Resistivity of an Infinite Three Dimensional Stationary Random Electric Conductor

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Abstract

In this paper, the pointwise degree 2 ergodic theorem for divergence-free stationary random fields (see [3]) is applied to the problem of the resistivity of a stationary random medium. It follows from this theorem that the macroscopic resistivity, which, according to the Ohm law, connects the flux of electrons to the dissipated power, is observable for almost all realizations.

1 Introduction

We consider a probability space $(\Omega, \mathcal{T}, \mu)$, and a three dimensional random tensor of resistivity $\tilde{\rho}$ defined on it, i.e. a random field

$$\begin{array}{rccc} \tilde{\rho}: & \mathbb{R}^3 \times \Omega & \rightarrow & \mathcal{S}_3^+ \\ & (x, \omega) & \mapsto & \tilde{\rho}(x, \omega) \end{array}$$

of symmetric positive definite 3×3-matrices. We suppose that its stochastic law is invariant under shifts in $\mathbb{R}^3$. Actually, we assume that there exists a measure preserving action T of $\mathbb{R}^3$

$$\begin{array}{rccc} T: & \mathbb{R}^3 \times \Omega & \rightarrow & \Omega \\ & (x, \omega) & \mapsto & T_x\omega \end{array}$$

and a positive definite symmetric random matrix $\omega \mapsto \rho(\omega)$ defined on Ω, such that $\tilde{\rho}(x, \omega) = \rho(T_x\omega)$. For details on this representation, which is not really restrictive, we refer the reader to Chapter 7 of the book of Jikov and al. [7]. We restrict ourself to the elliptic case, i.e. we suppose that there are two constants $0 < c < C$ such that the three eigenvalues of $\rho(\omega)$ are between c and C for any ω. We assume moreover than the dynamical system $(\Omega, \mathcal{T}, \mu, (T_x)_{x \in \mathbb{R}^3})$ on which ρ is defined is ergodic. The datum is the expectation $\vec{I} = (I_1, I_2, I_3)$ of the flux of the current in the medium. We seek a stationary flux which can be represented by a field $x \mapsto \vec{f}(T_x\omega)$ satisfying $\operatorname{div}_T \vec{f} = 0$, in the weak sense (law of conservation of the current), where the operator div_T is defined as following

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definition 1. Let $\vec{f}$ be an integrable random vector and g be an integrable random variable. We say that g is the divergence of $\vec{f}$ if, ω-almost surely, for any $\mathcal{C}^\infty$-function ϕ with compact support in $\mathbb{R}^3$ we have

$$\iiint \sum_{i=1}^{3} \frac{\partial \phi}{\partial x_i}(x) \cdot f_i(T_x\omega)\ d\lambda(x) \quad = \quad - \iiint \phi(x) \cdot g(T_x\omega)\ d\lambda(x),$$

where λ is the Lebesgue measure on $\mathbb{R}^3$. We denote this property by $\mathrm{div}_T(\vec{f}) = g$. When $\mathrm{div}_T(\vec{f}) = 0$, we say that the random vector $\vec{f}$ is divergence-free in the weak sense.

The dissipated power density $x \mapsto P(T_x\omega)$ is given by the Ohm formula $P = \langle \vec{f}, \rho\vec{f} \rangle$ (multidimensional and continuous version of the law "the power is equal to the resistance multiplied by the square of the intensity of current"). It is assume that electricity pass through the medium in such a way that the expectation of the power density is minimal. It can easily be checked, by a standard Lax-Milgram method of orthogonal projection, that the system satisfied by $\vec{f}$ can be written as

$$(\mathcal{S}_0) \begin{cases} \displaystyle\int_\Omega \vec{f}\ d\mu & = \quad \vec{I} \\ \mathrm{curl}_T \rho\vec{f} & = \quad 0 \\ \mathrm{div}_T \vec{f} & = \quad 0 \end{cases}$$

where the operator curl_T is defined as following.

definition 2. Let $\vec{g}, \vec{h}$ be integrable random vectors. We say that $\vec{h}$ is the curl of $\vec{g}$ if, ω-almost surely, we have for any $\mathcal{C}^\infty$-function ϕ with compact support,

$$\iiint \frac{\partial \phi}{\partial x_k}(x) g_j(T_x\omega) - \frac{\partial \phi}{\partial x_j}(x) g_k(T_x\omega)\ d\lambda(x) = \iiint \phi(x) h_i(T_x\omega)\ d\lambda(x),$$

for any cyclic permutation (i, j, k) of $(1, 2, 3)$. We will denote $\vec{h} = \mathrm{curl}_T \vec{g}$.

The field $x \mapsto \rho(T_x\omega)\vec{f}(T_x\omega)$ represents the gradient of the electric potential in the medium. The existence of a unique solution in $\mathbb{L}^2(\Omega)$ was proved by Golden and Papanicolaou (see [6]). In addition Jikov and al. showed in [7] the existence of a positive definite symmetric matrix ρ_0 such that

$$\int_\Omega P\ d\mu = \langle \vec{I}, \rho_0 \vec{I} \rangle.$$

Let ω be a realization. We want to calculate the effective resitivity of the medium. We consider, in the medium compressed with a factor R, a cubic cell, with edges parallel to the axes, length c and center x. From the macroscopic point of view, we can consider that it is crossed by a current

$$\vec{J}_R(c, x, \vec{I}, \omega) = \Big(\iint_{\mathcal{S}_i} \langle \vec{f}(T_{Ry}\omega), \vec{e}_i \rangle\ d\sigma(y) \Big)_{i=1,\ldots,3}$$

each integral being computed on the section $\mathcal{S}_i$ of the cell passing by its center x and with normal vector the i-th vector $\vec{e}_i$ of the canonical basis of $\mathbb{R}^3$. The power dissipated in this cell is given by

$$P_R(c, x, \vec{I}, \omega) = \iiint P(T_{Ry}\omega)\ d\lambda(y)$$

the domain of integration being the cell itself. According to the Ohm law, the resistance of the cell for a direction of current parallel to $\vec{J}_R(c, x, \vec{I}, \omega)$ is the quantity equal to the power divided by the square of the intensity of current

$$\frac{P_R(c, x, \vec{I}, \omega)}{\|\vec{J}_R(c, x, \vec{I}, \omega)\|^2}.$$

Its resistivity is its resistance multiplied by its cross sectional area c^2 and divided by its length c,

$$c\frac{P_R(c, x, \vec{I}, \omega)}{\|\vec{J}_R(c, x, \vec{I}, \omega)\|^2}.$$

Assume for the moment that, in order to impose the current in the cell, it is possible for any $\vec{J} \in \mathbb{R}^3$ to choose the datum $\vec{I}$ such that $\vec{J}_R(c, x, \vec{I}, \omega) = \vec{J}$ (the proof of this fact is given below). Denote by $\vec{I}_R(c, x, \vec{J}, \omega)$ the solution $\vec{I}$, and $\tilde{P}_R(c, x, \vec{J}, \omega)$ the corresponding power dissipated in the cell. The limit of the resistivity tensor of the cell is the matrix ρ_0 if, for any direction of current $\vec{J}$, the limit of the resistivity of the cell is $\frac{\langle \vec{J}, \rho_0 \vec{J}\rangle}{\|\vec{J}\|^2}$, hence if we have the convergence

$$c\frac{\tilde{P}_R(c, x, \vec{J}, \omega)}{\|\vec{J}\|^2} \longrightarrow \frac{\langle \vec{J}, \rho_0 \vec{J}\rangle}{\|\vec{J}\|^2}$$

for R tending to infinity. Moreover, the resistivity of the medium itself converges if the above convergence holds for any cell. As it is explained below, the existence of a full-probability set of realizations on which the above convergence holds for any cell (contained in the ball of center 0 and radius 1, see corollary 2) is a consequence of the pointwise degree 2 ergodic theorem for divergence-free random vectors (see [3]). In order to apply this theorem to the random vector $\vec{f}$, it should be checked that this last lies in the Lorentz space $\mathbb{L}_{2,1}$. This is what we prove in this paper.

