Degree two Ergodic Theorem for Divergence-Free Stationary Random Fields

Jérôme Depauw *

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Abstract

We prove the ergodic theorem for surface integrals of divergence-free stationary random fields of $\mathbb{R}^3$. Mean convergence in $\mathbb{L}^p$ spaces takes place as soon as the field is $\mathbb{L}^p$-integrable. The condition of integrability for the pointwise convergence is expressed by a Lorentz norm. This theorem is an ergodic theorem for cocycles of degree 2, analogous to the ergodic theorem for cocycles of degree 1 proved in [Boivin; Derriennic].

1 Introduction

We prove here the convergence of 2-dimensional averages of a 3-dimensional, stationary, divergence-free, random field. The averages are taken on triangles with the origin as one of the vertices, with angles bounded from below by a constant > 0. The convergence takes place in norm $\mathbb{L}^p$ or almost surely, according to the integrability of the random field, when the area of the triangles tends to 0 or to infinity. The family of integrals of such a field, on triangular surfaces, forms a degree 2 cocycle for the action of translations. Thus this theorem constitutes an ergodic theorem for degree 2 cocycles, analogous to the ergodic theorem for degree 1 cocycles of actions of $\mathbb{R}^d$, proved in [Boivin; Derriennic]. Similarly to this last reference, the required condition of integrability for the pointwise convergence is finiteness of a Lorentz norm.

We pass to a more detailed description: Let T be an action of the group $\mathbb{R}^3$ on a probability space $(\Omega, \mathcal{B}, \mu)$ such that $(\omega, x) \mapsto T_x\omega$ is measurable, and that T_x preserves the probability μ for all x. Let $\vec{f}(\omega) = (f_1(\omega), f_2(\omega), f_3(\omega))$ be an integrable random vector on Ω, with values in $\mathbb{R}^3$ *(Following [Wiener], we will denote the functions with their variables each time this is possible).*

Formally, the integral $\mathcal{F}(\vec{f})(x, y)(\omega)$ of the random field $(\vec{f}(T_m\omega))_{m\in\mathbb{R}^3}$ on the triangular surface $\Delta(0, x, y)$ with vertices 0, x, $x + y$ is written as

$$\mathcal{F}(\vec{f})(x, y)(\omega) = \iint_{0<s<t<1} \sum_{(i,j,k)} (x_iy_j - x_jy_i)f_k(T_{sx+ty}\omega)\, dsdt,$$

the sum being on cyclic permutations of $(1, 2, 3)$, (with $x = (x_1, x_2, x_3)$, and $y = (y_1, y_2, y_3)$). This integral is perhaps not well defined for individual ω. Indeed

*Laboratoire de Mathématiques et Physique Théorique, Faculté des Sciences et Techniques, Université de Tours, Parc de Grandmont, 37000 TOURS. email: depauw@univ-tours.fr

the Fubini theorem says that the function $m \mapsto \vec{f}(T_m\omega)$ is locally in $\mathbb{L}^1(\mathbb{R}^3)$, but it does not imply that this function is integrable on surfaces. However the function $\bar{f}_i(m)$ with values in $\mathbb{L}^1(\Omega)$, defined by $\bar{f}_i(m)(\omega) = f_i(T_m\omega)$, is continuous for the norm of $\mathbb{L}^1(\Omega)$. Thus the above integral expression makes sense for each x, y, as a two-dimensional Riemann integral, with values in $\mathbb{L}^1(\Omega)$.

The alea ω being fixed, the field $(\vec{f}(T_m\omega))_{m\in\mathbb{R}^3}$ is divergence-free if for each m

$$(\partial_1 f_1 + \partial_2 f_2 + \partial_3 f_3)(T_m\omega) = 0. \tag{1}$$

where ∂_i is the infinitesimal generator of the one parameter group $t \mapsto T_{te_i}$, and $(e_i)_{1\leq i\leq 3}$ is the canonical basis of $\mathbb{R}^3$. The divergence operator can be extended to fields which do not belong to the domain of the generators ∂_i. Thus this definition has to be taken in the following weak sense.

Definition 1 — *An integrable random vector $\vec{f}(\omega)$ admits as divergence an integrable random variable $g(\omega)$ if, ω-almost surely, for any $\mathcal{C}^\infty$ function $\phi(m)$ with compact support in $\mathbb{R}^3$ we have:*

$$\begin{aligned}\iiint \sum_{i=1}^{3} \frac{\partial \phi}{\partial m_i}(m)\cdot f_i(T_m\omega)\ d\lambda(m) \quad &= \\ -\iiint \phi(m)\cdot g(T_m\omega)\ d\lambda(m),&\end{aligned}$$

where λ is the Lebesgue measure on $\mathbb{R}^3$. We denote this property by $\operatorname{div}_T(\vec{f}) = g$. When $\operatorname{div}_T(\vec{f}) = 0$, we say that the random vector $\vec{f}(\omega)$ is divergence-free in weak sense, or is a weak functional cocycle of degree 2.

Equation (1) and its weak form given in Definition 1 have meaning in several domains.

- The concept of divergence-free random vector field in a weak sense has applications in the study of random media. It appears for instance in [Jikov; Kozlov; Oleinik] (p. 227), under the name of solenoidal field. Let us note that an integrable random vector $\vec{f}(\omega)$ is divergence-free in weak sense if and only if the random field $(\vec{f}(T_m\omega))_{m\in\mathbb{R}^3}$ is ω almost surely divergence-free on $\mathbb{R}^3$, in the sense of distributions.

- The analogue of Equation (1) for the discrete case (action of $\mathbb{Z}^3$) is studied in particular in [Katok; Katok] as a notion of functional cocycle of degree 2. That is the reason why a random vector, divergence-free in weak sense is also called a weak functional cocycle of degree 2.

- Finally, this notion makes sense from an algebraic point of view. Indeed, as in classical differential calculus, the integral of a weak functional cocycle of degree 2 on a closed surface is null. Expressed for the surface constituted by the four oriented faces of the tetrahedron with vertices 0, x, $x+y$, $x+y+z$, the free divergence property gives, ω-a.s.

$$\begin{aligned}\mathcal{F}(\vec{f})(x,y)(\omega) + \mathcal{F}(\vec{f})(x+y,z)(\omega) - \mathcal{F}(\vec{f})(x,y+z)(\omega)& \\ -\mathcal{F}(\vec{f})(y,z)(T_x\omega) = 0.&\end{aligned} \tag{2}$$

  This equation has a meaning from the point of view of group theory. It means that $\mathcal{F}(\vec{f})(x,y)(\omega)$ is an algebraic cocycle of degree 2 with values in $\mathbb{L}^1(\Omega)$, for the action U of the group $\mathbb{R}^3$ on $\mathbb{L}^1(\Omega)$, induced from T by: $(U_x\psi)(\omega) = \psi(T_x\omega)$ (see for example [Mac Lane] or [Feldman; Moore]). The concept of cocycle of degree 2 in ergodic theory is slightly different, because it is based on the orbits of T in Ω (see [Mackey] for the degree 1, or Definition 3 below for the degree 2). It is the reason why a function $\mathcal{F}(\vec{f})(x,y)(\omega)$ which verifies equation (2) will be called an algebraic pseudo-cocycle of degree 2.

The Lorentz norm of a random vector $\vec{f}(\omega)$ is defined, for $1 \leq p < +\infty$ and $1 \leq q \leq +\infty$ by

$$\|\vec{f}(\omega)\|_{p,q} = \begin{cases} \left(q\int_0^\infty \left(\mu(\|\vec{f}(\omega)\| > t)^{1/p}\cdot t\right)^q \dfrac{dt}{t}\right)^{1/q} & \text{if} \quad 1 \leq q < +\infty; \\ \sup\limits_{t>0}\left(\mu(\|\vec{f}(\omega)\| > t)^{1/p}\cdot t\right) & \text{if} \qquad q = +\infty. \end{cases} \tag{3}$$

This defines the Lorentz space $\mathbb{L}_{p,q}(\Omega)$. It is also possible to define Lorentz spaces in the case $p = +\infty$ and $1 \leq q < +\infty$, but it will not be used in our study. Lorentz spaces are interpolation spaces between Lebesgue spaces. We have for instance the identity $\mathbb{L}_{p,p}(\Omega) = \mathbb{L}^p(\Omega)$, and the inclusions

$$\bigcup_{p'>p} \mathbb{L}^{p'}(\Omega) \subset \mathbb{L}_{p,1}(\Omega) \subset \mathbb{L}^p(\Omega) \subset \mathbb{L}_{p,\infty}(\Omega) \subset \bigcap_{p'<p} \mathbb{L}^{p'}(\Omega)$$

for all $p \geq 1$ (see [Hunt]). Let us note moreover that the Markov inequality and the standard weak maximal inequality can be expressed using the Lorentz norms, respectively by

$$\|h(\omega)\|_{1,\infty} \leq \|h(\omega)\|_{1,1}, \tag{4}$$

and

$$\left\|\sup_R \frac{1}{R^3}\iiint_{\|m\|<R} h(T_m\omega)\ d\lambda(m)\right\|_{1,\infty} < C\|h(\omega)\|_{1,1}, \tag{5}$$

where λ is the Lebesgue measure on $\mathbb{R}^3$.

