On positivity properties in Hecke algebras of arbitrary Coxeter groups

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Coxeter groups and Artin-Tits groups

Hecke algebras

Positivity properties and Soergel bimodules

Main results
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Let \((W, S)\) be a **Coxeter system**, i.e., \(W\) is a group generated by \(S = \{s_1, \ldots, s_n\}\) with presentation

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W = \langle s_1, \ldots, s_n \mid s_i^2 = e, \quad s_is_j \cdots = s_js_i \cdots \text{ if } i \neq j \rangle,
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where \(m_{ij} = m_{ji} \in \{2, 3, \ldots\} \cup \{\infty\}\).
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Coxeter groups and their Artin-Tits groups

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where $m_{ij} = m_{ji} \in \{2, 3, \ldots\} \cup \{\infty\}$.

Denote by $\ell : W \to \mathbb{Z}_{\geq 0}$ the length function, by $T = \bigcup_{w \in W} ws w^{-1}$ the set of reflections of $W$ and by $\leq$ the (strong) Bruhat order.
Coxeter groups and their Artin-Tits groups

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Let \(B(W) = B(W, S)\) be the **Artin-Tits group** attached to \((W, S)\), that is, \(B(W)\) is generated by a copy \(\{s_1, \ldots, s_n\}\) of the elements of \(S\) and has a presentation
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Coxeter groups and their Artin-Tits groups

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- The symmetric group $W = \mathfrak{S}_n$, is a Coxeter group with
  $S = \{s_i = (i, i + 1) \mid i = 1, \ldots, n - 1\}$, $m_{ij} = 3$ if
  $|i - j| = 1$, $m_{ij} = 2$ if $|i - j| > 1$.
  $T = \{\text{transpositions}\}$. 
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Coxeter groups and their Artin-Tits groups

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- The corresponding group $B(W)$ is the Artin braid group $B_n$ on $n$ strands.

Given $w = s_1s_2 \cdots s_k$ with $\ell(w) = k$, the lift $s_1s_2 \cdots s_k$ in $B(W)$ is well-defined and denoted by $w$. 
Hecke algebra of a Coxeter system
Let $A = \mathbb{Z}[v, v^{-1}]$. Let $\mathcal{H}(W) = \mathcal{H}(W, S)$ be the Hecke algebra attached to $(W, S)$, that is, the associative unital $A$-algebra with a presentation

\[
\left\langle T_{s_1}, \ldots, T_{s_n}, s_i \in S \right\rangle \quad \begin{array}{l}
T_{s_i} T_{s_j} \cdots = T_{s_j} T_{s_i} \cdots \\
\quad \quad \quad \quad \quad m_{ij} \quad \quad \quad \quad m_{ij} \\
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Since the $T_{s_i}$ satisfy the braid relations, there is a group homomorphism $a : B(W) \to \mathcal{H}(W)^\times$, $a(s_i) = T_{s_i}$. 

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Main results
Let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. Let $\mathcal{H}(W) = \mathcal{H}(W, S)$ be the Hecke algebra attached to $(W, S)$, that is, the associative unital $\mathcal{A}$-algebra with a presentation

$$\langle T_{s_1}, \ldots, T_{s_n}, s_i \in S \rangle \left| \begin{align*}
T_{s_i}T_{s_j} \cdots &= T_{s_j}T_{s_i} \cdots \\
&\text{ for all } i, j = 1, \ldots, n, m_{ij} \\
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\end{align*} \right.$$ 

Since the $T_{s_i}$ satisfy the braid relations, there is a group homomorphism $a : B(W) \to \mathcal{H}(W)^\times$, $a(s_i) = T_{s_i}$.

For $w \in W$, let $T_w := a(w)$. The set $\{T_w\}_{w \in W}$ is a basis of $\mathcal{H}(W)$ as an $\mathcal{A}$-module, called standard.
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Each \( T_w \) is invertible and \( \{T_{w^{-1}}\}_{w \in W} \) is also a basis of \( \mathcal{H}(W) \), called costandard.
Kazhdan-Lusztig canonical bases

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There is an involution $^\sim : \mathcal{H}(W) \to \mathcal{H}(W)$ s.t. $\bar{v} = v^{-1}$, $T_w = (T_{w^{-1}})^{-1}$. For $w \in W$, set $H_w := v^{\ell(w)} T_w$. 
Theorem (Kazhdan-Lusztig, 1979)

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Theorem (Kazhdan-Lusztig, 1979)

