection with fixed point set the hyperplane perpendicular to α. We have $\sigma_\alpha(\Phi) = \Phi$.
- For any $\alpha, \beta \in \phi$ we have $\frac{2\langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle} \in \mathbb{Z}$.

Let Φ be an "irreducible" root system. The Weyl group of Φ is the group generated by $\{\sigma_\alpha | \alpha \in \Phi\}$. It has a presentation of the form:

$$\{\sigma_{\alpha_1}, ..., \sigma_{\alpha_r} | (\sigma_{\alpha_i}\sigma_{\alpha_j})^{m_{i,j}} = 1, \ \sigma_{\alpha_i}^2 = 1\}$$

where $m_{i,j} \in \{2, 3, 4, 6\}$ for $i \neq j$.

Let Q be the lattice generated by Φ:

$$\{n_1\alpha_1 + ... + n_k\alpha_k \,|\, n_i \in \mathbb{Z}, \alpha_i \in \Phi\}$$

The Weyl group W_0 of Φ acts on Q.

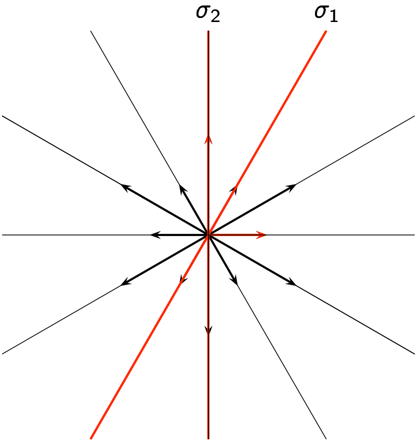
Thus we can form the semi-direct product:

$$W := W_0 \ltimes Q$$

This is the affine Weyl group associated to Φ.

V: Euclidean space of dimension r.

Φ: Irreducible root system of V.



For any $\alpha \in \Phi$ and $k \in \mathbb{Z}$ let :

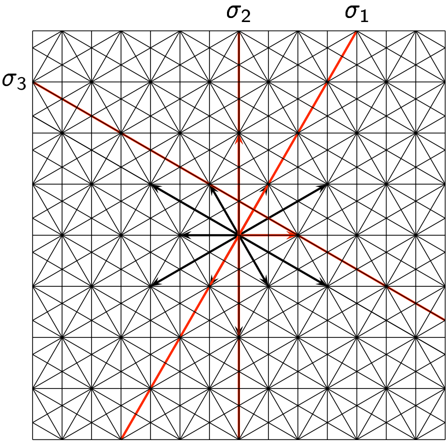
$$H_{\alpha,k} = \{x \in V \mid \frac{2\langle\alpha, x\rangle}{\langle\alpha,\alpha\rangle} = k\}$$

The Weyl group W_0 of Φ is generated the orthogonal reflections with fixed point set $H_{\alpha,0}$. We have:

$$W_0 = \langle\sigma_1, \sigma_2 | \ (\sigma_1\sigma_2)^6 = 1, \sigma_i^2 = 1\rangle$$

V: Euclidean space of dimension r.

Φ: Irreducible root system of V.



For any $\alpha \in \Phi$ and $k \in \mathbb{Z}$ let :

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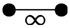
$$W_0 = \langle \sigma_1, \sigma_2 | \ (\sigma_1\sigma_2)^6 = 1, \sigma_i^2 = 1\rangle$$

The affine Weyl group W of Φ is generated by all the orthogonal reflections $\sigma_{H_{\alpha,k}}$ with fixed point set $H_{\alpha,k}$.

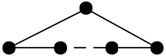
Here we have:

$$W = \langle \sigma_1, \sigma_2, \sigma_3 | \ (\sigma_1\sigma_2)^6 = 1, (\sigma_2\sigma_3)^3 = 1, (\sigma_1\sigma_3)^2 = 1, \sigma_i^2 = 1\rangle$$

$\tilde{A}_1$:

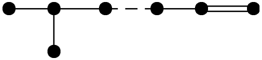


$\tilde{A}_n$:

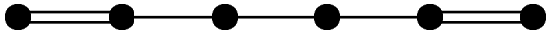


$\tilde{B}_2 = \tilde{C}_2$:

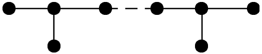
$\tilde{B}_n$:



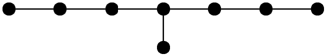
$\tilde{C}_n$:



$\tilde{D}_n$:

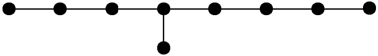


$\tilde{E}_7$:

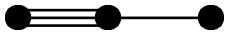


$\tilde{F}_4$:

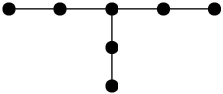
$\tilde{E}_8$:



$\tilde{G}_2$:



$\tilde{E}_6$:



Let W be an affine Weyl group with generating set S.