Let us pass to the ergodic theorem. For a constant field $\vec{f}(\omega) \equiv \int_\Omega \vec{f}\ d\mu$, we have

$$\mathcal{F}(\vec{f})(x,y)(\omega) = \frac{1}{2}\sum_{(i,j,k)} (x_iy_j - x_jy_i)\int_\Omega f_k\ d\mu,$$

the sum being taken on the (i,j,k) obtained by cyclic permutations of $(1,2,3)$, with $x = (x_1, x_2, x_3,)$ and $y = (y_1, y_2, y_3)$. This can be written as

$$\mathcal{F}(\vec{f})(x,y)(\omega) = \frac{1}{2}x \wedge y \cdot \int_\Omega \vec{f}\ d\mu,$$

where $x \wedge y$ is the vector whose k-th coordinate $(x\wedge y)_k$ is equal to $(x_iy_j - x_jy_i)$. Lastly, the area of the triangle $\Delta(0,x,y)$ is equal to

$$\frac{1}{2}\sqrt{\sum_{(i,j,k)} (x_iy_j - x_jy_i)^2} = \frac{1}{2}\|x \wedge y\|.$$

In the sequel, we will denote by $\vec{f}(\omega)$ the random vector defined on Ω, but also the random field $(\vec{f}(T_m\omega))_{m\in\mathbb{R}^3}$. Here is the ergodic theorem:

Theorem 1 — *Let $1 \leq p < +\infty$, and $1 \leq q \leq +\infty$. Let us suppose that the action T is ergodic with respect to μ. Let $\vec{f}(\omega)$ be a vector field in $\mathbb{L}_{p,q}(\Omega)$, divergence-free in weak sense.*

1. *We have the following mean convergence in $\mathbb{L}_{p,q}(\Omega)$:*

$$\lim_{x,y\to\infty} \frac{1}{\frac{1}{2}\|x\wedge y\|}\Big(\mathcal{F}(\vec{f})(x,y)(\omega) - \frac{1}{2}x\wedge y\cdot\int_\Omega \vec{f}\,d\mu\Big) = 0,$$

   when x and y tend to infinity, under the condition that the angles of the triangle $\Delta(0,x,y)$ are bounded from below by a constant $\theta_0 > 0$.

2. *If moreover $\vec{f}(\omega)$ belongs to $\mathbb{L}_{2,1}(\Omega)$, then the integral $\mathcal{F}(\vec{f})(x,y)(\omega)$ admits a pointwise version $\tilde{\mathcal{F}}(\vec{f})(\omega,x,y)$, continuous in x,y for almost all ω, and for which the above convergence is true almost surely.*

3. *These convergences are true for x and y tending to 0, always under the condition that the angles of the triangle $\Delta(0,x,y)$ are bounded from below by a constant $\theta_0 > 0$, if we put $\vec{f}(\omega)$ instead of the expectation value $\int_\Omega \vec{f}\,d\mu$.*

Similarly to the classical Wiener ergodic theorem, this theorem can be extended to the non ergodic case, replacing expectation by the conditional expectation given the σ-algebra of the invariant sets. This theorem has a second version, obtained by dividing the integral by the square of the radius of the smallest ball of center 0 containing the triangle, instead of the area thereof. For this convergence it is possible to relax the condition on the angles of the triangle. Moreover it is easier to interpret from the point of view of homogenization theory. We write it for pointwise convergence:

Theorem 2 — *Let $\vec{f}(\omega)$ be a vector field in $\mathbb{L}_{2,1}(\Omega)$, divergence-free in weak sense. The integral $\mathcal{F}(\vec{f})(x,y)(\omega)$ has a pointwise version $\tilde{\mathcal{F}}(\vec{f})(\omega,x,y)$, continuous in x,y for almost all ω, for which, for almost all ω, we have*

$$\begin{aligned}\lim_{R\to+\infty} \frac{1}{R^2}\tilde{\mathcal{F}}(\vec{f})(\omega,Rx,Ry) &= \frac{1}{2}x\wedge y\cdot\int_\Omega \vec{f}\,d\mu;\\ \lim_{R\to 0} \frac{1}{R^2}\tilde{\mathcal{F}}(\vec{f})(\omega,Rx,Ry) &= \frac{1}{2}x\wedge y\cdot\vec{f}(\omega),\end{aligned}$$

uniformly on the set $\{(x,y);\ \|x\|\leq 1$ and $\|x+y\|\leq 1\}$.

If we suppose moreover that $\max\{\|x\|,\|x+y\|\} = 1$ and that the angles of the triangles $\Delta(0,x,y)$ are bounded from below by a constant $\theta_0 > 0$, then R^2 has the same order than $\|Rx\wedge Ry\|$, and we find the pointwise convergence of Theorem 1.

The integrability $\mathbb{L}_{2,1}(\Omega)$ is optimal, in the following sense: given a probability ν on $\mathbb{R}_+$ such that the identity $s\mapsto s$ is not in $\mathbb{L}_{2,1}(\mathbb{R}_+,\nu)$, we can build a space $(\Omega,\mathcal{B},\mu)$, with an action T of $\mathbb{R}^3$, and a vector field $\vec{f}(\omega)$, divergence-free in weak sense, such that the Euclidean norm $\|\vec{f}(\omega)\|$ has the given distribution ν, and such that for any ω, the averages of Theorem 1 are not bounded, for x, y close to zero, or tending to infinity. A detailed construction of this counterexample is too long to appear here. This optimality property is not surprising when we know that Lorentz spaces $\mathbb{L}_{d,1}$ are optimal spaces for theorems on differentiation in Sobolev spaces (see [Stein]).

This article completes the preceding note [Depauw 1], in which proofs were presented without details. In that note we presented the discrete case (action of $\mathbb{Z}^3$), where the question of the definitions in weak sense, and the problem of the existence of a continuous version $\tilde{\mathcal{F}}(\vec{f})(\omega, x, y)$, do not appear. And, for the pointwise convergence, we use an integrability assumption stronger than here, without reference to Lorentz spaces.

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2 Cocycles and coboundaries of degree 2

The object of this paragraph is to study the concepts of weak functional cocycles and coboundaries of degree 2. We specify the connection between these concepts and those of algebraic cocycles and coboundaries of degree 2, and we prove that weak functional coboundaries of degree 2 in $\mathbb{L}_{p,q}(\Omega)$ are dense in the space of weak functional cocycles of degree 2 in $\mathbb{L}_{p,q}(\Omega)$. We start by exposing a phenomenon which exhibits the difference between the ergodic theorem for cocycle of degree 1 proved in [Boivin; Derriennic] and our theorem for the degree 2.

An algebraic pseudo-cocycle of degree 1 is a measurable function of two variables, denoted $F(x)(\omega)$ (preferably than $F(x, \omega)$, for reasons which will appear later), such that for any x, the function $F(x)(\omega)$ of ω belongs to $\mathbb{L}^1(\Omega)$, and such that for any $x, y \in \mathbb{R}^3$, ω-almost surely, we have

$$F(x + y)(\omega) = F(x)(\omega) + F(y)(T_x\omega).$$

The ergodic theorem for cocycle of degree 1 of [Boivin; Derriennic] has a converse statement, with the following weak version: if a pseudo-cocycle $F(x)(\omega)$ of degree 1 satisfies the local mean ergodic theorem in $\mathbb{L}^1(\Omega)$, and if we denote by $\vec{f}(\omega) = (f_i(\omega))_{1\le i\le 3}$ the limit field, defined by

$$\lim_{\|x\|\to 0} \frac{1}{\|x\|}(F(x)(\omega) - \sum_{i=1}^{3} x_i f_i(\omega)) = 0 ,$$

then the cocycle can be written as a line integral with value in $\mathbb{L}^1(\Omega)$:

$$F(x)(\omega) = \int_0^1 \sum_{i=1}^{3} x_i \bar{f}_i(sx)(\omega)\ ds,$$

where $\bar{f}_i(x)$ denotes the function with values in $\mathbb{L}^1(\Omega)$ defined by $\bar{f}_i(x)(\omega) = f_i(T_x\omega)$.

Because of purely algebraic phenomena, this is not true any more in degree 2. A pseudo-cocycle $F(x, y)(\omega)$ of degree 2 verifying the local ergodic theorem, with limit field $\vec{f}(\omega)$, is not necessarily of the form $\mathcal{F}(\vec{f})(x, y)(\omega)$. Here is an example:

$$F(x, y)(\omega) = \|x\|^3 + \|y\|^3 - \|x + y\|^3.$$

Equation (2) is satisfied by $F(x,y)(\omega)$, the function of ω is integrable for all x, y, and $\frac{1}{\|x\wedge y\|}F(x,y)(\omega)$ converges to 0, for x and y tending to 0 under the condition of Theorem 1, that is to say when the angles of the triangle $\Delta(0,x,y)$ are bounded from below. Hence, if there were a divergence-free field $\vec{f}(\omega)$ such that $F(x,y)(\omega) = \mathcal{F}(\vec{f})(x,y)(\omega)$, it would be zero. That is obviously contradictory, since $F(x,y)(\omega) \neq 0$. The characterization, in the set of the algebraic pseudo-cocycles of degree 2, of cocycles of the form $\mathcal{F}(\vec{f})(x,y)(\omega)$, is still work in progress.

In this paper we consider only algebraic cocycles of degree 2 deduced from functional cocycles of degree 2 by integration on triangles.

A functional coboundary of degree 2 for an action T of $\mathbb{R}^3$ is a vector field $\vec{f}(\omega)$ for which there exists a vector field $\vec{g}(\omega)$ such that $f_i = \partial_j g_k - \partial_k g_j$, where (i,j,k) is a cyclic permutation of $(1,2,3)$. It is again necessary to extend the concept of functional coboundary in a weak sense:

Definition 2 — *An integrable field $\vec{f}(\omega)$ is a weak functional coboundary of degree 2 if there exists an integrable field $\vec{g}(\omega)$ such that ω-almost surely, we have for any $\mathcal{C}^\infty$ function $\vec{\phi}(m)$ with compact support,*

$$\iiint \sum_{(i,j,k)} g_i(T_m\omega)\left(\frac{\partial \phi_k}{\partial m_j} - \frac{\partial \phi_j}{\partial m_k}\right)(m)\ d\lambda(m) = \iiint \sum_{i=1}^{3} \phi_i(m) f_i(T_m\omega)\ d\lambda(m),$$

the first sum being taken on cyclic permutations of $(1,2,3)$. We will denote $\vec{f}(\omega) = \mathrm{curl}_T \vec{g}(\omega)$.

