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An homogenization result for a deterministic regional periodic control problem

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Joint work with G. Barles & E. Chasseigne & N. Tchou

Model problem

I am on the beach and I want to take my boat in minimal time :

- How do I choose my trajectory ? How do I define my trajectory when I "swim and walk" ?

- How do I take into account the sea when I am on the beach and viceversa ?

Goal : To modellize optimal control problem with "different dynamics and/or costs in different domains"

Our first result was : Infinite Horizon, 2-domains

x_N

Dynamic : b_1

$H_1 = 0$ in Ω_1

$$\Omega_1 := \{x \in \mathbb{R}^N \ : \ x_N > 0\}$$

Running cost : l_1

$$\begin{cases} \min\{H_1, H_2\} & \leq \ 0 \\ \max\{H_1, H_2\} & \geq \ 0 \end{cases} \text{ on } \mathcal{H}$$

x'

Dynamic : b_2

$$\Omega_2 := \{x \in \mathbb{R}^N \ : \ x_N < 0\}$$

Running cost : l_2

$H_2 = 0$ in Ω_2

We answered to :

How to define the dynamic and running cost on $\mathcal{H} := \{x \in \mathbb{R}^N \ : \ x_N = 0\}$?

What is the Bellman problem satisfied by the value function(s) ?

What are the right viscosity inequalities to be satisfied on $\mathcal{H}$?

Same related works

– P. Dupuis : dimension 1, calculus of variations with discontinuous integrand.

– Soravia, Garavello and Soravia, De Zan and Soravia : problems with discontinuities but with a a special structure of the discontinuities.

– Camilli and Siconolfi : L^∞-framework, but special equations.

– Y. Achdou, F. Camilli, A. Cutri & N. Tchou and C. Imbert, R. Monneau & H. Zidani : problems on network. C. Imbert, R. Monneau : Nework in $\mathbb{R}^N$.

– Bressan and Hong, Wolenski , G. Barles and E. Chasseigne : optimal control problem on stratified domains.

–N. Forcadel, Z. Rao, A. Siconolfi, H. Zidani : same problem with "pure" control methods : they treat more general junctions but with more restictive controllability assumption and get less general stability result.

More general framework :

(H_Ω) $\mathbb{R}^N = \Omega_1 \cup \Omega_2 \cup \mathcal{H}$ with $\Omega_1 \cap \Omega_2 = \emptyset$
and $\mathcal{H} = \partial\Omega_1 = \partial\Omega_2$ is a $W^{2,\infty}$-hypersurface in $\mathbb{R}^N$;

(H_C^{1-2}) Regularity and boundedness for b_i, l_i $(i = 1, 2)$;

(H_C^3) For each $i = 1, 2$, $z \in \overline{\Omega_i}$, and $s \in [0, T]$, the set $\{(b_i(z, s, \alpha_i), l_i(z, s, \alpha_i)) : \alpha_i \in A_i\}$ is closed and convex.

(H_C^4) Controllability only in the normal direction :
There is a $\delta > 0$ such that for any $i = 1, 2$, $z \in \mathcal{H}$ and $s \in [0, T]$

$$\mathrm{B}_i(z, s) \cdot \mathrm{n}_i(z) \supset [-\delta, \delta]$$

where $\mathrm{B}_i(z, s) := \{b_i(z, s, \alpha_i) : \alpha_i \in A_i\}$.

Finite horizon control problems

Controlled trajectories :

$X_{x,t}(\cdot) = ((X_{x,t})_1, (X_{x,t})_2, \ldots, (X_{x,t})_N)(\cdot)$
are Lipschitz continuous functions which are solutions of the following differential inclusion

$$\dot{X}_{x,t}(s) \in \mathcal{B}(X_{x,t}(s), t-s) \quad \text{for a.e. } s \in [0,t); \quad X_{x,t}(0) = x$$

where

$$\mathcal{B}(z,s) := \begin{cases} \mathrm{B}_i(z,s) & \text{if } z \in \Omega_i\,, \\ \overline{\mathrm{co}}(\mathrm{B}_1(z,s) \cup \mathrm{B}_2(z,s)) & \text{if } z \in \mathcal{H}\,, \end{cases}$$

the notation $\overline{\mathrm{co}}(E)$ referring to the convex closure of the set $E \subset \mathbb{R}^N$.

