On Sums of Indicator Functions in Dynamical Systems

Olivier Durieu* & Dalibor Volný†

Laboratoire de Mathématiques Raphaël Salem,
UMR 6085 CNRS-Université de Rouen

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Abstract

In this paper, we are interested in the limit theorem question for sums of indicator functions. We show that in every invertible ergodic dynamical system, for every increasing sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $a_n \nearrow \infty$ and $\frac{a_n}{n} \to 0$ as $n \to \infty$, there exists a dense G_δ of measurable sets A such that the sequence of the distributions of the partial sums $\frac{1}{a_n}\sum_{i=0}^{n-1}(\mathbb{1}_A - \mu(A)) \circ T^i$ is dense in the set of the probability measures on $\mathbb{R}$.

Keywords: Dynamical system; Ergodic theorem; Sums of random variables; Limit theorem; Genericity.

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1 Introduction

In [1], Burton and Denker showed that in every aperiodic dynamical system there exists a process $(f \circ T^i)$ for which the CLT holds and they posed the question how big is the subset of $f \in L^2$ with this property. Clearly, we have to study the space L^2_0 of f with $E(f|\mathcal{I}) = 0$. As already observed by Burton and Denker, because the coboundaries are dense in L^2_0, this set is dense. In Volný [7] it has been proved that for any sequence $a_n \to \infty$, $\frac{a_n}{n} \to 0$, there exists a dense G_δ part G of L^2_0 such that for any $f \in G$ and any probability law ν there exists a sequence $n_k \to \infty$ such that $\frac{1}{a_{n_k}} S_{n_k}(f)$ converge in law to ν. The same result takes place for all spaces L^p, $1 \le p \le \infty$. Liardet and Volný [6] obtained the same result for the space of continuous functions for a uniquely ergodic continuous homeomorphism of a

*e-mail: olivier.durieu@univ-rouen.fr
†e-mail: dalibor.volny@univ-rouen.fr

metrizable compact space and similar results for spaces of smooth functions for irrational rotations of a circle. As a corollary we get that generically, the rate of convergence in the ergodic theorems (of Birkhoff and of von Neumann) may be arbitrarily slow. This gave a new proof of a result of A. del Junco and J. Rosenblatt [3]. In the paper of del Junco and Rosenblatt a similar result on the rate of convergence in the ergodic theorems was found for functions $\mathbb{1}_A - \mu(A)$, the genericity was studied in the space of $A \in \mathcal{A}$ equipped with the (pseudo)metric of the measure of symmetric difference.

In the present paper we shall study the distributional convergence for the functions $\mathbb{1}_A - \mu(A)$. The research was motivated by the study of the invariance principle of the empirical process of strictly stationary sequences $(X_i)_{i\in\mathbb{N}}$ in Dehling, Durieu and Volný [2].

2 Result

Let $(\Omega, \mathcal{A}, \mu)$ be a non-atomic Lebesgue probability space and T be an invertible measurable transformation from Ω to Ω. We say that T is measure preserving if for all $A \in \mathcal{A}$, $\mu(T^{-1}A) = \mu(A)$. In the sequel, we will often say that $(\Omega, \mathcal{A}, \mu, T)$ is a dynamical system when $(\Omega, \mathcal{A}, \mu)$ is a non-atomic Lebesgue probability space and T is an invertible measure preserving transformation of Ω.

Further, the transformation T is ergodic if $T^{-1}A = A$ implies that $\mu(A) = 0$ or 1. It is aperiodic if

$$\mu\{x \in \Omega \,/\, \exists n \geq 1, T^n x = x\} = 0.$$

On $\mathcal{A}$ we consider the pseudo-metric Θ defined by

$$\Theta(A, B) = \mu(A \triangle B), \quad A, B \in \mathcal{A}.$$

Our main result is the following, where $X_n \xrightarrow[n\to\infty]{\mathcal{D}} \nu$ means that the sequence of real random variables X_n converges in distribution to a real random variable having distribution ν.

Theorem 1 *Let $(\Omega, \mathcal{A}, \mu)$ be a non-atomic Lebesgue probability space and T be an ergodic invertible measure preserving transformation of Ω. Let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ be an increasing sequence such that $a_n \nearrow \infty$ and $\frac{a_n}{n} \to 0$ as $n \to \infty$.*

There exists a Θ-dense G_δ of sets $A \in \mathcal{A}$ having the property that for every probability ν on $\mathbb{R}$, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ satisfying

$$\frac{1}{a_{n_k}} S_{n_k}(\mathbb{1}_A - \mu(A)) \xrightarrow[k\to\infty]{\mathcal{D}} \nu.$$

To be complete, notice that, in the non-ergodic case, to find a set which satisfies the conclusion of Theorem 1, we have to consider the sets A such that $\mathbb{E}(\mathbb{1}_A|\mathcal{I})$ is almost surely constant. The class of such sets is not, in general, dense in $\mathcal{A}$. So, in the non-ergodic case, we cannot expect the result of genericity.

Nevertheless, we can prove the existence of such sets by an explict construction. This is the purpose of the paper by Durieu and Volný [5].