We define the concepts of algebraic cocycle and algebraic pseudo-cocycle of degree 2:

Definition 3 — *An algebraic cocycle of degree 2 is a measurable function of three variables $\tilde{F}(\omega,x,y)$ such that, for almost all ω, for all x,y,*

$$\tilde{F}(\omega,x,y) + \tilde{F}(\omega,x+y,z) - \tilde{F}(\omega,x,y+z) - \tilde{F}(T_x\omega,y,z) = 0. \qquad (6)$$

An integrable algebraic pseudo-cocycle of degree 2 is a measurable function of three variables, denoted $F(x,y)(\omega)$, such that the function $F(x,y)(\omega)$ of the variable ω belongs to $\mathbb{L}^1(\Omega)$ for any x,y, and the following equality holds in $\mathbb{L}^1(\Omega)$:

$$F(x,y)(\omega) + F(x+y,z)(\omega) - F(x,y+z)(\omega) - F(y,z)(T_x\omega) = 0. \qquad (7)$$

The concept of algebraic cocycle was studied in [Mackey], and the one of integrable algebraic pseudo-cocycle in [Feldman; Moore].

By the Fubini theorem, any algebraic cocycle, integrable in ω for all x,y, is an integrable pseudo-cocycle. Using the fact that $\mathbb{Q}$ is countable and dense in $\mathbb{R}$, we can easily check that a measurable function $\tilde{F}(\omega,x,y)$ which is ω-a.s. continuous in x, y, and such that $F(x,y)(\omega) = \tilde{F}(\omega,x,y)$ is an integrable algebraic pseudo-cocycle, is in fact an algebraic cocycle.

For any field $\vec{f}(\omega) \in \mathbb{L}^1(\Omega)$, we denote by $\bar{f}(x)$ the function defined on $\mathbb{R}^3$, with values in the space of random vector fields in $\mathbb{L}^1(\Omega)$, defined by $\bar{f}(x)(\omega) = \vec{f}(T_x\omega)$. We have

Lemma 1 — *Let $\vec{f}(\omega)$ be an integrable field and let $\mathcal{F}(\vec{f})(x,y)(\omega)$ be its integral on the triangles $\Delta(0,x,y)$:*

$$\mathcal{F}(\vec{f})(x,y)(\omega) = \iint_{\Delta(0,x,y)} \bar{f}(m)(\omega)\ d\vec{\sigma}(m). \tag{8}$$

Let us suppose that $\vec{f}(\omega)$ is divergence-free. Then the above integral defines an integrable algebraic pseudo-cocycle of degree 2. Moreover if the field $\vec{f}(\omega)$ is a weak functional coboundary $\vec{f} = \mathrm{curl}_T\vec{g}$, then the algebraic pseudo-cocycle $\mathcal{F}(\vec{f})(x,y)(\omega)$ is an integrable algebraic pseudo-coboundary: for all x, y, almost surely in ω,

$$\mathcal{F}(\vec{f})(x,y)(\omega) = \mathcal{H}(\vec{g})(x)(\omega) + \mathcal{H}(\vec{g})(y)(T_x\omega) - \mathcal{H}(\vec{g})(x+y)(\omega), \tag{9}$$

where $\mathcal{H}(\vec{g})(z)(\omega)$ is the line integral of $\bar{g}(u)(\omega)$ along the segment $[0,z]$:

$$\mathcal{H}(\vec{g})(z)(\omega) = \sum_{i=1}^{3} \int_0^1 z_i \bar{g}_i(sz)(\omega)\ ds,$$

for $z = (z_1, z_2, z_3)$.

Proof.— This is a consequence of the Stokes formula. Indeed, if the field $\vec{f}(\omega)$ is divergence-free, Equality (2) follows from

$$\iint_{\partial K} \bar{f}(m)\ d\vec{\sigma}(m) = \iiint_K \overline{\mathrm{div}_T\vec{f}}(m)\ d\lambda(m), \tag{10}$$

where K is the tetrahedron of vertices 0, x, $x+y$, $x+y+z$ (the variable ω is omitted; the functions are $\mathbb{L}^1(\Omega)$-valued). Similarly if the field $\vec{f}(\omega)$ is a functional coboundary $\mathrm{curl}_T\vec{g}(\omega)$, Equality (9) follows from

$$\iint_{\Delta} \overline{\mathrm{curl}_T\vec{g}}(m)\ d\vec{\sigma}(m) = \int_{\partial\Delta} \bar{g}(m)\ d\vec{\ell}(m), \tag{11}$$

where Δ is the triangular surface of vertices 0, x, $x+y$.
In order to check the two above formulas (10) and (11), the standard Stokes formula has to be adapted to the weak definitions by an argument of convolution. We consider the "T-convolution" of a $\mathcal{C}^\infty$ function $\phi(m)$ having compact support in $\mathbb{R}^3$ with an integrable function $g(\omega) \in \mathbb{L}^1(\Omega)$. For $\omega \in \Omega$, let us denote by $g_\omega(m)$ the function defined on $\mathbb{R}^3$ by $g_\omega(m) = g(T_m\omega)$. According to the Fubini theorem, the set $\Omega_1(g) \subset \Omega$ of ω's such that the function $g_\omega(m)$ is locally integrable in $\mathbb{R}^3$ has full probability. Thus the following integral is defined for $\omega \in \Omega_1(g)$:

$$(\phi \star g_\omega)(x) = \iiint_{\mathbb{R}^3} \phi(-m) g_\omega(x+m)\ d\lambda(m).$$

Moreover, since the function $\phi(x)$ is $\mathcal{C}^\infty$, the function $\phi \star g_\omega(x)$ is $\mathcal{C}^\infty$ with respect to $x \in \mathbb{R}^3$ for all $\omega \in \Omega_1(g)$. Hence its value on the origin is well defined and we can define the T-convolution $(\phi \star_T g)(\omega)$ by

$$(\phi \star_T g)(\omega) = (\phi \star g_\omega)(0).$$

This notion of T-convolution can be generalized with random fields by taking the convolution for each coordinate: $(\phi \star_T \vec{f})_i(\omega) = (\phi \star_T f_i)(\omega)$, where $\vec{f}(\omega) = (f_i(\omega))_{i=1,\ldots,3}$. We have

Lemma 2 — *Let $\vec{g}(\omega)$ and $\vec{f}(\omega)$ be two integrable fields, admitting respectively a curl and a divergence, in weak sense. Then we have ω-almost surely, for any $\mathcal{C}^\infty$ function $\phi(m)$ with compact support*

$$\begin{aligned} \mathrm{curl}(\phi \star \vec{g}_\omega)(x) &= (\phi \star_T \mathrm{curl}_T \vec{g})(T_x\omega); \\ \mathrm{div}(\phi \star \vec{f}_\omega)(x) &= (\phi \star_T \mathrm{div}_T \vec{f})(T_x\omega), \end{aligned} \tag{12}$$

where curl *and* div *without the index T are the classical operators of differential calculus.*

Proof — We begin with the second equality. For any $x \in \mathbb{R}^3$, Definition 1 can be rewritten as

$$-\iiint \sum_{i=1}^{3} \frac{\partial \phi}{\partial m_i}(x-m) f_i(T_m\omega)\ d\lambda(m) = \iiint \phi(x-m)\mathrm{div}_T \vec{f}(T_m\omega)\ d\lambda(m),$$

By derivation under the integral on the left-hand side, and on the right-hand side change of variables $m \mapsto m'$ defined by $m' = m - x$, it becomes

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \iiint \phi(x-m) f_i(T_m\omega)\ d\lambda(m) = \iiint \phi(-m')\mathrm{div}_T \vec{f}\big(T_{m'}(T_x\omega)\big)\ d\lambda(m').$$

This proves the second equality of Lemma 2. The first one follows similarly from Definition 2. □

Lemma 3 — *Let $\psi(m)$ be a $\mathcal{C}^\infty$ positive function with compact support, and integral equal to* 1*. For any integer $n \geq 0$, let $\psi_n(m) = n^3\psi(nm)$. Let $\vec{f}(\omega) \in \mathbb{L}^1(\Omega)$. We have the following mean convergence in $\mathbb{L}^1(\Omega)$:*

$$\iint_\Delta \bar{f}(m)\ d\vec{\sigma}(m) = \lim_{n\to+\infty} \iint_\Delta (\psi_n \star_T f)(T_m\omega)\ d\vec{\sigma}(m).$$

Remark.— For $\omega \in \Omega_1(\vec{f})$ the function $m \mapsto (\psi_n \star_T \vec{f})(T_m\omega)$ is $\mathcal{C}^\infty$ with respect to m, hence its integrals over triangular surfaces are well defined.

Proof.— Since the function $\bar{f}(m)$ is continuous, we have the following mean convergence in $\mathbb{L}^1(\Omega)$:

$$\vec{f}(\omega) = \lim_{n\to+\infty} (\psi_n \star_T \vec{f})(\omega).$$

Hence Lemma 3 will be proved if we show that the operator $\vec{f} \mapsto \iint_\Delta \vec{f}\ d\vec{\sigma}$ is continuous in $\mathbb{L}^1(\Omega)$. Since the generalization of the standard triangular inequality with integrals leads to:

$$\Big\| \iint_\Delta \bar{f}(m)\ d\vec{\sigma}(m) \Big\|_{\mathbb{L}^1(\Omega)} \leq \iint_\Delta \|\bar{f}(m)\|_{\mathbb{L}^1(\Omega)}\ \|d\vec{\sigma}\|(m),$$

where $\|d\vec{\sigma}\|$ is the infinitesimal area on Δ, and since T is stationary, we obtain

$$\Big\| \iint_\Delta \bar{f}(m)\ d\vec{\sigma}(m) \Big\|_{\mathbb{L}^1(\Omega)} \leq |\Delta| \|\vec{f}\|_{\mathbb{L}^1(\Omega)},$$

with $|\Delta|$ equal to the area of the triangular surface Δ. This achieves the proof of Lemma 3. □

Now Lemma 2 is a consequence of the classical Stokes formulae. Let $\vec{f}(\omega)$ be a vector field admitting a divergence $\mathrm{div}_T \vec{f}(\omega)$ in weak sense. Applying the classical Stokes formula to the left-hand side of Equality (12) with $\phi = \psi_n$, we obtain

$$\iint_{\partial K} (\psi_n \star \vec{f_\omega})(m)\ d\vec{\sigma}(m) = \iiint_K \big(\psi_n \star_T \mathrm{div}_T(\vec{f})\big)(T_m\omega)\ d\lambda(m).$$

for any tetrahedron K, with boundary ∂K. When n tends to infinity, and according to Lemma 3, we obtain Equality (10). A calculation very similar to the above one proves Equality (11). This completes the proof of Lemma 1. □

We will now prove a converse statement. It will not be used in the sequel, but it is interesting from the point of view of the connection between algebraic and functional cocycles of degree 2.