Theorem : (true without the controllability assumption)

(i) For each $x \in \mathbb{R}^N$, $t \in [0,T)$ there exists a Lipschitz function $X_{x,t} : [0,t] \to \mathbb{R}^N$ which is a solution of the differential inclusion.
(ii) For each solution $X_{x,t}(\cdot)$ there exists a control

$$a(\cdot) := (\alpha_1(\cdot), \alpha_2(\cdot), \mu(\cdot)) \in \mathcal{A} = L^\infty([0,T]; A_1 \times A_2 \times [0,1])$$

such that for a.e. $s \in (t,T)$

$$\dot{X}_{x,t}(s) = \sum_{i=1,2} b_i(X_{x,t}(s), t-s, \alpha_i(s)) \mathbb{1}_{\mathcal{E}_i}(s) + b_{\mathcal{H}}(X_{x,t}(s), t-s, a(s)) \mathbb{1}_{\mathcal{E}_{\mathcal{H}}}(s)$$

where

$$b_{\mathcal{H}}(x, t-s, a) = \mu b_1(x, t-s, \alpha_1) + (1-\mu) b_2(x, t-s, \alpha_2),$$

$$\mathcal{E}_i := \{ s \in (0,t) : X_{x,t}(s) \in \Omega_i \} \quad \mathcal{E}_{\mathcal{H}} := \{ s \in (0,t) : X_{x,t}(s) \in \mathcal{H} \}$$

(iii) We have

$$b_{\mathcal{H}}(X_{x,t}(s), t-s, a(s)) \cdot \mathrm{n}_i(X_{x,t}(s)) = 0 \text{ for a.e. } s \in \mathcal{E}_{\mathcal{H}}$$

Running cost : Define

$$\ell(X_{x,t}(s), t-s, a(s)) := \sum_{i=1,2} l_i(X_{x,t}(s), t-s, \alpha_i(s))\mathbb{1}_{\mathcal{E}_i}(s) + l_{\mathcal{H}}(X_{x,t}(s), t-s, a(s))\mathbb{1}_{\mathcal{E}_{\mathcal{H}}}(s)\,.$$

where

$$l_{\mathcal{H}}(x, t-s, a) := \mu l_1(x, t-s, \alpha_1) + (1-\mu) l_2(x, t-s, \alpha_2)\,.$$

Cost : associated to $(X_{x,t}(\cdot), a) \in \mathcal{T}_{x,t}$ is

$$J(x,t;(X_{x,t},a)) := \int_0^t \ell(X_{x,t}(s), t-s, a(s))ds + g(X_{x,t}(t))$$

with $g \in BUC(\mathbb{R}^N)$

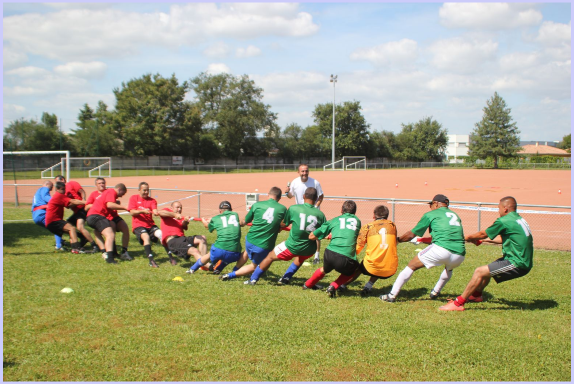
Regular and Singular dynamics on $\mathcal{H}$ the dynamic is :

$$b_{\mathcal{H}}(x, t-s, a) = \mu b_1(x, t-s, \alpha_1) + (1-\mu) b_2(x, t-s, \alpha_2),$$

$$b_{\mathcal{H}}(x, t-s, a) \cdot \mathrm{n}_i(z) = 0$$

The **regular dynamics**
(*"both pushes to be on* $\mathcal{H}$")

$$b_1(z, s, \alpha_1) \cdot \mathrm{n}_1(z) \geq 0$$
$$b_2(z, s, \alpha_2) \cdot \mathrm{n}_2(z) \geq 0$$



The **singular dynamics**
(*"both pull so we stay on* $\mathcal{H}$")

$$b_1(z, s, \alpha_1) \cdot \mathrm{n}_1(z) < 0\,,$$
$$b_2(z, s, \alpha_2) \cdot \mathrm{n}_2(z) < 0\,.$$