For $w \in W$, let $\ell(w)$ be the smallest integer $n \in \mathbb{N}$ such that $w = s_1 ... s_n$ with $s_i \in S$. The function ℓ is called the length function.

Let L be a weight function, that is a function $L : W \to \mathbb{N}$ such that :

$$L(ww') = L(w) + L(w') \text{ whenever } \ell(ww') = \ell(w) + \ell(w')$$
$$L(w) > 0 \text{ unless } w = 1$$

The case $L = \ell$ is known as the equal parameter case.

From the above relations, one can see that:

- A weight function L is completely determined by its values on S
- Let $s, t \in S$, if the order of (st) is odd, then we must have $L(s) = L(t)$.

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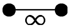

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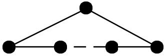

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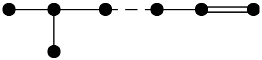
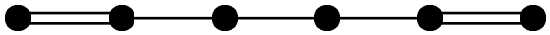
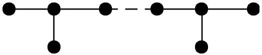


$\tilde{C}_n$:



$\tilde{D}_n$:



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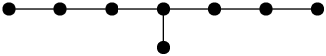

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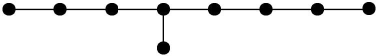

$\tilde{G}_2$:

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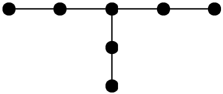

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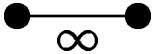

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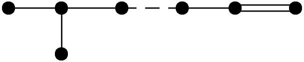

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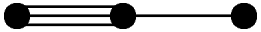

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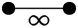
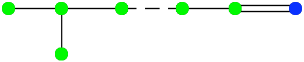


$\tilde{B}_n$:



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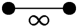

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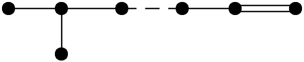

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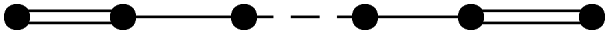

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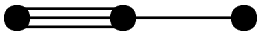

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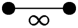

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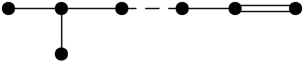

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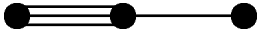

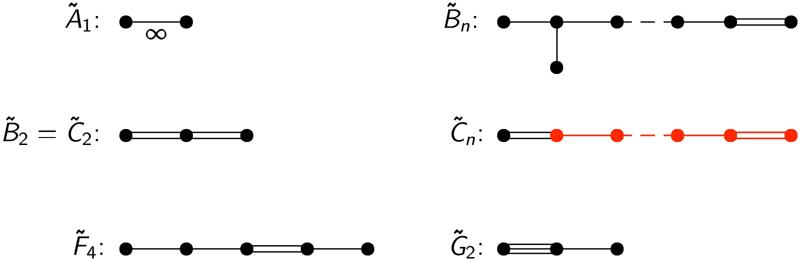
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Let (W, S) be an affine Weyl group and L a weight function on W.

Let $\mathcal{H}$ be the associated Iwahori-Hecke algebra over $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$.

Standard basis $\{T_w \mid w \in W\}$ with multiplication

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w) \\ T_{sw} + (v^{L(s)} - v^{-L(s)}) T_w & \text{if } \ell(sw) < \ell(w) \end{cases}$$

One can see that $T_s^{-1} = T_s - (v^{L(s)} - v^{-L(s)}) T_1$.

There is a unique ring involution $\mathcal{A} \to \mathcal{A}$, $a \mapsto \bar{a}$, such that $\bar{v} = v^{-1}$.

