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Asymptotic Estimates for a Variational Problem Involving a Quasilinear Operator in the Semi-Classical Limit

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Abstract. Let Ω be a domain of $\mathbb{R}^N$. We study the infimum $\lambda_1(h)$ of the functional $\int_\Omega |\nabla u|^p + h^{-p}V(x)|u|^p\, \mathrm{d}x$ in $W^{1,p}(\Omega)$ for $\|u\|_{L^p(\Omega)} = 1$ where $h > 0$ tends to zero and V is a measurable function on Ω. When V is bounded, we describe the behaviour of $\lambda_1(h)$, in particular when V is radial and 'slowly' decaying to zero. We also study the limit of $\lambda_1(h)$ when $h \to 0$ for more general potentials and show a necessary and sufficient condition for $\lambda_1(h)$ to be bounded. A link with the tunelling effect is established. We end with a theorem of existence for a first eigenfunction related to $\lambda_1(h)$.

Mathematics Subject Classifications: **Q1**

Key words: nonlinear equation, p-Laplacian, semi-classical limits, first eigenvalue, tunelling effect

1. Introduction

Let Ω be a connected open set of $\mathbb{R}^N$, $p \in (1, +\infty)$ and V a measurable function on Ω. We tackle with the first eigenvalue of the Neumann realisation of the quasilinear Schrödinger operator

$$u \mapsto -\Delta_p u + h^{-p}V(x)|u|^{p-2}u \tag{1.1}$$

for small positive h. We recall that $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. More precisely, we are interested in the minimization of the functional

$$F_V(\psi) = \int_\Omega (|\nabla\psi|^p + V(x)|\psi|^p)\, \mathrm{d}x \tag{1.2}$$

for $\|\psi\|_{L^p(\Omega)} = 1$. First, since V is only a measurable function which may take the value ∞ on sets of positive measure, we have to define

$$W^{1,p,V}(\Omega) = \{\psi \in W^{1,p}(\Omega) : V(x)|\psi|^p \in L^1(\Omega)\}, \tag{1.3}$$

and we assume that $W^{1,p,V}(\Omega) \neq \{0\}$, that is, it contains a least a function ψ such that $\|\psi\|_{L^p(\Omega)} = 1$. If V is bounded, $W^{1,p,V}(\Omega) = W^{1,p}(\Omega)$. For $h > 0$, we set

$$\lambda_1(h) = \inf\left\{F_{h^{-p}V}(\psi) : \psi \in W^{1,p,V}(\Omega), \int_\Omega |\psi|^p \mathrm{d}x = 1\right\} \tag{1.4}$$

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$\lambda_1(h)$ exists for all positive h since $W^{1,p,V}(\Omega) \neq \{0\}$ and belongs to $[-\infty, +\infty)$. If ω is a connected open subset of Ω, $\lambda_{1,\omega}$ is the first eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\omega)$ for the Dirichlet boundary condition and φ_ω a corresponding positive eigenfunction.

In the first part, we restrict our study for potentials V which satisfy $\operatorname{essinf}_\Omega V = 0$. In semi-classical limit, the first eigenfunction concentrates near the infimum of the potential when h goes to zero. This fact is the key-stone of the first part of the article. It leads to the following idea: u_h behaves like a first eigenfunction of $-\Delta_p$ for a certain open bounded connected set with the Dirichlet boundary condition, that is, outside it, the contribution of u_h can be neglected although u_h is positive on Ω. This observation leads to a new terminology.

Q2 DEFINITION 2.6. We said that V has an *almost nonempty interior* if by changing V on a zero measure set, the interior of $\{x : V(x) \leqslant z\}$ is not empty for all positive z.

V satisfies automatically this property if V is equal almost everywhere to a radial increasing function on Ω. There exists functions which do not hold the property of 'almost nonempty interior'. Henceforth, we take only potentials V which have an almost nonempty interior. Moreover, we manage to modify V on a zero measure set in such a way that the interior of $\{x : V(x) \leqslant z\}$ is not empty for all $z > 0$. Unfortunately, this set is not necessary connected. We prefer to take the largest connected component of the set $\{x : V(x) \leqslant z\}$ and its eigenvalue for the Dirichlet realization of $-\Delta_p$. But we can not explicitely calculate the first eigenvalue $\lambda_{1,\omega}$ for a great number of open connected sets ω when $N \geqslant 2$. The heart of the method in higher dimensions is based on translations and homotheties. More precisely, if ω is an open connected bounded subset of $\mathbb{R}^N$, T a translation of $\mathbb{R}^N$ and H_r an homothety of $\mathbb{R}^N$ of positive ratio r, then

$$\lambda_{1,T(H_r(\omega))} = r^{-p}\lambda_{1,\omega}. \tag{1.5}$$

We are interested in the rate of growth for $\lambda_1(h)$. That is why the fact that we can not calculate $\lambda_{1,\omega}$ does not matter for most applications [2, 3, 8]. If V has an almost nonempty interior, we define for all $z > 0$, $r(z, \omega)$ as the supremum of all positive r such that there exist a translation T and a homothety H_r of ratio r which satisfy, up to a zero measure set,

$$T(H_r(\omega)) \subset \{x \in \Omega : V(x) \leqslant z\}. \tag{1.6}$$

In other words, we try to find a bounded open connected set which is, up to a translation, the homothetic of ω. Since $\omega \mapsto \lambda_{1,\omega}$ is an increasing function with respect to the inclusion, a good upper bound for $\lambda_1(h)$ implies to choose large $H_r(\omega)$, that is, to have a great ratio r. This role is played by $r(z, \omega)$. It is clear that $r(z, \omega)$ exists if and only if V has an almost nonempty interior. $r(z, \omega)$ depends on the open set ω but if ω' is another open set then for all $z > 0$,

$$C_1\, r(z, \omega) \leqslant r(z, \omega') \leqslant C_2\, r(z, \omega) \tag{1.7}$$

for some positive constants C_1 and C_2. So we obtain an explicit upper bound for $\lambda_1(h)$.

Now, we deal with a lower bound. But before, we have to remark that from [2] and [3], $\lambda_1(h)$ appears through the measure. But on the other hand, we are working with homothetic open sets. It is crucial to link measure and homothetic open sets. With the above notations, if r is the ratio of the homothety H_r then $\text{meas}(T(H_r(\omega))) = r^N \text{meas}(\omega)$. It may happen that for some V when z goes to zero, the measure of $T(H_r(\omega))$ is too small with respect to the measure of $\{x \in \Omega : V(x) \leqslant z\}$, that is,

$$\lim_{z\to 0} \frac{r(z,\omega)^N}{\text{meas}\{x \in \Omega : V(x) \leqslant z\}} = 0. \tag{1.8}$$

Seemingly, in that case, the open set $T(H_r(\omega))$ does not 'fill up' correctly the set $\{x \in \Omega : V(x) \leqslant z\}$. This leads to a new definition.

DEFINITION 2.26. Let Ω be a domain of $\mathbb{R}^N$ and V a bounded measurable function on Ω. We say that V has a *large nonempty interior* if V has an almost nonempty interior, $\text{meas}\{x \in \Omega : V(x) \leqslant z\} < \infty$ for $z > 0$ small enough and there exists ω a connected open nonempty bounded subset of $\mathbb{R}^N$ such that

$$\liminf_{z\to 0} \frac{r(z,\omega)^N}{\text{meas}\{x \in \Omega : V(x) \leqslant z\}} > 0. \tag{1.9}$$

This definition does not depend on the open bounded connected set ω chosen. This condition is automatically satisfied for a radial function and more generally for a continuous function which vanishes in a finite number of points and which is radial in neighbourhoods of these points and greater than a positive constant elsewhere ('a finite number of radial wells'). Moreover, there exist some functions which have an almost nonempty interior but not a large nonempty interior. This new terminology provides us an explicit lower bound in a simple way. Then we describe the behaviour of $\lambda_1(h)$ for some classes of potentials. When V is a radial function, it is possible to derive a corollary for the first eigenvalue of the Neumann realization of the quasilinear operator $u \mapsto -\Delta_p u + h^{-p} V(|x|)u^{p-1}$.

COROLLARY 2.35. *Let Ω be a connected open set of $\mathbb{R}^N$, $O \in \Omega$, $N > p$, V a concave increasing radial function which is continuous on $[0, +\infty)$, $\mathcal{C}^1$ on $(0, +\infty)$ with $V(0) = 0$. Furthermore, we assume that*

$$r\, V'\big(r\, (V(r))^{\frac{1}{p}}\big) = o(V(r)) \tag{1.10}$$

when r goes to zero. Then

$$\lambda_1(h) \sim h^{-p} V(h). \tag{1.11}$$

Under some additional assumptions, we begin a two-terms asymptotic expansion.

In the next part, we deal with the study of the limit of $\lambda_1(h)$. We have already analysed the behaviour of $\lambda_1(h)$ but under strong assumptions. The goal of determining only the limit seems more modest but we are interested in more general potentials. We give a necessary and sufficient condition for $\lambda_1(h)$ to be bounded. When $h \mapsto \lambda_1(h)$ has a finite limit when $h \to 0$, we try to characterize the limit with the help of an optimization problem. We introduce the following definition.

DEFINITION 3.9. Let E be a measurable subset of Ω such that there exists a function ψ_0 in $W^{1,p}(\Omega)$ which satisfies $\|\psi_0\|_{L^p(\Omega)} = 1$ and $\psi_0 = 0$ almost everywhere in $\Omega\backslash E$. We denote by λ_E the quantity

$$\lambda_E = \inf\left\{ \int_\Omega |\nabla\psi|^p \,\mathrm{d}x : \psi \in W^{1,p}(\Omega), \|\psi\|_{L^p(\Omega)} = 1, \right.$$
$$\left. \psi = 0 \text{ a.e. on } \Omega\backslash E \right\}. \tag{1.12}$$

λ_E is the first eigenvalue of $-\Delta_p$ in $W^{1,p}(E)$ for the Dirichlet boundary condition if E is a bounded domain. But if E is any measurable set with a such ψ_0, this infimum is not necessary reached and if it is, there is no reason to have a unique positive minimizer. We remark that the function $E \mapsto \lambda_E$ is decreasing with respect to the inclusion, i.e., if $E_1 \subset E_2$, $\lambda_{E_1} \geqslant \lambda_{E_2}$ (if λ_{E_1} exists). For the sake of simplicity, we always suppose that $\operatorname{ess\,inf}_\Omega V = 0$. We define by (ψ_h) a sequence of functions of $W^{1,p,V}(\Omega)$ with $\|\psi_h\|_{L^p(\Omega)} = 1$ such that

$$\lim_{h\to 0} \lambda_1(h) - F_{h^{-p}V}(\psi_h) = 0. \tag{1.13}$$

In that way, we avoid the delicate problem of the existence of a minimizer for $\lambda_1(h)$ ($F_{h^{-p}V}$ is the functional related to $\lambda_1(h)$). Heuristically speaking, the sequence (ψ_h) concentrates near the infimum of the potential when h goes to zero. Therefore, the limit of $\lambda_1(h)$ should depend only on the behaviour of V in a neighbourhood of $\{x : V(x) = 0\}$. In fact, we shall see that the limit depends only on the set $\{x : V(x) = 0\}$. Now, we restrict our attention to potentials which satisfy

$$\begin{aligned} &\text{There exists } \varepsilon > 0 \text{ such that, up to a set of zero measure,} \\ &\quad \{x : V(x) < \varepsilon\} \text{ is bounded and } \lim_{h\to 0} h^p \lambda_1(h) = 0. \end{aligned} \tag{1.14}$$

The first assumption is close to the definition of a well in [7]. The assumption '$\{x : V(x) = 0\}$ is bounded' is not sufficient since this set may be empty. $\lim_{h\to 0} h^p\lambda_1(h) = 0$ shows that V does not take too large values in a neighbourhood of $\{x : V(x) = 0\}$. The main theorem is

THEOREM 3.13. *Suppose that* Ω *is a connected open set of* $\mathbb{R}^N$, V *a measurable function on* Ω *which satisfies* $W^{1,p,V}(\Omega) \neq \{0\}$, $\operatorname{ess\,inf}_\Omega V = 0$ *and* (1.14). *Then* $h \mapsto \lambda_1(h)$ *is bounded independently of* h *if and only if there exists a function*

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ψ_0 in $W^{1,p}(\Omega)$ which satisfies $\|\psi_0\|_{L^p(\Omega)} = 1$ and $\psi_0 = 0$ almost everywhere on $\Omega\backslash\{x : V(x) = 0\}$. Moreover, in this case, ψ_0 can be chosen as a cluster point of (ψ_h) for the weak topology of $W^{1,p}(\Omega)$. In fact, there exists a subsequence (ψ_{h_n}) of (ψ_h) which tends strongly in $W^{1,p}(\Omega)$ to ψ_0,

$$\lim_{n\to 0} h_n^{-p} \int_\Omega V(x)|\psi_{h_n}|^p \,\mathrm{d}x = 0, \tag{1.15}$$

the infimum $\lambda_{\{x:V(x)=0\}}$ is reached by ψ_0 and

$$\lim_{h\to 0} \lambda_1(h) = \lambda_{\{x:V(x)=0\}}. \tag{1.16}$$

We set V_0 the interior of $\{x : V(x) = 0\}$. $N < p$ implies that the injection of $W^{1,p}(\Omega)$ into the space of continuous functions on $\overline{\Omega}$ is continuous if $\partial\Omega$ is regular. The function ψ_0 in Definition 3.9 is taken such that it vanishes identically on $\Omega\backslash E$ and so on $\overline{\Omega}\backslash \overset{\circ}{K}$. When $N < p$, Theorem 3.13 remains true but with λ_{V_0} instead of $\lambda_{\{x:V(x)=0\}}$. We apply this result to the characteristic functions of the complementary of modified Cantor sets. Returning to our original problem, when the set V_0 is not connected, we get a refinement. We set

$$V_0 = \bigcup_{i=1}^{n_0} E_i,$$

where E_i are the connected components of V_0. We recall that if ω is a bounded domain of $\mathbb{R}^N$, $\lambda_{1,\omega}$ is the first eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\omega)$ for the Dirichlet boundary condition and φ_ω a first eigenfunction associated to $\lambda_{1,\omega}$.

COROLLARY 3.22. *Suppose that Ω is a connected open set of $\mathbb{R}^N$, $N < p$, V a measurable function on Ω which satisfies $W^{1,p,V}(\Omega) \neq \{0\}$, $\operatorname{ess\,inf}_\Omega V = 0$ and (1.14). Let $(\psi_h)_{h>0}$ be any family of nonnegative functions of $W^{1,p,V}(\Omega)$ which satisfies $\|\psi_h\|_{L^p(\Omega)} = 1$ and*

$$\lim_{h\to 0} F_{h^{-p}V}(\psi_h) - \lambda_1(h) = 0. \tag{1.17}$$

We assume that $h \mapsto \lambda_1(h)$ is bounded. Let $n_1 \in [1, +\infty)$ be an integer with $n_1 \leqslant n_0$ such that

$$\lambda_{1,E_1} = \lambda_{1,E_2} = \cdots = \lambda_{1,E_{n_1}}, \tag{1.18}$$

and for all integer i in $[n_1 + 1, n_0]$ (if they exist),

$$\lambda_{1,E_1} < \lambda_{1,E_i}. \tag{1.19}$$

We have the following alternative:

(1) *If $n_1 = 1$ then (ψ_h) tends to a first nonnegative eigenfunction of E_1 in the strong topology of $W^{1,p}(\Omega)$.*

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(2) *If* $n_1 \geqslant 2$ *and if* ψ_0 *is a cluster point of* (ψ_h) *for the weak topology of* $W^{1,p}(\Omega)$ *then there exist a sequence* (h_n) *of positive real numbers which tends to zero and* n_1 *nonnegative real numbers* α_i *such that* $\sum_{i=1}^{n_1} \alpha_i = 1$ *and*

$$\psi_{h_n} \to \psi_0 = \sum_{i=1}^{n_1} \alpha_i \varphi_{E_i}, \tag{1.20}$$

strongly in $W^{1,p}(\Omega)$ *where* φ_{E_i} *is a first nonnegative eigenfunction of* $-\Delta_p$ *in* $W_0^{1,p}(E_i)$ *for the Dirichlet boundary conditions with* $\|\varphi_{E_i}\|_{L^p(E_i)} = 1$. *In others words, all the cluster points of* (ψ_h) *are linear combinaisons of the* n_1 *eigenfunctions* φ_{E_i}, $1 \leqslant i \leqslant n_1$.

