An Indicator Function Limit Theorem in Dynamical Systems

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Abstract

We give a constructive proof of the following result: in all aperiodic dynamical system, for all sequences $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $a_n \nearrow \infty$ and $\frac{a_n}{n} \to 0$ as $n \to \infty$, there exists a set $A \in \mathcal{A}$ having the property that the sequence of the distributions of $(\frac{1}{a_n} S_n(\mathbb{1}_A - \mu(A)))_{n\in\mathbb{N}}$ is dense in the space of all probability measures on $\mathbb{R}$. This extends to the non-ergodic case a result of [5].

Keywords: Dynamical system; Ergodicity; Sums of random variables; Limit theorem.
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1 Introduction and result

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space where Ω is a Lebesgue space and let T be an invertible measure preserving transformation from Ω to Ω. For a random variable X from Ω to $\mathbb{R}$, we denote by $S_n(X)$ the partial sums $\sum_{i=0}^{n-1} X \circ T^i$, $n \geq 1$.

The present paper concerns the question of the limit behavior of partial sums in general aperiodic dynamical systems. In 1987, Burton and Denker [2] proved that in any aperiodic dynamical system, there exists a function in L_0^2 which verifies the central limit theorem (the same is true for the functional central limit theorem, see Volný [8]). In general, for functions in $\mathbb{L}^p$ spaces, Volný [7] proved that for any sequence $a_n \to \infty$, $\frac{a_n}{n} \to 0$, there exists a dense G_δ part G of L_0^p such that for any $f \in G$ the sequence of distributions of $\frac{1}{a_{n_k}} S_{n_k}(f)$ is dense in the set of all probability measures on $\mathbb{R}$, see also Liardet and Volný [6]. This work is also related to the question of the rate of convergence in the ergodic theorem (see del Junco and Rosenblatt [3]).

In Durieu and Volný [5], a similar result is shown for the class of centered indicator functions $\mathbb{1}_A - \mu(A)$, $A \in \mathcal{A}$ and for ergodic dynamical systems. Here we prove the following theorem concerning general aperiodic dynamical systems.

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Theorem 1 *Let $(\Omega, \mathcal{A}, \mu, T)$ be an aperiodic dynamical system and $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ be an increasing sequence such that $a_n \nearrow \infty$ and $\frac{a_n}{n} \to 0$ as $n \to \infty$. For all $\varepsilon > 0$, there exists a set $A \in \mathcal{A}$ with $\mu(A) < \varepsilon$ such that for every probability measure ν on $\mathbb{R}$, there exists a sequence $(n_k)_{k\in\mathbb{N}}$ such that*

$$\frac{1}{a_{n_k}} S_{n_k}(\mathbb{1}_A - \mu(A)) \xrightarrow[k\to\infty]{\mathcal{D}} \nu.$$

Remark. In the ergodic case, the family of sets A which satisfy the property is dense in $\mathcal{A}$ (for the pseudo-metric of the measure of the symmetric difference), see Durieu and Volný [5]. This permitted to use Baire arguments in proving the results in [5]. In the non-ergodic case, the density does not take place hence our proof here is constructive.

Using Lévy metric on the space $\mathcal{M}$ of all probability measures on $\mathbb{R}$, we can give an alternative statement of the preceding theorem. We denote by d the Lévy metric on $\mathcal{M}$. For all μ and ν in $\mathcal{M}$ with distribution functions F and G,

$$d(\mu, \nu) = \inf\{\varepsilon > 0 \,:\, G(t-\varepsilon) - \varepsilon \le F(t) \le G(t+\varepsilon) + \varepsilon, \forall t \in \mathbb{R}\}.$$

The space $(\mathcal{M}, d)$ is a complete separable metric space and convergence with respect to d is equivalent to weak convergence of distributions (see Dudley [4], pages 394-395). If $A \in \mathcal{A}$ with $\mu(A) > 0$, and $X : A \longrightarrow \mathbb{R}$ is a random variable, we denote by $\mathcal{L}_A(X)$ the distribution of X on $\mathbb{R}$ with respect to the induced probability measure $\mu_A(.) = \mu(A)^{-1}\mu(.)$. We also denote by $\mathcal{M}_0$ be the set of all centered probability measures on $\mathbb{R}$.