We can extend it to a ring involution $\mathcal{H} \to \mathcal{H}$, $h \mapsto \overline{h}$, such that:

$$\overline{\sum_{w \in W} a_w T_w} = \sum_{w \in W} \bar{a}_w T_{w^{-1}}^{-1} \qquad (a_w \in \mathcal{A}).$$

Theorem. KAZHDAN-LUSZTIG ($\sim$ 1979) LUSZTIG ($\sim$ 1983)

For any $w \in W$, there exists a unique $C_w \in \mathcal{H}$ such that:

- $\overline{C_w} = C_w$
- $C_w = T_w + \sum_{\ell(y)<\ell(w)} P_{y,w} T_y$ where $P_{y,w} \in v^{-1}\mathbb{Z}[v^{-1}]$

Furthermore, the C_w's form a basis of $\mathcal{H}$ known as the Kazhdan-Lusztig basis.

For example, we have :

$$C_1 = T_1 \quad \text{and } C_s = T_s + v^{-L(s)} T_1$$

Pre-order relation $\leq_L$ defined by :

$$\mathcal{H}C_w \subset \sum_{y \leq_L w} \mathcal{A}C_y$$

Let $s \in S$ and $w \in W$ such that $\ell(w) < \ell(sw)$, then:

$$C_s C_w = C_{sw} + \ldots \quad \text{so we have } sw \leq_L w$$

Corresponding equivalence relation $\sim_L$.

The equivalence classes are called left cells.

Similarly we define $\leq_R$, $\sim_R$ and right cells.

We say that $y \leq_{LR} w$ if there exists a sequence:

$$y = y_0, y_1, ..., y_n = w$$

such that for any $0 \leq i \leq n-1$ we have:

$$y_i \leq_L y_{i+1} \text{ or } y_i \leq_R y_{i+1}$$

We get the equivalence relation $\sim_{LR}$ and the two-sided cells.

Structure constants: Write

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z \text{ where } h_{x,y,z} \in \mathcal{A}.$$

G. LUSZTIG (1985): Define function $a\colon W \to \mathbb{N}_0$ by

$$a(z) = \min\{i \geq 0 \mid v^{-i} h_{x,y,z} \in \mathbb{Z}[v^{-1}]\ \forall x, y \in W\}.$$

If W is finite, then this function is clearly well defined. In the affine case, it is not clear that this minimum exists! But, it does... Let $\tilde{\nu} = L(w_0)$ where w_0 is the longest element of the Weyl group W_0 associated to W. We have :

$$v^{-\tilde{\nu}} h_{x,y,z} \in \mathbb{Z}[v^{-1}] \text{ for all } x, y, z \in W.$$

In other words, $a(z) \leq \tilde{\nu}$ for all $z \in W$.

The pre-order $\leq_{LR}$ induces a partial order on the two-sided cells.

Theorem.

Let

$$c_0 = \{w \in W \mid a(w) = \tilde{\nu}\}.$$

Then c_0 is a two-sided cell. Moreover, c_0 is the lowest two-sided cell.

Why lowest? Lusztig conjectures:

$$\text{if } z \leq_{LR} z' \text{ then } a(z') \leq a(z).$$

Let $z' \in c_0$. Let $z \leq_{LR} z'$. We have:

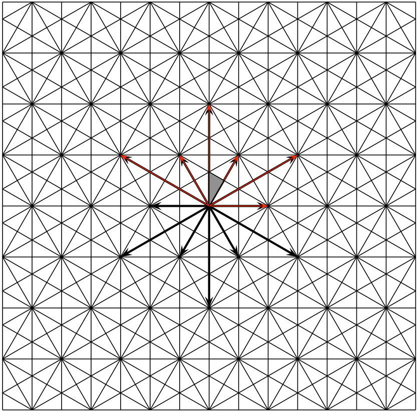
$$\tilde{\nu} = a(z') \leq a(z) \leq \tilde{\nu}$$

which implies $a(z) = \tilde{\nu}$ and $z \in c_0$.

- Shi ($\sim$ 1987): c_0 is a two-sided cell (equal parameter case).
- Shi ($\sim$ 1988): c_0 contains $|W_0|$ left cells (equal parameter).
- Bremke and Xi ($\sim$ 1996): c_0 is a two-sided cell (unequal parameter).
- Bremke ($\sim$ 1996): c_0 contains at most $|W_0|$ left cells.
- Bremke ($\sim$ 1996): c_0 contains $|W_0|$ left cells when the parameters are coming from a graph automorphism

When we know the exact number of left cells in c_0, it involves some deep properties of Kazhdan-Lusztig polynomials, such as positivity of the coefficient. Problem: Not true in general!