Theorem 1. *The solution of the system* $(\mathcal{S}_0)$ *lies in* $\mathbb{L}^p(\Omega)$ *for some* $p > 2$.

We state now the degree 2 ergodic theorem for $\vec{f}$. We denote by $\triangle(A, B, C)$ the surface delimited by the triangle with vertices A, B, C.

Theorem 2 (degree 2 ergodic theorem). *Let* $\vec{f}$ *be a divergence-free random field for an ergodic stationary* $\mathbb{R}^3$*-action. We suppose that* $\vec{f} \in \mathbb{L}_{2,1}$. *We denote by* $\mathcal{B}$ *the ball of center* 0 *and radius* 1. *Then, for almost all* ω, *the integral of the field* $x \mapsto \vec{f}(T_x\omega)$ *is defined through any triangle, and we have the uniform convergence*

$$\iint_{\triangle(A,B,C)} \langle \vec{f}(T_{Rx}\omega),\ d\vec{\sigma}(x)\rangle \underset{R\to+\infty}{\longrightarrow} \iint_{\triangle(A,B,C)} \langle \Big(\int_\Omega \vec{f}\ d\mu\Big),\ d\vec{\sigma}(x)\rangle$$

on A, B, $C \in \mathcal{B}$, *where* $d\vec{\sigma}$ *denotes the normal infinitesimal field of the triangle* $\triangle$.

We point out that a similar version can be stated for the usual Wiener ergodic theorem on balls (see [10]), or rather on tetrahedra, which appears as being the degree 3 ergodic theorem in [3]. This is the

Theorem 3 (degree 3 ergodic theorem). *Assume that P is an integrable function defined on Ω. For A, B, C, $D \in \mathcal{B}$, we denote the domain delimited by the tetrahedron of vertices A, B, C, D by $\tau(A,B,C,D)$. Then for almost all ω, the integral of the function $x \mapsto P(T_x\omega)$ is defined on any tetrahedron, and we have the uniform convergence*

$$\iiint_{\tau(A,B,C,D)} P(T_{Rx}\omega)\ d\lambda(x) \underset{R\to+\infty}{\longrightarrow} \iiint_{\tau(A,B,C,D)} \Big(\int_\Omega P\ d\mu\Big)\ d\lambda(x)$$

on A, B, C, $D \in \mathcal{B}$.

In our framework, these results mean that the flux of the current through triangles and the power dissipated in tetrahedra almost surely converge to their expectations, convergences being uniform on $\mathcal{B}$.

In these theorems, the datum $\vec{I}$ is fixed. Since the system of equations $(\mathcal{S})$ is linear, the principle of superposition of the solutions works, and $\vec{I}$ can vary. Convergences are now also uniform on $\vec{I}$, with the proviso of imposing that the norm $\|\vec{I}\|$ remains bounded. According to our notations for the power and current associated to a cell, the two above ergodic theorems can thus be rewritten as

Corollary 1. *Let M be a positive constant. For almost all ω, we have the following convergences*

$$\begin{array}{lcl} \vec{J}_R(c,x,\vec{I},\omega) & \underset{R\to+\infty}{\longrightarrow} & c^2\vec{I}; \\ P_R(c,x,\vec{I},\omega) & \underset{R\to+\infty}{\longrightarrow} & c^3\langle\vec{I},\rho_0\vec{I}\rangle, \end{array} \tag{1}$$

uniformly with respect to $\vec{I}$ satisfying $\|\vec{I}\| \le M$, and x, c such that the associated cell is contained in the ball $\mathcal{B}$.

In order to impose the current $\vec{J}_R(c,x,\vec{I},\omega)$ passing through the cell, it is possible to choose a suitable datum $\vec{I}$. More precisely, we have

Lemma 1. *Let c_0 be a positive constance. For almost all ω, there exists $R_0 > 0$ such that for any $c > c_0$ and x such that the associated cell belongs to the ball $\mathcal{B}$, for any $\vec{J} \in \mathbb{R}^3$, $R > R_0$, the equation*

$$\vec{J}_R(c,x,\vec{I},\omega) = \vec{J}$$

has a unique solution $\vec{I}$.

Proof. Since the map $\vec{I} \mapsto \vec{J}_R(c,x,\vec{I},\omega)$ is linear from $\mathbb{R}^3$ on itself, it is enough to check that the determinant

$$\det\Big(\vec{J}_R(c,x,\vec{e}_i,\omega)\Big)_{i=1,\dots 3}$$

does not vanish. According to Corollary 1, it is close to c^6. Now assumption $c > c_0$ guarantees that it is bounded below by $c_0^6/2$, for R large enough. □

We denote by $\vec{I}_R(c,x,\vec{J},\omega)$ the solution $\vec{I}$ given by the above lemma, and the corresponding power in the cell by $\tilde{P}_R(c,x,\vec{J},\omega)$. We have

Corollary 2. *Assume $c_0 > 0$. For almost all ω, we have*

$$c\frac{\tilde{P}_R(c,x,\vec{J},\omega)}{\|\vec{J}\|^2} \underset{R\to+\infty}{\longrightarrow} \frac{\langle \vec{J},\rho_0\vec{J}\rangle}{\|\vec{J}\|^2},$$

uniformly with respect to $\vec{J}$, and with respect to $c > c_0$ and x such that the associated cell is contained in the ball $\mathcal{B}$.

Proof. We first remark that multiplying $\vec{J}$ by a scalar does not change the ratios which appear in the convergence stated by our corollary. Hence we may assume that $\|\vec{J}\| = 1$. According to Corollary 1, the datum $\vec{I}$ is close to $\vec{J}/c^2$. Thus, the assumption $c > c_0$ guarantees that $\|\vec{I}\|$ remains bounded. Now it follows from Corollary 1 that the power $\tilde{P}(\vec{J}) = P(\vec{I})$ is close to $c^3\langle \vec{I},\rho_0\vec{I}\rangle$. This leads obviously to $c\tilde{P}(\vec{J}) \sim \langle \vec{J},\rho_0\vec{J}\rangle$, and completes the proof. □

As it was already said, this convergence means that the limit of the resistivity of any cell for a current of direction $\vec{J}$ is $\dfrac{\langle \vec{J},\rho_0\vec{J}\rangle}{\|\vec{J}\|^2}$, therefore the limit of the tensor of resistivity is ρ_0. Moreover the full-probability set of convergence depends neither of the cell nor of the direction of the current .