Therefore two "natural" value functions can be defined

$$\mathrm{U}^{-}(x,t) := \inf_{(X_{x,t},a)\in\mathcal{T}_{x,t}} J\big(x,t;(X_{x,t},a)\big)$$

$\mathcal{T}_{x,t}$ **: with regular and singular strategies on** $\mathcal{H}$

$$\mathrm{U}^{+}(x,t) := \inf_{(X_{x,t},a)\in\mathcal{T}^{\mathrm{reg}}_{x,t}} J\big(x,t;(X_{x,t},a)\big).$$

$\mathcal{T}^{\mathrm{reg}}_{x,t}$ **: without the singular strategies on** $\mathcal{H}$

NB : $\mathrm{U}^{-} \leq \mathrm{U}^{+}$ **in** $\mathbb{R}^N \times [0,T]$.

We will prove later that both are continuous but without controllability assumptions we do not know that they are Lipschitz continuous.

The "natural" Hamilton-Jacobi-Bellman system

$$u_t + H(x,t,Du) = 0$$

A subsolution (a supersolution) is a bounded usc function u (a bounded lsc function v) which satisfies ($i = 1,2$)

$$\begin{cases} u_t + H_i(x,t,Du) \leq 0 & \text{in } \Omega_i \times (0,T) \\ u_t + \min\{H_1(x,t,Du), H_2(x,t,Du)\} \leq 0 & \text{in } \mathcal{H} \times (0,T) \end{cases}$$

$$\left[\begin{cases} v_t + H_i(x,t,Dv) \geq 0 & \text{in } \Omega_i \times (0,T) \\ v_t + \max\{H_1(x,t,Dv), H_2(x,t,Dv)\} \geq 0 & \text{in } \mathcal{H} \times (0,T) \end{cases} \right]$$

where $H_i(x,t,p) := \sup_{\alpha_i \in A_i} \{-b_i(x,t,\alpha_i)\cdot p - l_i(x,t,\alpha_i)\}$.

Theorem : The value functions U^- and U^+ are both viscosity solutions of $u_t + H(x,t,Du) = 0$.

Two "tangential Hamiltonians" : H_T, H_T^{reg}

We consider the tangent bundle $T\mathcal{H} := \cup_{z\in\mathcal{H}}\big(\{z\}\times T_z\mathcal{H}\big)$ where $T_z\mathcal{H}$ is the tangent space to $\mathcal{H}$ at z.

For $((x,p),t) \in T\mathcal{H}\times[0,T]$ we define the Hamiltonians

$$H_T(x,t,p) := \sup_{A_0(x,t)} \big\{ -\langle b_{\mathcal{H}}(x,t,a),p\rangle - l_{\mathcal{H}}(x,t,a)\big\}\,,$$

$$A_0(x,t) := \Big\{ a = (\alpha_1,\alpha_2,\mu) : b_{\mathcal{H}}\big(x,t,(\alpha_1,\alpha_2,\mu)\big)\cdot \mathrm{n}_1(x) = 0 \Big\}\,,$$

and

$$H_T^{\text{reg}}(x,t,p) := \sup_{A_0^{\text{reg}}(x,t)} \big\{ -\langle b_{\mathcal{H}}(x,t,a),p\rangle - l_{\mathcal{H}}(x,t,a)\big\}\,,$$

$$A_0^{\text{reg}}(x,t) := \big\{ a \in A_0(x,t) : b_i(x,t,\alpha_i)\cdot \mathrm{n}_i(z) \geq 0,\ i = 1,2\big\}\,,$$

(*We do not allow singular strategies*).

Definition (same for H_T^{reg}) : A bounded usc function $u : \mathcal{H} \times [0,T] \to \mathbb{R}$ is a viscosity subsolution of

$$u_t(x,t) + H_T(x,t,D_{\mathcal{H}} u) = 0 \quad \text{on} \quad \mathcal{H} \times [0,T]$$

if, for any $\phi \in C^1(\mathcal{H} \times [0,T])$ and any maximum point (x,t) of $(z,s) \mapsto u(z,s) - \phi(z,s)$ in $\mathcal{H} \times [0,T]$, one has

$$\phi_t(x,t) + H_T(x,t,D_{\mathcal{H}}\phi(x,t)) \leq 0 \; .$$

Note that if $\phi \in C^1(\mathcal{H})$, and $x \in \mathcal{H}$, we denote by $D_{\mathcal{H}}\phi(x)$ the gradient of ϕ at x, which belongs to $T_x\mathcal{H}$.