Example: $\tilde{G}_2$



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An alcove is a connected component of :

$$V - \bigcup H_{\alpha,k}$$

Denote by X the set of alcoves.

Let $\Omega = \langle \sigma_{H_{\alpha,k}}, k \in \mathbb{Z}, \alpha \in \Phi\rangle$

Ω acts simply transitively on X.

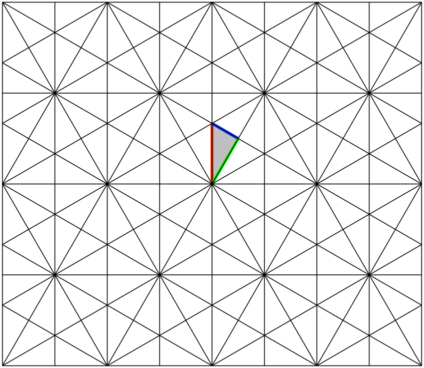
Let A_0 be the fundamental alcove :

$$A_0 = \{x \in V \mid 0 < \tfrac{2\langle\alpha,x\rangle}{\langle\alpha,\alpha\rangle} < 1\}$$

for all $\alpha \in \Phi^+$.

A face is a co-dimension 1 facet of an alcove.

Examples : The faces of A_0.



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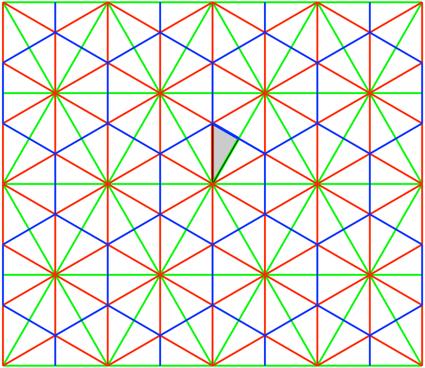

We look at the orbits of the faces under Ω.

Let S be the set of orbits.

Here we have 3 orbits, namely :

$$s_1 = \text{green}$$

$$s_2 = \text{red}$$

$$s_3 = \text{blue}$$

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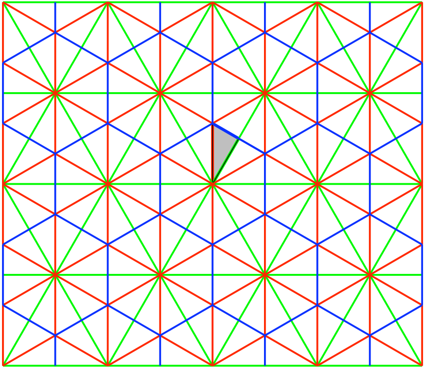

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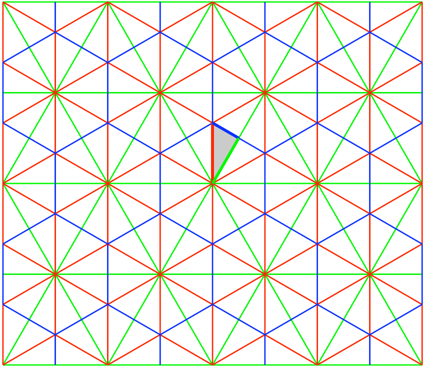
For $s \in S$, we define an involution $A \mapsto sA$ of X, where sA is the unique alcove which shares with A a face of type s. The set of such map is a group of permutation of X which is a Coxeter group W. We have $W \simeq \Omega$.

The action of W on X commutes with the action of Ω.

We identify $w \in W$ with the alcove wA_0.



Example:

- alcove $s_3 s_2 s_1 s_2 s_3 A_0$.

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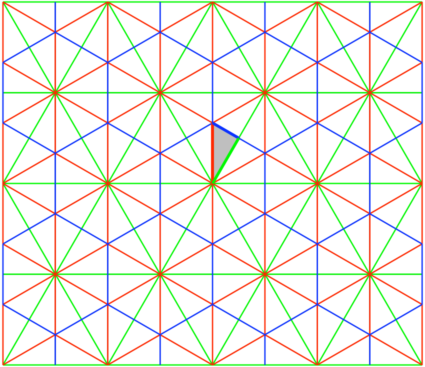