Lemma 4 — *Let $\vec{f}(\omega)$ be an integrable field, and let $\mathcal{F}(\vec{f})(x,y)(\omega)$ be the family of its integrals over triangles. If $\mathcal{F}(\vec{f})(x,y)(\omega)$ is an algebraic pseudo-cocycle of degree 2, then $\vec{f}(\omega)$ is divergence-free in weak sense.*

Remark.— It is a direct consequence of Equality (10) if the field $\vec{f}(\omega)$ is supposed to admit a divergence in weak sense.

Proof.— Let $\vec{f}(\omega)$ be an integrable field such that $\mathcal{F}(\vec{f})(x,y)(\omega)$ is an algebraic pseudo-cocycle of degree 2, that is to say such that the equality

$$\iint_{\partial K} \bar{f}(m)\ d\vec{\sigma}(m) = 0 \tag{13}$$

holds in $\mathbb{L}^1(\Omega)$ for any tetrahedron K. Let $\phi(y)$ be a $\mathcal{C}^\infty$ function with compact support, and $\psi_n(m)$ be the function defined in Lemma 3. The Fubini theorem and appropriate changes of variables lead to: for $\omega \in \Omega_1(\vec{f})$

$$\iint_{\partial K} \big(\psi_n \star (\phi \star \vec{f_\omega})\big)(m)\ d\vec{\sigma}(m) = \Big(\phi \star_T \big(\iint_{\partial K} (\psi_n \star_T \bar{f})\ d\vec{\sigma}\big)\Big)(\omega).$$

Thus, according to Lemma 3, the following equality holds in $\mathbb{L}^1(\Omega)$:

$$\iint_{\partial K} (\phi \star \vec{f_\omega})(m)\ d\vec{\sigma}(m) = \Big(\phi \star_T \big(\iint_{\partial K} \bar{f}\ d\vec{\sigma}\big)\Big)(\omega).$$

It follows from hypothesis (13) that the right-hand side is null, hence the Stokes formula applied to the left-hand side leads to

$$\iiint_K \mathrm{div}(\phi \star \vec{f_\omega})(m)\ d\lambda(m) = 0.$$

By derivation under the integral defining the convolution this can be rewritten as

$$\iiint_K (\mathrm{grad}(\phi) \star_T \vec{f})(T_m\omega)\ d\lambda(m) = 0.$$

Since it is true for any K, it follows from the classical local ergodic Wiener theorem that there exists a set E_ϕ with full measure such that for $\omega \in E_\phi$

$$(\mathrm{grad}(\phi) \star_T \vec{f})(\omega) = 0. \tag{14}$$

To prove that $\mathrm{div}_T \vec{f} = 0$, we have to find a full measure subset $E \subset \Omega$ such that Equation (14) holds for any $\omega \in E$ and any $\mathcal{C}^\infty$ function ϕ with compact support. Let $(K_n)_{n\geq 1}$ be a countable increasing family of compact sets covering $\mathbb{R}^3$. Let us recall that the space $\mathcal{D}_{K_n}$ of $\mathcal{C}^\infty$ functions with support belonging in K_n is separable for its standard topology. Let $(\phi_{k,n}(m))_{k\geq 1}$ be a dense countable family in $\mathcal{D}_{K_n}$, and $E \subset \Omega$ be a full measure set of ω's such that $\vec{f}_\omega(m)$ is locally in $\mathbb{L}^1(\mathbb{R}^3)$ and Equation (14) holds for any $\phi_{k,n}(m)$, $k, n \geq 1$. For any $\mathcal{C}^\infty$ function $\phi(m)$ with compact support, let n be an integer such that K_n contains the support of $\phi(m)$. For any $\omega \in E$ and any integer k we have

$$\Big| \iiint \sum_{i=1}^{3} \frac{\partial \phi}{\partial m_i}(-m) f_i(T_m\omega)\ d\lambda(m) \Big| \leq$$
$$\max_{1\leq i\leq 3} \Big\| \frac{\partial \phi}{\partial m_i} - \frac{\partial \phi_{k,n}}{\partial m_i} \Big\|_\infty \sum_{i=1}^{3} \iiint_{K_n} |\vec{f}_i(T_m\omega)|\ d\lambda(m).$$

Since k can be chosen so that the above maximum becomes as small as desired, Equation (14) holds in fact for the function $\phi(m)$ itself. This proves that $\mathrm{div}_T \vec{f} = 0$ and completes the proof of Lemma 4. □

According to the formula $\mathrm{div}_T(\mathrm{curl}_T) = 0$, any weak functional coboundary of degree 2, $\vec{f} = \mathrm{curl}_T \vec{g}$, with $\vec{f}$ and $\vec{g} \in \mathbb{L}_{p,q}(\Omega)$, is a weak functional cocycle. The converse statement is false (see [Depauw 2] for an example in the discrete case of an action of $\mathbb{Z}^3$). We have however the following proposition.

Proposition 1 — *Let $1 \leq p < +\infty$, and $1 \leq q \leq +\infty$. Let us suppose that the action T is ergodic. Any zero divergence field in the weak sense, in $\mathbb{L}_{p,q}(\Omega)$ and of null expectation is the limit in $\mathbb{L}_{p,q}(\Omega)$ of weak functional coboundaries, built from fields $\vec{g}(\omega)$ in $\mathbb{L}_{p,q}(\Omega)$.*

Proof.— Let $\vec{f}(\omega)$ be a zero divergence field in the weak sense, in $\mathbb{L}_{p,q}(\Omega)$, and of null expectation. A sequence of fields $\vec{g}_N(\omega)$, the curls of which tend to $\vec{f}(\omega)$, can be built explicitly. For $N \geq 1$, we define the field $\vec{g}_N(\omega) \in \mathbb{L}_{p,q}(\Omega)$ by

$$\vec{g}_N = \frac{1}{N^3} \iiint_{x\in[0,N]^3} \vec{g}_x\ d\lambda(x)$$

where $\vec{g}_x(\omega) = (g_{i,x}(\omega))_{i=1,\ldots,3}$ is the random field defined by

$$g_{i,x} = \int_0^1 x_j \bar{f}_k(sx) - x_k \bar{f}_j(sx)\ ds, \tag{15}$$

and (i, j, k) is a cyclic permutation of $(1, 2, 3)$. The lemma would be proved if we check that, in weak sense, we have

$$\mathrm{curl}_T \vec{g}_N = \vec{f}(\omega) - \frac{1}{N^3} \iiint_{x\in[0,N]^3} \vec{f}(T_x\omega)\ d\lambda(x). \tag{16}$$

Indeed, the standard mean ergodic theorem gives the convergence of the right-hand side to $\vec{f}(\omega)$, when N tends to infinity. This convergence is well known in Lebesgue spaces $\mathbb{L}^p(\Omega)$, and can be generalized with Lorentz spaces $\mathbb{L}_{p,q}(\Omega)$ by the density of $\mathbb{L}^{p'}(\Omega)$ in $\mathbb{L}_{p,q}(\Omega)$ for a $p' > p$. This would prove Proposition 1.
Let us come back to calculation of $\mathrm{curl}_T \vec{g}_N$. Formally, commuting the integration on x with curl_T, Equality (16) follows from

$$\mathrm{curl}_T \vec{g}_x(\omega) = \vec{f}(\omega) - \vec{f}(T_x\omega). \tag{17}$$

This calculation is easy if the field $\vec{f}(\omega)$ belongs to the domain of operators ∂_i. Indeed, Formula (15) leads to the following expression of the coordinate $(\mathrm{curl}_T \vec{g})_k = \partial_i g_{j,x} - \partial_j g_{i,x}$:

$$(\mathrm{curl}_T \vec{g})_k = \int_0^1 x_k(\partial_i \bar{f}_i(sx) + \partial_j \bar{f}_j(sx)) - x_i \partial_i \bar{f}_k(sx) - x_j \partial_j \bar{f}_k(sx)\ ds.$$

According to the hypothesis $\mathrm{div}_T \vec{f} = 0$, we have $\partial_i \bar{f}_i + \partial_j \bar{f}_j = -\partial_k \bar{f}_k$. Hence the above equality becomes

$$(\mathrm{curl} \vec{g})_k = -\int_0^1 \sum_{\ell=1}^3 x_\ell \partial_\ell \bar{f}_k(sx)\ ds.$$

Since the sum on ℓ is the derivative of the function $s \mapsto \bar{f}_k(sx)$, the integral on s equals $\bar{f}_k(0) - \bar{f}_k(x)$, which gives Formula (17).
In order to generalize this calculation to weak definitions, consider the following algebraic formula

$$\sum_{(j,k)} \big(x_j \bar{f}_k(sx+m) - x_k \bar{f}_j(sx+m)\big)\big(\frac{\partial \phi_k}{\partial s_j}(m) - \frac{\partial \phi_j}{\partial s_k}(m)\big) =$$
$$-\sum_{j=1}^3 x_j \sum_{k=1}^3 \bar{f}_k(sx+m) \frac{\partial \phi_j}{\partial s_k}(m) + \sum_{k=1}^3 \bar{f}_k(sx+m) \sum_{j=1}^3 x_j \frac{\partial \phi_k}{\partial s_j}(m),$$

where the first summation is over $(j,k) = (1,2),\ (2,3),\ (3,1)$. Let us consider the right-hand side. With an integration on m, the summation on k in the first double sum disappears, because $\mathrm{div}_T \vec{f} = 0$. Then, by a change of variables $m \mapsto m'$ defined by $m' = m + sx$, and integration on $s \in [0,1]$, the second double sum becomes

$$\int_{m'} \sum_{k=1}^3 \bar{f}_k(m')\big(\phi_k(m') - \phi_k(m'-x)\big)\ d\lambda(m').$$