Theorem 1 is obtained in this paper by applying methods from harmonic analysis. The point is to find $\mathbb{L}^p$ estimates ($p \geq 2$) for the solution $\vec{g}$ of the following auxiliary system

$$(\mathcal{S}'_{\vec{h}})\begin{cases} \displaystyle\int_\Omega \vec{g}\,d\mu &= \quad \vec{U} \\ \mathrm{curl}_T\vec{g} &= \quad 0 \\ \mathrm{div}_T\vec{g} &= \quad \mathrm{div}_T\vec{h}. \end{cases}$$

Golden-Papanicolaou's method to obtain $\mathbb{L}^2$ estimates was based on an orthogonal decomposition, which does not work on $\mathbb{L}^p$, for $p > 2$. In this paper we pass through an explicit expression of the solution $\vec{g}$, using a "Riesz's fashion" operator. This operator is an analog of the classical Riesz operator R_i, $i = 1,\ldots,3$ on $\mathbb{R}^3$, modified to be defined on dynamical systems $(\Omega,\mathcal{T},T)$. Similar methods were already used in our preceding work [4], which is related to discrete networks. Nevertheless, the proofs are different. Indeed, a fundamental property of Riesz operators is the equality $\sum_{i=1}^3 R_i^2 f = -f$. This property was obtain in paper [4] by using the fact that the square of our "Riesz's fashion operator" on discrete dynamical systems is a kernel operator. This is not true anymore on continuous dynamical systems. Hence in this paper the analogous equality is obtained by using a property of duality. See section 5 below for details on this remark.

2 Statement of the problem with the potential

The solution of our system is more simply expressed on the gradient $\vec{g}$ of the electrical potential $\vec{g} = \rho\vec{f}$. We denote by $\mathcal{A}$ the tensor of conductivity $\mathcal{A} = \rho^{-1}$.

The datum is now the expectation $\vec{U} = (U_1, U_2, U_3)$ of $\vec{g}$. The system satisfied by $\vec{g}$ is thus written as

$$(\mathcal{S}) \begin{cases} \displaystyle\int_\Omega \vec{g}\, d\mu &= \vec{U} \\ \mathrm{curl}_T \vec{g} &= 0 \\ \mathrm{div}_T \mathcal{A}\vec{g} &= 0 \end{cases}$$

The existence of a unique solution in $\mathbb{L}^2(\Omega)$, as the equality $\vec{U} = \rho_0 \vec{I}$, are proved for instance in [7] (section 7.2, equation 7.10). According to the ellipticity assumption, Theorem 1 can be rewritten as

Theorem 1. *There exists $p > 2$ such that the unique $\mathbb{L}^2(\Omega)$-solution of the above system $(\mathcal{S})$ lies in $\mathbb{L}^p(\Omega)$.*

The sequel of the paper is devoted to the proof of Theorem 1.

3 System with constant coefficients

We first study the system with constant coefficients, and with the right-hand side $\mathrm{div}_T \vec{h}$, where $\vec{h} \in \mathbb{L}^p$, namely

$$(\mathcal{S}'_{\vec{h}}) \begin{cases} \displaystyle\int_\Omega \vec{g}\, d\mu &= 0 \\ \mathrm{curl}_T \vec{g} &= 0 \\ \mathrm{div}_T \vec{g} &= \mathrm{div}_T \vec{h} \end{cases}$$

Since $\vec{h}$ is not supposed admitting a divergence, the last equation is to be understood as $\mathrm{div}_T(\vec{g} - \vec{h}) = 0$. We remark that for $p = 2$, the spaces

$$\begin{aligned} \mathcal{V}^2_{\mathrm{pot}} &= \{\vec{g} \in \mathbb{L}^2;\ \int_\Omega \vec{g}\, d\mu = 0 \text{ and } \mathrm{curl}_T \vec{g} = 0\} \\ \mathbb{L}^2_{\mathrm{sol}} &= \{\vec{g} \in \mathbb{L}^2;\ \mathrm{div}_T(\vec{g}) = 0\} \end{aligned}$$

constitute an orthogonal decomposition of $\mathbb{L}^2(\Omega)$ (see Lemma 7.3 in Jikov and Al [7]). This gives

Proposition 1 (Jikov and Al). *Assume that $\vec{h} \in \mathbb{L}^2(\Omega)$. The system $(\mathcal{S}'_{\vec{h}})$ has a unique solution $\vec{g} \in \mathbb{L}^2$, which is the orthogonal projection of $\vec{h}$ on $\mathcal{V}^2_{\mathrm{pot}}$. We have the inequality $\|\vec{g}\|_2 \le \|\vec{h}\|_2$.*

We want to solve the system $(\mathcal{S}'_{\vec{h}})$ for $p > 2$, and establish an inequality of the form $\|\vec{g}\|_p \le \kappa_p \|\vec{h}\|_p$ (in the vectorial case $\vec{h} = (h_1, h_2, h_3)$, the norm $\mathbb{L}^p(\Omega)$ is defined by

$$\|\vec{h}\|_p^p = \sum_{i=1}^3 \int_\Omega |h_i|^p\, d\mu \qquad).$$

The resolution of the similar question on $\mathbb{R}^3$ pass through the Riesz operator (see Giaquinta [5]). Hence we will define an operator of "Riesz's fashion" on the dynamical system (Ω, μ, T).

3.1 Operator of "Riesz's fashion"

Formally, the analog of the Riesz operator is defined for any $1 \leq i \leq 3$ by

$$\mathcal{R}_i(f)(\omega) = \iiint_y f(T_{-y}\omega) \frac{y_i}{\pi^2 \|y\|^4} \, d\lambda(y),$$

where λ denotes the Lebesgue measure on $\mathbb{R}^3$, and $y = (y_1, y_2, y_3)$. But the function under the integral does not belong to $\mathbb{L}^1(d\lambda)$ neither at 0 nor at infinity. The following section is devoted to giving sense to this integral. We note that in the discrete case of an action of $\mathbb{Z}^3$, there is no problem at 0 (See [1] or [4]), and for the traditional Riesz operator on $\mathbb{R}^3$, there is no problem at infinity, because the norm $\mathbb{L}^q(\mathbb{R}^3)$ of the kernel $\mathbf{1}_{\|y\|>\varepsilon} \frac{y_i}{\pi^2 \|y\|^4}$ is finite for all $1 < q < \infty$, and $y \mapsto f(y)$ is assumed belonging to $\mathbb{L}^p(d\lambda)$ at infinity (see [8]).

3.1.1 Definition, continuity

We start by defining the operator $\mathcal{R}_j$ on a space of functions having suitable regularity, then we will extend it by continuity. We denote by C_T the space of functions $h \in \mathbb{L}^\infty(\Omega)$ such that for almost all ω, the function h_ω defined on $\mathbb{R}^3$ by $h_\omega : x \mapsto h(T_x\omega)$ is a $\mathcal{C}^\infty$-function having partial derivatives of all orders bounded on x and ω. Finally let us denote by $\mathcal{E}$ the space defined by

$$\mathcal{E} = \left\{ f = \sum_{i=1}^{3} \partial_i h_i; \; h_i \in C_T, i = 1, \dots 3 \right\}$$

where $\partial_i h(\omega) = \frac{\partial h_\omega}{\partial x_i}(0)$.

Next, consider the operator $\mathcal{R}_{i,\varepsilon}$ defined on $\mathcal{E}$ by the following lemma

Lemma 2. *Assume that $f \in \mathcal{E}$ and $\varepsilon > 0$. The integral*

$$\mathcal{R}_{i,\varepsilon}^R(f)(\omega) = \iiint_{\varepsilon < \|y\| < R} f(T_{-y}\omega) \frac{y_i}{\pi^2 \|y\|^4} \, d\lambda(y)$$

converges in $\mathbb{L}^\infty(\Omega)$ for R tending to the infinity. Its limit is denoted in the sequel by $\mathcal{R}_{i,\varepsilon}(f)(\omega)$.