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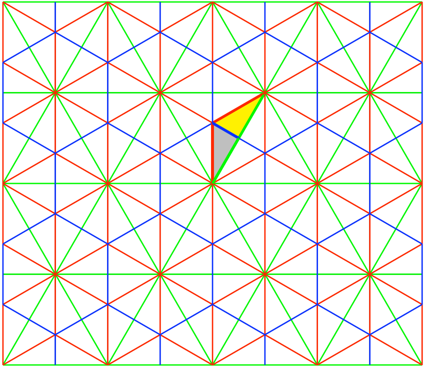

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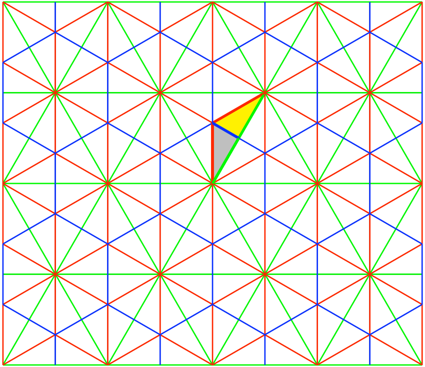

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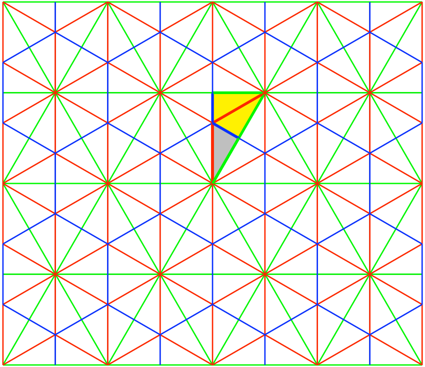

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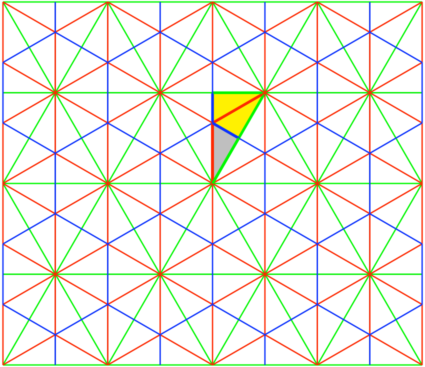

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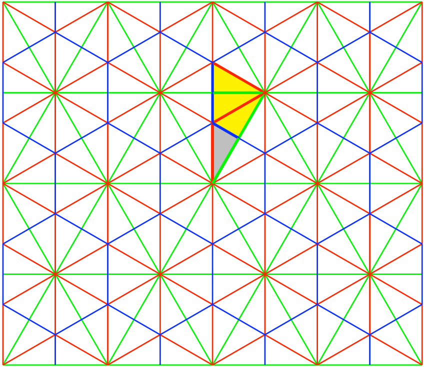

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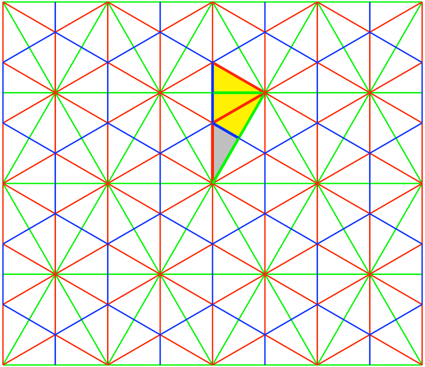

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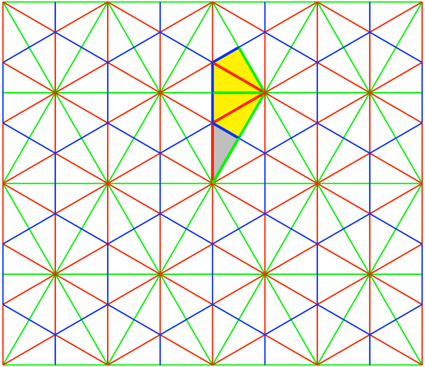

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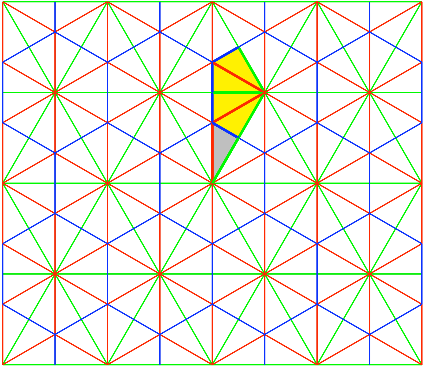