With the previous notations, suppose that $n_0 > 1$. If $n_1 < n_0$ then the cluster point of (ψ_h) vanishes on a set bigger than the set $\{x : V(x) > 0\}$. More precisely, if we set

$$\tilde{V}(x) = V(x) \text{ on } \{x : V(x) > 0\}, \quad \tilde{V}(x) = 1 \text{ on } E_i, i > n_1, \tag{1.21}$$

then $\tilde{\lambda}_1(h) - \lambda_1(h) = o(1)$ since they have the same limit where $\tilde{\lambda}_1(h)$ is associated to $\tilde{V}$. Open connected subsets of $\{x : V(x) = 0\}$ which have a large first eigenvalue do not intervene in the limit. In a same way, if we denote by $\overline{V}$ the function defined by

$$\overline{V}(x) = V(x) \text{ on } \{x : V(x) > 0\}, \quad \overline{V}(x) = 1 \text{ on } E_i, i > 1, \tag{1.22}$$

then $\overline{\lambda_1}(h) - \lambda_1(h) = o(1)$ for the same reason. We have chosen E_1, the vanishing set of $\overline{V}$ but any others open sets E_i, $1 \leqslant i \leqslant n_1$ matches. $\lambda_1(h)$ behaves, up to a small term, as if V vanishes only on E_1. In this case, there is a lack of uniqueness for the infimum $\lambda_{\{x:V(x)=0\}}$. In [7], the author proves that the difference between $\overline{\lambda_1}(h)$ and $\lambda_1(h)$ is exponentially small and that the phenomenon appears for a large range of situations, for instance, if the potential is continuous and vanishes in two points. It seems that the tunneling effect may appear in that particular case.

We end this paper with a theorem of existence for a minimizer for $\lambda_1(1)$ which has some analogy with the semi-classical limit.

In the sequel, we need the definition of a well. [2]

DEFINITION 1.1. V has a well in U if U is a $\mathcal{C}^1$ bounded, connected, nonempty set of Ω and if there exists ψ_0 in $W^{1,p,V}(\Omega)$ with $\|\psi_0\|_{L^p(\Omega)} = 1$ such that

$$\int_\Omega V(x)|\psi_0|^p \, \mathrm{d}x < \operatorname{essinf}_{\Omega \setminus U} V, \tag{1.23}$$

with $\operatorname{meas}(\Omega \setminus U) > 0$.

It is an extension of the notion used in [7]. For example, if $\{x : V(x) < a\}$ is a $\mathcal{C}^1$ bounded, connected, open nonempty subset of Ω, then V has a well in it.

We need also the following formulae.

THEOREM A. [2] *Assume one of the three following assumptions*:

$$\Omega = \mathbb{R}^N \quad \text{and} \quad p < N, \tag{1.24}$$

or

$$\Omega \text{ is a } \mathcal{C}^1 \text{ bounded connected open set}, \tag{1.25}$$

or

$$\Omega \text{ is a connected open set and } V \text{ has a well}, \tag{1.26}$$

with $\operatorname{essinf}_\Omega V = \sigma > -\infty$. *Then for* h *small enough*,

$$(\lambda_1(h) - h^{-p}\sigma)(\text{meas}\{x \in \Omega : V(x) \leqslant h^p \lambda_1(h)\})^{\frac{1}{\gamma}} \geqslant C, \tag{1.27}$$

where γ *holds*

$$\begin{cases} \gamma = \dfrac{N}{p} & \text{for } 1 < p < N, \\ \gamma \in (1, +\infty) & \text{for } p = N, \\ \gamma = 1 & \text{for } p > N, \end{cases} \tag{1.28}$$

and $C = C(p, N, \Omega, \gamma)$ *is a positive constant.*

When $p = 2$, we use a consequence of the Lieb-Thirring formula (3–16 in [3]) with the help of [7].

THEOREM B. *Assume that* Ω *is a smooth domain and* V *a locally bounded measurable function on* Ω *with* $\operatorname{essinf}_\Omega V = \sigma$ *and* $\lim_{|x|\to\infty} V(x) = +\infty$ *if* Ω *is not bounded. Then for* h *small enough*,

$$(\lambda_1(h) - h^{-2}\sigma)(\text{meas}\{x \in \Omega : V(x) \leqslant K h^2 \lambda_1(h)\})^{\frac{2}{N}} \geqslant C, \tag{1.29}$$

where $C = C(N, \Omega)$ *and* K *are positive constants.*

This last formula is sharper for $N = 1$ and $N = 2$, but unfortunately available only for $p = 2$.

2. Upper and Lower Bounds for the First Eigenvalue

2.1. ROUGH RESULTS

For the sake of simplicity, in this subsection, we assume that V is bounded. Later, we shall study more general potentials. Let us take $p > 1$. To begin with, we need a proposition about the behaviour of $h^p\lambda_1(h)$. We recall that $\operatorname{essinf}_\Omega V = \sigma > -\infty$.

PROPOSITION 2.1. *Assume that* Ω *is a domain of* $\mathbb{R}^N$. *If* V *is bounded, then*

$$\lim_{h\to 0} h^p \lambda_1(h) = \sigma. \tag{2.1}$$

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Proof. On one hand, by definition, it is obvious that $\lambda_1(h) \geqslant h^{-p}\sigma$. On the other hand, we pick up some bounded measurable set E of positive measure. We denote by χ_E its characteristic function. By density, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $W^{1,p}(\Omega)$ such that f_n goes to χ_E in $L^p(\Omega)$ strongly. Hence, $\lim_{n\to\infty} \int_\Omega |f_n|^p \,\mathrm{d}x = \mathrm{meas}(E)$. So without loss of generality, we assume that $\int_\Omega |f_n|^p \,\mathrm{d}x$ is always positive for all $n \geqslant 0$.

Step 1: We shall prove that $\limsup_{h\to 0} h^p \lambda_1(h) \leqslant (\int_E V(x)\,\mathrm{d}x/\mathrm{meas}(E))$. Let ε be a positive real. Using f_n in the definition $\lambda_1(h)$ leads to

$$\lambda_1(h) \leqslant \frac{\int_\Omega |\nabla f_n|^p \,\mathrm{d}x}{\int_\Omega |f_n|^p \,\mathrm{d}x} + h^{-p}\frac{\int_\Omega V(x)\,|f_n|^p \,\mathrm{d}x}{\int_\Omega |f_n|^p \,\mathrm{d}x}. \tag{2.2}$$

V is bounded so $\int_\Omega V(x)\,|f_n|^p\,\mathrm{d}x$ goes to $\int_E V(x)\,\mathrm{d}x$. Let n be a positive integer such that

$$\frac{\int_\Omega V(x)\,|f_n|^p \,\mathrm{d}x}{\int_\Omega |f_n|^p \,\mathrm{d}x} \leqslant \frac{\int_E V(x)\,\mathrm{d}x}{\mathrm{meas}(E)} + \varepsilon. \tag{2.3}$$

For this peculiar n,

$$h^p\lambda_1(h) \leqslant h^p \frac{\int_\Omega |\nabla f_n|^p \,\mathrm{d}x}{\int_\Omega |f_n|^p \,\mathrm{d}x} + \frac{\int_E V(x)\,\mathrm{d}x}{\mathrm{meas}(E)} + \varepsilon. \tag{2.4}$$

If h is small enough,

$$h^p\lambda_1(h) \leqslant \frac{\int_E V(x)\,\mathrm{d}x}{\mathrm{meas}(E)} + 2\varepsilon, \tag{2.5}$$

which completes the first step.
Step 2: We use the definition of σ. Let z be positive real number with $z > \sigma$. There exists a bounded measurable set E of positive measure included in $\{x : V(x) \leqslant z\}$. By the first step,

$$\limsup_{h\to 0} h^p\lambda_1(h) - \sigma \leqslant \frac{\int_E (V(x)-\sigma)\,\mathrm{d}x}{\mathrm{meas}(E)} \leqslant z - \sigma, \tag{2.6}$$

which ends the proof since $z - \sigma$ is arbitrary small. □

Remark 2.2. This proposition has already been proved in [3] for $p = 2$ and V locally bounded but the proof was extremely different. We shall see in the next section a generalization.

Remark 2.3. The assumption $V \in L^\infty$ allows us to pass to the limit in $\int_\Omega V(x)\,|f_n|^p\,\mathrm{d}x$ in the first step. This assumption does not seem natural in semi-classical analysis but if we suppose only that $\mathrm{essinf}_\Omega V > -\infty$ then the result may be false. See Example 3.5.

Until the end of this section, we always suppose that V is a bounded measurable function on Ω with $\mathrm{essinf}_\Omega V = 0$. The next proposition is of great interest for the

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behaviour of a first eigenfunction. Indeed, we try to give a meaning to the sentence ‘The first eigenfunction concentrates near the infimum of the potential when h goes to zero’. See the last section for a sufficient condition of the existence of a minimizer.

PROPOSITION 2.4. *Assume that V is a measurable and bounded function on Ω with* $\operatorname{essinf}_\Omega V = 0$ *and that a first eigenfunction u_h exists for h small enough with $\|u_h\|_{L^p(\Omega)} = 1$. Then for all $\varepsilon > 0$,*

$$\int_{\{x:V(x)\geqslant\varepsilon\}} |u_h|^p \,\mathrm{d}x \leqslant \frac{h^p \lambda_1(h)}{\varepsilon}, \tag{2.7}$$

and

$$\lim_{h\to 0} \int_{\{x:V(x)\geqslant\varepsilon\}} |u_h|^p \,\mathrm{d}x = 0. \tag{2.8}$$

Proof. We have

$$\begin{aligned} h^p\lambda_1(h) &= h^p \int_\Omega |\nabla u_h|^p \,\mathrm{d}x + \int_\Omega V(x)|u_h|^p \,\mathrm{d}x \geqslant \int_\Omega V(x)|u_h|^p \,\mathrm{d}x \\ &\geqslant \int_{\{x:V(x)\geqslant\varepsilon\}} V(x)|u_h|^p \,\mathrm{d}x \geqslant \varepsilon \int_{\{x:V(x)\geqslant\varepsilon\}} |u_h|^p \,\mathrm{d}x. \end{aligned} \tag{2.9}$$

We deduce (2.8) from Proposition 2.1. □

We show an upper bound for $\lambda_1(h)$, very useful in the sequel.

THEOREM 2.5. *If V is a measurable function locally bounded on Ω then for all positive h,*

$$\lambda_1(h) \leqslant \inf_{\omega\subset\Omega} (\lambda_{1,\omega} + h^{-p}\|V\|_{L^\infty(\omega)}), \tag{2.10}$$

where ω is an open connected bounded set of Ω.

Proof. Let φ_ω be a first eigenfunction of $-\Delta_p$ in $W_0^{1,p}(\omega)$ where ω is an open connected subset of Ω. φ_ω extended by 0 outside of ω is used in the definition of $\lambda_1(h)$.

$$\lambda_1(h) \int_\omega |\varphi_\omega|^p \mathrm{d}x \leqslant \int_\omega (|\nabla\varphi_\omega|^p + h^{-p} V(x)|\varphi_\omega|^p) \,\mathrm{d}x. \tag{2.11}$$

On one hand, $\int_\omega |\nabla\varphi_\omega|^p \,\mathrm{d}x = \lambda_{1,\omega} \int_\omega |\varphi_\omega|^p \,\mathrm{d}x$. On the other hand,

$$\int_\omega h^{-p} V(x)|\varphi_\omega|^p \,\mathrm{d}x \leqslant h^{-p}\|V\|_{L^\infty(\omega)} \int_\omega |\varphi_\omega|^p \,\mathrm{d}x, \tag{2.12}$$

which finishes the proof. □

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2.2. AN UPPER BOUND FOR $\lambda_1(h)$

An upper bound is estimated under a new hypothesis.

DEFINITION 2.6. We said that V has an *almost nonempty interior* if by changing V on a zero measure set, the interior of $\{x : V(x) \leqslant z\}$ is not empty for all positive z.

Changing V on a zero measure set does not change the value of $\lambda_1(h)$ and is important in the following example.

EXAMPLE 2.7. *$\Omega = \mathbb{R}$ and $V = \chi_{\mathbb{Q}}$ where $\chi_{\mathbb{Q}}$ is the characteristic function of $\mathbb{Q}$.*

One can ask whether there exists a function V which does not have an almost nonempty interior. The next example shows that such a V does exists.

EXAMPLE 2.8. *Set $\Omega = (0, 1)$ and $V = \chi_{\Omega \backslash K}$ where K satisfies the following property*:

$$\operatorname{meas}(K) > 0, \quad \overline{K} = K, \quad \text{and} \quad \mathring{K} = \emptyset. \tag{2.13}$$

Such K can be constructed as modified Cantor sets. Moreover, the measure of K is not equal to the measure of Ω. If it is, then $\Omega \backslash K$ is a nonempty open set of zero measure which is a contradiction. The lemma below is the heart of the construction.

LEMMA 2.9. *Let Ω be an open set of $\mathbb{R}^N$. If K satisfies* (2.13) *then for all measurable set E of zero measure, $K \cup E$ has an empty interior.*

Proof. Indeed, if not, let U be an open set of $K \mathring{\cup} E$. We consider its measure. On one hand,

$$\begin{aligned} \operatorname{meas}(U) &= \operatorname{meas}(U \cap (K \cup E)) \\ &= \operatorname{meas}((U \cap K) \cup (U \cap E)) = \operatorname{meas}(U \cap K), \end{aligned} \tag{2.14}$$

since E is of zero measure. On the other hand,

$$U = (U \cap K) \cup (U \cap (\Omega \backslash K)). \tag{2.15}$$

Therefore, $\operatorname{meas}(U \cap (\Omega \backslash K)) = 0$. But $U \cap (\Omega \backslash K)$ is an open set. Hence, $U \cap (\Omega \backslash K)$ is empty. U is an open subset of K. This leads to a contradiction since $\mathring{K} = \emptyset$. □

We end the example with the next remark: for all $0 < z < 1$, $\{x : V(x) \leqslant z\} = K$.

Now, we find an upper bound for $\lambda_1(h)$ by choosing suitable open bounded sets in Theorem 2.5. Suppose that V has an almost nonempty interior. Let D_z be the interior of $\{x : V(x) \leqslant z\}$. D_z is an open set so it has a finite number or a countable number of connected components by density of $\mathbb{Q}^N$ in $\mathbb{R}^N$. Let $n(z)$ in $\mathbb{N}^* \cup \{\infty\}$ be the number of connected component(s) of D_z and $D_{z,i}$ the connected component for $i = 1, \dots, n(z)$. So

$$D_z = \bigcup_{i=1}^{n(z)} D_{z,i}. \tag{2.16}$$

Let $\varphi_{z,i}$ be a first eigenfunction of $-\Delta_p$ on $D_{z,i}$ subject to the Dirichlet realization extended by zero outside of $D_{z,i}$ with $\|\varphi_{z,i}\|_{L^p(D_{z,i})} = 1$ and $\lambda_{1,z,i}$ its related eigenvalue. For $n \in \mathbb{N}^*$, we set

$$\mathcal{S}_{n,z} = \left\{ (a_i)_{i \in \mathbb{N}^*} \middle| \forall j > n,\ a_j = 0 \text{ and } \sum_{i=1}^{\infty} a_i^p = 1 \right\}, \tag{2.17}$$

where $(a_i)_{i \in \mathbb{N}^*}$ are sequences of nonnegative real numbers and

$$\mathcal{S}_{\infty,z} = \left\{ (a_i)_{i \in \mathbb{N}^*} \middle| \sum_{i=1}^{\infty} a_i^p = 1 \text{ and } \sum_{i=1}^{\infty} a_i^p \lambda_{1,z,i} < +\infty \right\}. \tag{2.18}$$

For example, $\mathcal{S}_{1,z} = \{a\}$ with $a = (1, 0, 0, \dots)$. Now, we define $\zeta_{z,a}$ by

$$\zeta_{z,a} = \sum_{i=0}^{n(z)} a_i\ \varphi_{i,z}, \tag{2.19}$$

where $a = (a_i)_{1 \leqslant i \leqslant n(z)}$ belongs to $\mathcal{S}_{n(z),z}$. By definition, connected components are disjoint and $\sum_{i=1}^{\infty} a_i^p = 1$, this leads to the following lemma.