3 Some preliminary results

Let $\mathcal{M}$ be the set of all probability measures on $\mathbb{R}$ and $\mathcal{M}_0$ be the set of all probability measures on $\mathbb{R}$ which have zero-mean. Recall that $\mathcal{M}_0$ is dense in $\mathcal{M}$ for the topology of the weak convergence. We denote by d the Lévy metric on $\mathcal{M}$. For all μ and ν in $\mathcal{M}$ with distribution functions F and G,

$$d(\mu,\nu) = \inf\{\varepsilon > 0 \,/\, G(t-\varepsilon) - \varepsilon \leq F(t) \leq G(t+\varepsilon) + \varepsilon, \forall t \in \mathbb{R}\}.$$

The space $(\mathcal{M}, d)$ is a complete separable metric space and convergence with respect to d is equivalent to weak convergence of distributions (see Dudley [4], pages 394-395).

If $X : \Omega \longrightarrow \mathbb{R}$ is a random variable, we denote by $\mathcal{L}_\Omega(X)$ the distribution of X on $\mathbb{R}$.

Lemma 3.1 *Let $(\Omega, \mathcal{A}, \mu)$ be a Lebesgue probability space and ν be a probability on $\mathbb{R}$. Then, there exists a random variable $X : \Omega \longrightarrow \mathbb{R}$, such that*

$$\mathcal{L}_\Omega(X) = \nu.$$

Proof.

It is well known that $(\Omega, \mathcal{A}, \mu)$ is isomorphic to $([0,1], \mathcal{B}[0,1], \lambda)$, where λ is the Lebesgue measure on $[0,1]$. If Q_ν denotes the pseudo-inverse of the distribution function of ν and U is the identity on $[0,1]$, it is classical that $\mathcal{L}_{[0,1]}(Q_\nu(U)) = \nu$. □

Let ν be a probability on $\mathbb{R}$. For $B \in \mathcal{B}(\mathbb{R})$ with $\nu(B) > 0$, ν_B denotes the probability on $\mathbb{R}$ defined by

$$\nu_B(A) = \nu(B)^{-1}\nu(A \cap B).$$

For $x \in \mathbb{R}$, ν_x denotes the probability on $\mathbb{R}$ defined by

$$\nu_x(B) = \nu(xB)$$

where $xB = \{xb \,/\, b \in B\}$.

Here are some properties of the Lévy metric which will be used in the sequel.

Lemma 3.2

(i) *For each probability ν on $\mathbb{R}$, for all Borel sets B,*

$$d(\nu_B, \nu) \leq \nu(\mathbb{R} \setminus B).$$

(ii) *For all probabilities ν and η on $\mathbb{R}$, for all $x \geq 1$,*

$$d(\nu_x, \eta_x) \leq d(\nu, \eta).$$

(iii) *For all probability ν on $\mathbb{R}$, for all mesurable functions f and g from Ω to $\mathbb{R}$,*

$$d(\mathcal{L}_\Omega(f+g), \nu) \leq (\mathcal{L}_\Omega(f), \nu) + d(\mathcal{L}_\Omega(g), \delta_0)$$

where δ_0 is the Dirac measure at 0.

(iv) *For all probability* ν *on* $\mathbb{R}$,

$$d(\nu, \delta_0) \leq A \text{ if and only if } \nu((-\infty, -A)) \leq A \text{ and } \nu((A, \infty)) \leq A.$$

The proof is an exercise which is left to the reader.

Lemma 3.3 *Let* $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ *be an increasing sequence such that* $a_n \nearrow \infty$ *as* $n \to \infty$. *For each probability* $\nu \in \mathcal{M}_0$ *and* $\varepsilon > 0$, *there exist* $C \geq 1$ *and* $n_0 \in \mathbb{N}$, *for all* $n \geq n_0$, *there exists a probability* η *on* $\mathbb{R}$ *with support* $S \subset [-a_nC, a_nC] \cap \mathbb{Z}$ *such that*

$$d(\eta_{a_n}, \nu) \leq \varepsilon$$

and

$$\mathbb{E}(\eta) := \int x \mathrm{d}\eta = 0.$$

Proof.

Let $\nu \in \mathcal{M}_0$ and $\varepsilon > 0$ be fixed and choose $\alpha > 0$ such that $6\alpha \leq \varepsilon$ and $\alpha < \frac{1}{2}$. There exists $C \geq 1$ such that

$$\int_{\mathbb{R}\smallsetminus[-C,C]} |x|\mathrm{d}\nu(x) \leq \alpha.$$

In particular, $\nu(\mathbb{R} \smallsetminus [-C, C]) \leq \alpha$. Define $\tau = \nu_{[-C,C]}$. By Lemma 3.2,

$$d(\tau, \nu) \leq \nu(\mathbb{R} \smallsetminus [-C, C]) \leq \alpha$$

and we have

$$|\mathbb{E}(\tau)| \leq \nu([-C, C])^{-1}|\mathbb{E}(\nu) - \int_{\mathbb{R}\smallsetminus[-C,C]} x\mathrm{d}\nu(x)| \leq \frac{\alpha}{1-\alpha}.$$

Now, choose $n_0 \in \mathbb{N}$ such that $\frac{1}{a_{n_0}} < \alpha$ and fix $n \geq n_0$. Then we define the probability η' on $\mathbb{R}$ with support in $\mathbb{Z}$, by $\eta'(\{k\}) := \tau\left(\left[\frac{k}{a_n}, \frac{k+1}{a_n}\right)\right)$, $k \in \mathbb{Z}$.