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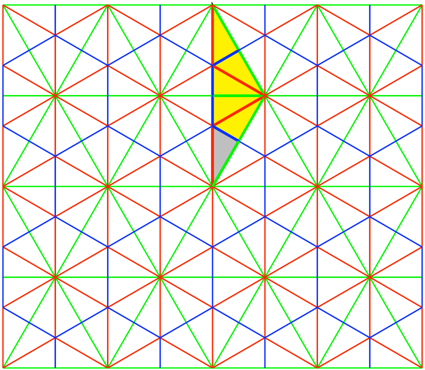

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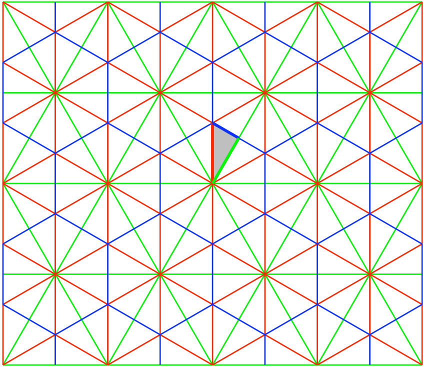

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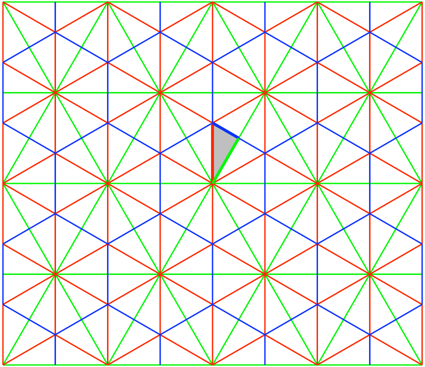

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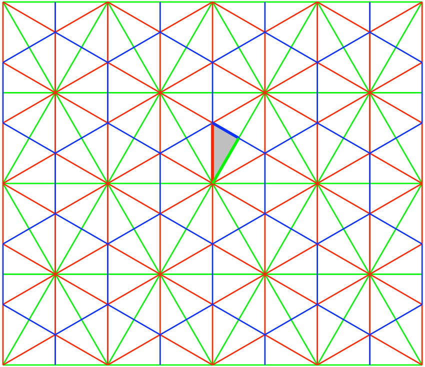

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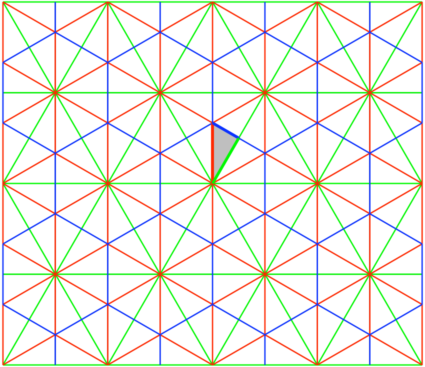

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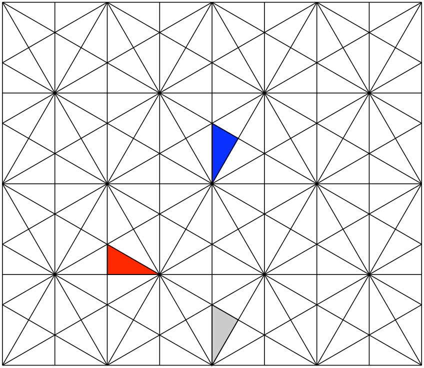
Let $s, t \in S$. If a hyperplane H supports a face of type s and a face of type t then s and t are conjugate in W. Therefore we can associate to any hyperplane H a weight $c_H = L(s)$ if H supports a face of type s.

Let $w \in W$, we have $\ell(w)$= number hyperplane which separate A_0 and wA_0.