- For any $w \in W$, there is a unique $C'_w \in \mathcal{H}(W)$ such that $\overline{C'_w} = C'_w$ and $C'_w \in H_w + \sum_{y < w} v \mathbb{Z}[v] H_y$. 
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Let \( C'_w = \sum_{y \leq w} h_{y,w} T_y \). Then \( h_{y,w} \in \mathbb{Z}_{\geq 0}[v] \).
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- Proven for (finite and affine) Weyl groups by KL in 1980; recently (2014) Elias and Williamson proved Soergel’s conjecture, which solves the general case.
Other positivity statements
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- Grojnowski and Haiman (2004): geometric proof of (D1) for affine Weyl groups.
About proof of KL positivity: Soergel bimodules
Soergel (1992) described the (equivariant) intersection cohomology of Schubert varieties using a remarkable family of graded bimodules over a polynomial algebra. He then generalized these bimodules to arbitrary Coxeter systems and linked them to KL positivity.
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$$B_s := R \otimes_{R^s} R(1).$$

It is an (indecomposable) graded $R$-bimodule.
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2. Let $\mathcal{B}$ be the Karoubian envelope of the category generated by (shifts of) products of the $B_s$. The indecomposables in $\mathcal{B}$ are (up to iso) given by the $B_w(i)$, $w \in W$, $i \in \mathbb{Z}$. 
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3. There is an isomorphism of rings \( \mathcal{E} : \mathcal{H}(W) \rightarrow \langle \mathcal{B}, \otimes_R \rangle \), \( \mathcal{E}(C'_s) = \langle B_s \rangle \), \( \mathcal{E}(v) = \langle R(1) \rangle \).
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3. There is an isomorphism of rings $\mathcal{E} : \mathcal{H}(W) \longrightarrow \langle \mathcal{B}, \otimes_R \rangle$, $\mathcal{E}(C'_s) = \langle B_s \rangle$, $\mathcal{E}(v) = \langle R(1) \rangle$. Its inverse is given by $\text{ch}(\langle B \in \mathcal{B} \rangle) = \sum_{x \in W} \sum_{i \in \mathbb{Z}} [B : R_x(i - \ell(x))] v^{i + \ell(x)} T_x$.

Conjecture (Soergel 2007; proven by Elias and Williamson 2014)

$\mathcal{E}(C'_w) = \langle B_w \rangle$ for all $w \in W$.
Standard filtrations of Soergel bimodules
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**Proposition (Soergel, 2007)**

Let $w_0 = e, w_1, w_2, \ldots$ be an enumeration of $W$ refining $\leq$. For $x \in W$, 
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Let $w_0 = e, w_1, w_2, \ldots$ be an enumeration of $W$ refining $\leq$. For $x \in W$, let $R_x$ be the graded bimodule $R$ with right operation twisted by $x$. Each $B \in \mathcal{B}$ has a unique filtration

$$0 = B^0 \subset B^1 \subset B^2 \subset \cdots \subset B^k = B$$

with $B^i / B^{i-1} \cong \bigoplus_p R_{w_i}(n_p)$. 
Standard filtrations of Soergel bimodules

- Soergel’s conjecture implies KL positivity for all $W$.
- The coefficients of the KL polynomials are interpreted as graded multiplicities; more precisely

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- It follows from Soergel’s conjecture that these multiplicities categorify the KL polynomials when $B = B_w$. 
Back to Dyer’s conjectures
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Rewrite

(D1) For all \( w, y \in W \), \( C'_w T_y \in \sum_{x \in W} \mathbb{Z}_{\geq 0}[v^{\pm 1}]T_x \).
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This suggests to interpret the coefficients in (D1) as graded multiplicities of alternative filtrations of Soergel bimodules $B_w$. 
On positivity properties in Hecke algebras of arbitrary Coxeter groups

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Coxeter groups and Artin groups

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Main results

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This suggests to interpret the coefficients in \((D1)\) as graded multiplicities of alternative filtrations of Soergel bimodules \(B_w\). What can we try to modify in Soergel’s approach to get alternative filtrations? Twist the Bruhat order by \(y\): define

\[u \leq_y v \iff uy \leq vy.\]
Theorem (G., 2016)

- \((D1')\) holds for arbitrary \(W\).
- \((D2)\) holds for arbitrary \(W\).

(In particular \((D1) - (D2)\) hold for arbitrary Coxeter groups).
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About the proof of $(D2)$: categorification of Artin groups
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On positivity properties in Hecke algebras of arbitrary Coxeter groups

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Positivity properties and Soergel bimodules

Main results
Let $K^b\mathcal{B}$ be the bounded homotopy category of $\mathcal{B}$. It is a triangulated category and as such, it has a Grothendieck group $\langle K^b\mathcal{B} \rangle_\Delta$. It is a general fact for an additive category $\mathcal{C}$ that $\langle \mathcal{C} \rangle \cong \langle K^b\mathcal{C} \rangle_\Delta$ (as abelian groups).
About the proof of (D2): categorification of Artin groups