LEMMA 2.10. *If V has an almost nonempty interior then $\zeta_{z,a}$ belongs to $W_0^{1,p}(\Omega)$ for z small enough and for all a in $\mathcal{S}_{n(z),z}$. Moreover, $\|\zeta_{z,a}\|_{L^p(D_z)} = 1$.*

The fact that $\sum_{i=1}^{\infty} a_i^p \lambda_{1,z,i} < +\infty$ implies that the gradient of $\zeta_{z,a}$ belongs to $L^p(\Omega)$ when D_z has an infinite number of connected components. The proof of the next theorem is straightforward.

THEOREM 2.11. *Assume that V has an almost nonempty interior. Then, with the above notations,*

$$\lambda_1(h) \leqslant \inf_{z>0} \inf_{a \in \mathcal{S}_{n(z),z}} \sum_{i=1}^{n(z)} a_i^p \left(\lambda_{1,z,i} + h^{-p} \|V\|_{L^\infty(D_{z,i})} \right). \tag{2.20}$$

We give a corollary based on the inequality $\|V\|_{L^\infty(D_{z,i})} \leqslant \|V\|_{L^\infty(D_z)} = z$, by taking $a = (1, 0, 0, \dots 0, 0)$ in the above theorem.

COROLLARY 2.12. *Assume that V has an almost nonempty interior. Then, for all h positive small enough,*

$$\lambda_1(h) \leqslant \inf_{z>0} (\lambda_{1,z,1} + h^{-p} z). \tag{2.21}$$

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The best choice is when $\lambda_{1,z,1} = \inf_{1 \leqslant i \leqslant n(z)} \lambda_{1,z,i}$. So it remains to prove that this infimum is a minimum if $n(z) = \infty$.

PROPOSITION 2.13. *Let ω be an open bounded set of $\mathbb{R}^N$. We set*

$$\omega = \bigcup_{i=1}^{n_0} \omega_i, \tag{2.22}$$

where ω_i are the connected components of ω. Then, there exists i_0 an integer such that

$$\lambda_{1,\omega_{i_0}} = \inf_{1 \leqslant i \leqslant n_0} \lambda_{1,\omega_i}. \tag{2.23}$$

Proof. Suppose that $n_0 = +\infty$.

(1) First step: $\lim_{i \to +\infty} \text{meas}(\omega_i) = 0$.
Indeed, if not, there exists a subsequence (ω_{i_k}) and $\varepsilon > 0$ such that for all k,

$$\text{meas}(\omega_{i_k}) \geqslant \varepsilon. \tag{2.24}$$

Hence,

$$\text{meas}(\omega) \geqslant \sum_k \text{meas}(\omega_{i_k}) = +\infty, \tag{2.25}$$

which is impossible since ω is bounded.

(2) Second step: we use a theorem proved by symetrisation in [10].

THEOREM 2.14. *Let Ω be a bounded domain of $\mathbb{R}^N$ and B be an open ball such that meas$(\Omega) =$ meas(B). Then*

$$\lambda_{1,\Omega} \geqslant \lambda_{1,B}. \tag{2.26}$$

Moreover, by a change of variables,

$$\lim_{\text{meas}(B) \to 0} \lambda_{1,B} = +\infty. \tag{2.27}$$

Finally, we deduce that

$$\lim_{i \to +\infty} \lambda_{1,\omega_i} = +\infty, \tag{2.28}$$

and so there exists only a finite number of connected components such that

$$\lambda_{1,\omega_i} \leqslant \lambda_{1,\omega_0}, \tag{2.29}$$

which ends the proof. □

Remark 2.15. The assumption 'ω is bounded' can be replaced by 'ω has a finite measure and all its connected components are bounded'. If $\operatorname{meas}(\omega) < +\infty$ is not satisfied, then Proposition 2.13 may be false. Take for instance a sequence of open balls (B_i) with a radius $r_i \to +\infty$ such that for all $i \neq j$, $B_i \cap B_j = \emptyset$. We get

$$\lim_{i \to +\infty} \lambda_{1,B_i} = 0, \tag{2.30}$$

and the infimum is not reached.

The main problem is to compute the first eigenvalue for $-\Delta_p$. In one dimension, up to a translation, all the intervals are homothetic. Hence, $\lambda_{1,I} = (\lambda_{1,(0,1)}/l(I)^p)$ where $l(I)$ is the length of an open interval I and $\lambda_{1,(0,1)}$ the first eigenvalue in $W_0^{1,p}(0,1)$ for $-\Delta_p$ on $(0,1)$ for the Dirichlet boundary condition. When $p = 2$, $\lambda_{1,(0,1)} = \pi^2$ and a first eigenfunction is $x \mapsto \sin \pi x$ in $W_0^{1,2}(0,1)$.

EXAMPLE 2.16. We take $\Omega = (0, \frac{2}{\pi})$ and

$$V(x) = \left| \sin \frac{1}{x} \right|^{\beta}, \tag{2.31}$$

where β is a positive real number. The choice of $\frac{2}{\pi}$ for the upper bound of Ω gives a simple expression for D_z. So

$$D_z = \bigcup_{i=1}^{\infty} \left(\frac{1}{i\pi + \arcsin z^{\frac{1}{\beta}}}, \frac{1}{i\pi - \arcsin z^{\frac{1}{\beta}}} \right), \tag{2.32}$$

for $0 < z < 1$. Therefore,

$$\begin{aligned} \lambda_{1,z,i} &= \frac{\lambda_{1,(0,1)}}{\left(\frac{1}{i\pi - \arcsin z^{\frac{1}{\beta}}} - \frac{1}{i\pi + \arcsin z^{\frac{1}{\beta}}} \right)^p} \\ &= \frac{\lambda_{1,(0,1)} \left(i^2\pi^2 - \left(\arcsin z^{\frac{1}{\beta}} \right)^2 \right)^p}{2^p \left(\arcsin z^{\frac{1}{\beta}} \right)^p}. \end{aligned} \tag{2.33}$$

Applying Corollary 2.12 gives an upper bound for $\lambda_1(h)$:

$$\begin{aligned} \lambda_1(h) &\leqslant \inf_{0<z<1} \frac{\lambda_{1,(0,1)} \left(\pi^2 - \left(\arcsin z^{\frac{1}{\beta}} \right)^2 \right)^p}{2^p \left(\arcsin z^{\frac{1}{\beta}} \right)^p} + h^{-p} z \\ &\leqslant \inf_{0<z<1} \frac{\lambda_{1,(0,1)} \pi^{2p}}{2^p \left(\arcsin z^{\frac{1}{\beta}} \right)^p} + h^{-p} z. \end{aligned} \tag{2.34}$$

Finally,

$$\lambda_1(h) \leqslant \inf_{0<z<1} \frac{\lambda_{1,(0,1)} \pi^{2p}}{2^p z^{\frac{p}{\beta}}} + h^{-p} z. \tag{2.35}$$

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The minimum is reached when h^p is proportionnal to $z^{1+p/\beta}$. We obtain the following estimate.

$$\lambda_1(h) \leqslant C\ h^{\frac{-p^2}{\beta+p}}, \tag{2.36}$$

where C is a positive constant.

The first eigenvalue may be estimated in some situations.

LEMMA 2.17. *Let ω be an open connected bounded subset of $\mathbb{R}^N$, T a translation of $\mathbb{R}^N$ and H_r an homothety of $\mathbb{R}^N$ of positive ratio r. Then*

$$\lambda_{1,T(H_r(\omega))} = r^{-p}\lambda_{1,\omega}. \tag{2.37}$$

Proof. It is clear that $\lambda_{1,T(H_r(\omega))} = \lambda_{1,H_r(\omega)}$. Without loss of generality, we assume that 0 is the center of H_r. H_r is defined by $H_r(x_1, \dots, x_N) = (rx_1, \dots, rx_N)$. Let ψ be a function in $W_0^{1,p}(\omega)$. By definition,

$$\lambda_{1,\omega} \int_\omega |\psi(x)|^p \,\mathrm{d}x \leqslant \int_\omega |\nabla\psi(x)|^p \,\mathrm{d}x. \tag{2.38}$$

We make the change of variables $y_1 = rx_1, \dots, y_N = rx_N$.

$$\lambda_{1,\omega} r^N \int_{H_r(\omega)} |\psi(y/r)|^p \,\mathrm{d}y \leqslant r^N r^p \int_{H_r(\omega)} |\nabla\psi(y/r)|^p \,\mathrm{d}y. \tag{2.39}$$

By taking the infimum, we have $\lambda_{1,\omega} \leqslant r^p \lambda_{1,H_r(\omega)}$. For the other inequality, we consider the homothety of ratio $1/r$. ☐

The interest of the lemma is shown in the following proposition.

PROPOSITION 2.18. *Let Ω be an open nonempty connected set of $\mathbb{R}^N$ and V a bounded measurable function on Ω with* $\operatorname{essinf}_\Omega V = 0$.

(1) *V has an almost nonempty interior if and only if there exists ω an open bounded nonempty connected set of $\mathbb{R}^N$ such that for all $z > 0$, there exist a translation T and an homothety H_r of ratio r such that*, *up to a zero measure set*,

$$T(H_r(\omega)) \subset \{x \in \Omega : V(x) \leqslant z\}. \tag{2.40}$$

In that case, *we set $r(z, \omega)$ the supremum of all positive r such that there exist a translation T and a homothety H_r of ratio r which satisfies* (2.40).

(2) *If the property is satisfied for an open bounded nonempty connected set ω, it remains valid for all the open bounded nonempty connected sets of $\mathbb{R}^N$. Moreover if ω_1 and ω_2 are such open sets, there exist two positive real constants C_1 and C_2 such that*

$$C_1\, r(z, \omega_1) \leqslant r(z, \omega_2) \leqslant C_2\, r(z, \omega_1), \tag{2.41}$$

for all $z > 0$.

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Proof. (1) Let $z > 0$. There exist x_0 and $r_0 > 0$ such that the open ball $B_{r_0}(x_0) \subset \{x : V(x) \leqslant z\}$. ω is a bounded set so there exists a ball B such that $\omega \subset B$. There exist a translation T and an homothety H_r with $r > 0$ such that $T(H_r(B)) = B_{r_0}(x_0)$. As a consequence, $T(H_r(\omega)) \subset \{x : V(x) \leqslant z\}$.

By similar arguments, the proof of (2) is straightforward. □

Remark 2.19. It is possible to consider others transformations such that rotations and symetries. More generally, orthogonal transformations do not change the value of the first eigenvalue of $-\Delta_p$. To simplify, we restrict our attention to translations and homotheties.

Remark 2.20. From a theoretical point of view, any bounded domain ω suits but for computations, it is often more practical to take a ball. On the opposite side, if the level sets of V are homothetic, the choice for ω is natural.

When V is radial and ω a ball, we get a simple expression for $r(z, \omega)$.

EXAMPLE 2.21. Let us take $\Omega = \mathbb{R}^N$ and V is a radial increasing continuous function with $V(0) = 0$. $B_1(O)$, the unit-ball of center O, is adapted to V. Indeed, $r(z, B_1(O)) = V^{-1}(z)$. Any other connected bounded nonempty open set ω can be taken but the function $r(z, \omega)$ is slightly different and may be not easy to calculate.

We notice that if the set $\{x : V(x) \leqslant z\}$ is not bounded, $r(z, \omega)$ may be infinite. The next theorem shows that the definition of almost nonempty interior appears in a natural way.

THEOREM 2.22. *Upper bound for* $\lambda_1(h)$.

Let Ω be an open nonempty connected set of $\mathbb{R}^N$ and V a bounded measurable function on Ω with $\operatorname{essinf}_\Omega V = 0$. *We assume that V has an almost nonempty interior. Then, for any open bounded nonempty connected set ω of $\mathbb{R}^N$, for any function $\varepsilon : h \mapsto \varepsilon(h)$ with values in* $(0, 1)$ *and for any $h > 0$, there holds*

$$\lambda_1(h)(r(h^p\lambda_1(h)(1 - \varepsilon(h)), \omega))^p \leqslant \frac{\lambda_{1,\omega}}{\varepsilon(h)}. \tag{2.42}$$

Proof. Let $z > 0$. V has an almost nonempty interior so there exist a translation T and an homothety H_r of ratio r such that

$$T(H_r(\omega)) \subset \{x \in \Omega : V(x) \leqslant z\}. \tag{2.43}$$

Let $\varphi_{T(H_r(\omega))}$ be a first eigenfunction for $-\Delta_p$ in $W_0^{1,p}(T(H_r(\omega)))$ extended by zero outside $T(H_r(\omega))$ with $\|\varphi_{T(H_r(\omega))}\|_{L^p(\Omega)} = 1$. The definition of $\lambda_1(h)$ gives

$$\lambda_1(h) \leqslant \int_{T(H_r(\omega))} |\nabla\varphi_{T(H_r(\omega))}|^p \,\mathrm{d}x + \int_{T(H_r(\omega))} h^{-p}V(x)|\varphi_{T(H_r(\omega))}|^p \,\mathrm{d}x. \tag{2.44}$$

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So,

$$\lambda_1(h) \leqslant \lambda_{1,T(H_r(\omega))} + h^{-p}\|V\|_{L^\infty(T(H_r(\omega)))}. \tag{2.45}$$

Hence,

$$\lambda_1(h) \leqslant r^{-p}\lambda_{1,\omega} + h^{-p}z. \tag{2.46}$$

We take the supremum with respect to r with the help of the definition of $r(z,\omega)$.

$$\lambda_1(h) \leqslant (r(z,\omega))^{-p}\lambda_{1,\omega} + h^{-p}z. \tag{2.47}$$

We take $z = h^p\lambda_1(h)(1-\varepsilon(h))$ which ends the proof. □

Let us denote by $\rho(z,\omega)$ the following function

$$\rho(.,\omega) : z \mapsto z(r(z,\omega))^p. \tag{2.48}$$

By definition, $r(.,\omega)$ is monotonous nondecreasing function on $(0,\infty)$ so $\rho(.,\omega)$ is an increasing function on $(0,\infty)$.

COROLLARY 2.23. *Explicit upper bound for* $\lambda_1(h)$.

Let Ω *be an open nonempty connected set of* $\mathbb{R}^N$ *and* V *a bounded measurable function on* Ω *with* $\operatorname{essinf}_\Omega V = 0$. *We assume that* V *has an almost nonempty interior and that* $\rho(.,\omega)$ *is continuous on* $(0,\infty)$. *Then, for any open bounded nonempty connected set* ω *of* $\mathbb{R}^N$, *for any function* $\varepsilon : h \mapsto \varepsilon(h)$ *with values in* $(0,1)$ *and for any* $h > 0$, *there holds*

$$\lambda_1(h) \leqslant \frac{1}{1-\varepsilon(h)} h^{-p}\rho^{-1}\left(\frac{h^p\lambda_{1,\omega}(1-\varepsilon(h))}{\varepsilon(h)}, \omega\right). \tag{2.49}$$

Proof. We start with (2.42).

$$\lambda_1(h)(r(h^p\lambda_1(h)(1-\varepsilon(h)),\omega))^p \leqslant \frac{\lambda_{1,\omega}}{\varepsilon(h)}. \tag{2.50}$$

We multiply by $h^p(1-\varepsilon(h))$. It appears $\rho(h^p\lambda_1(h)(1-\varepsilon(h)),\omega)$ in the left-hand side. Since $\rho(.,\omega)$ is an increasing continuous function, we obtain the result. □

Remark 2.24. Equation (2.49) in Corollary 2.23 can be written under the form

$$\lambda_1(h) \leqslant \inf_{\omega,\varepsilon(.)} \frac{1}{1-\varepsilon(h)} h^{-p}\rho^{-1}\left(\frac{h^p\lambda_{1,\omega}(1-\varepsilon(h))}{\varepsilon(h)}, \omega\right), \tag{2.51}$$

where the infimum is taken for all $(\omega, \varepsilon(.))$ such that ω is an open bounded nonempty connected set of $\mathbb{R}^N$ and $\varepsilon : h \mapsto \varepsilon(h)$ a function with values in $(0,1)$. Different choices of $\varepsilon(.)$ will be made in the sequel.