Theorem 2 *Let $(\Omega, \mathcal{A}, \mu, T)$ be an aperiodic dynamical system and $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ be an increasing sequence such that $a_n \nearrow \infty$ and $\frac{a_n}{n} \to 0$ as $n \to \infty$. For every $\varepsilon > 0$ and for every sequence $(\nu_k)_{k\in\mathbb{N}}$ in $\mathcal{M}_0$, there exist a set $A \in \mathcal{A}$, with $\mu(A) < \varepsilon$, and a sequence $(n_k)_{k\in\mathbb{N}}$ such that*

$$d(\mathcal{L}_\Omega(\frac{1}{a_{n_k}} S_{n_k}(\mathbb{1}_A - \mu(A))), \nu_k) \xrightarrow[k\to\infty]{} 0.$$

Using the separability of the set $\mathcal{M}_0$ which is dense in $\mathcal{M}$, one can see that Theorem 2 is equivalent to Theorem 1. The rest of the paper is devoted to the proof of Theorem 2.

2 Proof of Theorem 2

2.1 Auxiliary results

Let ν be a probability on $\mathbb{R}$. For $B \in \mathcal{B}(\mathbb{R})$ with $\nu(B) > 0$, ν_B denotes the probability on $\mathbb{R}$ defined by $\nu_B(A) = \nu(B)^{-1}\nu(A \cap B)$. For $x \in \mathbb{R}$, ν_x denotes the probability on $\mathbb{R}$ defined by $\nu_x(B) = \nu(\{xb \,/\, b \in B\})$. We give, without proof, some properties of the Lévy metric.

Lemma 2.1

(i) *For each probability ν on $\mathbb{R}$, for all Borel sets B, $d(\nu_B, \nu) \le \nu(\mathbb{R} \setminus B)$.*

(ii) *For all probabilities* ν *and* η *on* $\mathbb{R}$, *for all* $x \geq 1$, $d(\nu_x, \eta_x) \leq d(\nu, \eta)$.

(iii) *For all probability* ν *on* $\mathbb{R}$, *for all measurable functions* f *and* g *from* Ω *to* $\mathbb{R}$,

$$d(\mathcal{L}_\Omega(f+g), \nu) \leq d(\mathcal{L}_\Omega(f), \nu) + d(\mathcal{L}_\Omega(g), \delta_0)$$

where δ_0 *is the Dirac measure at* 0.

(iv) *For all probability* ν *on* $\mathbb{R}$, $d(\nu, \delta_0) \leq A$ *if and only if* $\nu((-\infty, -A)) \leq A$ *and* $\nu((A, \infty)) \leq A$.

The following lemma is a classical result.

Lemma 2.2 *For all probability* ν *on* $\mathbb{R}$, *for all* $\varepsilon > 0$, *if* $C = C(\nu, \varepsilon) \geq 1$ *and* $n = n(\nu, \varepsilon) \in \mathbb{N}$ *are large enough, there exists a probability* η *on* $\mathbb{R}$ *with support* $S \subset [-a_n C, a_n C] \cap \mathbb{Z}$ *such that for all* $i \in S$, $\eta(\{i\}) \in \mathbb{Q}$, $d(\eta_{a_n}, \nu) \leq \varepsilon$ *and* $\mathbb{E}(\eta) := \int x \mathrm{d}\eta(x) = 0$.

Recall that a set $F \in \mathcal{A}$ is the base of a Rokhlin tower of height n if the sets $F, TF, \ldots, T^{n-1}F$ are pairwise disjoint. The following lemma states that we can find a Rokhlin tower having the property that for all $x \in T^{n-1}F$, $Tx \in F$ or $T^2x \in F$.

Lemma 2.3 *In every aperiodic dynamical system, for all* $n \geq 1$ *and for all* $\varepsilon > 0$, *there exists a measurable set* F *such that* $\{F, \ldots, T^{n-1}F\}$ *is a Rokhlin tower of measure greater than* $1 - \varepsilon$ *and the sojourn time in the junk set* $J = \Omega \setminus (\cup_{i=0}^{n-1} T^i F)$ *is almost surely* 1, *i.e. for a.e.* $x \in J$, $Tx \in F$.

Proof. This can be view as a consequence of Alpern's theorem [1], by constructing a Rokhlin castle with two towers of height n and $n+1$ and the base of the second tower of measure less than ε. □

2.2 The main proposition

Let $(\Omega, \mathcal{A}, \mu, T)$ be an aperiodic dynamical system and $(a_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$ be an increasing sequence such that $a_n \nearrow \infty$ and $\frac{a_n}{n} \to 0$ as $n \to \infty$ which are fixed for all the sequel. Let the sequence $(\nu_k)_{k \geq 1}$ in $\mathcal{M}_0$ and the constant $\varepsilon > 0$ be also fixed. Let $(\varepsilon_k)_{k \geq 1}$ be a decreasing sequence of positive reals such that $\sum_{k \geq 1} \varepsilon_k < \varepsilon$ and $\sum_{k \geq 1} k\varepsilon_k < \infty$.

Theorem 2 is a consequence of the following proposition, which is proved in the next section.