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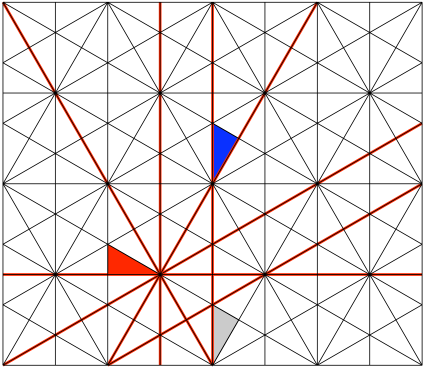

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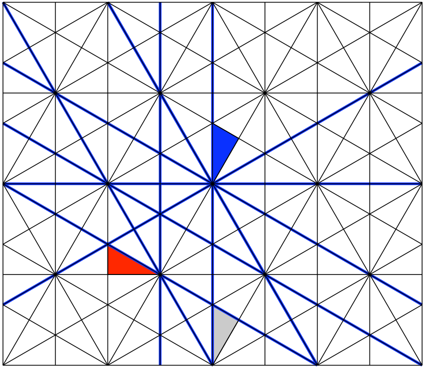

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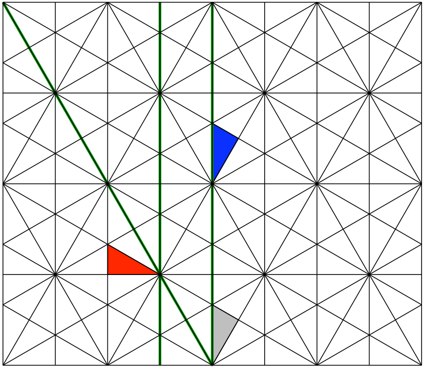

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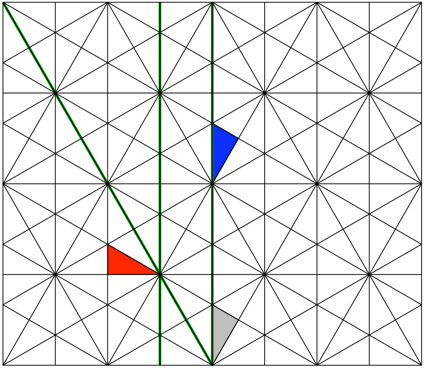

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On this example, we have

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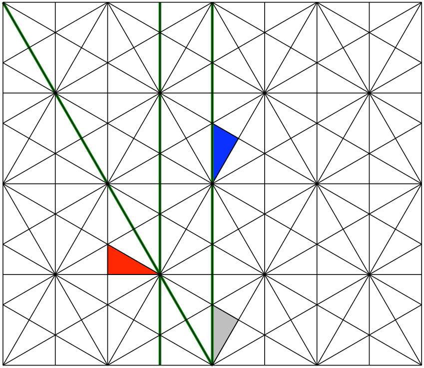

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Proposition. G. ($\sim$ 2006)

We have:

$$T_x T_y = \sum_{z \in W} f_{x,y,z} T_z \text{ where } \deg(f_{x,y,z}) \leq c_{x,y}.$$

Theorem. GECK ($\sim$ 2003)

Let $W' \subseteq W$ be a standard parabolic subgroup, and let X' be the set of all $w \in W$ such that w has minimal length in the coset wW'. Let $\mathcal{C}$ be a left cell of W'. Then $X'.\mathcal{C}$ is a union of left cells.

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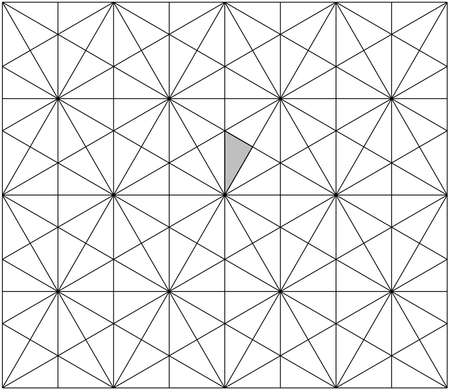
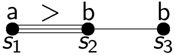


Let's take the example of $\tilde{G}_2$ with $W' = \langle s_1, s_2 \rangle$ and parameters :



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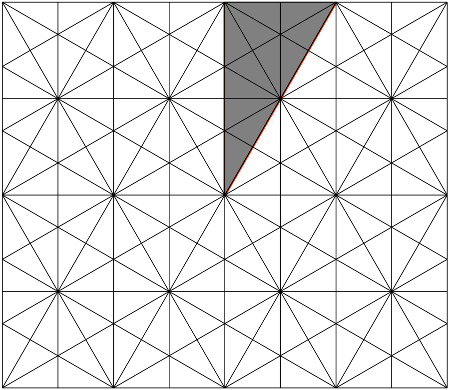

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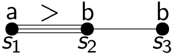