This is clearly equal, by stationarity of T, to

$$\int_{m'} \sum_{k=1}^3 \big(\bar{f}_k(m) - \bar{f}_k(m+x)\big)\phi_k(m)\ d\lambda(m).$$

Now, with an integration on $s \in [0,1]$, and an integration on $m \in \mathbb{R}^3$, the left-hand side of the above algebraic formula becomes $\int_m \sum_{(i,j,k)} \bar{g}_{i,x}(m)\big(\frac{\partial \phi_k}{\partial s_j}(m) -$

$\frac{\partial \phi_j}{\partial s_k}(m)\big)\ d\lambda(m)$. Finally we have

$$\int_m \sum_{(i,j,k)} \bar{g}_{i,x}(m)\big(\frac{\partial \phi_k}{\partial s_j}(m) - \frac{\partial \phi_j}{\partial s_k}(m)\big)\ d\lambda(m) =$$

$$\int_{m'} \sum_{k=1}^{3} \big(\bar{f}_k(m) - \bar{f}_k(m+x)\big)\phi_k(m)\ d\lambda(m).$$

According to Definition 2, this proves Formula (17).

To achieve the proof of Equality (16) it suffices to integrate the above formulas on $x \in [0,N]^3$, and to use the Fubini theorem. □

If the action T is not ergodic, the result remains true for functions with vanishing conditional expectation given the σ-algebra of the invariant sets.

3 Ergodic theorem

In this paragraph, we prove the ergodic theorem, for mean convergence in $\mathbb{L}_{p,q}(\Omega)$ and for pointwise convergence. For a vector field $\vec{f}(\omega) \in \mathbb{L}_{p,q}(\Omega)$, divergence-free in weak sense, let us denote $\mathcal{M}(\vec{f})(x,y)(\omega)$ the average integral of $\vec{f}$ on the triangle with vertices $(0,x,x+y)$:

$$\mathcal{M}(\vec{f})(x,y)(\omega) = \frac{1}{\frac{1}{2}\|x \wedge y\|}\mathcal{F}(\vec{f})(x,y)(\omega).$$

3.1 Mean convergence in $\mathbb{L}_{p,q}(\Omega)$.

Let us start with mean convergence in $\mathbb{L}_{p,q}(\Omega)$. It is clear that a constant field $\vec{f}(\omega) \equiv \int_\Omega \vec{f}\ d\mu$ is divergence-free, and verifies the mean ergodic theorem. Convergence when the field $\vec{f}(\omega)$ is a weak functional coboundary is stated in the following way:

Lemma 5 — *For a weak functional coboundary $\vec{f} = \mathrm{curl}_T \vec{g}$, with $\vec{g}(\omega) \in \mathbb{L}_{p,q}(\Omega)$, we have*

$$\|\mathcal{M}(\vec{f})(x,y)(\omega)\|_{p,q} \leq \frac{\|x\| + \|y\| + \|x+y\|}{\frac{1}{2}\|x \wedge y\|}\|\vec{g}\|_{p,q}. \tag{18}$$

This expression goes to 0 *when* x *and* y *tend to infinity, under the condition that the angles of the triangle* $\Delta(0,x,y)$ *are bounded from below by a constant* $\theta > 0$.

Proof.—The inequality follows from the generalization of triangular inequality for the norms $\mathbb{L}_{p,q}$ with integrals, which is written as

$$\Big\|\int_{m\in\partial\Delta} \vec{g}(T_m\omega)\ d\vec{\ell}(m)\Big\|_{\mathbb{L}_{p,q}(\Omega)} \leq \int_{m\in\partial\Delta} \|\vec{g}(T_m\omega)\|_{\mathbb{L}_{p,q}(\Omega)}\ \|d\vec{\ell}\|(m),$$

where $d\vec{\ell}$ and $\|d\vec{\ell}\|$ are respectively the infinitesimal tangential field and the infinitesimal length of $\partial\Delta(0,x,y)$. Since the action T preserves the measure μ, the last expression is bounded by $|\partial\Delta| \cdot \|\vec{g}\|_{p,q}$.

For convergence to 0 of the right-hand side in the inequality (18), we note that the fraction is equal to the ratio between the perimeter of the triangle and its area, so the condition that the angles of the triangle $\Delta(0,x,y)$ are bounded from below by a constant $\theta > 0$ implies that this fraction is $O(\|x\|^{-1})$. □

We also have uniform continuity in x, y of the operator $\mathcal{M}$ for the $\mathbb{L}_{p,q}(\Omega)$ norm:

Lemma 6 — *Let $\vec{f}(\omega)$ be a vector field, divergence-free in the weak sense. We have*

$$\|\mathcal{M}(\vec{f})(x,y)(\omega)\|_{p,q} \leq \|\vec{f}(\omega)\|_{p,q}$$

for all x, y.

It is again a consequence of the triangular inequality and stationarity. □

The density of the weak functional coboundaries and the two preceding lemmas give the mean ergodic theorem. Convergence for x, y tending to 0 is obvious, by continuity of the function $\bar{f}(x)$ with values in $\mathbb{L}_{p,q}(\Omega)$. This completes the proof of point 1 of Theorem 1. □

3.2 Pointwise convergence.

In order that the pointwise ergodic theorem makes sense, the integral of the zero divergence field $(\vec{f}(T_x\omega))_{x\in\mathbb{R}^3}$ has to be ω-almost surely defined on all triangles. It will work if there exists an ω-almost surely continuous version of the function $\mathcal{F}(\vec{f})(x,y)(\omega)$ of variables x, y.

3.2.1 Existence of a continuous version.

This ω-almost surely continuous version exists if $\vec{f}(\omega)$ belongs to $\mathbb{L}_{2,1}(\Omega)$:

Proposition 2 — *Let $\vec{f}(\omega)$ be a vector field in $\mathbb{L}_{2,1}(\Omega)$, weakly divergence-free. Let $\mathcal{F}(\vec{f})(x,y)(\omega)$ be the algebraic pseudo-cocycle of degree 2 which is obtained by integrating $\vec{f}(\omega)$ on triangles. Then there is an algebraic cocycle $\tilde{\mathcal{F}}(\vec{f})(\omega,x,y)$ of degree 2 such that $\mathcal{F}(\vec{f})(x,y)(\omega) = \tilde{\mathcal{F}}(\vec{f})(\omega,x,y)$ for all x, y, for almost all ω. Moreover, for almost all ω, the function $\tilde{\mathcal{F}}(\vec{f})(\omega,x,y)$ can be chosen continuous in the two variables x, y.*

Proof.— Let $\vec{f}(\omega)$ be a zero divergence field in weak sense in $\mathbb{L}_{2,1}(\Omega)$. We denote $K(0,x,y)$ the tetrahedron with the triangle $\Delta(0,x,y)$ as basis, with the length of the altitude equals to the radius R_I of the inscribed circle in the triangle $\Delta(0,x,y)$, and the foot of the altitude is the center c of this circle. This tetrahedron has vertices $0, x, x+y, x+y+z$ with z defined by

$$x + y + z = c + R_I\vec{n},$$

where $\vec{n}$ is the oriented normal vector of the triangle. The 2-dimensional integral $\mathcal{F}(\vec{f})(x,y)(\omega)$ can be expressed as a 3-dimensional integral on the tetrahedron $K(0,x,y)$. Indeed, let us consider for $\eta \in [0, \frac{\pi}{4}]$ the vertex $x + y + z(\eta)$ defined by

$$x + y + z(\eta) = c + R_I \tan\eta\ \vec{n}.$$

The equation of pseudo-cocycle of degree 2 can be written as the following equality in $\mathbb{L}_{2,1}(\Omega)$ for all x, y, η,

$$\mathcal{F}(\vec{f})(x,y)(\omega) = \mathcal{F}(\vec{f})(x, y+z(\eta))(\omega) - \mathcal{F}(\vec{f})(x+y, z(\eta))(\omega) + \mathcal{F}(\vec{f})(y, z(\eta))(T_x\omega)$$

and integrating on η, we obtain: for all x, y,

$$\begin{aligned}\mathcal{F}(\vec{f})(x,y)(\omega) \quad = \quad & \frac{4}{\pi}\int_0^{\frac{\pi}{4}} \mathcal{F}(\vec{f})(x, y+z(\eta))(\omega)\; d\eta\, - \\ & \frac{4}{\pi}\int_0^{\frac{\pi}{4}} \mathcal{F}(\vec{f})(x+y, z(\eta))(\omega)\; d\eta\, + \\ & \frac{4}{\pi}\int_0^{\frac{\pi}{4}} \mathcal{F}(\vec{f})(y, z(\eta))(T_x\omega)\; d\eta. \end{aligned} \tag{19}$$

Integrals $\mathcal{F}(\vec{f})$ being double integrals with values in $\mathbb{L}_{2,1}(\Omega)$, we obtain a triple integral on the tetrahedron $K(0, x, y)$, with value in $\mathbb{L}_{2,1}(\Omega)$. Let us denote $\mathcal{F}_i(\vec{f})(x,y)(\omega)$, for $i = 1, 2, 3$, the three triple integrals corresponding to the above decomposition. Let us consider for instance the first one:

$$\mathcal{F}_1(\vec{f})(x,y)(\omega) = \frac{4}{\pi}\int_0^{\frac{\pi}{4}} \mathcal{F}(\vec{f})(x, y+z(\eta))(\omega)\; d\eta.$$