Remark 2.25. If ρ is not continuous (i.e. r is not continuous) on $(0,\infty)$ then we have to find a continuous and nondecreasing function $\tilde{r}$ such that $\tilde{r} \leqslant r$. We set $\tilde{\rho}(z) = z(\tilde{r}(z,\omega))^p$ and Corollary 2.23 remains valid but for $\tilde{\rho}$ instead of ρ.

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2.3. A LOWER BOUND FOR $\lambda_1(h)$

To find a lower bound, we introduce the next definition.

DEFINITION 2.26. Let Ω be a domain of $\mathbb{R}^N$ and V a bounded measurable function on Ω. We say that V has a *large nonempty interior* if V has an almost nonempty interior, $\text{meas}\{x \in \Omega : V(x) \leqslant z\} < \infty$ for $z > 0$ small enough and there exists ω a connected open nonempty bounded subset of $\mathbb{R}^N$ such that

$$\liminf_{z\to 0} \frac{r(z,\omega)^N}{\text{meas}\{x \in \Omega : V(x) \leqslant z\}} > 0. \tag{2.52}$$

From the second assertion of Proposition 2.18, the definition does not depend on ω, that is, if ω' is another connected open nonempty bounded subset of $\mathbb{R}^N$ then

$$\liminf_{z\to 0} \frac{r(z,\omega')^N}{\text{meas}\{x \in \Omega : V(x) \leqslant z\}} > 0. \tag{2.53}$$

There exists functions with a large nonempty interior but which are not radial.

EXAMPLE 2.27. $\Omega = (0, \frac{2}{\pi})$, $V(x) = |\sin(1/x)|^\beta$ with $\beta > 0$. We already know that

$$\{x \in \Omega : V(x) \leqslant z\} = \bigcup_{i \geqslant 1}^{\infty} \left[\frac{1}{i\pi + \arcsin z^{\frac{1}{\beta}}} \, , \, \frac{1}{i\pi - \arcsin z^{\frac{1}{\beta}}} \right], \tag{2.54}$$

for $0 < z < 1$. So,

$$\begin{aligned} \text{meas}\{x \in \Omega : V(x) \leqslant z\} &= \sum_{i=1}^{\infty} \frac{1}{i\pi - \arcsin z^{\frac{1}{\beta}}} - \frac{1}{i\pi + \arcsin z^{\frac{1}{\beta}}} \\ &= \sum_{i=1}^{\infty} \frac{2 \arcsin z^{\frac{1}{\beta}}}{i^2\pi^2 - \left(\arcsin z^{\frac{1}{\beta}}\right)^2} \leqslant C_1 \arcsin z^{\frac{1}{\beta}} \leqslant C_2 \, z^{\frac{1}{\beta}}, \end{aligned} \tag{2.55}$$

as z goes to 0. On the other hand, $r(z)$ is the half of the length of the biggest connected component of the interior of $\{x \in \Omega : V(x) \leqslant z\}$ which gives

$$r(z) = \frac{\arcsin z^{\frac{1}{\beta}}}{\pi^2 - \left(\arcsin z^{\frac{1}{\beta}}\right)^2} \geqslant C_3 \, z^{\frac{1}{\beta}}. \tag{2.56}$$

where C_1, C_2 and C_3 are positive constants. So V has a large nonempty interior.

Remark 2.28. Assume that V is a continuous and periodic function on $\mathbb{R}$ with $\text{essinf}_\Omega V = 0$ then V does not hold this property. Indeed, for z small enough, the measure of the set $\{x : V(x) \leqslant z\}$ is not finite. On the other hand, if V is not identically equal to zero, $r(z,\omega)$ is finite.

The freedom for the choice of ω is important when for instance, V is a radial continuous increasing function which vanishes only in a point of the boundary

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of Ω. If this point satisfied ‘an interior cone condition’, then V has a large nonempty interior. Indeed, the ratio between the measure of the cone and the measure of the ball generated by the cone is constant. There exists some potentials V which have an almost nonempty interior but not a large nonempty interior.

EXAMPLE 2.29. $N = 1$, $\Omega = (0, 2)$, $V(x) = x$ on $(0, 1)$ and $V(x) = \chi_{\Omega\setminus K}$ where K is a measurable closed set of $[1, 2)$ such that $\mathring{K} = \emptyset$ and $\mathrm{meas}(K) > 0$. It is clear that for $z \in (0, 1)$,

$$\{x \in \Omega : V(x) \leqslant z\} = (0, z] \cup K.$$

So the interior of $\{x \in \Omega : V(x) \leqslant z\}$ is $(0, z) \cup \mathring{K} = (0, z)$. We take $\omega = (0, 1)$ hence $r(z) = z$. Finally,

$$\lim_{z\to 0} \frac{r(z, \omega)}{\mathrm{meas}\{x \in \Omega : V(x) \leqslant z\}} = 0,$$

since for all positive z, $\mathrm{meas}\{x \in \Omega : V(x) \leqslant z\} \geqslant \mathrm{meas}(K) > 0$.

The property of ‘large nonempty interior’ is directly linked to $\lambda_1(h)$.

THEOREM 2.30. *Lower bound for* $\lambda_1(h)$.
Assume that Ω *is an open nonempty connected open set of* $\mathbb{R}^N$ *and* V *a measurable bounded function on* Ω *with* $\mathrm{essinf}_\Omega V = 0$ *which has a large nonempty interior. Let* ω *be a connected bounded nonempty open set of* $\mathbb{R}^N$.

(1) *Assume* (1.24) *or* (1.25) *or* (1.26). *Then for* h *small enough*,

$$\lambda_1(h)(r(h^p\lambda_1(h), \omega))^{\frac{N}{\gamma}} \geqslant C, \tag{2.57}$$

where $C = C(\omega)$ *is positive constant which does not depend on* h *and* γ *satisfies* (1.28).

(2) *Assume that* Ω *is smooth and* $p = 2$. *Then for* h *small enough*,

$$\lambda_1(h)(r(Kh^2\lambda_1(h), \omega))^2 \geqslant C, \tag{2.58}$$

where $C = C(\omega)$ *is positive constant which does not depend on* h *and* K *is defined in Theorem B*.

Proof. For (1), we start with (1.27)

$$\lambda_1(h)(\mathrm{meas}\{x \in \Omega : V(x) \leqslant h^p\lambda_1(h)\})^{\frac{1}{\gamma}} \geqslant C' > 0. \tag{2.59}$$

Using the fact that V has a large nonempty interior ends the proof. We have a similar proof for (2). □

Once again, the following corollary has a straighforward proof.

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COROLLARY 2.31. *Explicit lower bound for $\lambda_1(h)$.*

Assume that Ω is an open nonempty connected open set of $\mathbb{R}^N$ and V a measurable bounded function on Ω with $\operatorname{essinf}_\Omega V = 0$ which has a large nonempty interior. Let ω be a connected bounded nonempty open set of $\mathbb{R}^N$ and assume that $\rho(., \omega)$ is continuous on $(0, \infty)$.

(1) *Under (1.24) or (1.25) or (1.26), if $N > p$ then there exists a positive constant C such that for h small enough,*

$$\lambda_1(h) \geqslant h^{-p} \rho^{-1}(C h^p, \omega). \tag{2.60}$$

(2) *If $p = 2$ and $N \geqslant 1$, then there exists a constant $C = C(\omega)$ such that for h small enough,*

$$\lambda_1(h) \geqslant \frac{h^{-2}}{K} \rho^{-1}(C h^2, \omega), \tag{2.61}$$

where K is defined in Theorem B.

2.4. ASYMPTOTIC ESTIMATES FOR $\lambda_1(h)$

Lower bounds and upper bounds have similar expressions. So we derive the following corollary.

COROLLARY 2.32. *Assume that Ω is an open nonempty connected open set of $\mathbb{R}^N$ and V a measurable bounded function on Ω with $\operatorname{essinf}_\Omega V = 0$ which has a large nonempty interior. ω is a connected bounded nonempty open set of $\mathbb{R}^N$ and assume that $\rho(., \omega)$ is continuous on $(0, \infty)$. Under (1.24) or (1.25) or (1.26), if $N > p$, then there exists $C > 0$ such that*

$$h^{-p} \rho^{-1}(C h^p, \omega) \leqslant \lambda_1(h) \leqslant 2 h^{-p} \rho^{-1}(h^p \lambda_{1,\omega}, \omega), \tag{2.62}$$

for $h > 0$ small enough.

Proof. We use Corollaries 2.23 and 2.31 with $\varepsilon(h) = 1/2$. □

We take three examples.

EXAMPLE 2.33. C_1, C_2, C_3 and C_4 are positive constants and $N > p$.

(1) Ω is the unit ball of $\mathbb{R}^N$, $V(x) = |x|^\beta$ with $\beta > 0$ and ω is the unit ball. Then $r(z) = z^{\frac{1}{\beta}}$, $\rho(z) = z^{\frac{\beta}{\beta+p}}$ and

$$C_1\, h^{\frac{p\beta}{p+\beta}} \leqslant \lambda_1(h) \leqslant C_2\, h^{\frac{p\beta}{p+\beta}}. \tag{2.63}$$

This is highly important in [2].

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(2) Ω is the unit ball of $\mathbb{R}^N$, $V(x) = \exp(-1/|x|^\beta)$ with $\beta > 0$ and ω is the unit ball. Then $r(z) = 1/(-\ln z)^{1/\beta}$ and $\rho(z) = (z)(1/(-\ln z)^{p/\beta}) = \alpha$. When $z \to 0$, we make an asymptotic development for ρ^{-1}. We get

$$C_3\ (-\ln h)^{\frac{p}{\beta}} \leqslant \lambda_1(h) \leqslant C_4\ (-\ln h)^{\frac{p}{\beta}}\ . \tag{2.64}$$

This is crucial in [3].

(3) $\Omega = (0, 2/\pi)$ and $V(x) = |\sin(1/x)|^\beta$ with $\beta > 0$. For $p = 2$, we use Corollary 2.31 coming from the Lieb-Thirring formula and we obtain

$$C_1\ h^{\frac{-4}{\beta+2}} \leqslant \lambda_1(h) \leqslant C_2\ h^{\frac{-4}{\beta+2}}. \tag{2.65}$$

For $p \neq 2$, our formulae are not enough sharp for $N = 1$.

We give below two-side estimates for $\lambda_1(h)$ as $h \to 0$.

THEOREM 2.34. *We assume the hypothesis of Corollary* 2.31 *and* $N > p$.

(1) *If for all* $K > 0$, $r(Kz, \omega) \sim r(z, \omega)$ *when* $z \to 0$ *then*

$$0 < \liminf_{h\to 0} \lambda_1(h)(r(h^p\lambda_1(h), \omega))^p \leqslant \limsup_{h\to 0} \lambda_1(h)(r(h^p\lambda_1(h), \omega))^p \leqslant \lambda_{1,\omega}. \tag{2.66}$$

(2) *If for all* $K > 0$, $\rho^{-1}(K\alpha, \omega) \sim \rho^{-1}(\alpha, \omega)$ *when* $\alpha \to 0$ *then*

$$\lambda_1(h) \sim h^{-p}\rho^{-1}(h^p, \omega), \tag{2.67}$$

when $h \to 0$.

Proof. Let $\varepsilon \in (0, 1)$ fixed and $\varepsilon(h) = \varepsilon$ for all h small enough.

(1) On one hand, for h small enough, $(r(h^p\lambda_1(h)(1-\varepsilon), \omega))^p \geqslant \varepsilon(r(h^p\lambda_1(h), \omega))^p$ since $r(z(1-\varepsilon), \omega) \sim r(z, \omega)$ when $z \to 0$. Hence, from (2.42),

$$\lambda_1(h)(r(h^p\lambda_1(h), \omega))^p \leqslant \frac{\lambda_{1,\omega}}{\varepsilon^2}. \tag{2.68}$$

On the other hand, from (2.57) with $\frac{N}{\gamma} = p$ since $N > p$,

$$\lambda_1(h)(r(h^p\lambda_1(h), \omega))^p \geqslant C. \tag{2.69}$$

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By passing to the limit, we have

$$C \leqslant \liminf_{h\to 0} \lambda_1(h)(r(h^p\lambda_1(h), \omega))^p \leqslant \limsup_{h\to 0} \lambda_1(h)(r(h^p\lambda_1(h), \omega))^p \leqslant \frac{\lambda_{1,\omega}}{\varepsilon^2}. \tag{2.70}$$

But ε is chosen arbitrary in $(0, 1)$, hence (2.66) satisfies.

(2) For h small enough,

$$\rho^{-1}\left(\frac{h^p\lambda_{1,\omega}(1-\varepsilon)}{\varepsilon}, \omega\right) \leqslant (1+\varepsilon)\rho^{-1}(h^p, \omega), \tag{2.71}$$

since for all $K > 0$, $\rho^{-1}(K\alpha, \omega) \sim \rho^{-1}(\alpha, \omega)$ when $\alpha \to 0$. As a consequence, from Corollary 2.23,

$$\lambda_1(h) \leqslant \frac{1+\varepsilon}{1-\varepsilon} h^{-p}\rho^{-1}(h^p, \omega). \tag{2.72}$$

We have also for h small enough,

$$\lambda_1(h) \geqslant h^{-p}\rho^{-1}(Ch^p, \omega) \geqslant (1-\varepsilon)h^{-p}\rho^{-1}(h^p, \omega). \tag{2.73}$$

So we derive the second assertion by letting $\varepsilon \to 0$. □

Now, we apply it when V is a radial function and vanishes in Ω. We write $V(x) = V(|x|)$ for simplicity.

COROLLARY 2.35. *Let Ω be a connected open set of $\mathbb{R}^N$, $O \in \Omega$, $N > p$, V a concave increasing radial function which is continuous on $[0, +\infty)$, $\mathcal{C}^1$ on $(0, +\infty)$ with $V(0) = 0$. Furthermore, we assume that*

$$r\, V'\big(r\, (V(r))^{\frac{1}{p}}\big) = o(V(r)), \tag{2.74}$$

when r goes to zero. Then

$$\lambda_1(h) \sim h^{-p}V(h). \tag{2.75}$$

Proof. Assumptions of Theorem 2.30 are obviously satisfied for ω the unit ball of $\mathbb{R}^N$. There are two steps. First, we shall prove that $\rho^{-1}(\alpha, \omega) \sim V(\alpha^{\frac{1}{p}})$ and then $V(Kr) \sim V(r)$ for all $K > 0$. For the sake of simplicity, we write $\rho(z)$ instead of $\rho(z, \omega)$.

First step: We start with the definition of $\rho(z)$.

$$\rho(z) = z(r(z))^p = \alpha, \tag{2.76}$$

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which gives

$$V\left(z^{\frac{1}{p}}r(z)\right) = V\left(\alpha^{\frac{1}{p}}\right). \tag{2.77}$$

Since V is a concave function, for $z \leqslant 1$,

$$0 \leqslant V(r(z)) - V\left(z^{\frac{1}{p}}r(z)\right) \leqslant r(z)V'\left(z^{\frac{1}{p}}r(z)\right). \tag{2.78}$$

But $r(z) = V^{-1}(z)$ for radial function when ω is the unit ball. Therefore,

$$z - V\left(z^{\frac{1}{p}}r(z)\right) = \mathcal{O}\left(r(z)V'\left(z^{\frac{1}{p}}r(z)\right)\right). \tag{2.79}$$

From (2.74), since r is continuous and bijective function with $r(0) = 0$,

$$r(z)\ V'\left(r(z)\ z^{\frac{1}{p}}\right) = o(z), \tag{2.80}$$

when z tends to zero which gives

$$z \sim V\left(z^{\frac{1}{p}}r(z)\right). \tag{2.81}$$

This means that

$$\rho^{-1}(\alpha) \sim V\left(\alpha^{\frac{1}{p}}\right), \tag{2.82}$$

for $z = \rho^{-1}(\alpha)$ when $\alpha \to 0$. Indeed, once again, ρ^{-1} is a continuous and bijective function with $\rho^{-1}(0) = 0$.