Proposition 2.4 *There exist a sequence of pairwise disjoint sets* $A_k \in \mathcal{A}$ *and a sequence of integers* $(n_k)_{k \geq 1}$ *such that,*

(i) $\mu(A_1) \leq \varepsilon_1$ *and for all* $k > 1$, $\mu(A_k) \leq \frac{a_{n_{k-1}}}{n_{k-1}} \varepsilon_k$;

(ii) *for all* $k \geq 1$, $d(\mathcal{L}_\Omega(\frac{1}{a_{n_k}} S_{n_k}(\mathbb{1}_{A_k} - \mu(A_k))), \nu_k) \leq \varepsilon_k$;

(iii) *for all* $k \geq 1$ *and for all* $j > k$, $d(\mathcal{L}_\Omega(\frac{1}{a_{n_j}} S_{n_j}(\mathbb{1}_{A_k} - \mu(A_k))), \delta_0) \leq \varepsilon_j$.

Theorem 2 follows immediately from Proposition 2.4 and Lemma 2.1 by setting $A = \bigcup_{k \geq 1} A_k$.

2.3 Proof of Proposition 2.4

The proof shall be done by induction.

The set A_1. The goal is to find a set A_1 and an integer n_1 such that

$$d(\mathcal{L}_\Omega(\frac{1}{a_{n_1}}S_{n_1}(\mathbb{1}_{A_1} - \mu(A_1))), \nu_1) \leq \varepsilon_1.$$

We need, moreover, that the set A_1 becomes negligible for the partial sums of length n_k, $k \geq 2$ (condition (iii)). There are several steps. First, we will define a set $A_{1,1}$ which satisfies (ii) and (iii) for $j = 2$. Then, we will modify this set, step by step, to have (iii) for all $j > 2$.

The set $A_{1,1}$. We consider the probability ν_1, the constant ε_1 and we set $\alpha_1 := \frac{\varepsilon_1}{8}$. Applying Lemma 2.2 to ν_1 and α_1, we get two constants $C_1 := C(\nu_1, \alpha_1)$ and $n(\nu_1, \alpha_1)$ and we choose $n_1 \geq n(\nu_1, \alpha_1)$ such that

$$\frac{d_1}{n_1} \leq \alpha_1 \quad \text{where } d_1 := \lfloor a_{n_1} C_1 \rfloor + 1. \tag{1}$$

We get a corresponding centered probability η_1 (given by Lemma 2.2) with support in $\{-d_1 + 1, \ldots, d_1 - 1\}$ such that $d(\eta_{1\,a_{n_1}}, \nu_1) \leq \alpha_1$. Since for all $i \in \{1, \ldots, 2d_1 - 1\}$, $\eta_1(\{i - d_1\}) \in \mathbb{Q}$, there exist $q_1 \in \mathbb{N}$ and $q_1^{(i)} \in \mathbb{N}$, with $\sum_{i=1}^{2d_1-1} q_1^{(i)} = q_1$, such that $\eta_1(\{i - d_1\}) = \frac{q_1^{(i)}}{q_1}$, for all $i = 1, \ldots, 2d_1 - 1$.

Now, we consider the probability ν_2 and the constant ε_2. We define $\alpha_2 := \frac{a_{n_1}}{2n_1}\varepsilon_2$. Applying Lemma 2.2 to ν_2 and α_2, we get two constants $C(\nu_2, \alpha_2)$ and $n(\nu_2, \alpha_2)$. Set $C_2 := \max\{C(\nu_2, \alpha_2), C_1\}$ and let $n_2 \geq n(\nu_2, \alpha_2)$ be a multiple of $q_1 n_1$ such that

$$\frac{q_1 n_1}{a_{n_2}} \leq \alpha_2. \tag{2}$$

By Lemma 2.3, we can consider a set $F_1 \in \mathcal{A}$ such that $\{F_1, TF_1, \ldots, T^{n_2-1}F_1\}$ is a Rokhlin tower of height n_2, with the sojourn time in the junk set almost surely equal to 1 and the measure of the junk set smaller than $\gamma_1 := \min\{\frac{a_{n_2}}{n_2}\alpha_2, \alpha_1\}$. Since n_2 is a multiple of n_1, this tower contains the $p_1 := \frac{n_2}{n_1}$ towers $\{T^{ln_1}F_1, \ldots, T^{(l+1)n_1-1}F_1\}$ of height n_1. Notice that by definition of n_2, p_1 is a multiple of q_1.