Let c' be the orthogonal projection of c, on the axis generated by x. Let us consider the orthonormal basis $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ of $\mathbb{R}^3$ defined by $\vec{u}_1 = \frac{x}{\|x\|}$, $\vec{u}_2 = \frac{c-c'}{\|c-c'\|}$ and $\vec{u}_3 = \vec{n}$. We denote by $M(s, r, \eta)$ the point whose coordinates in this basis are $(s, r\cos\eta, r\sin\eta)$. The integral $\mathcal{F}_1(\vec{f})(x,y)(\omega)$ can be rewritten as

$$\frac{4}{\pi}\int_{\eta=0}^{\frac{\pi}{4}} \iint_{(r,s)\in\Delta_\eta} \vec{f}(M(s,r,\eta))(\omega)\cdot((\sin\eta)\vec{u}_2 - (\cos\eta)\vec{u}_3)\; drds\; d\eta,$$

where the domain Δ_η corresponds to the parameterization by (s, r) of the triangle of vertices $(0, x, x+y+z(\eta))$. In order to fix ω, consider the operators $\mathcal{G}_i$, similar to the $\mathcal{F}_i$, but for the functions $\vec{\phi}(m)$ of the variable $m \in \mathbb{R}^3$. For instance, for $i = 1$: $\mathcal{G}_1(\vec{\phi})(x, y) =$

$$= \frac{4}{\pi}\int_{\eta=0}^{\frac{\pi}{4}} \iint_{(r,s)\in\Delta_\eta} \vec{\phi}(M(s,r,\eta))\cdot((\sin\eta)\vec{u}_2 - (\cos\eta)\vec{u}_3)\; drds\; d\eta. \tag{20}$$

Let us denote $\mathcal{G} = \sum_{i=1}^{3} \mathcal{G}_i$. We have

Lemma 7 — *Fix $\rho > 0$, and denote by B the ball of center 0 and radius ρ. For any field $\vec{\phi}(m) \in \mathbb{L}_{2,1}(B)$, for any x, y, such that x and $x+y \in B$ we have $|\mathcal{G}(\vec{\phi})(x,y)| \leq 24\sqrt{\frac{2\rho}{\pi}}\|\vec{\phi}(m)\|_{\mathbb{L}_{2,1}(B)}$. Moreover, the function $\mathcal{G}(\vec{\phi})(x,y)$ is continuous on the set $\{(x,y),\; x, x+y \in B\}$.*

Proof of Lemma 7.— Let us first consider the function $\mathcal{G}_1(\phi)(x,y)$. The volume element is $d\lambda = rdrdsd\eta$. Hence the integral (20) which defines $\mathcal{G}_1(\vec{\phi})(x,y)$ is absolutely convergent if

$$\frac{4}{\pi}\iiint \|\vec{\phi}(M(s,r,\eta))\|\frac{1}{r}\; d\lambda(s,r,\eta) < \infty.$$

However, by the analogue of Hölder inequality for Lorentz norms (see [Hunt]), this last integral is bounded by

$$2 \cdot \|\vec{\phi}(m)\|_{\mathbb{L}_{2,1}(K_1(0,x,y))} \cdot \Big\|\frac{1}{r}\Big\|_{\mathbb{L}_{2,\infty}(K_1(0,x,y))}$$

where $K_1(0,x,y)$ is the set of points m of $\mathbb{R}^3$ corresponding to the triple integral above. Let us calculate the second factor of the above product: $\|r^{-1}\|_{\mathbb{L}_{2,\infty}(K_1)}$. The parameter r is the distance from the point m to the axis $\vec{u}_1$. Hence the set $K_1(0,x,y) \cap \{\frac{1}{r} > t\}$ has a volume lower than the one of a cylinder of axis $\vec{u}_1$, length $\|x\|$ and radius $\frac{1}{t}$, so

$$\lambda\Big(K_1(0,x,y) \cap \{\frac{1}{r} > t\}\Big) \leq \pi\|x\|(\frac{1}{t})^2.$$

By the definition of $\mathbb{L}_{2,\infty}$ norm, it follows that

$$\|\frac{1}{r}\|_{\mathbb{L}_{2,\infty}(K_1(0,x,y))} \leq \sqrt{\pi\|x\|}.$$

The same argument can be applied to the two other integrals corresponding to the terms $\mathcal{G}_i(\vec{\phi})(x,y)$, $i = 2,3$, which are bounded respectively by $\sqrt{\pi\|x+y\|}$ and $\sqrt{\pi\|y\|}$. This proves the announced inequality, since x and $x+y$ belong to B. To achieve the proof, note that $\mathcal{G}(\vec{\phi})(x,y)$ is continuous on the set $\{(x,y), x \text{ and } x+y \in B\}$ when $\vec{\phi}(m)$ in continuous, and conclude by a density argument in $\mathbb{L}_{2,1}(B)$ norm. □

In order to apply Lemma 7 to the field $m \mapsto \vec{f}(T_m\omega)$, we have to prove the following lemma:

Lemma 8 — *Let $\vec{f}(\omega)$ be a vector field in $\mathbb{L}_{2,1}(\Omega)$. For a fixed ω, let $\vec{f}_\omega(m)$ be the field defined on $\mathbb{R}^3$ by $\vec{f}_\omega(m) = \vec{f}(T_m\omega)$. This field is locally in $\mathbb{L}_{2,1}(\mathbb{R}^3)$ for almost all ω.*

Proof.— Let $B \subset \mathbb{R}^3$ be a ball centered at 0. To prove that $\|\vec{f}_\omega(m)\|_{\mathbb{L}_{2,1}(B)}$ is ω-a.s. finite, it is enough to check that it belongs to $\mathbb{L}_{2,\infty}(\Omega)$. Replacing the $\mathbb{L}_{2,1}(B)$-Lorentz norm by its expression (3), we have

$$\Big\|\|\vec{f}_\omega(m)\|_{\mathbb{L}_{2,1}(B)}\Big\|_{\mathbb{L}_{2,\infty}(\Omega)} = \Big\|\int_0^\infty \sqrt{\lambda(m \in B;\ \|\vec{f}_\omega(m)\| > t)}\ dt\Big\|_{\mathbb{L}_{2,\infty}(\Omega)}.$$

According to the triangular inequality generalized with integrals, an upper bound is obtained by commuting the above Lorentz norm and the integral. From the standard equality $\|\sqrt{h}\|_{\mathbb{L}_{2,\infty}(\Omega)} = \sqrt{\|h\|_{\mathbb{L}_{1,\infty}(\Omega)}}$, we obtain

$$\int_0^\infty \sqrt{\Big\|\lambda(m \in B;\ \|\vec{f}_\omega(m)\| > t)\Big\|_{\mathbb{L}_{1,\infty}(\Omega)}}\ dt. \tag{21}$$

The random variable $\omega \mapsto \lambda(\ldots)$ under the above Lorentz norm can be rewritten as

$$\iiint_{m \in B} \mathbf{1}_{\|\vec{f}\| > t}(T_m\omega)\ d\lambda(m).$$

Moreover, according to the Markov inequality (4), its $\mathbb{L}_{1,\infty}(\Omega)$-Lorentz norm is bounded by its $\mathbb{L}^1(\Omega)$-Lebesgue norm, which is equal, by stationarity of T, to

$$\lambda(B) \cdot \mu(\|\vec{f}(\omega)\| > t).$$

Replacing in (21) leads to

$$\left\| \|\vec{f}_\omega(m)\|_{\mathbb{L}_{2,1}(B)} \right\|_{\mathbb{L}_{2,\infty}(\Omega)} \leq \sqrt{\lambda(B)} \|\vec{f}(\omega)\|_{\mathbb{L}_{2,1}(\Omega)},$$

which is finite. Thus for almost all ω, the norm $\|\vec{f}_\omega(m)\|_{\mathbb{L}_{2,1}(B)}$ is finite. Considering a countable covering of $\mathbb{R}^3$ with balls, we obtain that the set of ω's for which $\vec{f}_\omega(m)$ is locally in $\mathbb{L}_{2,1}(\mathbb{R}^3)$ has full measure (we denote this set $\Omega_{2,1}(\vec{f})$ in the sequel). This proves Lemma 8. □

For $\omega \in \Omega_{2,1}(\vec{f})$, let us define $\tilde{\mathcal{F}}_i(\vec{f})(\omega, x, y)$ by

$$\tilde{\mathcal{F}}_i(\vec{f})(\omega, x, y) = \mathcal{G}_i(\vec{f}_\omega)(x, y).$$

For all x, y, we have clearly, $\tilde{\mathcal{F}}_i(\vec{f})(\omega, x, y) = \mathcal{F}_i(\vec{f})(x, y)(\omega)$, where this equality holds in $\mathbb{L}_{2,1}(\Omega)$. According to (19), and setting $\tilde{\mathcal{F}} = \sum_{i=1}^3 \tilde{\mathcal{F}}_i$, the functions $\tilde{\mathcal{F}}(\vec{f})(\omega, x, y)$ and $\mathcal{F}(\vec{f})(x, y)(\omega)$ coincide in $\mathbb{L}_{2,1}(\Omega)$ for all x, y.
Lastly, by continuity of $\tilde{\mathcal{F}}(\vec{f})(\omega, x, y)$ (see Lemma 7), the cocycle equation, satisfied by $\tilde{\mathcal{F}}(\vec{f})(\omega, x, y)$ a priori for all x, y, z, for almost all ω, is in fact satisfied, for almost all ω, and all x, y, z. This proves that $\tilde{\mathcal{F}}(\vec{f})(\omega, x, y)$ is a "true" algebraic cocycle of degree 2, and completes the proof of Proposition 2. □

3.2.2 Weak maximal inequality.

A standard way to prove the pointwise ergodic theorem would be to use the convergence for weak functional coboundaries and a maximal inequality. This maximal inequality implies that the set of weakly divergence-free fields $\vec{f}(\omega)$, which verify pointwise convergence, is closed in $\mathbb{L}_{2,1}(\Omega)$. In fact, the convergence for weak functional coboundaries is not so easy to obtain, and we will not use this technique. But the maximal inequality is interesting for itself. That is the reason why we prove it now.