Second step: Let K be a positive real number. We apply for the second time the inequality of the mean value. Always by concavity,

$$|V(K\alpha) - V(\alpha)| \leqslant \alpha\ |K - 1|\ \max(V'(K\alpha), V'(\alpha)). \tag{2.83}$$

If r is positive real number close to zero, $V(r)$ is less than 1. By concavity

$$0\ \leqslant r\ V'(r) \leqslant r\ V'\left(r\ (V(r))^{\frac{1}{p}}\right) = o(V(r)). \tag{2.84}$$

Hence,

$$V(K\alpha) \sim V(\alpha). \tag{2.85}$$

The end of the proof is a straighforward consequence of Theorem 2.34. □

This corollary applies to potentials decaying ‘very slowly’ to zero.

EXAMPLE 2.36. Ω a small ball of center O, $V(r) = \frac{1}{-\ln r}$. Then

$$\lambda_1(h) \sim h^{-p}\frac{1}{-\ln h}. \tag{2.86}$$

Now, we present under new assumptions an estimate of the difference between $\lambda_1(h)$ and $h^{-p}\rho^{-1}(h^p, \omega)$.

THEOREM 2.37. *Assume that* Ω *is an open nonempty connected open set of* $\mathbb{R}^N$, $N > p$, V *a measurable bounded function on* Ω *with* $\mathrm{essinf}_\Omega V = 0$ *which has a large nonempty interior and* ω *is a connected bounded nonempty open set of*

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$\mathbb{R}^N$. *We suppose that there exists* $\alpha_0 \in (0, +\infty]$ *such that* $\rho^{-1}(., \omega)$ *is Lipschitz on* $[\alpha, \alpha_0)$ *for all* $\alpha \in (0, \alpha_0)$, *that is,*

$$\mu(\alpha, \omega) = \sup_{(x,y)\in[\alpha,\alpha_0)^2, x\neq y} \left| \frac{\rho^{-1}(x, \omega) - \rho^{-1}(y, \omega)}{x - y} \right| < +\infty. \tag{2.87}$$

(1) *If*

$$\lim_{\alpha\to 0} \frac{\alpha\mu(\alpha, \omega)}{\rho^{-1}(\alpha, \omega)} = 0, \tag{2.88}$$

then

$$0 \leqslant \lambda_1(h) - h^{-p}\rho^{-1}(Ch^p, \omega) = \mathcal{O}(\sqrt{\mu(Ch^p, \omega)h^{-p}\rho^{-1}(h^p, \omega)}), \tag{2.89}$$

when $h \to 0$ *where* C *is the positive constant of Corollary* 2.31.

(2) *Moreover, if for all* $K > 0$

$$\mu(K\alpha, \omega) \sim \mu(\alpha, \omega), \tag{2.90}$$

when $\alpha \to 0$ *then*

$$\lambda_1(h) - h^{-p}\rho^{-1}(h^p, \omega) = \mathcal{O}(\sqrt{\mu(h^p, \omega)h^{-p}\rho^{-1}(h^p, \omega)}), \tag{2.91}$$

when $h \to 0$.

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We notice that (2.88) implies that $\sqrt{\mu(h^p, \omega)h^{-p}\rho^{-1}(h^p, \omega)} = o(h^{-p}\rho^{-1}(h^p, \omega))$ when h goes to zero. For the sake of simplicity, we do not write the dependance of ω in ρ and μ. It is always possible to suppose that C, the positive constant of Corollary 2.31, satisfies $C \leqslant 1$. We begin with a proposition.

PROPOSITION 2.38. *Under the same assumptions, if* $h \mapsto \varepsilon(h)$ *is a function with values in* $(0, 1)$ *which satisfies*

$$\frac{\lambda_{1,\omega}(1 - \varepsilon(h))}{\varepsilon(h)} > C, \tag{2.92}$$

and

$$0 < h^p \frac{\lambda_{1,\omega}(1 - \varepsilon(h))}{\varepsilon(h)} < \alpha_0, \tag{2.93}$$

for h small enough, then

$$0 \leqslant \lambda_1(h) - h^{-p}\rho^{-1}(Ch^p) \leqslant \lambda_{1,\omega}\frac{\mu(Ch^p)}{\varepsilon(h)} + \frac{\varepsilon(h)}{1 - \varepsilon(h)}h^{-p}\rho^{-1}(Ch^p), \tag{2.94}$$

and

$$\begin{aligned} -(1-C)\mu(Ch^p) &\leqslant \lambda_1(h) - h^{-p}\rho^{-1}(h^p) \\ &\leqslant \lambda_{1,\omega}\frac{\mu(Ch^p)}{\varepsilon(h)} + \frac{\varepsilon(h)}{1-\varepsilon(h)} h^{-p}\rho^{-1}(Ch^p). \end{aligned} \tag{2.95}$$

Proof of the proposition: From Corollaries 2.23 and 2.31,

$$\begin{aligned} 0 &\leqslant \lambda_1(h) - h^{-p}\rho^{-1}(Ch^p) \\ &\leqslant h^{-p}\left[\frac{1}{1-\varepsilon(h)}\rho^{-1}\left(h^p\frac{\lambda_{1,\omega}(1-\varepsilon(h))}{\varepsilon(h)}\right) - \rho^{-1}(Ch^p)\right]. \end{aligned} \tag{2.96}$$

We estimate $\rho^{-1}(h^p)((\lambda_{1,\omega}(1-\varepsilon(h))/\varepsilon(h))$ with the help of $\rho^{-1}(Ch^p)$ and μ.

$$\begin{aligned} \rho^{-1}\left(h^p\frac{\lambda_{1,\omega}(1-\varepsilon(h))}{\varepsilon(h)}\right) &\leqslant \rho^{-1}(Ch^p) + h^p\left(\frac{\lambda_{1,\omega}(1-\varepsilon(h))}{\varepsilon(h)} - C\right)\mu(Ch^p) \\ &\leqslant \rho^{-1}(Ch^p) + h^p\frac{\lambda_{1,\omega}(1-\varepsilon(h))}{\varepsilon(h)}\mu(Ch^p), \end{aligned} \tag{2.97}$$

which leads to (2.94). The next assertion comes from

$$0 \leqslant \rho^{-1}(h^p) - \rho^{-1}(Ch^p) \leqslant (1-C)h^p\mu(Ch^p), \tag{2.98}$$

since $C \leqslant 1$. □

Proof of the theorem. The key-stone is the choice of $\varepsilon(h)$. We set

$$\varepsilon(h) = \sqrt{\frac{\lambda_{1,\omega}\mu(Ch^p)}{h^{-p}\rho^{-1}(Ch^p)}}. \tag{2.99}$$

We deduce that

$$\begin{aligned} 0 &\leqslant \lambda_1(h) - h^{-p}\rho^{-1}(Ch^p) \\ &\leqslant \left(1 + \frac{1}{1-\sqrt{\frac{\lambda_{1,\omega}\mu(Ch^p)}{h^{-p}\rho^{-1}(Ch^p)}}}\right)\sqrt{\lambda_{1,\omega}\mu(Ch^p)h^{-p}\rho^{-1}(Ch^p)}. \end{aligned} \tag{2.100}$$

h^p tends to zero so does $\varepsilon(h)$ and (2.92) satisfies for h small enough. For (2.93),

$$\frac{h^p}{\varepsilon(h)} = \sqrt{\frac{h^p\rho^{-1}(Ch^p)}{\lambda_{1,\omega}\mu(Ch^p)}}. \tag{2.101}$$

$h^p \to 0$, ρ^{-1} is a positive increasing function and $1/\mu$ is also a positive increasing function so they have finite limits in zero. (2.93) is valid for h small enough, hence

$$0 \leqslant \lambda_1(h) - h^{-p}\rho^{-1}(Ch^p) = \mathcal{O}\left(\sqrt{\mu(Ch^p)h^{-p}\rho^{-1}(Ch^p)}\right). \tag{2.102}$$

Now, we have

$$|\rho^{-1}(Ch^p) - \rho^{-1}(h^p)| \leqslant |C-1|h^p \max(\mu(Ch^p), \mu(h^p)). \tag{2.103}$$

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But we know that $\alpha\mu(\alpha) = o(\rho^{-1}(\alpha))$ so $\rho^{-1}(Ch^p) \sim \rho^{-1}(h^p)$ which yields to 1). We get rid of the constant C both in $\rho^{-1}(Ch^p)$ and under $\mathcal{O}$. To begin with,

$$\mathcal{O}(\sqrt{\mu(Ch^p)h^{-p}\rho^{-1}(h^p)}) = \mathcal{O}(\sqrt{\mu(h^p)h^{-p}\rho^{-1}(h^p)}), \tag{2.104}$$

since $\mu(Ch^p) \sim \mu(h^p)$. Once again, we use that $\mu(h^p) = o(h^{-p}\rho^{-1}(h^p))$ which gives

$$\mu(h^p) = o(\sqrt{\mu(h^p)h^{-p}\rho^{-1}(h^p)}). \tag{2.105}$$

In a same way,

$$\mu(Ch^p) = o(\sqrt{\mu(h^p)h^{-p}\rho^{-1}(h^p)}), \tag{2.106}$$

which ends the proof. □

A corollary like Corollary 2.35 in that case will be somewhat technical, it is better to make the asymptotic expansion of the function itself.

3. Study of the Limit of $\lambda_1(h)$

The key-objective is to determine precisely the limit of the first eigenvalue when h tends to zero. To begin with, once again, we deal with the limit of $h^p\lambda_1(h)$ but for general potentials under weak assumptions on the potential V. We have already proved that if V is bounded (or locally bounded [3]) then $h^p\lambda_1(h)$ tends to the essential infimum of V. This is not true for 'pathologic' potentials. We start with σ the essential infimum of V. We set

$$L^{p,V}(\Omega) = \{\psi \in L^p(\Omega) : V(x)|\psi|^p \in L^1(\Omega)\}. \tag{3.1}$$

The proof of the next lemma is straightforward.

LEMMA 3.1. *Let Ω be a domain of $\mathbb{R}^N$ and V a measurable function on Ω which satisfies $W^{1,p,V}(\Omega) \neq \{0\}$ and one of the following assumptions*:

$$\sigma > -\infty, \tag{3.2}$$

or

$$\sigma = -\infty \text{ and } \forall z \in \mathbb{R},\ \text{meas}\{x \in \Omega : -\infty < V(x) < z\} > 0. \tag{3.3}$$

where $\sigma = \text{essinf}_\Omega V$. Then

$$\sigma = \inf\left\{\int_\Omega V(x)|\psi|^p\,\mathrm{d}x : \psi \in L^{p,V}(\Omega), \int_\Omega |\psi|^p\,\mathrm{d}x = 1\right\}. \tag{3.4}$$

There exists a measurable function V which does not satisfy the assumptions of Lemma 3.1.

EXAMPLE 3.2.

$$V(x) = \begin{cases} -\infty & \text{for } x \in [0,1] \\ 0 & \text{for } x \in \mathbb{R}\backslash[0,1] \end{cases}$$

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We return to (3.4). By analogy, we define

$$\sigma^* = \inf\left\{\int_\Omega V(x)|\psi|^p \,\mathrm{d}x : \psi \in W^{1,p,V}(\Omega), \int_\Omega |\psi|^p \,\mathrm{d}x = 1\right\}. \tag{3.5}$$

Obviously, $\sigma^* \geqslant \sigma$ and σ^* may be take the value $-\infty$. We recall that $\lambda_1(h)$ is the first eigenvalue of $-\Delta_p + h^{-p}V(x)$ in $W^{1,p,V}(\Omega)$ for the Neumann boundary condition (if $\partial\Omega$ is not empty), i.e.,

$$\lambda_1(h) = \inf\left\{\int_\Omega |\nabla\psi|^p + h^{-p}V(x)|\psi|^p \,\mathrm{d}x : \psi \in W^{1,p,V}(\Omega),\right.$$
$$\left.\times \int_\Omega |\psi|^p \,\mathrm{d}x = 1\right\}, \tag{3.6}$$

for $h > 0$. The link with semi-classical analysis is shown in the next theorem.

THEOREM 3.3. *Let Ω be a domain of $\mathbb{R}^N$ and V a measurable function on Ω which satisfies $W^{1,p,V}(\Omega) \neq \{0\}$ and $\lambda_1(1) > -\infty$. Then*

$$\lim_{h\to 0} h^p \lambda_1(h) = \sigma^*. \tag{3.7}$$

Proof. Obviously, $\lambda_1(h)$ exists for all $h \leqslant 1$ and $h^p\lambda_1(h) \geqslant \sigma^*$. Conversely, let ψ be in $W^{1,p,V}(\Omega)$ with $\|\psi\|_{L^p(\Omega)} = 1$. The definition of $\lambda_1(h)$ implies that $\lambda_1(h) \leqslant \int_\Omega |\nabla\psi|^p + h^{-p}V(x)|\psi|^p \,\mathrm{d}x$ and so $\limsup_{h\to 0} h^p\lambda_1(h) \leqslant \int_\Omega V(x)|\psi|^p \,\mathrm{d}x$. ☐

We need the following lemma for the next example. $B_r(x_0)$ is the open ball of center x_0 and radius $r > 0$.

LEMMA 3.4. *We assume the hypothesis of Theorem* 3.3. *Suppose that $p > N$ and $\psi \in W^{1,p,V}(\Omega)$. If there exists a point x_0 in Ω with $\psi(x_0) \neq 0$, then there exists $r > 0$ such that $V \in L^1(B_r(x_0))$.*

Proof. ψ can be taken continuous so one can find two positive real numbers η and r which satisfy $\forall x \in B_r(x_0)$, $|\psi(x_0)| \geqslant \eta$. Hence, $|V(x)\psi|^p \geqslant \eta^p|V(x)|$. $\psi \in W^{1,p,V}(\Omega)$ ends the proof. ☐

The definition of σ^* is rather implicit. The difference between σ and σ^* appears only for 'irregular' potentials. For a better understanding of this close relationship, a more complicated example is provided.

EXAMPLE 3.5. We consider $\Omega = \mathbb{R}$ and K a measurable set of $[0, 1]$ with an empty interior and $0 < \text{meas } K < 1$. (K can be constructed as a modified Cantor set) Let

$$V(x) = \begin{cases} 0 & \text{for } x \in K \\ +\infty & \text{for } x \in [0, 1]\backslash K \\ 1 & \text{otherwise} \end{cases}$$

By the previous lemma, if $\psi \in W^{1,p,V}(\mathbb{R})$, $\psi(x) = 0$ on $[0, 1]$, so $\sigma^* = 1$ but $\sigma = 0$. Moreover, $\lambda_1(1) = 1$ because $\inf(\int_{\mathbb{R}} |\nabla\psi|^p \, \mathrm{d}x \, : \, \psi \in W^{1,p,V}(\mathbb{R}), \int_{\mathbb{R}} |\psi|^p \, \mathrm{d}x = 1) = 0$. The value of V on K is not taking into account for $\lambda_1(1)$ and σ^*, it can be replaced by any real number even $-\infty$ or $+\infty$.

We now established a sufficient condition on V for the equality of σ and σ^*.

THEOREM 3.6. *Under the assumptions of Theorem* 3.3, *if for all* $z > \sigma$, *there exists* $s > 1$, $x_0 \in \mathbb{R}^N$, $r > 0$ *such that*

$$\operatorname{meas}\{x : V(x) \leqslant z\} \cap B_r(x_0) > 0, \tag{3.8}$$

and

$$V \in L^s(B_r(x_0)), \tag{3.9}$$

then $\sigma = \sigma^*$.