Proof. If $f \in \mathcal{E}$, we consider $\vec{h} \in C_T$ such that $f = \mathrm{div}_T \vec{h}$. Assume $\varepsilon < R < R'$. To apply the Cauchy criterion, we want to estimate

$$\mathcal{R}_{i,\varepsilon}^R(f)(\omega) - \mathcal{R}_{i,\varepsilon}^{R'}(f)(\omega) = \iiint_{R < \|y\| < R'} f(T_{-y}\omega) \frac{y_i}{\pi^2 \|y\|^4} \, d\lambda(y).$$

By integration by parts (integrating $\frac{\partial h_{j,\omega}}{\partial x_j}$ and deriving $\frac{y_i}{\pi^2 \|y\|^4}$), we obtain the following upper bound

$$\iiint_{R < \|y\| < R'} \sum_{j=1}^{3} h_{j,\omega}(-y) \frac{\partial}{\partial x_j} \frac{y_i}{\pi^2 \|y\|^4} \, d\lambda(y) +$$
$$\iint_{\|y\|=R} \frac{y_i}{\pi^2 \|y\|^4} \langle \vec{h}_\omega(-y), \; d\vec{\sigma}_R(y) \rangle - \iint_{\|y\|=R'} \frac{y_i}{\pi^2 \|y\|^4} \langle \vec{h}_\omega(-y), \; d\vec{\sigma}_{R'}(y) \rangle$$

where $d\vec{\sigma}_R$ is the infinitesimal normal field of the sphere of radius R. Passing to polar coordinates we have $\frac{|y_i|}{\|y\|^4} \leq \frac{1}{r^3}$, and $|\frac{\partial}{\partial x_j} \frac{y_i}{\|y\|^4}| \leq \frac{4}{r^4}$. Integrating (again in polar coordinates), we thus obtain the upper bound

$$\frac{1}{\pi^2}\|\vec{h}\|_\infty 4\pi\Big(\int_{R<r<R'} 4r^{-2}dr + R^{-1} + R'^{-1}\Big) \times 3.$$

Since $R < R'$, this is still bounded by $\|\vec{h}\|_\infty 72\pi^{-1}R^{-1}$. It shows the convergence, in the sense of generalized Riemann integrals, according to the Cauchy criterion. □

Our next claim is

Lemma 3. *For all $1 < p < \infty$, there exists a constant c_p such that for all $f \in \mathcal{E}$ and all $\varepsilon > 0$ we have $\|\mathcal{R}_{i,\varepsilon}(f)\|_p \leq c_p\|f\|_p$.*

Proof. According to the "transfer principle", the proof consists in using the analogous result on $\mathbb{R}^3$, due to Calderon and Zygmund (see [2], an other good reference is Stein's book [8], Theorem 3 in Chapter II)

Theorem 4 (Calderon and Zygmund). *For all $1 < p < \infty$, there exists a constant c_p such that, for all $\varepsilon > 0$, the operator $R_{i,\varepsilon}$ defined on $\mathbb{L}^p(\mathbb{R}^3)$ by*

$$R_{i,\varepsilon}(\phi)(x) = \iiint_{\|y\|>\varepsilon} \frac{y_i}{\pi^2\|y\|^4}\phi(x-y)\ d\lambda(y)$$

satisfies $\|R_{i,\varepsilon}\phi\|_p \leq c_p\|\phi\|_p$. Moreover the limit $\lim_{\varepsilon\to 0} R_{i,\varepsilon}(\phi)(x) = R_i(\phi)(x)$ exists in $\mathbb{L}^p(\mathbb{R}^3)$ norm, and satisfies $\|R_i\phi\|_p \leq c_p\|\phi\|_p$.

Application of the Transfer principle can be done by using the usual pointwise ergodic theorem on balls (see Wiener [10]). We present this method now. Thus assume $\varepsilon > 0$ and $f \in \mathcal{E}$. For any ω, for all $R > 0$, the function $\phi_{\omega,R}$ defined by $\phi_{\omega,R}(x) = f(T_x\omega) \cdot \mathbf{1}_{\|x\|<R}$ belongs to $\mathbb{L}^p(\mathbb{R}^3)$. We denote $K(y) = \dfrac{y_i}{\pi^2\|y\|^4}$ (dropping the index i). We have

$$R_\varepsilon(\phi_{\omega,R})(x) = \iiint_{\substack{\|y\|>\varepsilon \\ \|x-y\|<R}} K(y)f(T_{x-y}\omega)\ d\lambda(y)$$

and, for $\rho > 0$,

$$\mathcal{R}^\rho_\varepsilon(f)(T_x\omega) = \iiint_{\varepsilon<\|y\|<\rho} K(y)f(T_{x-y}\omega)\ d\lambda(y).$$

If $\rho < R$ and $\|x\| < R - \rho$, the first domain of integration contains the second one. Hence we have

$$R_\varepsilon(\phi_{\omega,R})(x) = \mathcal{R}^\rho_\varepsilon(f)(T_x\omega) + \iiint_{\substack{\|y\|>\rho \\ \|x-y\|<R}} K(y)f(T_{x-y}\omega)\ d\lambda(y).$$

According to the proof of Lemma 2, and considering $\vec{h} \in C_T$ such that $f = \mathrm{div}\vec{h}$, the last above integral is bounded by $\|\vec{h}\|_\infty 72\pi^{-1}\rho^{-1}$. Hence by considering the norm $\mathbb{L}^p(\mathcal{D}, d\lambda)$ of the domain $\mathcal{D} = \{x,\ \|x\| < R - \rho\}$, we have, according to the triangular inequality,

$$\Big(\iiint_{\|x\|<R-\rho} |\mathcal{R}_\varepsilon^\rho(f)(T_x\omega)|^p \ d\lambda(x)\Big)^{1/p} \leq$$
$$\Big(\iiint_{\|x\|<R-\rho} |R_\varepsilon(\phi_{\omega,R})(x)|^p \ d\lambda(x)\Big)^{1/p} + \Big(\frac{4\pi(R-\rho)^3}{3}\Big)^{1/p} \|\vec{h}\|_\infty 72\pi^{-1}\rho^{-1}. \tag{2}$$

An upper-bound for the integral which appears in the right-hand side is obtained by integrating on the whole space $\mathbb{R}^3$. Thus it is bounded, according to the continuity of the operator R_ε on $\mathbb{L}^p(\mathbb{R}^3)$ stated in the above Calderon-Zygmund's Theorem, by

$$c_p\Big(\iiint_{\mathbb{R}^3} |\phi_{\omega,R}(x)|^p \ d\lambda(x)\Big)^{1/p}.$$

Since $\phi_{\omega,R}$ is zero except on the domain $\|x\| < R$, this integral is equal to

$$c_p\Big(\iiint_{\|x\|<R} |f(T_x\omega)|^p \ d\lambda(x)\Big)^{1/p}.$$

Plugging this in Inequality (2), dividing each term by $(4\pi R^3/3)^{1/p}$ and using the usual ergodic theorem, we obtain for R tending to infinity

$$\|\mathcal{R}_\varepsilon^\rho(f)\|_p \leq c_p\|f\|_p + \|\vec{h}\|_\infty 72\pi^{-1}\rho^{-1}.$$

Finally, for ρ tending to infinity, we obtain the desired bound $\|\mathcal{R}_\varepsilon(f)\|_p \leq c_p\|f\|_p$. □

This lemma allows us to extend the operator $\mathcal{R}_{i,\varepsilon}$ to the closure of $\mathcal{E}$ in $\mathbb{L}^p(\Omega)$. However we have