Theorem : U^- is a subsolution of

$$u_t(x,t) + H_T(x,t,D_{\mathcal{H}} u) = 0 \quad \text{on} \quad \mathcal{H} \times [0,T]$$

while U^+ is a subsolution of

$$u_t(x,t) + H_T^{reg}(x,t,D_{\mathcal{H}} u) = 0 \quad \text{on} \quad \mathcal{H} \times [0,T]$$

Notation :
$u_t + \mathbb{H}^-(x,t,Du) = 0$ ($u_t + \mathbb{H}^+(x,t,Du) = 0$)
will denote system $u_t + H(x,t,Du) = 0$ and condition
$u_t + H_T(x,t,D_{\mathcal{H}}u) \leq 0$ ($u_t + H_T^{reg}(x,t,D_{\mathcal{H}}u) \leq 0$).

From the PDE point of view :

Properties of subsolutions :

If u is subsolution of $u_t + H(x,t,Du) = 0$, then u is a subsolution of $u_t + \mathbb{H}^+(x,t,Du) = 0$.

This means that inequality $u_t(x,t) + H_T^{\text{reg}}(x,t,D_{\mathcal{H}}u) \le 0$ is encoded in the original problem and not an additional property

MAGIC LEMMA Properties of supersolutions :

Let v be a supersolution of $v_t + H(x,t,Dv) = 0$. Let $\phi \in C^1(\mathcal{H} \times [0,T])$ and (x,t) be a minimum point of $(z,s) \mapsto v(z,s) - \phi(z,s)$. Then, the following alternative holds :

A) either there exist $\eta > 0$, $i \in \{1,2\}$ and a sequence $x_k \in \overline{\Omega}_i$ converging to x such that $v(x_k,t) \to v(x,t)$ and, for each k, there exists a control $\alpha_i^k(\cdot)$ such that the corresponding trajectory $Y^i_{x_k,t}(s) \in \overline{\Omega}_i$ for all $s \in [0,\eta]$ and

$$v(x_k,t) \ge \int_0^\eta l_i\big(Y^i_{x_k,t}(s), t-s, \alpha_i^k(s)\big)ds + v\big(Y^i_{x_k,t}(\eta), t-\eta\big) \ ;$$

B) or there holds

$$\phi_t(x,t) + H_T\big(x,t,D_{\mathcal{H}}\phi(x,t)\big) \ge 0.$$

With the additional $H_T \leq 0$-inequality, we have a uniqueness result for $u_t + \mathbb{H}^-(x,t,Du) = 0$

Theorem (Strong Comparison Result) :

1) Assume that u and v are respectively bounded usc sub a bounded lsc supersolution of $w_t + H(x,t,Dw) = 0$ and that

$$w_t(x,t) + H_T\big(x,t,D_{\mathcal{H}}w(x,t)\big) \leq 0 \quad \text{on } \mathcal{H} \times (0,T)$$

If $u(x,0) \leq v(x,0)$ in $\mathbb{R}^N$ then $u \leq v$ in $\mathbb{R}^N \times (0,T)$.

2) The value function U^- is continuous and the unique solution of

$$u_t + \mathbb{H}^-(x,t,Du) = 0 \text{ in } \mathbb{R}^N \times (0,T) \qquad u(x,0) = g(x) \text{ in } \mathbb{R}^N \ .$$

3) U^- is the minimal supersolution of and U^+ is the maximal subsolution of $u_t + H(x,t,Du) = 0$.

Remark : 1) is based on a local comparison result.

3) is based on the fact that U^+ verify an alternative property as for the supersolutions with H_T^{reg} instead of H_T.