Let $\{A_{F_1,0}, \ldots, A_{F_1,q_1-1}\}$ be a partition of F_1 into q_1 sets of measure $\frac{1}{q_1}\mu(F_1)$. For each $x \in F_1$, we define an associated word $\omega(x) = (\omega_i(x))_{i=0,\ldots,n_2-1} \in \{0,1\}^{n_2}$ in the following way:

For $i = 1, \ldots, p_1$, let $u_i = (1, \ldots, 1, 0, \ldots, 0)$ be the word of length n_1 composed with k_i ones placed at first positions, where k_i is the smallest integer $k \geq 1$ such that $i \mod q_1 \in \{\sum_{j=1}^{k-1} q_1^{(j)}, \ldots, \sum_{j=1}^{k} q_1^{(j)}\}$.

Let σ be the cyclic permutation $(1 \cdots p_1)$. For each $l \in \{0, \ldots, q_1 - 1\}$, if $x \in A_{F_1,l}$, we define by concatenation $\omega(x) := u_{\sigma^l(1)}u_{\sigma^l(2)} \ldots u_{\sigma^l(p_1)}$.

Then the set $A_{1,1}$ is defined as the union, over all $x \in F_1$, of the iterates of x which correspond to a 1 in its associated word, i.e.

$$A_{1,1} := \bigcup_{x \in F_1} \bigcup_{\substack{i \in \{0,\dots,n_2-1\} \\ \omega_i(x)=1}} \{T^i x\}.$$

Remark that for each $l \in \{0, \dots, p_1-1\}$, $\mu_{T^{ln_1}F_1}(\{x : S_{n_1}(\mathbb{1}_{A_{1,1}})(x) = i\}) = \eta_1(\{i-d_1\})$. Moreover, for each $l \in \{1, \dots, \frac{n_2}{n_1 q_1}\}$, for any $x \in T^{ln_1q_1}F_1$, we have $S_{n_1q_1}(\mathbb{1}_{A_{1,1}})(x) = d_1q_1$. In particular, for any $x \in F_1$, $S_{n_2}(\mathbb{1}_{A_{1,1}})(x) = d_1p_1$.

Lemma 2.5

(i) $\mu(A_{1,1}) \leq \alpha_1$;

(ii) *for all* $x \in F_1$, *for each* $l \in \{0, \dots, p_1-1\}$, $S_{n_1}(\mathbb{1}_{A_{1,1}})(T^{ln_1}x) \leq 2d_1$;

(iii) $d(\mathcal{L}_\Omega(\frac{1}{a_{n_1}} S_{n_1}(\mathbb{1}_{A_{1,1}} - \mu(A_{1,1}))), \nu_1) \leq \varepsilon_1$;

(iv) $d(\mathcal{L}_\Omega(\frac{1}{a_{n_2}} S_{n_2}(\mathbb{1}_{A_{1,1}} - \mu(A_{1,1}))), \delta_0) \leq \alpha_2$.

Proof. Since for any $x \in F_1$, $S_{n_2}(\mathbb{1}_{A_{1,1}})(x) = d_1p_1$, we have

$$\mu(A_{1,1}) = p_1 d_1 \mu(F_1) \leq \frac{p_1}{n_2} d_1 = \frac{d_1}{n_1}.$$

Therefore, (i) follows by (1).

By construction, (ii) is clear.

Let $\Omega_1 := \bigcup_{l=0}^{p_1-1} \bigcup_{i=0}^{n_1-2d_1-1} T^{ln_1-i} F_1$. For $i = 0, \dots, 2d_1-1$, we have

$$\mu_{\Omega_1}(\{x : S_{n_1}(\mathbb{1}_{A_{1,1}}) = i\}) = \eta_1(\{i-d_1\})$$

and by centering, $\mathcal{L}_{\Omega_1}(S_{n_1}(\mathbb{1}_{A_{1,1}} - \mu(A_{1,1}))) = \eta_1$. Now, since $\gamma_1 \leq \alpha_1$,

$$\mu(\Omega_1) = p_1(n_1 - 2d_1)\mu(F_1) \geq (n_2 - 2p_1d_1)\frac{(1-\gamma_1)}{n_2} = 1 - \gamma_1 - \frac{2d_1}{n_1} \geq 1 - 3\alpha_1.$$

Thus, by Lemma 2.1 (i),

$$d(\mathcal{L}_\Omega(S_{n_1}(\mathbb{1}_{A_{1,1}} - \mu(A_{1,1}))), \eta_1) \leq 3\alpha_1.$$

and by Lemma 2.1 (ii),

$$d(\mathcal{L}_\Omega(\frac{1}{a_{n_1}} S_{n_1}(\mathbb{1}_{A_{1,1}} - \mu(A_{1,1}))), \eta_{1a_{n_1}}) \leq 3\alpha_1.$$

We infer, by triangular inequality, that (iii) holds.