We have, for all $t \in \mathbb{R}$,

$$\begin{aligned}
\eta'_{a_n}((-\infty, t]) &= \eta'((-\infty, \lfloor ta_n \rfloor]) \\
&= \tau\left(\left(-\infty, \frac{\lfloor ta_n \rfloor + 1}{a_n}\right)\right) \\
&\leq \tau\left(\left(-\infty, t + \frac{1}{a_n}\right]\right)
\end{aligned}$$

and

$$\tau((-\infty, t]) \leq \eta'_{a_n}((-\infty, t]).$$

Thus $d(\eta'_{a_n}, \tau) \leq \frac{1}{a_n} \leq \alpha$ and $d(\eta'_{a_n}, \nu) \leq 3\alpha$.

So, if $\mathbb{E}(\eta') = 0$, η' verifies the conclusion of the proposition. If it is not the case, we proceed as follows. Observe that

$$a_n\mathbb{E}(\tau) - 1 \leq \mathbb{E}(\eta') \leq a_n\mathbb{E}(\tau),$$

and thus, since $\alpha < \frac{1}{2}$ and $a_n\alpha > 1$,

$$|\mathbb{E}(\eta')| \leq a_n|\mathbb{E}(\tau)| + 1 \leq a_n\frac{\alpha}{1-\alpha} + 1 \leq 3a_n\alpha.$$

We denote by s the sign of $\mathbb{E}(\eta')$ and we set $p = 1 + \frac{|\mathbb{E}(\eta')|}{\lfloor a_n C\rfloor}$.

Now we denote by η the probability on $\mathbb{R}$ with support in $\{-\lfloor a_n C\rfloor, \ldots, \lfloor a_n C\rfloor\}$ defined by

$$\eta(\{i\}) = \begin{cases} \frac{1}{p}(\eta'(\{i\}) + \frac{|\mathbb{E}(\eta')|}{\lfloor a_n C\rfloor}) & \text{if} \quad i = -s\lfloor a_n C\rfloor \\ \frac{1}{p}\eta'(i) & \text{otherwise} \end{cases}$$

Then $\mathbb{E}(\eta) = 0$ and by Lemma 3.2 (ii),

$$d(\eta'_{a_n}, \eta_{a_n}) \leq d(\eta', \eta) \leq \frac{|\mathbb{E}(\eta')|}{\lfloor a_n C\rfloor} \leq 3\alpha.$$

Therefore $d(\eta_{a_n}, \nu) \leq \varepsilon$. ☐

Proposition 3.4 *Let $(\Omega, \mathcal{A}, \mu, T)$ be an ergodic dynamical system, $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ be an increasing sequence such that $a_n \nearrow \infty$ and $\frac{a_n}{n} \to 0$ as $n \to \infty$ and $\varepsilon > 0$. Let $A \in \mathcal{A}$ be a set such that $\mu(A) < 1$ and ν be a probability in $\mathcal{M}_0$. There exists $N \in \mathbb{N}$ such that for any $n \geq N$, there exists a set $B_n \in \mathcal{A}$ such that $\mu(B_n) \leq \varepsilon$, $A \cap B_n = \emptyset$ and*

$$d(\mathcal{L}_\Omega(\frac{1}{a_n}S_n(\mathbb{1}_{B_n} - \mu(B_n))), \nu) \leq \varepsilon$$

Proof.

Fix $\varepsilon > 0$ and $A \in \mathcal{A}$ such that $\mu(A) < 1$. Let α be a positive constant such that $\alpha \leq \frac{\varepsilon}{5}$ and $\mu(A) + 2\alpha < 1$.

By Lemma 3.3 applied to ν and α we get the constants $C \geq 1$ and $n_0 \geq 1$ for which the conclusion of the lemma holds. Set $\gamma := \frac{\alpha}{C+1}$. Let n_1 be an integer such that, for all $n \geq n_1$,

$$2\frac{a_n C + 1}{n} \leq \alpha. \tag{1}$$

Applying Birkhoff's ergodic theorem and Egorov's theorem, we get that there exist a set $E \in \mathcal{A}$ of measure greater than $1 - \frac{\gamma}{2}$ and an integer $n_2 \geq 1$ such that for all $n \geq n_2$, for all $x \in E$,

$$\left|\frac{1}{n}S_n(\mathbb{1}_A - \mu(A))(x)\right| \leq \alpha. \tag{2}$$

We denote by $\overline{n}$ the maximum of n_0, n_1 and n_2 and we choose $N \in \mathbb{N}$ such that

$$\frac{\overline{n}}{N} \leq \alpha.$$

For any $n \geq N$, there exists a Rokhlin tower of height n with base $F \subset E$ and junk set of measure smaller than γ.