The proof of our weak maximal inequality uses Lemma 7, and the standard weak maximal inequality, which was pointed out in Introduction. We set

$$\tilde{\mathcal{M}}(\vec{f})(\omega, x, y) = \frac{1}{\frac{1}{2}\|x \wedge y\|} \tilde{\mathcal{F}}(\vec{f})(\omega, x, y)$$

and $\tilde{\mathcal{M}}^*_\theta(\vec{f})(\omega) = \sup_{(x,y) \in \mathcal{T}_\theta} |\tilde{\mathcal{M}}(\vec{f})(\omega, x, y)|$, where $\mathcal{T}_\theta$ is the set of the (x, y) such that the three angles of the triangle $\Delta(0, x, y)$ are $\geq \theta$.

Proposition 3 (Weak maximal inequality).— *Let θ be in $]0, \frac{\pi}{2}[$. There exists a constant c_θ such as for any weakly divergence-free field in the weak sense $\vec{f}(\omega)$ in $\mathbb{L}_{2,1}(\omega)$ we have*

$$\|\tilde{\mathcal{M}}^*_\theta(\vec{f})\|_{2,\infty} \leq c_\theta \|\vec{f}\|_{2,1}.$$

Proof.— By definition of $\tilde{\mathcal{F}}$, the average $\tilde{\mathcal{M}}(\vec{f})(\omega, x, y)$ is written as

$$\tilde{\mathcal{M}}(\vec{f})(\omega, x, y) = \frac{1}{\frac{1}{2}\|x \wedge y\|}\mathcal{G}(\vec{f}_\omega)(x, y).$$

In the integral (20) defining $\mathcal{G}_1$ we make the change of variables $r = Rr'$, $s = Rs'$, $\eta' = \eta$. We make similar change of variables in the integrals defining $\mathcal{G}_i$, $i = 2, 3$, and we obtain

$$\frac{R^2}{\frac{1}{2}\|x \wedge y\|}\mathcal{G}(\vec{f}_{R,\omega})(\frac{x}{R}, \frac{y}{R}),$$

where $\vec{f}_{R,\omega}(m)$ is defined by $\vec{f}_{R,\omega}(m) = \vec{f}_\omega(Rm)$. If we take $R = R(x, y) = \max(\|x\|, \|x + y\|)$, and if we consider the upper bound on the $(x, y) \in \mathcal{T}_\theta$, it follows

$$\tilde{\mathcal{M}}^*_\theta(\vec{f})(\omega) \leq \sup_{(x,y)\in\mathcal{T}_\theta} \frac{R(x,y)^2}{\frac{1}{2}\|x \wedge y\|} \cdot \sup_{(x,y)\in\mathcal{T}_\theta} \Big|\mathcal{G}(\vec{f}_{R(x,y),\omega})(\frac{x}{R(x,y)}, \frac{y}{R(x,y)})\Big|.$$

Let κ_θ be the left-hand-side factor of the above product. The condition on the angles guarantees that the area $\frac{1}{2}\|x \wedge y\|$ of the triangle $\Delta(0, x, y)$ is of order $R(x, y)^2$. The constant κ_θ is thus finite. Let us consider the right-hand-side factor. Setting $x' = x/R(x, y)$, $y' = y/R(x, y)$ and $R = R(x, y)$, it is bounded by

$$\sup_R \sup_{x', x'+y' \in B} \mathcal{G}(\vec{f}_{R,\omega})(x', y'),$$

where B is the ball of radius $\rho = 1$ and center 0. By Lemma 7, it follows

$$\tilde{\mathcal{M}}^*_\theta(\vec{f})(\omega) \leq \kappa_\theta \cdot 24\sqrt{\frac{2}{\pi}} \sup_R \|\vec{f}_{R,\omega}\|_{\mathbb{L}_{2,1}(B)}.$$

Replacing the Lorentz norm $\|\cdot\|_{\mathbb{L}_{2,1}(B)}$ by its expression (3), we obtain

$$\tilde{\mathcal{M}}^*_\theta(\vec{f})(\omega) \leq \kappa_\theta 24\sqrt{\frac{2}{\pi}} \sup_R \int_0^\infty \sqrt{\lambda\big\{\|m\| < 1,\ \|\vec{f}_\omega(Rm)\| > s\big\}}\ ds.$$

Commuting the supremum and the integral, and then using the triangular inequality generalized with integrals, it readily follows

$$\|\tilde{\mathcal{M}}^*_\theta(\vec{f})\|_{2,\infty} \leq \kappa_\theta 24\sqrt{\frac{2}{\pi}} \int_0^\infty \Big\|\sqrt{\sup_R\ \lambda\big\{\|m\| < 1,\ \|\vec{f}_\omega(Rm)\| > s\big\}}\Big\|_{2,\infty}\ ds. \tag{22}$$

According to the standard equality $\|\sqrt{h}\|_{2,\infty} = \sqrt{\|h\|_{1,\infty}}$, the above upper bound becomes

$$\kappa_\theta 24\sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\Big\|\sup_R\ \lambda\big\{\|m\| < 1,\ \vec{f}_\omega(Rm)\| > s\big\}\Big\|_{1,\infty}}\ ds.$$

Replacing $\lambda\{A\}$ by $\iiint_m \mathbf{1}_A(m)\ d\lambda(m)$ in the above supremum on R, this last becomes the classical maximal function of the function $\mathbf{1}_{(\|\vec{f}\|>s)}(\omega)$:

$$(\mathbf{1}_{\|\vec{f}\|>s})^*(\omega) = \sup_R\ \frac{1}{R^3} \iiint_{\|m\|<R} \mathbf{1}_{(\|\vec{f}\|>s)}(T_m\omega)\ d\lambda(m).$$

By the standard ergodic weak maximal inequality (5), we have the following inequality:

$$\left\| (\mathbf{1}_{\|\vec f\|>s})^*(\omega) \right\|_{1,\infty} \le C\mu(\|\vec f(\omega)\| > s).$$

Replacing in (22), this leads to

$$\|\tilde{\mathcal{M}}^*_\theta(\vec f)\|_{2,\infty} \le \kappa_\theta 24\sqrt{\frac{2}{\pi}}\sqrt{C}\int_0^\infty \sqrt{\mu(\|\vec f(\omega)\| > s)}\ ds,$$

which proves Proposition 3 with $c_\theta = \kappa_\theta 24\sqrt{\frac{2}{\pi}}\sqrt{C}$. □

3.2.3 Proof of the pointwise ergodic theorem.

A refinement of the weak maximal inequality allows one to avoid the step of the proof of pointwise convergence for a dense family in $\mathbb{L}_{2,1}(\Omega)$:

Proposition 4 — *Let $\vec f(\omega)$ be a vector field in $\mathbb{L}_{2,1}(\Omega)$, weakly divergence-free. Then for almost all ω, the family of functions defined by*

$$\tilde{\mathcal{F}}_R(\vec f)(\omega, x, y) = \frac{1}{R^2}\tilde{\mathcal{F}}(\vec f)(\omega, Rx, Ry)$$

is equicontinuous on the set $\{(x, y), \|x\|,\ \|x + y\| \le 1\}$.

Proof.— For a given ω, let us calculate the modulus of continuity of the function $\tilde{\mathcal{F}}_R(\vec f)(\omega, x, y)$ on the set $\{(x, y), \|x\|$ and $\|x + y\| \le 1\}$. Let x, x', y, y' be such that the vertices x, x', $x + y$, $x' + y'$ are in the ball of radius 1. Let Δ and Δ' be the oriented triangular surfaces respectively with 0, Rx, $R(x + y)$ and 0, Rx', $R(x' + y')$ as vertices. The oriented path $\partial\Delta - \partial\Delta'$ is the boundary of an oriented surface constituted by 4 triangles (drawn between Δ and Δ' on figure 1). The cocycle equation, and shifts of the vertices of triangles give (let us recall that the variables of $\mathcal{F}(\vec f)$ are the edges of the triangles, and not the vertices),

$$\begin{aligned}\tilde{\mathcal{F}}(\vec f)(\omega, Rx, Ry) - \tilde{\mathcal{F}}(\vec f)(\omega, Rx', Ry') =\\ +\tilde{\mathcal{F}}(\vec f)(\omega, Rx, Rx' - Rx)\\ -\tilde{\mathcal{F}}(\vec f)(T_{R(x+y)}\omega, -Ry, R(x' - x))\\ -\tilde{\mathcal{F}}(\vec f)(T_{Rx'}\omega, Ry', R(x + y - x' - y'))\\ +\tilde{\mathcal{F}}(\vec f)(\omega, R(y' + x'), R(x + y - x' - y')).\end{aligned}$$

In the four terms of the above decomposition, the third variable tends to zero when x, y tend respectively to x', y'. Consequently it is sufficient to bound $\frac{1}{R^2}\tilde{\mathcal{F}}(\vec f)(T_{Ru}\omega, Rv, Rw)$, independently in u, v in the ball of center 0 and radius $\rho = 2$, and in $R > 0$, by a quantity $c^*_\omega(\|w\|)$ decreasing to 0 when $\|w\|$ decreases to 0. We have by definition

$$\tilde{\mathcal{F}}(\vec f)(T_{Ru}\omega, Rv, Rw) = \mathcal{G}(\vec f_{T_{Ru}\omega})(Rv, Rw),$$

hence, by change of variables $r = Rr'$, $s = Rs'$, $\eta' = \eta$ in the integral (20) defining $\mathcal{G}_1$ and similar change of variables in the integrals defining $\mathcal{G}_i$, $i = 2, 3$, we obtain

$$\frac{1}{R^2}\tilde{\mathcal{F}}(\vec f)(T_{Ru}\omega, Rv, Rw) = \mathcal{G}(\tau_u \vec f_{R,\omega})(v, w),$$