Recall that n_2 is a multiple of n_1q_1 and, by definition of $A_{1,1}$, $S_{n_1q_1}(\mathbb{1}_{A_{1,1}})(x) = d_1q_1$ whenever x belongs to one of the $T^{ln_1q_1}F_1$ for $l = 0, \ldots, \frac{n_2}{n_1q_1}$. Since the sojourn time in the junk set is 1, we infer that for any $x \in \Omega$,

$$(p_1 - q_1)d_1 \leq S_{n_2}(\mathbb{1}_{A_{1,1}})(x) \leq (p_1 + q_1)d_1.$$

Using $\mu(A_{1,1}) = p_1d_1\mu(F_1)$, we get

$$|S_{n_2}(\mathbb{1}_{A_{1,1}} - \mu(A_{1,1}))| \leq p_1d_1|1 - n_2\mu(F_1)| + q_1d_1 \leq p_1d_1\gamma_1 + q_1d_1.$$

Thus, since $\gamma_1 \leq \frac{a_{n_2}}{n_2}\alpha_2$ and by (2), we have

$$\frac{1}{a_{n_2}}|S_{n_2}(\mathbb{1}_{A_{1,1}} - \mu(A_{1,1}))| \leq \frac{d_1}{n_1}\alpha_2 + \frac{d_1}{n_1}\frac{q_1n_1}{a_{n_2}} \leq 2\alpha_1\alpha_2 \leq \alpha_2$$

and (iv) follows from application of Lemma 2.1 (iv). ☐

At this point, the set is not well enough defined to be negligible for higher partial sums. So, we need to modify a small part of $A_{1,1}$. Thus we introduce a sequence of sets $A_{1,k}$, $k \geq 2$, which give the successive adjustments.

The sets $A_{1,k}$, $k \geq 2$. We shall give here the general algorithm to deduce the set $A_{1,k}$ from $A_{1,k-1}$. To do that, we need first to define the entire sequence $(n_k)_{k\geq 1}$ and all the related sequences. These sequences will also be used in a later stage for the construction of the sets A_k.

By induction, we define the sequences $(\alpha_k)_{k\geq 2}$, $(C_k)_{k\geq 2}$, $(n_k)_{k\geq 2}$, $(q_k)_{k\geq 2}$ as follows. We choose $\alpha_k \leq \frac{a_{n_{k-1}}}{2n_{k-1}}\varepsilon_k$ such that $\sum_{j=1}^{k}\alpha_j \leq \frac{\varepsilon_k}{4}$. Applying Lemma 2.2 to ν_k and α_k, we get two constants $C(\nu_k, \alpha_k)$ and $n(\nu_k, \alpha_k)$. Set $C_k := \max\{C(\nu_k, \alpha_k), C_{k-1}\}$ and let $n_k \geq n(\nu_k, \alpha_k)$ be a multiple of $q_{k-1}n_{k-1}$ such that

$$\frac{d_k}{n_k} \leq \alpha_k \quad \text{where } d_k := \lfloor a_{n_k}C_k \rfloor + 1. \tag{3}$$

and

$$\frac{q_{k-1}n_{k-1}}{a_{n_k}} \leq \alpha_k. \tag{4}$$

By Lemma 2.2, we get a corresponding centered probability η_k with support contained in $\{-d_k + 1, \ldots, d_k - 1\}$ such that $d(\eta_{k\,a_{n_k}}, \nu_k) \leq \alpha_k$. There also exist $q_k \in \mathbb{N}$ and $q_k^{(i)} \in \mathbb{N}$, $i = 1\ldots, 2d_k - 1$, such that $\eta_k(\{i - d_k\}) = \frac{q_k^{(i)}}{q_k}$.

Further, for $k \geq 1$, we set $p_k := \frac{n_{k+1}}{n_k} \in \mathbb{N}$ and $\beta_k := \alpha_k - \alpha_{k+1}$. Thus, for all $k \geq 1$,

$$\sum_{j\geq k}\beta_j \leq \alpha_k. \tag{5}$$

We define the sequence $(\gamma_k)_{k\geq 1}$ by

$$\gamma_k := \min\left\{\frac{\beta_{k+1}}{2p_{k+1}}, \frac{a_{n_{k+1}}}{n_{k+1}}\alpha_{k+1}\right\}. \tag{6}$$

Finally, for all $k \geq 1$, by application of Lemma 2.3, we obtain a set $F_k \in \mathcal{A}$ such that $\{F_k, TF_k, \dots, T^{n_{k+1}-1}F_k\}$ is a Rokhlin tower of height n_{k+1} and the junk set $J_k := \Omega \setminus \bigcup_{i=0}^{n_{k+1}-1} T^i F_k$ is a set with sojourn time 1 and $\mu(J_k) \leq \gamma_k$.