Lemma 4. *The closure of $\mathcal{E}$ for the $\mathbb{L}^p(\Omega)$ norm is the space of $\mathbb{L}^p(\Omega)$-functions of vanishing expectation. This space is denoted by $\mathbb{L}^p_0(\Omega)$ in the sequel.*

Proof. Let f be a function in $\mathbb{L}^p(\Omega)$ with vanishing expectation. We want to approach it, for the $\mathbb{L}^p(\Omega)$ norm, by a function belonging to $\mathcal{E}$. Since $\mathbb{L}^\infty(\Omega)$ is dense in $\mathbb{L}^p(\Omega)$, we can suppose that f is bounded. Consider the function $\vec{h}_R$ defined by

$$h_{j,R}(\omega) = \frac{3}{4\pi R^3}\iiint_{\|x\|<R} h_{j,x}(\omega)\ d\lambda(x)$$

where $\vec{h}_x = (h_{j,x})_{1\leq j\leq 3}$ is defined by

$$h_{j,x}(\omega) = -x_j\int_0^1 f(T_{sx}\omega)\ ds.$$

It can be easily checked that, in the weak sense, we have $\mathrm{div}_T\vec{h}_x = f - f\circ T_x$. This leads to

$$\mathrm{div}_T\ \vec{h}_R(\omega) = f(\omega) - \frac{3}{4\pi R^3}\iiint_{\|x\|<R} f(T_x\omega)\ d\lambda(x).$$

This function is denoted by f_R. The ergodic theorem in $\mathbb{L}^p$ norm on balls gives next the convergence of f_R to f. Moreover, since f is bounded, so is

$\vec{h}_R$. It remains to approach coordinates of $\vec{h}_R$ by functions belonging to C_T. This will be done by convolution with smooth functions. Consider a positive $\mathcal{C}^\infty$-function ψ with compact support containing a neighborhood of 0. Assume moreover that its integral is equal to 1. For $\eta > 0$, denote by ψ_η the function defined by $\psi_\eta(x) = \frac{1}{\eta^3}\psi(\frac{x}{\eta})$. Consider finally the function $(\psi_\eta \star_T \vec{h}_R)$ defined by

$$(\psi_\eta \star_T \vec{h}_R)(\omega) = \iiint \psi_\eta(y)\vec{h}_R(T_{-y}\omega)\ d\lambda(y).$$

The following formula can be easily checked a.s. ω

$$\partial_i(\psi_\eta \star_T \vec{h}_R) = (\frac{\partial \psi_\eta}{\partial x_i} \star_T \vec{h}_R) \tag{3}$$

(by dominated convergence, since h_R is bounded and ψ has compact support). It is now clear that a.s. ω, the function $x \mapsto (\psi_\eta \star_T h_R)(T_x\omega)$ admits partial derivatives of order 1, which are bounded on ω. By induction, we show in a similar way that this function is $\mathcal{C}^\infty$ and has all its partial derivatives bounded on ω. Thus the function $f_{\eta,R} = \mathrm{div}_T(\psi_\eta \star_T \vec{h}_R)$ lies in $\mathcal{E}$. It remains to be checked that $f_{\eta,R}(\omega)$ converges to $f_R(\omega)$ for the $\mathbb{L}^p(\Omega)$ norm when η tends to 0. However we also have, according to formula (3) and Definition 1

$$\mathrm{div}_T(\psi_\eta \star_T \vec{h}_R) = \psi_\eta \star_T \mathrm{div}_T(\vec{h}_R)$$

what tends to $\mathrm{div}_T(\vec{h}_R)$ in $\mathbb{L}^p$ norm for η tending to 0, according to the usual Wiener local ergodic theorem on balls. Since f_R is as close as we want to f in $\mathbb{L}^p(\Omega)$, the lemma is proved. □

We always denote by $\mathcal{R}_{i,\varepsilon}$ the operator extended to $\mathbb{L}_0^p(\Omega)$.

Proposition 2. *For any $p > 1$ and for any function $f \in \mathbb{L}_0^p(\Omega)$, the limit*

$$\mathcal{R}_i(f)(\omega) = \lim_{\epsilon \to 0} \mathcal{R}_{i,\varepsilon}(f)$$

exists in $\mathbb{L}_0^p(\Omega)$ norm. We have moreover $\|\mathcal{R}_i(f)\|_p < c_p\|f\|_p$.

Proof. Since the norm of the operator $\mathcal{R}_{i,\varepsilon}$ does not depend on ε, it is enough to check convergence for functions $f \in C_T$ of null expectation. However we have $\mathcal{R}_{i,\varepsilon}(f)(\omega) - \mathcal{R}_{i,\varepsilon'}(f)(\omega) =$

$$\iiint_{\varepsilon<\|y\|<\varepsilon'} f(T_y\omega)K(-y)\ d\lambda(y).$$

Since the integral of the kernel $K(y)$ on any sphere is zero, it is still equal to

$$\iiint_{\varepsilon<\|y\|<\varepsilon'} \frac{f(T_y\omega) - f(\omega)}{\|y\|}\|y\|K(-y)\ d\lambda(y).$$

The above ratio being bounded (because $f \in C_T$), and function $y \mapsto \|y\|K(y)$ being integrable on a neighborhood of 0, the family of functions $\mathcal{R}_{i,\varepsilon}(f)$ constitutes a Cauchy sequence in $\mathbb{L}_0^p(\Omega)$. This proves the desired convergence, and proposition. □

3.1.2 Properties of duality

Weak operators curl_T and div_T which appear in the system $(\mathcal{S}'_{\bar{h}})$ were defined, by analogy with the theory of distributions, by formulas of duality. To solve the system $(\mathcal{S}'_{\bar{h}})$ using $\mathcal{R}_i$, it is thus necessary to study the properties of duality of these operators. We start by extending the convolution $\phi \star_T f$. As it was seen in the proof of Lemma 4, it is defined by the formula

$$\phi \star_T f(\omega) = \iiint \phi(-x) f(T_x \omega)\ d\lambda(x).$$

This integral converges for almost all ω if ϕ is integrable on $\mathbb{R}^3$, and f is bounded. Moreover, if $f \in C_T$ and if ϕ and all its derivatives are bounded by $\frac{1}{\|x\|^4}$ up to a constant, we have, by dominated convergence, the two formulas

$$\begin{aligned} \partial_\ell(\phi \star_T f) &= \phi \star_T \partial_\ell f \\ &= \frac{\partial \phi}{\partial x_\ell} \star_T f. \end{aligned} \tag{4}$$

We extend now this convolution for $f \in \mathbb{L}^p(\Omega)$.

Lemma 5. *Assume* $1 \le p \le \infty$. *Assume* $\phi \in \mathbb{L}^1(\mathbb{R}^3)$ *and* $f \in \mathbb{L}^p(\Omega)$. *The formula*

$$\phi \star_T f(\omega) = \iiint \phi(-x) f(T_x \omega)\ d\lambda(x)$$

defines an $\mathbb{L}^p(\Omega)$*-function, and we have* $\|\phi \star_T f\|_p \le \|\phi\|_1 \|f\|_p$.

Proof. The inequality is an easy consequence of the Jensen inequality and of the stationarity of T. Since $\mathbb{L}^p(\Omega)$ is complete, this proves that $\phi \star_T f$ is well defined. □

We state now the first duality property of $\mathcal{R}_i$.