Indeed, let G be the base of a Rokhlin tower of height n and of measure greater than $1 - \frac{\gamma}{2}$. Because $\mu(\Omega \setminus E) \leq \frac{\gamma}{2}$, there exists an integer $i_0 \in \{0, \ldots, n-1\}$ such that

$$\mu((T^{i_0}G) \cap E) \geq \frac{1}{n}(1 - \frac{\gamma}{2} - \frac{\gamma}{2})) = \frac{1-\gamma}{n}.$$

If $F = T^{i_0}G \cap E$, then $F \subset E$ and the sets F, TF,...,$T^{n-1}F$ are disjoint. So, F is the base of a Rokhlin tower of height n with a junk set of measure smaller than γ.

From now on, n is fixed. By Lemma 3.3, there exists a centered probability η with support in $S \subset [-a_nC, a_nC] \cap \mathbb{Z}$ such that

$$d(\eta_{a_n}, \nu) \leq \alpha.$$

By Lemma 3.1, there exists a function $h : F \longrightarrow \mathbb{Z}$ such that $\mathcal{L}_F(h) = \eta$. In particular, $\mathbb{E}_F(h) := \int_F h \mathrm{d}\mu_F = 0$. We set $d = \lfloor a_nC \rfloor + 1$ and

$$\begin{array}{rcl} g : F & \longrightarrow & \mathbb{Z} \\ x & \longmapsto & h(x) + d. \end{array}$$

Note that $1 \leq g \leq 2d$ almost surely and $\mathbb{E}_F(g) = d$. We now set

$$F_i := g^{-1}(\{i\}),\ i = 1, \ldots, 2d.$$

Note that the F_i's depend on ν, α, C and n. Further $\{F_1, F_2, \ldots, F_{2d}\}$ is a partition of the set F.

By (2), for each $x \in F$, the sub-orbit $\{x, Tx, \ldots, T^{\overline{n}-1}x\}$ hits A at most $\overline{n}(\mu(A) + \alpha)$ times (*i.e.* $\#(\{x, Tx, \ldots, T^{\overline{n}-1}x\} \cap A) \leq \overline{n}(\mu(A) + \alpha)$. Since by (1), $\frac{2d}{\overline{n}} \leq \alpha$ and $\mu(A) + 2\alpha < 1$, we can find $2d$ points in this sub-orbit which are not in A.

Then, for each $i = 1, \ldots, 2d$ and for each $x \in F$, we can denfine the set $b_i(x)$ composed by the i first points of the sub-orbit $\{x, Tx, \ldots, T^{\overline{n}-1}x\}$ which are not in A.

We now set, for each $i = 1, \ldots, 2d$,

$$B_i = \bigcup_{x \in F_i} b_i(x).$$

Thus the B_i's are disjoint measurable sets. For each $i = 1, \ldots, 2d$, $B_i \subset \bigcup_{j=0}^{\overline{n}-1} T^j F_i$, $B_i \cap A = \emptyset$, $\mu(B_i) = i\mu(F_i)$ and for any $x \in F_i$,

$$S_{\overline{n}}(\mathbb{1}_{B_i})(x) = i.$$

Finally, we set

$$B = \bigcup_{i=1}^{2d} B_i.$$

We have $B \in \mathcal{A}$ and $A \cap B = \emptyset$.

From the construction of B and (1),

$$\mu(B) = \mathbb{E}_F(g)\mu(F) = d\mu(F) \leq \frac{a_n C + 1}{n} \leq \alpha \leq \varepsilon.$$

We define

$$\Omega_k = \bigcup_{i=0}^{n-\overline{n}-1} T^{-i}F_k, \quad k = 1, \ldots, 2d$$

and

$$\overline{\Omega} = \bigcup_{k=1}^{2d} \Omega_k.$$

Since the $T^{-i}F_k$ are disjoint, using (1) and the fact that $\gamma \leq \alpha$, we have

$$\mu(\overline{\Omega}) = (n - \overline{n})\mu(F) \geq 1 - \gamma - \frac{\overline{n}}{n} \geq 1 - 3\alpha. \tag{3}$$

For $x \in \overline{\Omega}$, by construction and by disjointness of the $T^{-i}F_k$, we have

$$S_n(\mathbb{1}_B)(x) = k \text{ if and only if } x \in \Omega_k.$$

Therefore, for all $k = 1, \ldots, 2d$,

$$\mu_{\overline{\Omega}}(S_n(\mathbb{1}_B) = k) = \mu_{\overline{\Omega}}(\Omega_k) = \mu_F(F_k).$$