Remark that α_2, n_2 and γ_2 have been previously defined but they respect this new definition.

Now, specially for the construction of the $A_{1,k}$, we introduce the sequence of sets F'_k defined by induction by $F'_1 := F_1$ and $F'_k := \bigcup_{x \in F_k} T^{n(x)}x$, where for all x in F_k, $n(x) := \inf\{n \geq 0 \,/\, T^n x \in F'_{k-1}\}$ is the time of the first visit in F'_{k-1}.

Lemma 2.6 *There exists a sequence of measurable sets* $(A_{1,k})_{k\geq 1}$ *such that*

(i) $\mu(A_{1,k-1} \triangle A_{1,k}) \leq \beta_k$;

(ii) *for all* $x \in F'_k$, *for all* $l \in \{0, \dots, \frac{n_k}{n_1} - 1\}$, $S_{n_1}(\mathbb{1}_{A_{1,k}})(T^{ln_1}x) \leq 2d_1$;

(iii) *for all* $x \in F'_k$, *for all* $i \in \{0, \dots, p_k - 1\}$, $S_{n_k}(\mathbb{1}_{A_{1,k}} \circ T^{in_k})(x) = \frac{n_k}{n_1}d_1$;

(iv) $d(\mathcal{L}_\Omega(\frac{1}{a_{n_{k+1}}} S_{n_{k+1}}(\mathbb{1}_{A_{1,k}} - \mu(A_{1,k})), \delta_0) \leq \alpha_{k+1}$.

Proof. We prove the lemma by induction. The set $A_{1,1}$ is already defined. Now, for a fixed k, we shall deduce the set $A_{1,k}$ from $A_{1,k-1}$.

For $x \in F'_k$, we associate to it a word $\omega(x) = (\omega_i(x))_{i=0,\dots,n_k-1} \in \{0,1\}^{n_k}$ by setting $\omega_i(x) = 1$ if and only if $T^i x \in A_{1,k-1}$. We shall transform the word $\omega(x)$ to a word $\omega'(x)$ using the following technique. For $i = 0, \dots, p_k - 1$, let

$$\rho_i(x) := \sum_{j=in_k}^{(i+1)n_k - 1} \omega_i(x) - \frac{n_k}{n_1}d_1.$$

By hypothesis, for all $x \in F'_k$, $\rho_0(x) = S_{n_k}(\mathbb{1}_{A_{1,k-1}})(x) - \frac{n_k}{n_1}d_1 = 0$ but for $i > 0$ it can be different. The differences appear when the orbit of the point x meets the junk set J_{k-1}. Nevertheless, since the Rokhlin tower respect Lemma 2.3, it can meet J_{k-1} only one time in every n_k consecutive iterates by T. So we have, for all $x \in F'_k$ and $i \in \{0, \dots, p_k - 1\}$, $\rho_i(x) \in \{-\frac{n_{k-1}}{n_1}d_1, \dots, \frac{n_{k-1}}{n_1}d_1\}$.

We deduce $\omega'(x)$ from $\omega(x)$ as follows. For each $i \in \{0, \dots, p_k - 1\}$, we consider the sub-word $(\omega_{in_k}(x), \dots, \omega_{(i+1)n_k-1}(x))$ of the word $\omega(x)$. If $\rho_i(x) \geq 0$, we replace $\rho_i(x)$ ones by zeros. If $\rho_i(x) < 0$, we replace $|\rho_i(x)|$ zeros by ones in such a way that every sub-word $(\omega_{in_k+ln_1}(x), \dots, \omega_{in_k+(l+1)n_1-1}(x))$, $l = 0, \dots, \frac{n_k}{n_1} - 1$, contains at most $2d_1$ ones.

Now, we can define

$$A_{1,k} := \bigcup_{x \in F'_k} \bigcup_{\substack{i \in \{1,\dots,n_k\} \\ \omega'_i(x)=1}} \{T^i x\}.$$

Since the orbit of a point x can only meet J_{k-1} one time every n_k and using (6), we have

$$\mu(A_{1,k-1} \triangle A_{1,k}) \leq 2p_k \mu(J_{k-1}) \leq 2p_k \gamma_{k-1} \leq \beta_k.$$

Remark that (ii) and (iii) are guaranteed by construction of $A_{1,k}$.