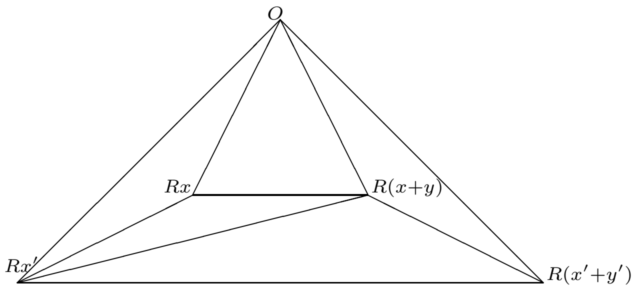

Figure 1: Triangles decomposition

where the function $\vec{f}_{R,\omega}(m)$ is defined by $\vec{f}_{R,\omega}(m) = \vec{f}_\omega(R \cdot m)$, and τ is the translation of functions of the variable $m \in \mathbb{R}^3$, defined by $(\tau_u \vec{\phi})(m) = \vec{\phi}(u+m)$. Therefore, by Lemma 7, for almost all ω, it holds that

$$\frac{1}{R^2} |\tilde{\mathcal{F}}(\vec{f})(T_{Ru}\omega, Rv, Rw)| \leq \frac{48}{\sqrt{\pi}} \cdot \|\vec{f}_{R,\omega}(m)\|_{\mathbb{L}_{2,1}(K(u,v,w))} \tag{23}$$

where $K(u,v,w)$ is obtained by shift of vector u of $K(0,v,w)$. However by the choice of u, v, w, the tetrahedron $K(u,v,w)$ is in the ball B of center 0 and radius 3. Hence we have

$$\lambda\big\{m \in K(u,v,w),\ \|\vec{f}_{R,\omega}(m)\| > t\big\} \leq$$
$$\min\Big[\lambda\big(K(u,v,w)\big)\ ,\ \lambda\big\{\|m\| \leq 3, \|\vec{f}_{R,\omega}(m)\| > t\big\}\Big].$$

It follows that the right-hand side of Inequality (23) is bounded independently in u, v, R by

$$c^*_\omega(\|w\|) =$$
$$\frac{48}{\sqrt{\pi}} \cdot \int_0^\infty \min\Big[\sup_{u,v} \sqrt{\lambda\big(K(u,v,w)\big)}\ ,\ \sqrt{\sup_R \lambda\big\{\|m\| \leq 3, \|\vec{f}_{R,\omega}(m)\| > t\big\}}\Big]\, dt.$$

Hence we need to study the convergence to 0 of the quantity $c^*_\omega(\|w\|)$, for $\|w\|$ decreasing to 0. Let us first consider the supremum on u, v. The tetrahedron $K(u,v,w)$ has edges v, w and altitude R_I — the radius of the circle inscribed in the triangle $\Delta(0,v,w)$. Hence its volume equals

$$\lambda\big(K(u,v,w)\big) = \frac{1}{6}\|v \wedge w\| R_I.$$

Since $\|v\| \leq 2$ and $R_I \leq \|w\|/2$, it follows

$$\sup_{u,v} \lambda\big(K(u,v,w)\big) \leq \frac{1}{6}\|w\|^2.$$

This tends to 0 when $\|w\|$ decreases to 0. Thus, according to the monotone convergence theorem, the above convergence of $c^*_\omega(\|w\|)$ will be true if the integral

$$i(\omega) = \int_0^\infty \sqrt{\sup_R \lambda\big\{\|m\| \leq 3,\ \|\vec{f}_\omega(R \cdot m)\| > t\big\}}\, dt$$

is a.s.-ω finite. It thus suffices to check that $i(\omega) \in \mathbb{L}_{2,\infty}(\Omega)$. According to the triangular inequality generalized with integrals, we have

$$\|i(\omega)\|_{2,\infty} \leq \int_0^\infty \left\| \sqrt{\sup_R \lambda\{\|m\| \leq 3,\ \|\vec{f}_\omega(R \cdot m)\| > t\}} \right\|_{2,\infty} dt.$$

Up to a constant, the above upper bound is the one of the inequality (22). We saw, in the proof of the weak maximal inequality, that this last is bounded by the norm $\mathbb{L}_{2,1}(\Omega)$ of $\vec{f}(\omega)$, up to a multiplicative constant. This prove that the function $i(\omega)$ is ω-a.s. finite, and completes the proof of the uniform equicontinuity, with modulus of continuity

$$2(c^*_\omega(\|x - x'\|) + c^*_\omega(\|x + y - x' - y'\|)).$$

□

Remark.— When the field $\vec{f}(\omega)$ belongs to $\mathbb{L}^p(\Omega)$ for a $p > 2$, we can prove that for almost all ω, the functions $\tilde{\mathcal{F}}_R(\vec{f})(\omega, x, y)$ are uniformly Hölder continuous in x, y, with exponent $1 - 2/p$, that is to say:

Proposition 5 — *Let $\vec{f}(\omega)$ be a weakly divergence-free field in $\mathbb{L}^p(\Omega)$ for some $p > 2$. Then, for almost all ω, there is a constant k^*_ω such as for all R, we have*

$$|\tilde{\mathcal{F}}_R(\vec{f})(\omega, x', y') - \tilde{\mathcal{F}}_R(\vec{f})(\omega, x, y)| \leq k^*_\omega \cdot (\|x - x'\|^{1-\frac{2}{p}} + \|y - y'\|^{1-\frac{2}{p}})$$

on the set $\{(x, y),\ \|x\|$ and $\|x + y\| \leq 1\}$.

The proof is similar to that of the preceding proposition, but based on the Hölder inequality.

Let us return to the case $\vec{f}(\omega)$ in $\mathbb{L}_{2,1}(\Omega)$. According to the Ascoli theorem, the family of functions $\tilde{\mathcal{F}}_R(\vec{f})(\omega, x, y)$ admits limit points for the topology of the uniform continuity in the space of continuous functions, and it is enough to prove uniqueness to have convergence. The identification of the limit value, for each x, y, with rational coordinates, is easily done with the mean ergodic theorem in $\mathbb{L}_{2,1}(\Omega)$. This concludes the proof of the pointwise ergodic theorem in the form stated in Theorem 2. □

4 About higher dimension and higher degree

The notions of "functional cocycle" and "functional coboundary" of degree k of a stationary action of $\mathbb{R}^d$, with $k \leq d$, follow directly by analogy with differential calculus (see [Katok; Katok]). The weak versions of these definitions follow, by a standard method, from an integration by parts. Similarly, the notions of "algebraic cocycle" and "pseudo-cocycle" of degree k follow by analogy with group theory (see [Feldman; Moore], [Mackey], [Mac Lane]). All previous statements of this paper can be generalized with these cases.
For example, the connection between the functional and the algebraic notions is the following. Let $f = (f_{i_1,\ldots,i_k}(\omega))_{1 \leq i_1 < \ldots < i_k \leq d}$ be a "random differential

form" of degree k. Let $x_1, \ldots, x_k$ be points of $\mathbb{R}^d$. Consider the k-dimensional $\mathbb{L}^1(\Omega)$-valued integral defined by

$$\mathcal{F}(f)(x_1, \ldots, x_k)(\omega) = \sum_{\substack{(i_1,\ldots,i_k) \\ 1 \le i_1 < \ldots < i_k \le d}} \int_{s_1} \cdots \int_{s_k} (f_{i_1,\ldots,i_k}(T_{s_1 x_1 + \cdots + s_k x_k}\omega)) \det_{i_1,\ldots,i_k}(x_1, \ldots, x_k)\ ds_1 \cdots ds_k$$

where

- $\det_{i_1,\ldots,i_k}(x_1, \ldots, x_k)$ is the determinant of the $k \times k$ matrix obtained by taking the lines $i_1, \ldots, i_k$ of the $d \times k$ matrix whose columns are $x_1, \ldots, x_k$;
- the domain of integration is the simplex of points $(s_1, \ldots s_k)$ such that $0 \le s_1 \le \cdots \le s_k \le 1$.

Then $\mathcal{F}(f)(x_1, \ldots, x_k)(\omega)$ is an algebraic pseudo-cocycle of degree k if and only if the random differential form $f(\omega)$ is a weak functional cocycle of degree k. As in the proof of Lemma 1, the argument is reduced to the Stokes formula.
Let us determine the integrability condition for the pointwise convergence. We have to express the analogue of the right-hand side integrals of the equality (19). For instance, the first term is given by the formula

$$\mathcal{F}_1(f)(x_1, \ldots, x_k)(\omega) = \int_\eta \mathcal{F}(f)(x_1, \ldots, x_k + z(\eta))(\omega)\ d\eta.$$

Because $\mathcal{F}(f)$ is a k-dimensional integral and because we need $\mathcal{F}_1(f)$ to be a d-dimensional integral, the integral on η has to be on a suitable subset of a $(d-k)$ dimensional sub-space. Then, in the d-dimensional integral $\mathcal{F}_1(f)$, we make the change of variables $(s_1, \ldots, s_k, \eta) \mapsto (s_1, \ldots, s_{k-1}, r, \eta)$ where the parameter r is the distance to the $(k-1)$-dimensional sub-space containing the points $0, x_1, \ldots, x_{k-1}$. Lebesgue measure of $\mathbb{R}^d$ has the same order than:

$$ds_1 \ldots ds_{k-1} dr \cdot r^{d-k} d\eta.$$

We thus need to find the smallest locally Lorentz space containing $\frac{1}{r^{d-k}}$. An elementary calculation shows that it is $\mathbb{L}_{p,\infty}(\mathbb{R}^d)$ with $p = \frac{d-k+1}{d-k}$. The dual exponent is $p^* = \frac{p}{p-1} = d - k + 1$. Hence the required integrability for the pointwise convergence is $\mathbb{L}_{d-k+1,1}(\Omega)$. This generalizes the theorem stated for $k = 1$ in [Boivin; Derriennic].

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