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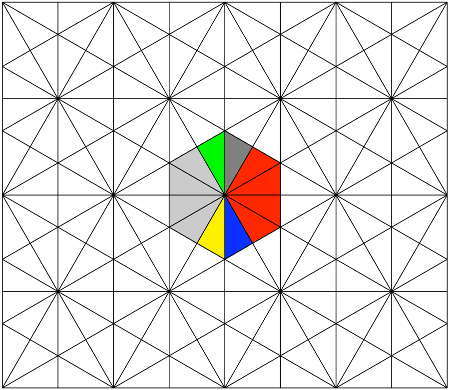

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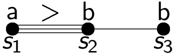

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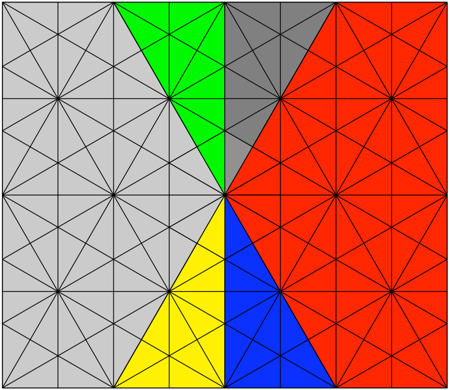

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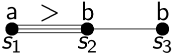

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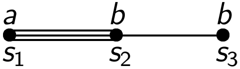
$W = \tilde{G}_2$:



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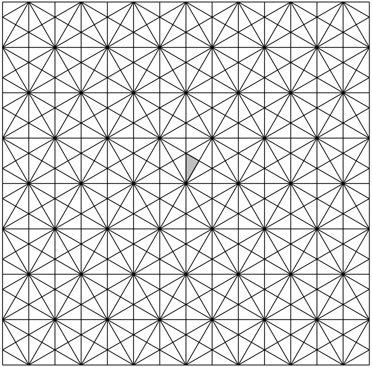

For $J \subset S$, we denote by W_J the group generated by J and by w_J the longest element of W_J. We look at the subsets J of S such that the group generated by J is isomorphic to W_0. Here, we find just $J = \{s_1, s_2\}$ and $w_J = s_1 s_2 s_1 s_2 s_1 s_2$ Then:

$$c_0 = \{w \in W | w = z.w_J.z', \ z, z' \in W\}$$

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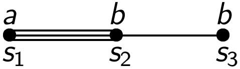

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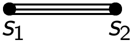

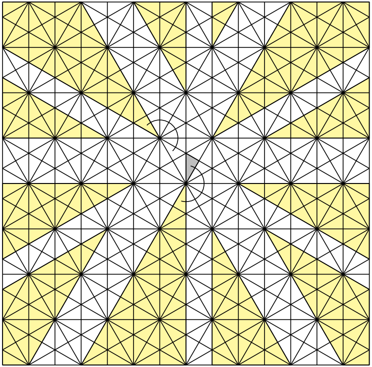

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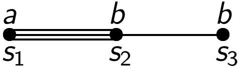
$W = \tilde{G}_2 :$ 

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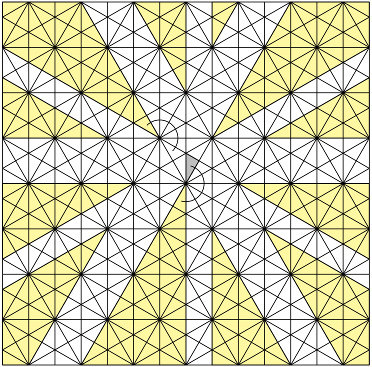

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Then:

$$c_0 = \{w \in W | w = z.w_J.z', \ z, z' \in W\}$$

Moreover, let $M_J = \{z \in W | sw_Jz \notin c_0, \text{ for all } s \in J\}$. We have:

$$c_0 = \bigcup_{z \in M_J} \{w \in W | w = x.w_J.z, \ x \in W\}$$

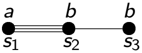
We have:

Theorem. G. ($\sim$ 2007)

Let $z \in M_J$. The set $\{w \in W | w = x.w_J.z,\ x \in W\}$ is a union of left cells.

This implies that:

- c_0 contains exactly $|W_0|$ left cells.
- For $z \in M_J$, the set $\{w \in W | w = x.w_J.z,\ x \in W\}$ is a left cell.



, for all $a > 3b$