Further for all $x \in F'_k$, we have

$$S_{n_{k+1}}(\mathbb{1}_{A_{1,k}})(x) = p_k \frac{n_k}{n_1} d_1 = \frac{n_{k+1}}{n_1} d_1.$$

We deduce that $|\mu(A_{1,k}) - \frac{n_{k+1}}{n_1} d_1 \mu(F'_k)| \leq \mu(J_k)$ and

$$(p_k - 1)\frac{n_k}{n_1} d_1 \leq S_{n_{k+1}}(\mathbb{1}_{A_{1,k}}) \leq (p_k + 1)\frac{n_k}{n_1} d_1.$$

Then,

$$|S_{n_{k+1}}(\mathbb{1}_{A_{1,k}} - \mu(A_{1,k}))| \leq \frac{n_k}{n_1} d_1 + (1 + \frac{n_{k+1}}{n_1} d_1)\mu(J_k)$$

and by (6) and (4),

$$\frac{1}{a_{n_{k+1}}} |S_{n_{k+1}}(\mathbb{1}_{A_{1,k}} - \mu(A_{1,k}))| \leq \frac{n_k}{a_{n_{k+1}}} \frac{d_1}{n_1} + (\frac{1}{n_{k+1}} + \frac{d_1}{n_1})\alpha_{k+1} \leq \alpha_{k+1}.$$

By Lemma 2.1 (iv), we get (iv). □

The set A_1. Now, we can define the set $A_1 \in \mathcal{A}$ as $A_1 := \lim_{k\to\infty} A_{1,k}$. It is well defined because the sequence $(\mu(A_{1,k} \triangle A_{1,k+1}))_{k\geq 1}$ is summable.

Lemma 2.7

(i) *$\mu(A_1) \leq 2\alpha_1$;*

(ii) *$S_{n_1}(\mathbb{1}_{A_1}) \leq 4d_1$;*

(iii) *$d(\mathcal{L}_\Omega(\frac{1}{a_{n_1}} S_{n_1}(\mathbb{1}_{A_1} - \mu(A_1))), \nu_1) \leq 2\varepsilon_1$;*

(iv) *For all $k \geq 2$, $d(\mathcal{L}_\Omega(\frac{1}{a_{n_k}} S_{n_k}(\mathbb{1}_{A_1} - \mu(A_1))), \delta_0) \leq \varepsilon_k$.*

Proof. For all $k \geq 1$, we have

$$\mu(A_1 \triangle A_{1,k}) \leq \sum_{j=k+1}^{\infty} \mu(A_{1,j-1} \triangle A_{1,j}) \leq \sum_{j=k+1}^{\infty} \beta_j \leq \alpha_{k+1} \tag{7}$$

and then $\mu(A_1) \leq \mu(A_{1,1}) + \mu(A_1 \triangle A_{1,1}) \leq 2\alpha_1$.

Assertion (ii) comes from Lemma 2.6 (ii).

Further (7) and Lemma 2.1 (iv) imply that for all n,

$$d(\mathcal{L}_\Omega(\frac{1}{a_n}(S_n(\mathbb{1}_{A_{1,k}} - \mu(A_{1,k})) - S_n(\mathbb{1}_{A_1} - \mu(A_1)))), \delta_0) \leq \frac{n}{a_n}\alpha_{k+1}.$$

Using Lemma 2.1 (iii), we can deduce (ii) from Lemma 2.5 (iii) and (iii) from Lemma 2.6 (iv). □

Induction process. Now we show by induction that we can find a sequence of measurable sets A_k, $k \geq 1$, such that for all $k \geq 1$,

(a) A_k disjoint of $A_1 \cup \cdots \cup A_{k-1}$,

(b) $\mu(A_k) \leq 2\alpha_k$,

(c) $S_{n_k}(\mathbb{1}_{A_k}) \leq 4d_k$,

(d) $d(\mathcal{L}_\Omega(\frac{1}{a_{n_k}} S_{n_k}(\mathbb{1}_{A_k} - \mu(A_k))), \nu_k) \leq 2\varepsilon_k$,

(e) For all $j \geq k+1$, $d(\mathcal{L}_\Omega(\frac{1}{a_{n_j}} S_{n_j}(\mathbb{1}_{A_k} - \mu(A_k))), \delta_0) \leq \varepsilon_j$,

which proves Proposition 2.4.

The set A_1 is already defined. Here we describe the construction of the set A_k knowing $A_1, \ldots, A_{k-1}$. The integer $k > 1$ is now fixed. As we did for A_1, the set A_k will be the limit of a sequence of sets $A_{k,j}$, $j \geq k$.

The set $A_{k,k}$. We consider $F_k \in \mathcal{A}$ and we shall use almost the same technique as to find the set $A_{1,1}$, working with n_k, q_k, p_k, d_k instead of n_1, q_1, p_1, d_1. The difference comes to the fact that we want $A_k \cap (A_1 \cup \cdots \cup A_{k-1}) = \emptyset$.

Recall that η_k is a probability measure with finite support in $\{-d_k+1, \ldots, d_k-1\}$. Let $\{A_{F_k,0}, \ldots, A_{F_k,q_k-1}\}$ be a partition of F_k into q_k sets of measure $\frac{1}{q_k}\mu(F_k)$.