Thus $\mathcal{L}_{\overline{\Omega}}(S_n(\mathbb{1}_B)) = \mathcal{L}_F(g)$ and by (3) and Lemma 3.2 (i),

$$d(\mathcal{L}_\Omega(S_n(\mathbb{1}_B)), \mathcal{L}_F(g)) \leq 3\alpha.$$

So, by Lemma 3.2 (ii),

$$d(\mathcal{L}_\Omega(\frac{1}{a_n} S_n(\mathbb{1}_B)), \mathcal{L}_F(\frac{g}{a_n})) \leq 3\alpha$$

and

$$d(\mathcal{L}_\Omega(\frac{1}{a_n}(S_n(\mathbb{1}_B) - d)), \mathcal{L}_F(\frac{g-d}{a_n})) \leq 3\alpha. \tag{4}$$

Now, remark that

$$d(\mathcal{L}_\Omega(\frac{1}{a_n}(S_n(\mathbb{1}_B) - d)), \mathcal{L}_\Omega(\frac{1}{a_n} S_n(\mathbb{1}_B - \mu(B)))) \leq \alpha. \tag{5}$$

Indeed, since $\mu(B) = d\mu(F)$, we have

$$-\gamma d \leq n\mu(B) - d \leq 0$$

and then

$$\left|\frac{1}{a_n}S_n(\mathbb{1}_B - \mu(B)) - \frac{1}{a_n}(S_n(\mathbb{1}_B) - d)\right| \leq \frac{\gamma d}{a_n} \leq \alpha.$$

To conclude, using (4), (5) and the fact that $d(\mathcal{L}_F(\frac{h}{a_n}), \nu) \leq \alpha$, we get

$$d(\mathcal{L}_\Omega(\frac{1}{a_n}S_n(\mathbb{1}_B - \mu(B))), \nu) \leq 5\alpha \leq \varepsilon.$$

□

Proposition 3.5 *Let $(\Omega, \mathcal{A}, \mu, T)$ be an ergodic dynamical system, $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ be an increasing sequence such that $a_n \nearrow \infty$ and $\frac{a_n}{n} \to 0$ as $n \to \infty$ and $\varepsilon > 0$. For any set $A \in \mathcal{A}$ such that $\mu(A) < 1$, there exists a set $B \in \mathcal{A}$ such that*

(i) *$\mu(A \triangle B) \leq \varepsilon$,*

(ii) *there exists a sequence $(n_k)_{k\geq 1}$ such that for all $k \geq 1$,*

$$d(\mathcal{L}_\Omega(\frac{1}{a_{n_k}}S_{n_k}(\mathbb{1}_B - \mu(B))), \delta_0) \leq \varepsilon.$$

This proposition will be proved as a corollary of the following lemma.

Lemma 3.6 *Let $(\Omega, \mathcal{A}, \mu, T)$ be an ergodic dynamical system, $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ be an increasing sequence such that $a_n \nearrow \infty$ and $\frac{a_n}{n} \to 0$ as $n \to \infty$ and $\varepsilon > 0$. For any set $A \in \mathcal{A}$ such that $\mu(A) < 1$, for any $N \in \mathbb{N}$, there exist $n \geq N$ and a set $C \in \mathcal{A}$ such that $\mu(A \triangle C) \leq \varepsilon$ and*

$$d(\mathcal{L}_\Omega(\frac{1}{a_n}S_n(\mathbb{1}_C - \mu(C))), \delta_0) \leq \varepsilon.$$

Proof.
Let $\varepsilon > 0$, $A \in \mathcal{A}$ such that $\mu(A) < 1$ and $N \in \mathbb{N}$ be fixed and let α be a positive constant such that $5\alpha \leq \varepsilon$ and $\mu(A) + 2\alpha < 1$.

By Birkhoff's ergodic theorem and Egorov's theorem, there exist a set $G \in \mathcal{A}$ with $\mu(G) > 1 - \alpha$ and an integer M such that for all $k \geq M$ and for all $x \in G$,

$$\left|\frac{1}{k}S_k(\mathbb{1}_A - \mu(A))(x)\right| \leq \alpha. \tag{6}$$

Furthermore, we can choose M such that $\frac{1-\alpha}{M} \leq \alpha$.

There exists an integer $n \geq N$, such that

$$\frac{M}{a_n} \leq \alpha. \tag{7}$$

Let F be the base of a Rokhlin tower of height Mn with a junk set of measure smaller than α. For $k = 0, \ldots, M-1$, we define the sets

$$F_k = \bigcup_{i=0}^{n-1} T^{iM+k} F.$$

Since $\mu(G) > 1 - \alpha$, there exists $k_0 \in \{0, \ldots, M-1\}$ such that

$$\mu(F_{k_0} \setminus G) \leq \frac{2\alpha}{M}.$$

Further, $H = T^{k_0} F$ is the base of a Rokhlin tower of height Mn with a junk set J such that $\mu(J) \leq \alpha$.

For $x \in \Omega$, we denote by $s_l(x) = \{x, Tx, \ldots, T^{l-1}x\}$ the sub-orbit of length l which begins at x. The set A will be modified in a way that for each orbit $s_M(x)$, $x \in F_{k_0}$, the average of visits of the set along the orbit be close to the measure $\mu(A)$.