Proof. It suffices to prove that $\sigma^* \leqslant \sigma$. Let $z > \sigma$ and s, x_0 and r the related quantities of the theorem. Let $r' \in (0, r)$. We set

$$E = \{x : V(x) \leqslant z\} \cap B_{r'}(x_0). \tag{3.10}$$

There exists a sequence (f_n) of $\mathcal{C}^\infty(\mathbb{R}^N)$ such that $f_n \to \chi_E$ for the norm of $L^{s'}(\mathbb{R}^N)$. (s' is the conjuguate exponent, $\frac{1}{s} + \frac{1}{s'} = 1$)

As $\| f_n| - |\chi_E\| = \| f_n| - \chi_E| \leqslant | f_n - \chi_E|$, f_n can be taken nonnegative. Let φ be a $\mathcal{C}^\infty$ function defined by:

$$\varphi(x) = \begin{cases} 0 & \text{for } x \in \mathbb{R}^N \backslash B_r(x_0) \\ 1 & \text{for } x \in B_{r'}(x_0) \end{cases}, \tag{3.11}$$

with $0 \leqslant \varphi \leqslant 1$. We set

$$g_n = \left(f_n + \frac{1}{n} \right)^{\frac{1}{p}} \varphi. \tag{3.12}$$

Thank to $1/n$ and φ, g_n belongs to $\mathcal{C}_0^\infty(B_r(x))$. As a consequence, $g_n \in W^{1,p,V}(\mathbb{R}^N)$. We shall show that

$$\int_{\mathbb{R}^N} g_n^p \, \mathrm{d}x \to \int_{\mathbb{R}^N} \chi_E \, \mathrm{d}x, \tag{3.13}$$

and

$$\int_{\mathbb{R}^N} V(x) g_n^p \, \mathrm{d}x \to \int_{\mathbb{R}^N} V(x) \chi_E \, \mathrm{d}x, \tag{3.14}$$

when $n \to \infty$. Indeed,

$$g_n^p = \left(f_n + \frac{1}{n} \right) \varphi^p, \tag{3.15}$$

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and moreover, since $f_n \to \chi_E$ in $L^{s'}(\mathbb{R}^N)$, $f_n\varphi^p$ tends to $\chi_E\varphi^p = \chi_E$ in $L^{s'}(\mathbb{R}^N)$ and so in $L^{s'}(B_r(x_0))$ because the support of φ is a subset of $B_r(x_0)$. Finally, $f_n\varphi^p \to \chi_E\varphi^p = \chi_E$ in $L^1(B_r(x_0))$ which gives

$$\int_{\mathbb{R}^N} g_n^p \,\mathrm{d}x \to \int_{\mathbb{R}^N} \chi_E \,\mathrm{d}x = \text{meas } E. \tag{3.16}$$

For the second limit, we begin with

$$\int_{\mathbb{R}^N} V(x)g_n^p \,\mathrm{d}x - \int_{\mathbb{R}^N} V(x)\chi_E \,\mathrm{d}x = \int_{B_r(x_0)} V(x)\left(f_n + \frac{1}{n} - \chi_E\right)\varphi^p \,\mathrm{d}x, \tag{3.17}$$

which yields to

$$\int_{\mathbb{R}^N} V(x)g_n^p \,\mathrm{d}x - \int_{\mathbb{R}^N} V(x)\chi_E \,\mathrm{d}x = \frac{1}{n}\int_{B_r(x_0)} V(x)\varphi^p \,\mathrm{d}x \\ + \int_{B_r(x_0)} V(x)(f_n - \chi_E)\varphi^p \,\mathrm{d}x. \tag{3.18}$$

The right-hand side tends to zero since $V \in L^s(B_r(x_0))$ and $(f_n - \chi_E)\varphi^p \to 0$ in $L^{s'}(B_r(x_0))$. So

$$\int_{\mathbb{R}^N} V(x)g_n^p \,\mathrm{d}x \to \int_{\mathbb{R}^N} V(x)\chi_E \,\mathrm{d}x. \tag{3.19}$$

We use the sequence (g_n) in the definition of σ^*.

$$\sigma^* \leqslant \frac{\int_{\mathbb{R}^N} V(x)g_n^p \,\mathrm{d}x}{\|g_n\|_{L^p(\mathbb{R}^N)}} = \frac{\int_{B_r(x_0)} V(x)g_n^p \,\mathrm{d}x}{\|g_n\|_{L^p(B_r(x_0))}}. \tag{3.20}$$

Letting n go to infinity implies

$$\sigma^* \leqslant \frac{\int_E V(x) \,\mathrm{d}x}{\text{meas } E} \leqslant z. \tag{3.21}$$

z is chosen arbitrary, hence $\sigma^* = \sigma$. □

We deduce

COROLLARY 3.7. *Under the assumptions of Theorem* 3.3, *if* $V \in L^s_{\text{loc}}(\Omega)$ *for some* $s > 1$ *then* $\sigma^* = \sigma$.

Proof. Let $z > \sigma$. We use an argument of Lemma 3.2 in [3]. The measure of the set $\{x : V(x) \leqslant z\}$ is positive so almost all its points have density one with respect to Lebesgue measure. Hence, there exists a point x_0 and a positive real number r such that meas $\{x : V(x) \leqslant z\} \cap B_r(x_0) > 0$. Theorem 3.6 finishes the proof. □

We now prove a partial converse.

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PROPOSITION 3.8. *Under the assumptions of Theorem* 3.3, *if* $p > N$ *and* $\sigma = \sigma^* > -\infty$ *then for all* $z > \sigma$, *there exists* $x_0 \in \mathbb{R}^N$ *and* $r > 0$ *such that*

$$\text{meas}\,\{x : V(x) \leqslant z\} \cap B_r(x_0) > 0 \textit{ and } V \in L^1(B_r(x_0)). \tag{3.22}$$

Proof. Let $z > \sigma$ be a real number. In virtue of the definition of σ^*, we can find a function $\psi \in W^{1,p,V}(\Omega)$ with $\|\psi\|_{L^p(\Omega)} = 1$ such that,

$$\int_\Omega (V(x) - \sigma)|\psi|^p \, dx < z - \sigma. \tag{3.23}$$

Then,

$$\int_{\{x:V(x)>z\}} (V(x) - \sigma)|\psi|^p \, dx < z - \sigma, \tag{3.24}$$

therefore,

$$\int_{\{x:V(x)>z\}} (z - \sigma)|\psi|^p \, dx < z - \sigma, \tag{3.25}$$

and

$$\int_{\{x:V(x)>z\}} |\psi|^p \, dx < 1. \tag{3.26}$$

Finally,

$$\int_{\{x:V(x)\leqslant z\}} |\psi|^p \, dx > 0. \tag{3.27}$$

Since the set $E = \{x : V(x) \leqslant z\} \cap \{x : |\psi(x)| > 0\}$ is measurable with positive measure, almost all its points have density 1 with respect to Lebesgue measure that is

$$\lim_{r \to 0} \frac{\text{meas } B_r(x) \cap E}{\text{meas } B_r(x)} = 1 \text{ for a.e. } x \in E. \tag{3.28}$$

We choose a such point x_0 in E. Since $|\psi(x_0)| > 0$, thank to Lemma 3.4, one can find a positive real number r' which satisfies $V \in L^1(B_{r'}(x_0))$. So there exists $r \in (0, r']$ such that $V \in L^1(B_r(x_0))$ and meas $B_r(x) \cap E \geqslant 1/2$ meas $B_r(x) > 0$. □

σ^* seems more important than σ for semi-classical applications. To simplify, for the study of the limit of $\lambda_1(h)$, we suppose that $\text{ess inf}_\Omega V = 0$. Let us introduce a new definition.

DEFINITION 3.9. Let E be a measurable subset of Ω such that there exists a function ψ_0 in $W^{1,p}(\Omega)$ which satisfies $\|\psi_0\|_{L^p(\Omega)} = 1$ and $\psi_0 = 0$ almost everywhere in $\Omega\backslash E$. We denote by λ_E the quantity

$$\lambda_E = \inf\left\{ \int_\Omega |\nabla\psi|^p \, dx : \psi \in W^{1,p}(\Omega), \|\psi\|_{L^p(\Omega)} = 1, \right.$$
$$\left. \psi = 0 \text{ a.e. on } \Omega\backslash E \right\}. \tag{3.29}$$

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Remark that $\psi_0 \chi_{\Omega\backslash E} = 0$ almost everywhere on Ω where $\chi_{\Omega\backslash E}$ is the characteristic function of $\Omega\backslash E$. This definition is enlightened by the following proposition.

PROPOSITION 3.10. *Suppose that* Ω *is a connected open set of* $\mathbb{R}^N$, V *a measurable function on* Ω *with* $W^{1,p,V}(\Omega) \neq \{0\}$ *and ess inf*$_\Omega V = 0$. *If there exists a function* ψ_0 *in* $W^{1,p}(\Omega)$ *which satisfies* $\|\psi_0\|_{L^p(\Omega)} = 1$ *and* $\psi_0 = 0$ *almost everywhere in* $\Omega\backslash\{x : V(x) = 0\}$ *then* $h \mapsto \lambda_1(h)$ *is bounded independently of* h. *Furthermore*,

$$\lim_{h\to 0} \lambda_1(h) \leqslant \lambda_{\{x:V(x)=0\}}. \tag{3.30}$$

Proof. First, $V|\psi_0|^p = 0$ a.e. on Ω so ψ_0 is in $W^{1,p,V}(\Omega)$. We use definition of $\lambda_1(h)$.

$$\lambda_1(h) \leqslant \int_\Omega |\nabla\psi|^p \,\mathrm{d}x. \tag{3.31}$$

A passage to the infimum ends the proof. □

Let us denote by (ψ_h) a sequence of functions of $W^{1,p,V}(\Omega)$ with $\|\psi_h\|_{L^p(\Omega)} = 1$ such that

$$\lim_{h\to 0} \lambda_1(h) - F_{h^{-p}V}(\psi_h) = 0. \tag{3.32}$$

Proposition 2.4 remains true for the sequence (ψ_h).

PROPOSITION 3.11. *With the notations of Proposition* 3.10, *if* $\lim_{h\to 0} h^p \lambda_1(h) = 0$ *then for all* $\varepsilon > 0$, *there holds*

$$\int_{\{x:V(x)\geqslant\varepsilon\}} |\psi_h|^p \,\mathrm{d}x \leqslant \frac{h^p F_{h^{-p}V}(\psi_h)}{\varepsilon}, \tag{3.33}$$

and

$$\lim_{h\to 0} \int_{\{x:V(x)\geqslant\varepsilon\}} |\psi_h|^p \,\mathrm{d}x = 0. \tag{3.34}$$

Since the sequence (ψ_h) is bounded in $L^p(\Omega)$, it has a limit for the weak topology of $L^p(\Omega)$.

LEMMA 3.12. *With the notations of Proposition* 3.10, *let* ψ_0 *be a cluster point of the set* (ψ_h) *for the weak topology of* $L^p(\Omega)$. *If* $\lim_{h\to 0} h^p \lambda_1(h) = 0$ *then*

$$\int_{\{x:V(x)>0\}} |\psi_0|^p \,\mathrm{d}x = 0. \tag{3.35}$$

Proof. (ψ_{h_n}) is a subsequence such that $\psi_{h_n} \rightharpoonup \psi_0$ weakly in $L^p(\Omega)$. This allows us to use a function ξ of $L^{p'}(\Omega)$ where p' is the conjugate of p. Let z be a positive real number. The Hölder's inequality leads to

$$\left|\int_{\{x:V(x)\geqslant z\}} \psi_{h_n}\xi \,\mathrm{d}x\right| \leqslant \left(\int_{\{x:V(x)\geqslant z\}} |\psi_{h_n}|^p\right)^{\frac{1}{p}} \left(\int_{\{x:V(x)\geqslant z\}} |\xi|^{p'}\right)^{\frac{1}{p'}}. \tag{3.36}$$

The left-hand side goes to $|\int_{\{x:V(x)\,\geqslant\, z\}} \psi_0\xi \;\mathrm{d}x|$ when n goes to infinity and the right-hand side tends to zero. Consequently,

$$\int_{\{x:V(x)\geqslant z\}} \psi_0\xi \,\mathrm{d}x = 0, \tag{3.37}$$

and it is available for all ξ in $L^{p'}(\Omega)$ which gives

$$\int_{\{x:V(x)\geqslant z\}} |\psi_0|^p \,\mathrm{d}x = 0. \tag{3.38}$$

Obviously, we have

$$\{x : V(x) > 0\} = \bigcup_{n\in\mathbb{N}^*} \left\{x : V(x) \geqslant \frac{1}{n}\right\}, \tag{3.39}$$

and the function $A \mapsto \int_A |\psi_0|^p \,\mathrm{d}x$ is a measure on the collection of measurable sets of Ω. As a consequence,

$$\int_{\{x:V(x)>0\}} |\psi_0|^p \,\mathrm{d}x = \lim_{n\to+\infty} \int_{\{x:V(x)\geqslant\frac{1}{n}\}} |\psi_0|^p \,\mathrm{d}x = 0. \tag{3.40}$$

□

Proposition 3.11. leads to a new assumption on V.

There exists $\varepsilon > 0$ such that, up to a set of zero measure,

$$\{x : V(x) < \varepsilon\} \text{ is bounded and } \lim_{h\to 0} h^p \lambda_1(h) = 0. \tag{3.41}$$

From now on, we forget the set of zero measure and suppose directly that $\{x : V(x) < \varepsilon\}$ is bounded. We give a necessary and sufficient condition for $\lambda_1(h)$ to be bounded.

THEOREM 3.13. *Suppose that* Ω *is a connected open set of* $\mathbb{R}^N$, V *a measurable function on* Ω *which satisfies* $W^{1,p,V}(\Omega) \neq \{0\}$, $\operatorname{essinf}_\Omega V = 0$ *and* (3.41). *Then* $h \mapsto \lambda_1(h)$ *is bounded independently of* h *if and only if there exists a function* ψ_0 *in* $W^{1,p}(\Omega)$ *which satisfies* $\|\psi_0\|_{L^p(\Omega)} = 1$ *and* $\psi_0 = 0$ *almost everywhere on* $\Omega\backslash\{x : V(x) = 0\}$. *Moreover, in this case,* ψ_0 *can be chosen as a cluster point of* (ψ_h) *for the weak topology of* $W^{1,p}(\Omega)$. *In fact, there exists a subsequence* (ψ_{h_n}) *of* (ψ_h) *which tends strongly in* $W^{1,p}(\Omega)$ *to* ψ_0,

$$\lim_{n\to 0} h_n^{-p} \int_\Omega V(x)|\psi_{h_n}|^p \,\mathrm{d}x = 0, \tag{3.42}$$

the infimum $\lambda_{\{x:V(x)=0\}}$ *is reached by* ψ_0 *and*

$$\lim_{h\to 0} \lambda_1(h) = \lambda_{\{x:V(x)=0\}}. \tag{3.43}$$

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Proof. ($\Leftarrow$) has already been proved. Assume that $h \mapsto \lambda(h)$ is bounded. Therefore, (ψ_h) is bounded in $W^{1,p}(\Omega)$ so there exist ψ_0 in $W^{1,p}(\Omega)$ and a sequence (h_n) of positive real numbers which tends to zero such that $\psi_{h_n} \rightharpoonup \psi_0$ weakly in $W^{1,p}(\Omega)$. Since $\{x : V(x) < \varepsilon\}$ is bounded, there exists a bounded $\mathcal{C}^1$ domain U such that $\{x : V(x) < \varepsilon\} \subset U$. So, $\psi_{h_n} \rightharpoonup \psi_0$ weakly in $W^{1,p}(U)$. From the Rellich-Kondrachov theorem, the injection of $W^{1,p}(U)$ into $L^p(U)$ is compact. Up to a subsequence, (ψ_{h_n}) tends strongly in $L^p(U)$ to ψ_0. Thus,

$$\lim_{n\to\infty} \int_U |\psi_{h_n}|^p \, \mathrm{d}x = \int_U |\psi_0|^p \, \mathrm{d}x. \tag{3.44}$$

But since $\Omega \backslash U \subset \{x : V(x) \geqslant \varepsilon\}$, from Proposition 3,

$$\lim_{n\to\infty} \int_{\Omega\backslash U} |\psi_{h_n}|^p \, \mathrm{d}x = 0. \tag{3.45}$$