For each $x \in F_k$ and $i = 1, \ldots, p_k$, let $u_i(x) = (u_{i,j}(x))_{j=1,\ldots,n_k} \in \{0,1\}^{n_k}$ be the word of length n_k constructed as follows. First, we put zero at position j if $T^{(i-1)n_k+j-1}x \in A_1 \cup \cdots \cup A_{k-1}$. Then, let l_i be the integer $l \geq 1$ such that $i \mod q_k \in \{\sum_{j=1}^{l-1} q_k^{(j)}, \ldots, \sum_{j=1}^{l} q_k^{(j)}\}$, we put ones on the l_i first empty positions. And then we complete by zeros. Notice that, since for $j < k$, $S_{n_j}(\mathbb{1}_{A_j}) \leq 4d_j$, the ones are contained in the $2d_k + \sum_{j=1}^{k-1} \frac{n_k}{n_j} 4d_j$ first positions of the word $u_i(x)$. Notice that $2d_k + \sum_{j=1}^{k-1} \frac{n_k}{n_j} 4d_j \leq n_k(2\alpha_k + \varepsilon_{k-1})$.

Let σ be the cyclic permutation $(1 \cdots p_k)$. For each $l \in \{0, \ldots, q_k - 1\}$, if $x \in A_{F_1,l}$, we define the associated word of length n_{k+1} by

$$\omega(x) := u_{\sigma^l(1)}(x) u_{\sigma^l(2)}(x) \ldots u_{\sigma^l(p_1)}(x).$$

Then the set $A_{k,k}$ is defined as the union, over all $x \in F_k$, of the iterates of x which correspond to a 1 in its associated word, i.e.

$$A_{k,k} := \bigcup_{x \in F_k} \bigcup_{\substack{i \in \{0,\ldots,n_{k+1}-1\} \\ \omega_i(x)=1}} \{T^i x\}.$$

Notice that the set $A_{k,k}$ is contained in $\bigcup_{l=0}^{p_k-1} \bigcup_{i=0}^{\lfloor n_k(2\alpha_k+\varepsilon_{k-1})\rfloor} T^{ln_k+i} F_k$. Further, on the set $\Omega_k = \bigcup_{l=0}^{p_k-1} \bigcup_{i=0}^{n_{k+1}-\lfloor n_k(2\alpha_k+\varepsilon_{k-1})\rfloor-1} T^{ln_k-i} F_k$, we have $\mathcal{L}_{\Omega_k}(S_{n_k}(\mathbb{1}_{A_{k,k}} - \mu(A_{k,k}))) = \eta_k$. Since $\mu(\Omega_k) \geq 1 - \varepsilon_k$ we infer the following lemma.

Lemma 2.8

(i) $\mu(A_{k,k}) \leq \alpha_k$;

(ii) *for all* $x \in F_k$, *for all* $l \in \{0, \dots, p_k - 1\}$, $S_{n_k}(\mathbb{1}_{A_{k,k}})(T^{l n_k} x) \leq 2d_k$;

(iii) $d(\mathcal{L}_\Omega(\frac{1}{a_{n_k}} S_{n_k}(\mathbb{1}_{A_{k,k}} - \mu(A_{k,k}))), \nu_k) \leq \varepsilon_k$;

(iv) $d(\mathcal{L}_\Omega(\frac{1}{a_{n_{k+1}}} S_{n_{k+1}}(\mathbb{1}_{A_{k,k}} - \mu(A_{k,k}))), \delta_0) \leq \alpha_{k+1}$.

Proof. The proof follows the one of Lemma 2.5 and is left to the reader. □

The sets $A_{k,j}$, $j \geq k+1$. Now we define a sequence $A_{k,j}, j \geq k+1$ using the same iterative technique as in Lemma 2.6 and preserving the fact that for all $j \geq k$, $A_{k,j}$ is disjoint of $A_1 \cup \cdots \cup A_{k-1}$. In particular, the sets $A_{k,j}$ satisfy $\mu(A_{k,j-1} \triangle A_{k,j}) \leq \beta_j$ and

$$d(\mathcal{L}_\Omega(\frac{1}{a_{n_{j+1}}} S_{n_{j+1}}(\mathbb{1}_{A_{k,j}} - \mu(A_{k,j}))), \delta_0) \leq \alpha_{j+1}.$$

The set A_k. The set $A_k := \lim_{j\to\infty} A_{k,j}$ is well defined and disjoint of $A_1 \cup \cdots \cup A_{k-1}$. Following the proof of Lemma 2.7, we can easily check that A_k satisfies the condition (b), (c), (d), (e). □

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