For $x \in H$, we modify the set A along the sub-orbit $s_M(x)$ in the following way. We write $a(x) := A \cap s_M(x)$. There are two situations:

• If $x \in G$, by (6), along the sub-orbit $s_M(x)$, the number of visits of the set A belongs to $[M(\mu(A) - \alpha), M(\mu(A) + \alpha)]$ (*i.e.* $\#a(x) \in [M(\mu(A) - \alpha), M(\mu(A) + \alpha)]$). Further, $M(\mu(A) + \alpha) \leq (1 - \alpha)M$. So by adding or removing at most αM points to $a(x)$, we can modify the set $a(x)$ to get a set $a_0(x)$ (with $a_0(x) \subset s_M(x)$) such that along the sub-orbit $s_M(x)$, the number of visits of the set $a_0(x)$ belongs to $[M\mu(A) - 1, M\mu(A) + 1]$. A way to do that is to remove or add the $|\#a(x) - \lfloor M\mu(A) \rfloor|$ first points of $a(x)$ depending on whether $\#a(x) - \lfloor M\mu(A) \rfloor$ is positive or not.

• If $x \notin G$, we can also modify $a(x)$ in order to have that the number of visits of the set $a_0(x)$ along the segment $s_M(x)$ belongs to $[M\mu(A) - 1, M\mu(A) + 1]$. Here, to do that we possibly need to add or remove M points to $a(x)$ along $s_M(x)$.

To summarize, for each $x \in H$, we can modify the set $A \cap s_M(x)$ to get a set $a_0(x) \subset s_M(x)$ such that $\#a_0(x) \in [M\mu(A) - 1, M\mu(A) + 1]$ and if $x \in G$, $\#(a_0(x) \triangle a(x)) \leq \alpha M$, if $x \notin G$, $\#(a_0(x) \triangle a(x)) \leq M$.

We then have a set $A_0 = \bigcup_{x \in H} a_0(x)$ having the property that for all $x \in H$,

$$|S_M(\mathbb{1}_{A_0} - \mu(A))(x)| \leq 1 \tag{8}$$

(the problem of measurability for A_0 is not discussed, but we can see that the modifications can be done in a measurable way).

Now we will do almost the same modifications on sub-orbits of length M starting from $T^M H$.

For each $x \in T^M H$, we modify the set $a(x) = A \cap s_M(x)$ to get a set $a_1(x) \subset s_M(x)$ such that $\#a_1(x) \in [M\mu(A) - 1, M\mu(A) + 1]$.

Notice that for each $x \in T^M H$, considering the number $\#a_0(T^{-M}x)$, we can further define $a_1(x)$ in such a way that $\#(a_0(T^{-M}x) \cup a_1(x)) \in [2M\mu(A) - 1, 2M\mu(A) + 1]$. We keep the property that if $x \in T^M H \cap G$, $\#(a_1(x) \triangle a(x)) \leq \alpha M$, if $x \in T^M H \setminus G$, $\#(a_1(x) \triangle a(x)) \leq M$.

We can define $A_1 = \bigcup_{x \in T^M H} a_1(x)$. Then for all $x \in T^M H$,

$$|S_M(\mathbb{1}_{A_1} - \mu(A))(x)| \leq 1 \tag{9}$$

and for all $x \in H$,

$$|S_{2M}(\mathbb{1}_{A_0 \cup A_1} - \mu(A))(x)| \leq 1.$$

Now we do the same modifications for all points of $T^{2M}H$, $T^{3M}H$, ... , and $T^{(n-1)M}H$. Finally, we can get a measurable set $B = \bigcup_{i=0}^{n-1} A_i$ deduced from A and having the property that for all $k \in \{1, \dots, n\}$ and for each $x \in T^{kM}H$,

$$|S_M(\mathbb{1}_B - \mu(A))(x)| \leq 1 \tag{10}$$

Further, for all $x \in H$ and for all $k \in \{1, \dots, n\}$,

$$|S_{kM}(\mathbb{1}_B - \mu(A))(x)| \leq 1.$$

Note that we did not change the set A on the junk set J, so $A \cap J = B \cap J$. Recall that we did at most αM changes for points of $F_{k_0} \cap G$ and at most M for points of $F_{k_0} \setminus G$, so

$$\begin{aligned} \mu(A \triangle B) &\leq \alpha M \mu(F_{k_0} \cap G) + M\mu(F_{k_0} \setminus G) \\ &\leq 3\alpha. \end{aligned} \tag{11}$$

We also have

$$\begin{aligned} |\mu(B) - \mu(A)| &\leq \left| \int_H S_{Mn}(\mathbb{1}_B)(x) \mathrm{d}\mu(x) + \mu(B \cap J) - \mu(A) Mn \mu(H) - \mu(A)\mu(J) \right| \\ &\leq \int_H |S_{Mn}(\mathbb{1}_B - \mu(A))(x)| \mathrm{d}\mu(x) + |\mu(B \cap J) - \mu(A)\mu(J)| \\ &\leq \mu(H) + |\mu(B \cap J) - \mu(A)\mu(J)| \\ &\leq \mu(H) + \mu(J). \end{aligned}$$

Then, by changing the set B on only one level of the tower and on the junk set, we can obtain a new set $C \in \mathcal{A}$ such that $\mu(A) = \mu(C)$ and $\mu(B \triangle C) \leq \mu(H) + \mu(J) \leq 2\alpha$.