Stability Results

Theorem. Fix $\varepsilon > 0$, let H_i^ε $(i = 1, 2)$ and H_T^ε be defined trought $b_1^\varepsilon, b_2^\varepsilon, l_1^\varepsilon, l_2^\varepsilon$ satysfing $[(\mathrm{H}_\mathrm{C}^{1-2})]$ uniformly with respect to ε. If

$$(b_1^\varepsilon, b_2^\varepsilon, l_1^\varepsilon, l_2^\varepsilon) \to (b_1, b_2, l_1, l_2) \text{ locally uniformly}$$

then :

(i) if, for all $\varepsilon > 0$, v_ε is a lsc supersolution of

$$u_t + \mathbb{H}_\varepsilon^-(x, t, Du) = 0 \text{ in } \mathbb{R}^N \times (0, T), \tag{1}$$

then $\underline{v} = \liminf_* v_\varepsilon$ is a lsc supersolution of

$$u_t + \mathbb{H}^-(x, t, Du) = 0 \text{ in } \mathbb{R}^N \times (0, T), \tag{2}$$

(ii) If, for $\varepsilon > 0$, u_ε is an usc subsolution of (1) and if b_1, b_2 satisfy the normal controllability assumption $[(\mathrm{H}_\mathrm{C}^4)]$ then $\bar{u} = \limsup^* u_\varepsilon$ is a subsolution of (2).

(iii) Moreover, $\mathrm{U}_\varepsilon^- \to \mathrm{U}^-$ and $\mathrm{U}_\varepsilon^+ \to \mathrm{U}^+$.

Based on :

A fundamental Lemma. For any $(z,t),(z',t') \in \mathcal{H} \times [0,T]$ and for each control $a \in A_0(z,t)$ $(A_0^{reg}(z,t))$, there exists a control $a' \in A_0(z',t')$ $(A_0^{reg}(z,t))$ such that,

$$|b_{\mathcal{H}}(z,t,a) - b_{\mathcal{H}}(z',t',a'))| \leq C|(z,t)-(z',t')|$$

$$|l_{\mathcal{H}}(z,t,a) - l_{\mathcal{H}}(z',t',a'))| \leq C|(z,t)-(z',t')| + m_l(|(z,t)-(z',t')|$$

This implies the Lipschitz regularity of the tangential Hamiltonians $H_T(x,t,p)$ and $H_T^{\mathrm{reg}}(x,t,p)$ with respect $x \in \mathcal{H}$ and $p \in \mathbb{R}^N$.

Theorem (convergence of trajectories).

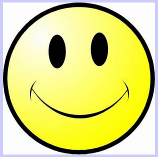
Fix $\varepsilon > 0$, let $(X^\varepsilon, a^\varepsilon) \in \mathcal{T}^\varepsilon_{x,t}$,

i) There exists a subsequence $(X^{\varepsilon_n}, a^{\varepsilon_n})_n \to (X, a) \in \mathcal{T}_{x,t}$. More precisely, $X^{\varepsilon_n} \to X$ uniformly in $[0, T]$ and

$$J(x, t; (X^{\varepsilon_n}, a^{\varepsilon_n})) \to J(x, t, (X, a)) \text{ uniformly in } [0, T]\,.$$

ii) If, moreover the trajectories $(X^\varepsilon, a^\varepsilon) \in \mathcal{T}^\varepsilon_{x,t}$ are regular for any $\varepsilon > 0$ then we have a subsequence for which the limit trajectory is also regular.

Extensions and open problems



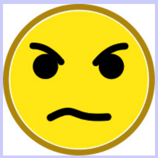
A regularly time dependent Ω_i.

An additional control problem on $\mathcal{H}$.

Infinite horizon control problems.

Triple junctions or chessboard : stratified ! !

(G.Barles- E. Chasseigne :
"(Almost) everything you always wanted to know about deterministic control problems in stratified domains".)
Almost because "only" U^-.



The characterization of U^+ on stratified domains.
More general interface.
Viscosity approximation.
Second order, etc..etc...