Lemma 6. *Let* ψ *be a* $\mathcal{C}^\infty$*-function We suppose that* ψ *and all its derivatives are bounded by* $\frac{1}{\|x\|^4}$ *up to a constant. Then*

1. $R_i(\frac{\partial \psi}{\partial x_j})$ *and all its derivatives are* $\mathcal{C}^\infty$*-functions bounded by* $x \mapsto \|x\|^{-4}$ *up to a constant.*

2. *We have, for any* $h \in C_T$, *the formula of duality a.s.* ω

$$\psi \star_T \mathcal{R}_i(\partial_j h)(\omega) = R_i(\frac{\partial \psi}{\partial x_j}) \star_T h(\omega). \tag{5}$$

Remark. We can not extend this lemma for $h \in \mathbb{L}^p_0(\Omega)$, since the operator ∂_j is not continuous.

Proof. In the sequel we denote by a "prime" the derivatives with respect to x_j. To begin with, we study $\mathcal{R}_i(\partial_j h)(T_x \omega)$. This quantity is the limit, for R tending to infinity and ε tending to 0, of

$$\mathcal{R}^R_{i,\varepsilon}(\partial_j h)(T_x \omega) = \iiint_{\varepsilon < \|y\| < R} \partial_j h(T_{x-y}\omega) K(y)\ d\lambda(y)$$

Splitting this integral according to $\|y\| < 1$ and $\|y\| > 1$, we have, for R tending to infinity,

$$\begin{aligned}\mathcal{R}_{i,\varepsilon}\partial_j h(T_x\omega) = &\iiint_{\varepsilon<\|y\|<1} \partial_j h(T_{x-y}\omega)K(y)\ d\lambda(y)+ \\ &\iiint_{1<\|y\|} h(T_{x-y}\omega)K'(y)\ d\lambda(y)+ \\ &-\iint_{\|y\|=1} h(T_{x-y}\omega)K(y)\ d\sigma_j(y),\end{aligned} \tag{6}$$

where $d\vec{\sigma} = (d\sigma_k)_{k=1,\dots,3}$ is the normal infinitesimal field of the sphere, and the two last terms are coming from an integration by parts. Hence $\mathcal{R}_{i,\varepsilon}\partial_j h$ is bounded, since all these integrals are convergent, and $h \in C_T$. Similarly, computing derivatives by dominated convergence, we have $\mathcal{R}_{i,\varepsilon}\partial_j h \in C_T$. Now, since $\psi \in \mathbb{L}^1(\mathbb{R}^3)$, the convolution $\psi \star_T \mathcal{R}_{i,\varepsilon}\partial_j h$ is bounded, and is equal to

$$\begin{aligned}&\psi \star_T \mathcal{R}_{i,\varepsilon}\partial_j h(\omega) = \\ &\iiint_x \iiint_{\varepsilon<\|y\|<1} \partial_j h(T_{x-y}\omega)K(y)\psi(-x)\ d\lambda(y)d\lambda(x)+ \\ &\quad\iiint_x \iiint_{1<\|y\|} h(T_{x-y}\omega)K'(y)\psi(-x)\ d\lambda(y)d\lambda(x)+ \\ &\quad\quad\iiint_x \iint_{\|y\|=1} h(T_{x-y}\omega)K(y)\psi(-x)\ d\sigma_j(y)d\lambda(x).\end{aligned}$$

By the change of variables $u = x - y$, $v = y$ in these three integrals, and then integrating by parts with respect to u in the first one, we obtain

$$\psi \star_T \mathcal{R}_{i,\varepsilon}\partial_j h(\omega) = R_{i,\varepsilon}\psi' \star_T h(\omega), \tag{7}$$

where

$$\begin{aligned}R_{i,\varepsilon}\psi'(-u) = &\iiint_{\varepsilon<\|v\|<1} K(v)\psi'(-u-v)\ d\lambda(v)+ \\ &\iiint_{1<\|v\|} K'(v)\psi(-u-v)\ d\lambda(v)+ \\ &\iint_{\|v\|=1} K(v)\psi(-u-v)\ d\sigma_j(v).\end{aligned}$$

Since the domains of the first and third integral are contained in $\|u - v\| \leq 1$, these integrals have the same order as $\psi'(-u)$ and $\psi(-u)$ for $\|u\|$ tending to infinity, hence are bounded by $M_\varepsilon \max(\|u\|^{-4}, 1)$. The analogue upper bound on the second integral follows from the classical formula

$$\iiint_{\substack{\|v\|>1 \\ \|u-v\|>1}} \|v\|^{-4}\|u-v\|^{-4}\ d\lambda(v) \leq M\|u\|^{-4}$$

for $\|u\|$ large enough. Now, since the integral of K on any sphere vanishes, the first term of the above decomposition of $R_{i,\varepsilon}\psi'(-u)$ can be rewritten as

$$\iiint_{\varepsilon<\|v\|<1} \|v\|K(v)\frac{\psi'(-u-v) - \psi'(-u)}{\|v\|}\ d\lambda(v).$$

This is bounded by $M \max(\|u\|^{-4}, 1)$ independently of ε, since $v \mapsto \|v\|K(v)$ is integrable on $\|v\| < 1$. Hence, for ε tending to 0, we have

$$|R_i\psi'(u)| < M \max(\|u\|^{-4}, 1).$$

By similar arguments, this kind of upper bound holds for any derivatives of $R_i\psi'$, and Paragraph 1 is proved. Moreover, it follows from the fact that M does not depend on ε, that the convergence $R_{i,\varepsilon}\psi' \to R_i\psi'$ for ε tending to 0 is dominated, hence holds in $\mathbb{L}^1(\mathbb{R}^3)$. By an analogue computation, the convergence $\mathcal{R}_{i,\varepsilon}\partial_\ell h \to \mathcal{R}_i\partial_\ell h$ for ε tending to 0 holds in $\mathbb{L}^\infty(\Omega)$. Hence we can take ε tending to infinity in (7), which gives (5). This achieves the proof of Lemma 6. $\square$

In order to obtain a duality formula for $\mathcal{R}_i^2$, we have to generalise Lemma 6 for functions $f \in \mathbb{L}_0^p$.

Lemma 7. *Let ϕ be a $\mathcal{C}^\infty$-function with compact support. Then*

1. *$R_i(\frac{\partial\phi}{\partial x_j})$ and all its partial-derivatives are $\mathcal{C}^\infty$-functions bounded by $x \mapsto \|x\|^{-4}$ up to a constant.*

2. *We have, for all $f \in \mathbb{L}_0^p(\Omega)$, the duality formula a.s. ω*

$$\frac{\partial\phi}{\partial x_j} \star_T \mathcal{R}_i(f)(\omega) = R_i(\frac{\partial\phi}{\partial x_j}) \star_T f(\omega). \tag{8}$$

Proof. Paragraphs 1 follows from Lemma 6. Next, applying ∂_ℓ to (5) we have

$$\partial_\ell(\phi \star_T \mathcal{R}_i(\partial_j h)) = \partial_\ell(R_i(\frac{\partial\phi}{\partial x_j}) \star_T h).$$

Applying (4), we obtain

$$\phi \star_T \partial_\ell(\mathcal{R}_i(\partial_j h)) = R_i(\frac{\partial\phi}{\partial x_j}) \star_T \partial_\ell h. \tag{9}$$