Thus, by (11),

$$\mu(A \triangle C) \leq 5\alpha \leq \varepsilon$$

and we have the following property: for all $x \in H$, for all $k, k' \in \{1, \ldots, n\}$,

$$|S_{kM}(\mathbb{1}_C - \mu(C))(x) - S_{k'M}(\mathbb{1}_C - \mu(C))(x)| \leq 3. \tag{12}$$

Let $\tilde{\Omega} = \bigcup_{i=0}^{n(M-1)} T^i H$. We have

$$\mu(\Omega \setminus \tilde{\Omega}) \leq n\mu(H) + \mu(J) \leq 2\alpha$$

and, by (12) and (7), for all $x \in \tilde{\Omega}$,

$$\frac{1}{a_n} |S_n(\mathbb{1}_C - \mu(C))(x)| \leq \frac{3 + 2M}{a_n} \leq 3\alpha$$

From these two inequalities, by Lemma 3.2 (iv), we deduce that

$$d(\mathcal{L}_\Omega(\frac{1}{a_n} S_n(\mathbb{1}_C - \mu(C))), \delta_0) \leq 3\alpha \leq \varepsilon.$$

□

Proof of Proposition 3.5.
Let $\varepsilon > 0$ and $A \in \mathcal{A}$ such that $\mu(A) < 1$ be fixed and choose $\varepsilon_1 \leq \frac{\varepsilon}{2}$.
By Lemma 3.6, there exist $n_1 \in \mathbb{N}$ and a set $C_1 \in \mathcal{A}$ such that $\mu(A \triangle C_1) \leq \varepsilon_1$ and

$$d(\mathcal{L}_\Omega(\frac{1}{a_{n_1}} S_{n_1}(\mathbb{1}_{C_1} - \mu(C_1))), \delta_0) \leq \varepsilon_1.$$

We will proceed by induction. After step $k-1$, we choose $\varepsilon_k = \frac{\varepsilon_{k-1}}{2n_{k-1}}$. By application of Lemma 3.6, there exist an integer $n_k \geq n_{k-1}$ and a set $C_k \in \mathcal{A}$ such that $\mu(C_{k-1} \triangle C_k) \leq \varepsilon_k$ and

$$d(\mathcal{L}_\Omega(\frac{1}{a_{n_k}} S_{n_k}(\mathbb{1}_{C_k} - \mu(C_k))), \delta_0) \leq \varepsilon_k.$$

Note that for all $i, j > 0$, $\varepsilon_{i+j} \leq \frac{\varepsilon_i}{2^j}$.

Finally, we set $B = \bigcup_{n \geq 1} \bigcap_{k \geq n} C_k$. Noticing that $D \triangle (E \cap F) \subset (D \triangle E) \cup (E \triangle F)$ for all sets D,E,F, we get

$$\mu(A \triangle B) \leq \sum_{k \geq 1} \varepsilon_k \leq \varepsilon$$

and for all $k \geq 1$, using Lemma 3.2 (iii) and (iv),

$$\begin{aligned}
d(\mathcal{L}_\Omega(\frac{1}{a_{n_k}}S_{n_k}(\mathbb{1}_B - \mu(B))), \delta_0) &\leq d(\mathcal{L}_\Omega(\frac{1}{a_{n_k}}S_{n_k}(\mathbb{1}_{C_k} - \mu(C_k))), \delta_0) \\
&\quad + d(\mathcal{L}_\Omega(\frac{1}{a_{n_k}}S_{n_k}(\mathbb{1}_{B \setminus C_k} - \mu(B \setminus C_k))), \delta_0) \\
&\quad + d(\mathcal{L}_\Omega(\frac{1}{a_{n_k}}S_{n_k}(\mathbb{1}_{C_k \setminus B} - \mu(C_k \setminus B))), \delta_0) \\
&\leq \varepsilon_k + n_k \mu(B \triangle C_k) \\
&\leq \varepsilon_k + n_k \sum_{i \geq k+1} \mu(C_{i-1} \triangle C_i) \\
&\leq \varepsilon_k + n_k \sum_{i \geq k+1} \varepsilon_i \\
&\leq \varepsilon.
\end{aligned}$$

□

4 Proof of Theorem 1

Let $(\Omega, \mathcal{A}, \mu, T)$ be an ergodic dynamical system. Let $(a_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$ be an increasing sequence such that $a_n \nearrow \infty$ and $\frac{a_n}{n} \to 0$ as $n \to \infty$.

Let $(\varepsilon_k)_{k \geq 1}$ be a decreasing sequence of positive reals such that ε_k goes to 0 as k goes to ∞.