Consequently,

$$\lim_{n\to\infty} \int_U |\psi_{h_n}|^p \, \mathrm{d}x = 1. \tag{3.46}$$

The integral of $|\psi_0|^p$ over U is equal to 1. So does the integral over Ω. Indeed, one can prove $\int_{\Omega\backslash U} |\psi_0|^p \, \mathrm{d}x = 0$ by Lemma 3.12, since $\Omega\backslash U \subset \{x : V(x) \geqslant \varepsilon\}$. Finally, we have proved that there exists a function ψ_0 in $W^{1,p}(\Omega)$ with $\|\psi_0\|_{L^p(\Omega)} = 1$ and $\psi_0 = 0$ almost everywhere on $\Omega\backslash U$. Once again from Lemma 3.12, $\psi_0 = 0$ on $\Omega\backslash\{x : V(x) = 0\}$. Now, we shall deal with the others claims. To begin with,

$$\lim_{n\to\infty} \int_\Omega |\psi_{h_n}|^p \, \mathrm{d}x = \int_\Omega |\psi_0|^p \, \mathrm{d}x. \tag{3.47}$$

The norm of $L^p(\Omega)$ is uniformly convex so (ψ_{h_n}) tends to ψ_0 for the strong topology of $L^p(\Omega)$. We study the norm of gradient ψ_0. (ψ_{h_n}) tends weakly to ψ_0 in $W^{1,p}(\Omega)$ so on one hand,

$$\int_\Omega |\nabla\psi_0|^p \, \mathrm{d}x \leqslant \liminf_{n\to\infty} \int_\Omega |\nabla\psi_{h_n}|^p \, \mathrm{d}x. \tag{3.48}$$

On the other hand, from the minimax principle,

$$\lambda_1(h_n) \leqslant \int_\Omega |\nabla\psi_0|^p \, \mathrm{d}x, \tag{3.49}$$

and with the help of $F_{h_n^{-p}V}(\psi_{h_n})$ leads to

$$\begin{aligned}\int_\Omega |\nabla\psi_{h_n}|^p \, \mathrm{d}x + h_n^{-p} \int_\Omega V(x)|\psi_{h_n}|^p \, \mathrm{d}x + \lambda_1(h_n) - F_{h_n^{-p}V}(\psi_{h_n})\\ \leqslant \int_\Omega |\nabla\psi_0|^p \, \mathrm{d}x.\end{aligned} \tag{3.50}$$

Consequently, since $h_n^{-p} \int_\Omega V(x)|\psi_{h_n}|^p \, \mathrm{d}x \geqslant 0$,

$$\int_\Omega |\nabla\psi_{h_n}|^p \, \mathrm{d}x + \lambda_1(h_n) - F_{h_n^{-p}V}(\psi_{h_n}) \leqslant \int_\Omega |\nabla\psi_0|^p \, \mathrm{d}x. \tag{3.51}$$

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We take the limit superior. From the definition of the sequence (ψ_h), the term $\lambda_1(h_n) - F_{h_n^{-p}V}(\psi_{h_n})$ tends to zero. We obtain

$$\limsup_{n\to\infty} \int_\Omega |\nabla \psi_{h_n}|^p \, \mathrm{d}x \leqslant \int_\Omega |\nabla \psi_0|^p \, \mathrm{d}x \leqslant \liminf_{n\to\infty} \int_\Omega |\nabla \psi_{h_n}|^p \, \mathrm{d}x. \tag{3.52}$$

The limit of the $L^p(\Omega)$-norm of the gradient of ψ_{h_n} is the $L^p(\Omega)$-norm of the gradient of ψ_0 and once again by uniformly convexity of the norm, $\nabla \psi_{h_n}$ goes to $\nabla \psi_0$ strongly in $L^p(\Omega)$. Finally, the subsequence (ψ_{h_n}) converges to ψ_0 strongly in $W^{1,p}(\Omega)$. Furthermore, (3.50) implies that $h_n^{-p} \int_\Omega V(x)|\psi_{h_n}|^p \, \mathrm{d}x$ vanishes as $n \to \infty$. Hence,

$$\lim_{n\to\infty} F_{h_n^{-p}V}(\psi_{h_n}) = \int_\Omega |\nabla \psi_0|^p \, \mathrm{d}x, \tag{3.53}$$

which yields to

$$\lim_{n\to\infty} \lambda_1(h_n) = \int_\Omega |\nabla \psi_0|^p \, \mathrm{d}x, \tag{3.54}$$

and as $h \mapsto \lambda_1(h)$ is a decreasing function,

$$\lim_{n\to\infty} \lambda_1(h) = \int_\Omega |\nabla \psi_0|^p \, \mathrm{d}x. \tag{3.55}$$

It suffices to prove that ψ_0 achieves the infimum. The Proposition 3.10 and the previous limit give

$$\int_\Omega |\nabla \psi_0|^p \, \mathrm{d}x \leqslant \lambda_{\{x:V(x)=0\}}, \tag{3.56}$$

but as ψ_0 is a suitable function for the minimax principle of $\lambda_{\{x:V(x)=0\}}$,

$$\lambda_{\{x:V(x)=0\}} \leqslant \int_\Omega |\nabla \psi_0|^p \, \mathrm{d}x. \tag{3.57}$$

These last two inequalities finish the proof. □

In the next example, we point out how important is assumption (3.41).

EXAMPLE 3.14. (Counter examples)

(1) We take $\Omega = \mathbb{R}$ and $V = 0$. Then $\lambda_1(h) = 0$ for all $h > 0$, $\lambda_{\{x:V(x)=0\}} = 0$ and

$$\lim_{h\to 0} \lambda_1(h) = \lambda_{\{x:V(x)=0\}} = 0. \tag{3.58}$$

But this infimum is not reached.

(2) We take $\Omega = \mathbb{R}$ and $V(x) = 1/(1 + x^2)$ or any continuous positive function which tends to zero at $+\infty$ or at $-\infty$. Then by translations and dilatations of a compact support $\mathcal{C}^1$ function, one obtains $\lambda_1(h) = 0$. But $\lambda_{\{x:V(x)=0\}}$ does not exist since the vanishing set of V is empty.

Now, we are interested in potentials which vanishes on zero measure set.

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COROLLARY 3.15. *Under the assumptions of Theorem* 3, *if* $V > 0$ *almost everywhere then* $h \mapsto \lambda_1(h)$ *tends to infinity.*

Proof. It comes from the nonzero L^p-norm of ψ_0. $\square$

If there exists a unique nonnegative function which reaches the infimum $\lambda_{\{x:V(x)=0\}}$ then the whole family (ψ_h) converges.

COROLLARY 3.16. *We assume the hypothesis of Theorem* 3 *and that there exists a unique nonnegative function* ψ_0 *with* $\|\psi_0\|_{L^p(\Omega)} = 1$ *which achieves* $\lambda_{\{x:V(x)=0\}}$. *If* $(\psi_h)_{h>0}$ *is any family of nonnegative functions of* $W^{1,p,V}(\Omega)$ *which satisfies* $\|\psi_h\|_{L^p(\Omega)} = 1$ *and*

$$\lim_{h\to 0} F_{h^{-p}V}(\psi_h) - \lambda_1(h) = 0, \tag{3.59}$$

then (ψ_h) *tends to* ψ_0 *in the strong topology of* $W^{1,p}(\Omega)$.

Proof. It remains to prove that the family (ψ_h) has a unique cluster point for the weak topology of $W^{1,p}(\Omega)$. The existence comes from the fact that $h \mapsto \lambda_1(h)$ is bounded. Let ψ_0' be a such point. From Theorem 3.13, ψ_0' reaches the infimum of $\lambda_{\{x:V(x)=0\}}$. Moreover, ψ_0' is nonnegative since the ψ_h are. By uniqueness, $\psi_0' = \psi_0$. $\square$

Returning to Definition 3.9, when $N < p$, we now compare λ_E and $\lambda_{\overset{\circ}{E}}$.

LEMMA 3.17. *Let* U *be a* $\mathcal{C}^1$ *open connected bounded subset of* $\mathbb{R}^N$ *with* $N < p$, E *a measurable subset of* U, ψ_0 *a function of* $W^{1,p}(U)$ *which satisfies* $\|\psi_0\|_{L^p(U)} = 1$ *and* $\psi = 0$ *almost everywhere on* $U\backslash E$. *Then, up to a zero measure set,* ψ_0 *vanishes on* $\overline{U}\backslash \overset{\circ}{E}$ *and* $\lambda_E = \lambda_{\overset{\circ}{E}}$.

Proof. From the Sobolev-Morrey-Ascoli theorem, the injection of $W^{1,p}(U)$ into $\mathcal{C}^0(\overline{U})$ is compact and so continuous. By changing ψ_0 on a set of zero measure, ψ_0 vanishes identically on $U\backslash E$. By continuity, $\psi_0 = 0$ on $\overline{U}\backslash \overset{\circ}{E}$ which proves the existence of $\lambda_{\overset{\circ}{E}}$. We end the proof by claiming that there exists a bijection between the functions of norm $L^p(U)$ equal to 1 which vanish on $U\backslash E$ and those of norm $L^p(U)$ equal to 1 which vanish on $\overline{U}\backslash \overset{\circ}{E}$. $\square$

In the remaining part of this section, it will be assumed that $N < p$. We set V_0 the interior of $\{x : V(x) = 0\}$.

COROLLARY 3.18. *Let* Ω *be an open connected bounded set of* $\mathbb{R}^N$, $N < p$, V *a measurable function on* Ω *which satisfies* $W^{1,p,V}(\Omega) \neq \{0\}$, ess $\inf_\Omega V = 0$ *and* (3.41). *Then* $h \mapsto \lambda_1(h)$ *is bounded independently of* h *if and only if the interior of* $\{x : V(x) = 0\}$ *is not empty. Moreover,*

$$\lim_{h\to 0} \lambda_1(h) = \lambda_{V_0}. \tag{3.60}$$

Proof. If the interior is not empty, taking the first eigenfunction for $-\Delta_p$ in any ball of it suffices to prove that $h \mapsto \lambda_1(h)$ is bounded. Conversely, we apply successively Theorem 3 and Lemma 3.17 since $\{x : V(x) = 0\}$ is a subset of U. □

In order to illustrate Corollary 3, we choose V such that it vanishes on modified Cantor sets.

COROLLARY 3.19. *Let Ω be a $\mathcal{C}^1$ open connected bounded set of $\mathbb{R}^N$ and $N < p$. We denote by K a measurable subset of Ω which satisfies*

$$\text{meas}(K) > 0, \quad \overline{K} = K, \text{ and } \overset{\circ}{K} = \emptyset. \tag{3.61}$$

If V is the characteristic function of $\Omega \backslash K$, i.e., V vanishes on K and equal to 1 on $\Omega \backslash K$ then

$$\lim_{h \to 0} \lambda_1(h) = +\infty, \tag{3.62}$$

where $\lambda_1(h)$ is defined by (1.4).

Proof. V is bounded so $W^{1,p,V}(\Omega) = W^{1,p}(\Omega) \neq \{0\}$. Moreover, ess $\inf_\Omega V = 0$ and since V is bounded, $\lim_{h \to 0} h^p \lambda_1(h) = 0$. Last but not least, $\{x : V(x) < 2\}$ is Ω, a $\mathcal{C}^1$ open connected bounded set. Applying the previous Corollary gives the conclusion. □

Here, we have taken the convention that we forget negligeable sets especially in that case since Lemma 2.2 holds, i.e., if E is any zero measure set, $K \cup E$ has always an empty interior. Now, we introduce some notations. If E is a measurable set with a nonempty interior, we denote by E_i the connected components of $\overset{\circ}{E}$ where $1 \leqslant i \leqslant n_0$ and $n_0 \in [1, +\infty]$ the number of connected components. We associed to each E_i, λ_{1,E_i} the first eigenvalue of $-\Delta_p$ in $W_0^{1,p}(E_i)$ for the Dirichlet boundary conditions.

LEMMA 3.20. *Suppose that U is a $\mathcal{C}^1$ connected bounded open set of $\mathbb{R}^N$, $N < p$, E a measurable subset of U such that there exists a function ψ_0 in $W^{1,p}(U)$ with $\|\psi_0\|_{L^p(U)} = 1$ and $\psi_0 = 0$ almost everywhere on $U \backslash E$. Then, with the notations above, $\lambda_E = \inf_{1 \leqslant i \leqslant n_0} \lambda_{1,E_i}$ and the infimum for λ_E is achieved.*

Proof. From Lemma 3.17, E has a nonempty interior and $\lambda_E = \lambda_{\overset{\circ}{E}}$. Dealing with E_i has a sense. Moreover, by Proposition 2.13, it is always possible to assume that the sequence $(\lambda_{1,E_i})_{1 \leqslant i \leqslant n_0}$ is monotonous and nondecreasing. Obviously, we can write that $\lambda_E \leqslant \lambda_{1,E_1}$. Conversely, pick up any function ψ_0 of $W^{1,p}(U)$ with $\|\psi_0\|_{L^p(U)} = 1$ and $\psi_0 = 0$ a.e. on $U \backslash E$. Also with the help of Lemma 3.17, $\psi_0 = 0$ a.e. on $\overline{U} \backslash \overset{\circ}{E}$. We set $\psi_{0,i}$ the function equal to ψ_0 on E_i and equal to zero elsewhere. Consequently,

$$\psi_0 = \sum_{i=1}^{n_0} \psi_{0,i}, \tag{3.63}$$

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and since the connected components are disjoint,

$$\int_\Omega |\nabla \psi_0|^p \,\mathrm{d}x = \sum_{i=1}^{n_0} \int_\Omega |\nabla \psi_{0,i}|^p \,\mathrm{d}x. \tag{3.64}$$

We use the definition of λ_{1,E_i}.

$$\int_\Omega |\nabla \psi_0|^p \,\mathrm{d}x \geqslant \sum_{i=1}^{n_0} \lambda_{1,E_i} \int_\Omega |\psi_{0,i}|^p \,\mathrm{d}x. \tag{3.65}$$

Since $\lambda_{1,E_i} \geqslant \lambda_{1,E_1}$,

$$\int_\Omega |\nabla \psi_0|^p \,\mathrm{d}x \geqslant \lambda_{1,E_1} \sum_{i=1}^{n_0} \int_\Omega |\psi_{0,i}|^p \,\mathrm{d}x = \int_\Omega |\psi_0|^p \mathrm{d}x = 1. \tag{3.66}$$

As a consequence, $\lambda_E \geqslant \lambda_{1,E_1}$. ☐

The final theorem is written under the following form.

THEOREM 3.21. *Suppose that* Ω *is a connected open set of* $\mathbb{R}^N$, $N < p$, V *a measurable function on* Ω *which satisfies* $W^{1,p,V}(\Omega) \neq \{0\}$, $\operatorname{ess\,inf}_\Omega V = 0$ *and* (3.41). *Then* $h \mapsto \lambda_1(h)$ *is bounded independently of* h *if and only if the interior of* $\{x : V(x) = 0\}$ *is not empty. Moreover*, *in this case*, *if*

$$V_0 = \bigcup_{i=1}^{n_0} E_i, \tag{3.67}$$

where E_i *are the connected components and if the sequence* $(\lambda_{1,E_i})_{1 \leqslant i \leqslant n_0}$ *is monotonous and nondecreasing then*

$$\lim_{h \to 0} \lambda_1(h) = \lambda_{1,E_1}. \tag{3.68}$$

Proof. We use Corollary 3.18 and Lemma 3.20 by setting $E = \{x : V(x) = 0\}$. ☐

This theorem has justified the notation λ_E.