Since $\partial_\ell(\mathcal{R}_i(\partial_j h)) = \mathcal{R}_i(\partial_j\partial_\ell h) = \partial_j(\mathcal{R}_i(\partial_\ell h))$, the left hand side of (9) is $\partial_j(\phi \star_T \mathcal{R}_i(\partial_\ell h))$. From (4), it is still $\frac{\partial\phi}{\partial x_j} \star_T (\mathcal{R}_i(\partial_\ell h)$. It follows

$$\frac{\partial\phi}{\partial x_j} \star_T \mathcal{R}_i(\partial_\ell h)(\omega) = R_i(\frac{\partial\phi}{\partial x_j}) \star_T \partial_\ell h(\omega).$$

By density of $\mathcal{E}$ in $\mathbb{L}_0^p(\Omega)$, continuity of $\mathcal{R}_i$, and Lemma 5, this gives (8). $\square$

These lemmas will be applied in particular in the following way: if $\psi = R_i(\frac{\partial\phi}{\partial x_k})$, where ϕ is a $\mathcal{C}^\infty$-function with compact support, and $h \in C_T$, it follows from Lemma 6

$$R_i(\frac{\partial\phi}{\partial x_k}) \star_T \mathcal{R}_j(\partial_\ell h) = R_j(\frac{\partial}{\partial x_\ell}R_i(\frac{\partial\phi}{\partial x_k})) \star_T h$$

and from Lemma 7 with $f = \mathcal{R}_j\partial_\ell h$

$$\frac{\partial\phi}{\partial x_k} \star_T \mathcal{R}_i(\mathcal{R}_j\partial_\ell h) = R_i(\frac{\partial\phi}{\partial x_k}) \star_T \mathcal{R}_j\partial_\ell h. \tag{10}$$

Finally we have

Corollary 3. *For any $\mathcal{C}^\infty$-function ϕ with compact support and for any $h \in C_T$, we have*

$$R_j(\frac{\partial}{\partial x_\ell}R_i(\frac{\partial\phi}{\partial x_k})) \star_T h = \frac{\partial\phi}{\partial x_k} \star_T \mathcal{R}_i(\mathcal{R}_j\partial_\ell h).$$

3.2 Resolution of $(\mathcal{S}'_{\vec{h}})$

These results have as an immediate consequence the expression of the solution of the system $(\mathcal{S}'_{\vec{h}})$ using the operators $\mathcal{R}_j$.

Proposition 3. *Assume that $p \geq 2$ and that $\vec{h} \in \mathbb{L}^p_0$. The system $(\mathcal{S}'_{\vec{h}})$ admits as unique solution in $\mathbb{L}^p(\Omega)$ the field $\vec{g}$ defined by*

$$g_i = -\sum_{j=1}^{3} \mathcal{K}_{ij}(h_j),$$

with $\mathcal{K}_{ij} = \mathcal{R}_i \mathcal{R}_j$. Moreover there exists a constant κ_p not depending of $\vec{h}$ such that $\|\vec{g}\|_p \leq \kappa_p \|\vec{h}\|_p$.

Proof. The case $p = 2$ was done on page 6. It was also seen that $\kappa_2 = 1$. By inclusion $\mathbb{L}^p \subset \mathbb{L}^2$, this gives uniqueness for $p > 2$. To check that the suggested solution is appropriate, we can suppose that $h_j \in \mathcal{E}$, i.e. $h_j = \sum_\ell \partial_\ell h_{\ell,j}$ with $h_{\ell,j} \in C_T$. The first equation $\int_\Omega \vec{g}\ d\mu = 0$ is trivial. We check first that $\mathrm{curl}_T \vec{g} = 0$. Let ϕ be a $\mathcal{C}^\infty$-function with compact support; let us compute

$$\iiint \frac{\partial \phi}{\partial x_i}(x) g_k(T_x\omega) - \frac{\partial \phi}{\partial x_k}(x) g_i(T_x\omega)\ d\lambda(x).$$

According to the expression of $\vec{g}$ given by Proposition 3, and Equality (10), it is equal to

$$\iiint \Big(R_k \frac{\partial \phi}{\partial x_i}(x) - R_i \frac{\partial \phi}{\partial x_k}(x) \Big) \Big(\sum_{j=1}^{3} \mathcal{R}_j h_j(T_x\omega) \Big)\ d\lambda(x).$$

The first bracket vanish (this is a classical property of the Riesz operator R_i on $\mathbb{R}^3$, which follows from an obvious computation on Fourier transforms). This shows that $\mathrm{curl}_T \vec{g} = 0$.
We compute $\mathrm{div}_T \vec{g}$. This random variable is defined by

$$\iiint \phi(-x) \cdot \mathrm{div}_T \vec{g}(T_x\omega)\ d\lambda(x) = \iiint \sum_{i=1}^{3} \frac{\partial \phi}{\partial x_i}(-x) g_i(T_x\omega)\ d\lambda(x).$$

According to the expression of $\vec{g}$ given by Proposition 3, and Corollary 3, it is equal to

$$-\iiint \sum_{\ell=1}^{3} \sum_{j=1}^{3} \Big(\sum_{i=1}^{3} R_j \frac{\partial}{\partial x_\ell} R_i \frac{\partial \phi}{\partial x_i}(-x) \Big) h_{\ell,j}(T_x\omega)\ d\lambda(x),$$

which is still equal, according to a property of the Riesz operator R_i on $\mathbb{R}^3$ easily checkable on Fourier transforms, to

$$\iiint \sum_{\ell=1}^{3} \sum_{j=1}^{3} \Big(\frac{\partial^2 \phi}{\partial x_j \partial x_\ell}(-x) \Big) h_{\ell,j}(T_x\omega)\ d\lambda(x).$$

Integrating by parts, we obtain

$$\iiint \sum_{j=1}^{3} \frac{\partial \phi}{\partial x_j}(-x)\Big(\sum_{\ell=1}^{3} \partial_\ell h_{\ell,j}(T_x\omega)\Big)\ d\lambda(x).$$

According to the equality $h_j = \sum_{\ell=1}^{3} \partial_\ell h_{\ell,j}$, this shows that $\mathrm{div}_T\vec{g} = \mathrm{div}_T\vec{h}$, and proves Proposition 3. □

4 Initial system

Let us return to the system $(\mathcal{S})$. We consider the sequence of fields $\vec{g}_n$ defined by induction by $\vec{g}_0 \equiv 0$, and for $n+1$, the field $\vec{g}_{n+1}$ is the solution of the system $(\mathcal{S}'_{\vec{h}_n})$ of right-hand side $\vec{h}_n = \vec{g}_n - \frac{\mathcal{A}}{C}(\vec{g}_n + \vec{U})$. The system $(\mathcal{S}'_{\vec{h}_n})$ being linear, the field $\vec{g}_{n+1} - \vec{g}_n$ is the solution corresponding to the second member $\vec{h}_n - \vec{h}_{n-1}$. According to what precedes, we have then

$$\|\vec{g}_{n+1} - \vec{g}_n\|_p \le \kappa_p \|\vec{h}_n - \vec{h}_{n-1}\|_p.$$

It follows from the expression of $\vec{h}_n$ and from the assumption of ellipticity on $\mathcal{A}$ that it is still bounded by

$$\kappa_p(1 - \frac{c}{C})\|\vec{g}_n - \vec{g}_{n-1}\|_p,$$

and therefore by induction still by $(\kappa_p(1-\frac{c}{C}))^n\|\vec{g}_1\|_p$. However according to Riesz interpolation theorem (see [9]), the norm κ_p is convex, therefore continuous, with respect to p. However $\kappa_2 = 1$, therefore, for p enough close to 2, we have $\kappa_p(1-\frac{c}{C}) < 1$. The series $\sum_{n=1}^{\infty}\|\vec{g}_{n+1} - \vec{g}_n\|_p$ is thus convergent, and therefore the sequence of the fields $\vec{g}_n$ is a Cauchy sequence, hence converges to a field denoted $\vec{g}_\infty$. Now, for $n+1$, the third equation of the system $(\mathcal{S}'_{\vec{h}_n})$ is written $\mathrm{div}_T(\vec{g}_{n+1} - \vec{h}_n) = 0$, i.e.