Let $K^b(B)$ be the bounded homotopy category of $B$. It is a triangulated category and as such, it has a Grothendieck group $\langle K^b(B) \rangle_\Delta$. It is a general fact for an additive category $C$ that $\langle C \rangle \cong \langle K^b(C) \rangle_\Delta$ (as abelian groups). Here $\otimes_R$ induces a total tensor product of complexes $\otimes_R^{\text{tot}}$ compatible with this isomorphism. Hence $\langle K^b(B) \rangle_\Delta \cong \langle B \rangle$ (as $A$-algebras).
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Rouquier showed that the complexes $F_s := 0 \to B_s \to R(1) \to 0$, $s \in S$ (with $B_s$ in cohom. degree zero) admit an inverse $E_s$ for $\otimes^\text{tot}_R$ in $K^b(B)$ and that they satisfy the braid relations of $W$. 
About the proof of \((D2)\): categorification of Artin groups

Let \(K^{b}(\mathcal{B})\) be the bounded homotopy category of \(\mathcal{B}\). It is a triangulated category and as such, it has a Grothendieck group \(\langle K^{b}(\mathcal{B}) \rangle_{\Delta}\). It is a general fact for an additive category \(\mathcal{C}\) that \(\langle \mathcal{C} \rangle \cong \langle K^{b}(\mathcal{C}) \rangle_{\Delta}\) (as abelian groups). Here \(\otimes_{R}\) induces a total tensor product of complexes \(\otimes^{\text{tot}}_{R}\) compatible with this isomorphism. Hence \(\langle K^{b}(\mathcal{B}) \rangle_{\Delta} \cong \langle \mathcal{B} \rangle\) (as \(\mathcal{A}\)-algebras).

Rouquier showed that the complexes \(F_{s} := 0 \to B_{s} \to R(1) \to 0\), \(s \in S\) (with \(B_{s}\) in cohom. degree zero) admit an inverse \(E_{s}\) for \(\otimes^{\text{tot}}_{R}\) in \(K^{b}(\mathcal{B})\) and that they satisfy the braid relations of \(W\). In fact, viewed as functors on \(K^{b}(\mathcal{B})\) via \(F_{s} \otimes^{\text{tot}}_{R} \), they provide a categorical action of \(B(W)\) on \(K^{b}(\mathcal{B})\). This action is conjecturally faithful (proven for finite \(W\)).
Categorifications of Mikado braids
In particular, we get complexes of Soergel bimodules categorifying every element $\beta \in B(W)$ (defined only up to homotopy).
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- Every complex $C^\bullet$ in $K^b(B)$ admits a minimal complex $C^\bullet,\text{min}$, that is, with no contractible summand of the form $0 \to M \xrightarrow{\text{isom.}} M' \to 0$. 
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Every complex $C^\bullet$ in $K^b(B)$ admits a *minimal complex* $C^\bullet,\text{min}$, that is, with no contractible summand of the form $0 \to M \xrightarrow{\text{isom.}} M' \to 0$. This complex is unique up to isomorphism of complexes.
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Let $x, y \in W$, $\beta(x, y) := xy^{-1}$, $T_x T_y^{-1} = \sum_{w \in W} q_{x,w}^y C_w$. 

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1. Let $w \in W$. The bimodule $B_w$ appears as a direct summand in $C_{\beta(x,y)}^\bullet,\text{min}$ either only in odd cohomological degrees or only in even degrees.
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**Theorem (G., 2016)**

Let $x, y \in W$, $\beta(x, y) := xy^{-1}$, $T_x T_y^{-1} = \sum_{w \in W} q^y_{x, w} C_w$.

1. Let $w \in W$. The bimodule $B_w$ appears as a direct summand in $C^\bullet,\text{min}_{\beta(x, y)}$ either only in odd cohomological degrees or only in even degrees.

2. The coefficient $q^y_{x, w}$ gives the multiplicity of $B_w$ in all cohom. degrees of $C^\bullet,\text{min}_{\beta(x, y)}$ together. $\Rightarrow q^A_{x, w} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$. 


Linearity of complexes
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A key point in the proof of the theorem above is to show that the complex $C_{x,y}^{\bullet, \text{min}}$ is linear, that is, that every indecomposable summand in cohomological degree $i$ has graduation shift equal to $i$. 
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A key point in the proof of the theorem above is to show that the complex $C_{\beta(x,y)}^{\bullet, \min}$ is linear, that is, that every indecomposable summand in cohomological degree $i$ has graduation shift equal to $i$. It precisely means that $C_{\beta(x,y)}^{\bullet, \min}$ lies in the heart of the canonical $t$-structure on $K^b(\mathcal{B})$.

Open problem: Understand the perverse cohomology groups of the Rouquier complexes $C_{\beta(x,y)}^{\bullet, \min}$.  

Thank you!