An homogeneisation result : the infinite horizon problem

$$\begin{cases} \lambda u(x) + H_1(x, \frac{x}{\varepsilon}, Du) = 0 & \text{in } \varepsilon\Omega_1\,, \\ \\ \lambda u(x) + H_2(x, \frac{x}{\varepsilon}, Du) = 0 & \text{in } \varepsilon\Omega_2\,, \end{cases}$$

with tangential condition

$$\lambda u(x) + H_T(x, \frac{x}{\varepsilon}, D_{\mathcal{H}} u) \leq 0 \text{ in } \varepsilon\mathcal{H}$$

or

$$\lambda u(x) + H_T^{\text{reg}}(x, \frac{x}{\varepsilon}, D_{\mathcal{H}} u) \leq 0 \text{ in } \varepsilon\mathcal{H}.$$

We define

$$U_\varepsilon^-(x_0) := \inf_{(X_{x_0},a)\in\mathcal{T}_{x,t}} \int_0^{+\infty} \ell(X_{x_0}(t), \frac{X_{x_0}}{\varepsilon}(t), a) e^{-\lambda t} dt$$

$$U_\varepsilon^+(x_0) := \inf_{(X_{x_0},a)\in\mathcal{T}_{x,t}^{\text{reg}}} \int_0^{+\infty} \ell(X_{x_0}(t), \frac{X_{x_0}}{\varepsilon}(t), a) e^{-\lambda t} dt$$

Assumption : the Ω_i are $\mathbb{Z}^N$ periodic.

The result for U_ε^-

Comparison and stability and "classical" tools :

Theorem : **The sequence $(U_\varepsilon^-)_{\varepsilon>0}$ converges locally uniformly in $\mathbb{R}^N$ to a function U which is the unique solution of**

$$\lambda U(x) + \bar{H}^-(x, DU) = 0 \quad \text{in } \mathbb{R}^N .$$

**where $\bar{H}^- : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is defined as follows :
for any $x, p \in \mathbb{R}^N$, there exist a unique constant $\lambda = \bar{H}^-(x,p)$ such that the following cell problem has a Lipschitz continuous, $\mathbb{Z}^N$-periodic viscosity solution v**

$$\mathbb{H}^-(x, y, Dv + p) = \lambda \text{ in } \mathbb{R}^N .$$

The result for U_ε^+

Since we do not have a comparison result but only "half" (U_ε^+ is the maximal subsolution) the result is "half" based on the control formulation of the problem.

The cell problem. For any $x, p \in \mathbb{R}^N$, there exists a unique constant $\bar{H}^+(x,p) \in \mathbb{R}$ such that there exists a Lipschitz continuous, periodic function V^+ satisfying,
for any $\tau \geq 0$ and $y_0 \in \mathbb{R}^N$

$$V^+(y_0) = \inf_{(Y_{y_0},a)\in\mathcal{T}^{reg}_{y_0}} \Big\{ \int_0^\tau \Big(\tilde{l}(x,p,Y_{y_0}(t),a(t)) + \bar{H}^+(x,p)\Big)\,dt + V^+(Y_{y_0}(\tau))\Big\}$$

where

$$\tilde{l}(x,p,Y_{y_0}(t),a(t)) = l(x,Y_{y_0}(t),a(t)) + b(x,Y_{y_0}(t),a(t))\cdot p.$$

Moreover V^+ is a viscosity subsolution of

$$\mathbb{H}^+(x,y,DV^+ + p) = \bar{H}^+(x,p) \text{ in } \mathbb{R}^N.$$

Finally, for all $y_0 \in \mathbb{R}^N$ we have

$$\bar{H}^+(x,p) = \lim_{t\to+\infty}\Big(-\inf_{(Y_{y_0},a)\in\mathcal{T}^{reg}_{y_0}}\{\frac{1}{t}\int_0^t \tilde{l}(x,p,Y_{y_0}(t),a(t))dt\}\Big)$$

The convergence result for U_ε^+

The sequence $(U_\varepsilon^+)_{\varepsilon>0}$ converges locally uniformly in $\mathbb{R}^N$ to a continuous function U^+, which is the unique viscosity solution of

$$\lambda u(x) + \bar{H}^+(x, Du(x)) = 0 \text{ in } \mathbb{R}^N.$$

U^+ is a supersolution. Complete PDE argument.

U^+ is a subsolution. Follows closely the PDE ideas but perform all the arguments on the control formulas.
A real technical difficulty appears due to the x dependence on the dynamics. This is solved by an approximation of the cell problem as in :

G. Barles, F. Da Lio, P.L. Lions, P. Souganidis "Ergodic problems and periodic homogenization for fully non linear equation in half-space time domanis with Neumann boundary conditions. " 2008

.........*grazie per l'attenzione.*