$$\mathrm{div}_T\Big((\vec{g}_{n+1} - \vec{g}_n) - \frac{\mathcal{A}}{C}(\vec{g}_n + \vec{U})\Big) = 0.$$

The operator div_T is not continuous, however its kernel is closed. We thus obtain, for n tending to infinity, $\mathrm{div}_T(\mathcal{A}(\vec{g}_\infty + \vec{U})) = 0$. In the same way, the second equation of the system $(\mathcal{S}'_{\vec{h}_n})$ gives, for n tending to infinity, $\mathrm{curl}_T(\vec{g}_\infty) = 0$. Thus $\vec{g}_\infty + \vec{U}$ is an $\mathbb{L}^p$-solution of $(\mathcal{S})$. Hence by uniqueness, the $\mathbb{L}^2$-solution lies in $\mathbb{L}^p(\Omega)$ for p close enough to 2. And the proof of Theorem 1 is complete.

5 Integral expression of $\mathcal{K}_{ij}$ operators

Since the operators $\mathcal{R}_i$ are defined by an integral expression, a natural question is: does such an expression exists for $\mathcal{K}_{ij} = \mathcal{R}_i\mathcal{R}_j$. The answer is stated in the following proposition.

Proposition 4. *For any $f \in \mathbb{L}_0^p(\Omega)$, we have*

$$\mathcal{K}_{ij} f(\omega) = \begin{cases} \iiint \dfrac{3y_i y_j}{4\pi\|y\|^5} f(T_{-y}\omega)\ d\lambda(y) & \text{if} \quad i \neq j \\ \dfrac{-1}{3} f(\omega) + \iiint \dfrac{2y_i^2 - y_k^2 - y_\ell^2}{4\pi\|y\|^5} f(T_{-y}\omega)\ d\lambda(y) & \text{if} \quad i = j, \end{cases}$$

where $\{k, \ell\} = \{1, 2, 3\} \setminus \{i\}$.

Remark. The kernel under both the integrals is the derivative $\frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{4\pi\|x\|}$ of the Green function. The term $-f(\omega)/3$ may be understood as following: at $x = 0$, the derivative $\frac{\partial^2}{\partial x_i^2} \frac{1}{4\pi\|x\|}\Big|_{x=0}$ is the Dirac distribution $\phi \mapsto -\phi(0)/3$.

Proof. Consider the case $i = j$, which is more interesting. To begin with, we check the analogous formula for the classical Riesz operator. For any $\mathcal{C}^\infty$-function ϕ, we denote by $\tilde{K}_{ii}(\phi)$ the function defined by

$$\tilde{K}_{ii}(\phi)(x) = \iiint \frac{2y_i^2 - y_k^2 - y_\ell^2}{4\pi\|y\|^5} \phi(x - y)\ d\lambda(y).$$

According to Theorem 5 Chapter III of [8], its Fourier transform is

$$\eta \mapsto \frac{-2\eta_i^2 + \eta_k^2 + \eta_\ell^2}{3\|\eta\|^2} \hat{\phi}(\eta).$$

Since the Fourier transform of $R_i^2(\phi)$ is the function $\eta \mapsto \frac{-\eta_i^2}{\|\eta\|^2} \hat{\phi}(\eta)$, this leads to $(R_i^2(\phi) - \tilde{K}_{ii}(\phi))\hat{}(\eta) = \frac{-1}{3} \hat{\phi}(\eta)$ and proves the desired equality

$$R_i^2 \phi = -\frac{1}{3}\phi + \tilde{K}_{ii}\phi. \tag{11}$$

Next, we consider the operator $\tilde{\mathcal{K}}_{ii}$ defined for any $f \in \mathbb{L}_0^p(\Omega)$ by

$$\tilde{\mathcal{K}}_{ii}(f)(\omega) = \iiint \frac{2y_i^2 - y_k^2 - y_\ell^2}{4\pi\|y\|^5} f(T_{-y}\omega)\ d\lambda(y).$$

We have the duality formula

$$\tilde{K}_{ii}(\frac{\partial \phi}{\partial x_j}) \star_T f = \frac{\partial \phi}{\partial x_j} \star_T \tilde{\mathcal{K}}_{ii}(f) \tag{12}$$

(the proof of Lemma 7 works without modification). Thus it follows from this equality, Corollary 3, and equality (4), that for any $\mathcal{C}^\infty$-function ϕ with compact support, any $j \in \{1, 2, 3\}$, and any $h \in C_T$,

$$\frac{\partial \phi}{\partial x_j} \star_T \Big(\mathcal{R}_i^2(\partial_\ell h) - \tilde{\mathcal{K}}_{ii}(\partial_\ell h) + \frac{1}{3}\partial_\ell h\Big) =$$
$$\Big(R_i \frac{\partial}{\partial x_\ell} R_i \frac{\partial \phi}{\partial x_j} - \frac{\partial}{\partial x_\ell} \tilde{K}_{ii} \frac{\partial \phi}{\partial x_j} + \frac{1}{3} \frac{\partial^2 \phi}{\partial x_\ell \partial x_j}\Big) \star_T h.$$

It is zero, according to the equality (11). This imply by density of $\mathcal{E}$ that for any $f \in \mathbb{L}_0^p(\mathbb{R}^3)$

$$\frac{\partial \phi}{\partial x_j} \star_T \Big(\mathcal{R}_i^2(f) - \tilde{\mathcal{K}}_{ii}(f) + \frac{1}{3} f\Big) = 0.$$

According to the assumption $\int_\Omega f\ d\mu = 0$, the proof of Proposition 4 is completed by the following lemma.

Lemma 8. *Assume $f \in \mathbb{L}^p(\Omega)$. Assume that for any $\mathcal{C}^\infty$-function ϕ with compact support, and any $j \in \{1,2,3\}$, we have*

$$\frac{\partial \phi}{\partial x_j} \star_T f = 0. \tag{13}$$

Then f is constant: $f = \int_\Omega f \, d\mu$.

Proof. It follows from the ergodicity assumption that it is sufficient to prove that f is T-invariant. Assume $x \in \mathbb{R}^3$. According to the Wiener local ergodic theorem, it is sufficient to prove that $\phi \star_T (f - f \circ T_x) = 0$, for any $\mathcal{C}^\infty$-function ϕ with compact support. We have

$$\iiint \phi(y)\big(f(T_{-y}\omega) - f(T_{x-y}\omega)\big) \, d\lambda(y) = \iiint \big((\phi(y) - \phi(x+y)\big) f(T_{-y}\omega) \, d\lambda(y).$$

By computation of the derivative of $s \mapsto \phi(sx+y)$, this is equal to

$$\iiint \Big(\int_{s=0}^{1} \sum_{j=0}^{1} x_j \frac{\partial \phi}{\partial x_j}(sx+y) \, ds\Big) f(T_{-y}\omega) \, d\lambda(y),$$

which is zero, according to the assumption (13) and the Fubini theorem. This completes the proof of Lemma 8 and therefore of Proposition 4. □

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