For each $\nu \in \mathcal{M}_0$ and for each $k \geq 1$, we define

$$H_k^\nu = \{A \in \mathcal{A} \,/\, \exists n \geq k \text{ such that } d(\mathcal{L}_\Omega(\frac{1}{a_n}S_n(\mathbb{1}_A - \mu(A))), \nu) < \varepsilon_k\}.$$

For each $\nu \in \mathcal{M}_0$ and for each $k \geq 1$, it is clear that H_k^ν is an open set in $\mathcal{A}$. We now prove that it is dense.

Assume that ν and k are fixed and let $\varepsilon > 0$ and $A \in \mathcal{A}$. By Proposition 3.5, there exists a set $B \in \mathcal{A}$ such that $\mu(A \triangle B) < \frac{\varepsilon}{2}$ and there exists a sequence $(n_i)_{i \geq 1}$ such that for all $i \geq 1$,

$$d(\mathcal{L}_\Omega(\frac{1}{a_{n_i}}S_{n_i}(\mathbb{1}_B - \mu(B))), \delta_0) \leq \frac{\varepsilon_k}{2}. \tag{13}$$

By Proposition 3.4, there exists an integer i_0 such that, for the integer $n = n_{i_0} \geq k$, there exists a set $C \in \mathcal{A}$ satisfying $\mu(C) < \frac{\varepsilon}{2}$, $C \cap B = \emptyset$ and

$$d(\mathcal{L}_\Omega(\frac{1}{a_n}S_n(\mathbb{1}_C - \mu(C))), \nu) < \frac{\varepsilon_k}{2}. \tag{14}$$

Hence, $\mu((B \cup C)\triangle A) < \varepsilon$ and since B and C are disjoint, by Lemma 3.2 (iii) and by (13) and (14), we get

$$\begin{aligned}
&d(\mathcal{L}_\Omega(\frac{1}{a_n}S_n(\mathbb{1}_{B\cup C} - \mu(B \cup C))), \nu) \\
&\quad \leq d(\mathcal{L}_\Omega(\frac{1}{a_n}S_n(\mathbb{1}_C - \mu(C))), \nu) + d(\mathcal{L}_\Omega(\frac{1}{a_n}S_n(\mathbb{1}_B - \mu(B))), \delta_0) \\
&\quad < \varepsilon_k,
\end{aligned}$$

i.e. $B \cup C$ belongs to H_k^ν. Therefore H_k^ν is dense in $\mathcal{A}$ for the pseudo-metric Θ.

Let M be a countable subset of $\mathcal{M}_0$ which is dense in $\mathcal{M}$ and set

$$H = \bigcap_{\nu\in M} \bigcap_{k=1}^{\infty} H_k^\nu.$$

By Baire's theorem, H is a dense G_δ (for the metric of the measure of the symmetric difference).

Further, for each $A \in H$, the sequence $(\mathcal{L}_\Omega(\frac{1}{a_n}S_n(\mathbb{1}_A - \mu(A))))_{n\geq 1}$ is dense in $\mathcal{M}$ for the Lévy metric d.

Indeed, let $A \in H$, $\eta \in \mathcal{M}$ and $\varepsilon > 0$. By density of M, there exist $\nu \in M$ such that

$$d(\nu, \eta) < \frac{\varepsilon}{2}.$$

But $A \in H_k^\nu$ for all $k \geq 1$, then there exists an increasing sequence $(n_k)_{k\geq 1}$ such that

$$d(\mathcal{L}_\Omega(\frac{1}{a_{n_k}}S_{n_k}(\mathbb{1}_A - \mu(A))), \nu) \leq \varepsilon_k.$$

Thus, there exists $K \in \mathbb{N}$ such that

$$d(\mathcal{L}_\Omega(\frac{1}{a_{n_K}}S_{n_K}(\mathbb{1}_A - \mu(A))), \nu) \leq \frac{\varepsilon}{2}$$

and

$$d(\mathcal{L}_\Omega(\frac{1}{a_{n_K}}S_{n_K}(\mathbb{1}_A - \mu(A))), \eta) \leq \varepsilon.$$

□

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References

[1] Robert Burton and Manfred Denker. On the central limit theorem for dynamical systems. *Trans. Amer. Math. Soc.*, 302(2):715–726, 1987.

[2] Herold Dehling, Olivier Durieu, and Dalibor Volný. New techniques for empirical processes of dependent data. *submitted,* ArXiv: 0806.2941.

[3] Andrés del Junco and Joseph Rosenblatt. Counterexamples in ergodic theory and number theory. *Math. Ann.*, 245(3):185–197, 1979.

[4] R. M. Dudley. *Real analysis and probability*, volume 74 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002. Revised reprint of the 1989 original.

[5] Olivier Durieu and Dalibor Volný. An indicator function limit theorem in dynamical systems. *preprint,* ArXiv: 0810.2452.

[6] Pierre Liardet and Dalibor Volný. Sums of continuous and differentiable functions in dynamical systems. *Israel J. Math.*, 98:29–60, 1997.

[7] Dalibor Volný. On limit theorems and category for dynamical systems. *Yokohama Math. J.*, 38(1):29–35, 1990.