COROLLARY 3.22. *Suppose that* Ω *is a connected open set of* $\mathbb{R}^N$, $N < p$, V *a measurable function on* Ω *which satisfies* $W^{1,p,V}(\Omega) \neq \{0\}$, $\operatorname{ess\,inf}_\Omega V = 0$ *and* (3.41). *Let* $(\psi_h)_{h>0}$ *be any family of nonnegative functions of* $W^{1,p,V}(\Omega)$ *which satisfies* $\|\psi_h\|_{L^p(\Omega)} = 1$ *and*

$$\lim_{h \to 0} F_{h^{-p}V}(\psi_h) - \lambda_1(h) = 0. \tag{3.69}$$

We assume that $h \mapsto \lambda_1(h)$ *is bounded. Let* $n_1 \in [1, +\infty)$ *be an integer with* $n_1 \leqslant n_0$ *such that*

$$\lambda_{1,E_1} = \lambda_{1,E_2} = \cdots = \lambda_{1,E_{n_1}}, \tag{3.70}$$

and for all integer i *in* $[n_1+1, n_0]$ (*if they exist*),

$$\lambda_{1,E_1} < \lambda_{1,E_i}. \tag{3.71}$$

We have the following alternative:

1. *If* $n_1 = 1$ *then* (ψ_h) *tends to a first nonnegative eigenfunction of* E_1 *in the strong topology of* $W^{1,p}(\Omega)$.
2. *If* $n_1 \geqslant 2$ *and if* ψ_0 *is a cluster point of* (ψ_h) *for the weak topology of* $W^{1,p}(\Omega)$ *then there exist a sequence* (h_n) *of positive real numbers which tends to zero and* n_1 *nonnegative real numbers* α_i *such that* $\sum_{i=1}^{n_1}\alpha_i = 1$ *and*

$$\psi_{h_n} \to \psi_0 = \sum_{i=1}^{n_1} \alpha_i \varphi_{E_i}, \tag{3.72}$$

strongly in $W^{1,p}(\Omega)$ *where* φ_{E_i} *is a first nonnegative eigenfunction of* $-\Delta_p$ *in* $W_0^{1,p}(E_i)$ *for the Dirichlet boundary conditions with* $\|\varphi_{E_i}\|_{L^p(E_i)} = 1$. *In others words, all the cluster points of* (ψ_h) *are linear combinaisons of the* n_1 *eigenfunctions* φ_{E_i}, $1 \leqslant i \leqslant n_1$.

Proof. By Proposition 2.13, n_1 exists.

1. We use Theorem 3.21 and Corollary 3.16.
2. We apply Theorems 3.13 and 3.21 which gives the sequence and the strong convergence. Moreover, ψ_0 is a minimizer of $\lambda_{V_0} = \lambda_{1,E_1}$, i.e.,

$$\int_\Omega |\nabla\psi_0|^2\,\mathrm{d}x = \lambda_{V_0} = \lambda_{1,E_1}. \tag{3.73}$$

We define $\psi_{0,i}$ by $\psi_{0,i} = \psi_0$ on E_i and zero elsewhere, that is,

$$\psi_0 = \sum_{i=1}^{n_0} \psi_{0,i}. \tag{3.74}$$

We estimate the L^p-norm of the gradient of ψ_0.

$$\begin{aligned}\int_\Omega |\nabla\psi_0|^p\,\mathrm{d}x &= \sum_{i=1}^{n_0}\int_\Omega |\nabla\psi_{0,i}|^p\,\mathrm{d}x\\ &\geqslant \lambda_{1,E_1}\sum_{i=1}^{n_1}\int_\Omega |\psi_{0,i}|^p\,\mathrm{d}x + \sum_{i=n_1+1}^{n_0}\lambda_{1,E_i}\int_\Omega |\psi_{0,i}|^p\,\mathrm{d}x.\end{aligned} \tag{3.75}$$

But,

$$\int_\Omega |\nabla\psi_0|^p\,\mathrm{d}x = \lambda_{1,E_1}, \tag{3.76}$$

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since $\|\psi_0\|_{L^p(\Omega)} = 1 = \sum_{i=1}^{n_0} \int_\Omega |\psi_{0,i}|^p \, \mathrm{d}x$. So

$$\lambda_{1,E_1} \sum_{i=1}^{n_0} \int_\Omega |\psi_{0,i}|^p \, \mathrm{d}x \geqslant \lambda_{1,E_1} \sum_{i=1}^{n_1} \int_\Omega |\psi_{0,i}|^p \, \mathrm{d}x + \sum_{i=n_1+1}^{n_0} \lambda_{1,E_i} \int_\Omega |\psi_{0,i}|^p \, \mathrm{d}x. \tag{3.77}$$

Therefore,

$$\sum_{i=n_1+1}^{n_0} (\lambda_{1,E_1} - \lambda_{1,E_i}) \int_\Omega |\psi_{0,i}|^p \, \mathrm{d}x \geqslant 0. \tag{3.78}$$

As $\lambda_{1,E_i} > \lambda_{1,E_1}$ for $i \geqslant n_1 + 1$,

$$\int_\Omega |\psi_{0,i}|^p \, \mathrm{d}x = 0, \tag{3.79}$$

for $i \geqslant n_1 + 1$. By a similar argument, we shall prove that $\psi_{0,i}$ is the first eigenfunction of $-\Delta_p$ in $W_0^{1,p}(E_i)$ extended by zero outside for $1 \leqslant i \leqslant n_1$. Indeed,

$$\lambda_{1,E_1} = \int_\Omega |\nabla \psi_0|^p \, \mathrm{d}x = \sum_{i=1}^{n_1} \int_\Omega |\nabla \psi_{0,i}|^p \, \mathrm{d}x, \tag{3.80}$$

and if ψ_{0,i_0} is not a first eigenfunction of $-\Delta_p$ for some $i_0 \in [1, n_1] \cap \mathbb{N}$ then

$$\int_\Omega |\nabla \psi_{0,i_0}|^p \, \mathrm{d}x > \lambda_{1,E_1} \int_\Omega |\psi_{0,i_0}|^p \, \mathrm{d}x, \tag{3.81}$$

and so

$$\lambda_{1,E_1} = \sum_{i=1}^{n_1} \int_\Omega |\nabla \psi_{0,i}|^p \, \mathrm{d}x > \lambda_{1,E_1} \sum_{i=1}^{n_1} \int_\Omega |\psi_{0,i}|^p \, \mathrm{d}x = \lambda_{1,E_1}, \tag{3.82}$$

which is a contradiction. □

Remark 3.23. The converse for (2) is true, in the sense that, if $(\alpha_1, \ldots \alpha_{n_1})$ are n_1 real nonnegative numbers with $\sum_{i=1}^{n_1} \alpha_i = 1$ then there exists a sequence (ψ_h) with $\|\psi_h\|_{L^p(\Omega)} = 1$ and $\lim_{h \to 0} F_{h^{-p}V}(\psi_h) - \lambda_1(h) = 0$ which converges to $\sum_{i=1}^{n_1} \alpha_i \varphi_{E_i}$. Indeed, it suffices to take $\psi_h = \sum_{i=1}^{n_1} \alpha_i \varphi_{E_i}$.

One can ask whether ψ_h has a limit if $\lambda_1(h)$ is not bounded and in which sense. We try to answer this question in a special case. Assume that Ω is a regular bounded domain of $\mathbb{R}^N$ with $O \in \bar{\Omega}$ and V a function such that

$$\begin{cases} V \text{ is continuous on } \bar{\Omega}, \\ V > 0 \text{ on } \bar{\Omega} \backslash \{O\}, \\ V(O) = 0. \end{cases} \tag{3.83}$$

From Lemma 3.12, ψ_h tends to zero for the weak topology of $L^1(\Omega)$, so ψ_h does not seem interesting. But by assumption, $\|\psi_h\|_{L^p(\Omega)} = 1$. A good candidate is $|\psi_h^p|$.

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THEOREM 3.24. *Suppose that* Ω *is a regular domain of* $\mathbb{R}^N$ *with* $O \in \bar{\Omega}$ *and* V *a function which satisfies* (3.83). *If* $(\psi_h)_{h>0}$ *is any family of functions of* $W^{1,p}(\Omega)$ *with*

$$\|\psi_h\|_{L^p(\Omega)} = 1, \tag{3.84}$$

and

$$\lim_{h\to 0} F_{h^{-p}V}(\psi_h) - \lambda_1(h) = 0, \tag{3.85}$$

then $|\psi_h|^p$ *tends to* δ_O, *the measure of Dirac in* O, *i.e.,*

$$\lim_{h\to 0} \int_\Omega |\psi_h|^p f(x)\,\mathrm{d}x = f(O), \tag{3.86}$$

for all continuous functions f *on* $\bar{\Omega}$.

Proof. We claim that for all relatively open subset ω of $\bar{\Omega}$ (i.e. $\omega = \bar{\Omega} \cap U$ where U is an open subset of $\mathbb{R}^N$) with $O \in \omega$,

$$\inf_{x\in\bar{\Omega}\backslash\omega} V(x) > 0. \tag{3.87}$$

It is clear since $\bar{\Omega}\backslash\omega$ is a compact and V is continuous and positive on it. Let $\varepsilon > 0$. By continuity of f in O, there exists a relatively open set ω such that $|f(x) - f(O)| < \varepsilon$ on ω. Now, from (3.87) and by Proposition 3.11, for h small enough,

$$\int_{\bar{\Omega}\backslash\omega} |\psi_h|^p \mathrm{d}x < \varepsilon. \tag{3.88}$$

So,

$$\begin{aligned}\left|\int_\Omega |\psi_h|^p f(x)\,\mathrm{d}x - f(O)\right| &= \left|\int_\Omega |\psi_h|^p (f(x) - f(O))\,\mathrm{d}x\right| \\ &\leqslant \left|\int_{\bar{\Omega}\backslash\omega} |\psi_h|^p (f(x) - f(O))\,\mathrm{d}x\right| + \left|\int_\omega |\psi_h|^p (f(x) - f(O))\,\mathrm{d}x\right| \\ &\leqslant (2\|f\|_{L^\infty(\Omega)} + 1)\varepsilon. \end{aligned} \tag{3.89}$$

for h small enough which completes the proof. □

This is not a surprising result at all since $|\psi_h|^p$ concentrates near O.

Remark 3.25. Theorem 3.24 remains true if Ω is a regular domain, V a nonnegative mesurable function on $\bar{\Omega}$ such that $V(x) \geqslant \eta > 0$ a.e. on $\bar{\Omega}\backslash\omega$ and V continuous on ω with $O \in \omega$ and $V > 0$ on $\omega\backslash\{O\}$ where ω is a relatively open subset of $\bar{\Omega}$ and $\lim_{h\to 0} h^p \lambda_1(h) = 0$. Moreover, the conclusion holds if f is bounded on $\bar{\Omega}$ and continuous only in O.

Remark 3.26. A regular domain Ω means that $\mathrm{meas}(\partial\Omega) = 0$, i.e., all the integrals over Ω are the same over $\bar{\Omega}$. It is inevitable if O belongs to the boundary of Ω, but of no use if O is in Ω.

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4. Existence for a Minimizer

We show a theorem of existence of a minimizer for $-\Delta_p + V(x)$. We assume that Ω is an open connected set of $\mathbb{R}^N$ and (U_m) be a family of $\mathcal{C}^1$ open bounded connected sets such that

$$\Omega = \bigcup_{m=0}^{\infty} U_m \quad \text{and} \quad \overline{U_m} \subset U_{m+1}, \tag{4.1}$$

for all $m \geqslant 0$. We suppose that U_0 is not empty. We shall play with weak and strong convergence for a minimizing sequence which is illustrated in the following lemma.

LEMMA 4.1. *Let (u_n) be a sequence of $L^p(\Omega)$, u a function of $L^p(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $L^p(\Omega)$, and for all U_m, there exists a subsequence of (u_n) which goes to u strongly in $L^p(U_m)$. Then the sequence (u_n) goes strongly to u in $L^p(\Omega)$ if and only if for all $\varepsilon > 0$, there exists two integers m and N such that for all $n \geqslant N$,*

$$\int_{U_m} |u_n|^p \, \mathrm{d}x \geqslant 1 - \varepsilon. \tag{4.2}$$

Proof. First, suppose that $u_n \to u$ strongly in $L^p(\Omega)$. Let $\varepsilon > 0$. Since u belongs to $L^p(\Omega)$, there exists an integer m such that

$$\int_{\Omega \backslash U_m} |u|^p \, \mathrm{d}x \leqslant \varepsilon. \tag{4.3}$$

u_n goes to u strongly in $L^p(\Omega)$ so in $L^p(\Omega \backslash U_m)$. Therefore, there exists N such that for all $n \geqslant N$,

$$\int_{\Omega \backslash U_m} (|u_n|^p - |u|^p) \, \mathrm{d}x \leqslant \varepsilon. \tag{4.4}$$

Finally,

$$\int_{U_m} |u_n|^p \, \mathrm{d}x \geqslant 1 - 2\varepsilon, \tag{4.5}$$

for $n \geqslant N$. Conversely, let $\varepsilon > 0$ and m, N the two related integers. For a subsequence (u_{n_k}) of (u_n),

$$\int_{U_m} |u_{n_k}|^p \, \mathrm{d}x \to \int_{U_m} |u|^p \, \mathrm{d}x. \tag{4.6}$$

Hence,

$$\int_{U_m} |u|^p \, \mathrm{d}x \geqslant 1 - \varepsilon. \tag{4.7}$$

We deduce that

$$\int_{\Omega} |u|^p \, \mathrm{d}x \geqslant 1 - \varepsilon, \tag{4.8}$$

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and since ε is arbitrary small,

$$\int_\Omega |u|^p \, \mathrm{d}x = 1. \tag{4.9}$$

The norm of L^p is uniformly convex so $u_n \to u$ strongly in $L^p(\Omega)$. □

For the sake of simplicity, we assume that V has its essential infimum equal to 0. We denote by a_m the essential infimum of V on $\Omega \backslash U_m$. The assumption is the following.

$$\lim_{m \to +\infty} a_m = +\infty. \tag{4.10}$$

THEOREM 4.2. *Assume that* Ω *is an open connected set of* $\mathbb{R}^N$, V *a measurable function on* Ω *with* $\operatorname{ess\,inf}_\Omega V = 0$ *and* $V \in L^\infty_{loc}(\Omega)$. *Under* (4.1) *and* (4.10), *there exists a minimizer for the infimum*

$$\lambda_1 = \inf \left\{ \int_\Omega |\nabla u|^p \, \mathrm{d}x + \int_\Omega V(x)|u|^p \, \mathrm{d}x : u \in W^{1,p,V}(\Omega), \ \|u\|_{L^p(\Omega)} = 1 \right\}. \tag{4.11}$$

Proof. We use the notations of the beginning of the section. λ_1 exists since $W^{1,p,V}(\Omega) \neq \{0\}$ and is finite since $V \geqslant 0$. Let (u_n) be a minimizing sequence with $\|u_n\|_{L^p(\Omega)} = 1$ and m an integer that we shall choose later.

$$F_V(u_n) = \int_\Omega |\nabla u_n|^p \, \mathrm{d}x + \int_\Omega V(x)|u_n|^p \, \mathrm{d}x \geqslant \int_{\Omega \backslash U_m} V(x)|u_n|^p \, \mathrm{d}x. \tag{4.12}$$

Hence,

$$F_V(u_n) \geqslant a_m \int_{\Omega \backslash U_m} |u_n|^p \, \mathrm{d}x. \tag{4.13}$$

So

$$\int_{\Omega \backslash U_m} |u_n|^p \, \mathrm{d}x \leqslant \frac{F_V(u_n)}{a_m}. \tag{4.14}$$

The limit of $F_V(u_n)$ is λ_1 and $a_m \to +\infty$. If $\varepsilon > 0$, there exists an integer m such that

$$\frac{\sup_{n \in \mathbb{N}} F_V(u_n)}{a_m} < \varepsilon, \tag{4.15}$$

and consequently,

$$\int_{U_m} |u_n|^p \, \mathrm{d}x \geqslant 1 - \varepsilon, \tag{4.16}$$

for all $n \geqslant 0$. (u_n) is a bounded sequence of $W^{1,p}(\Omega)$ so there exists a function u in $W^{1,p}(\Omega)$ such that, up to a subsequence, (u_n) converges weakly to u in $W^{1,p}(\Omega)$.

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Moreover, for all $\mathcal{C}^1$ bounded open connected set of Ω, from the Rellich-Kondrachov theorem, there exists a subsequence of (u_n) which converges strongly for the L^p-norm of this open set. We apply the Lemma. So (u_n) tends strongly in $L^p(\Omega)$ to u. For the gradient, we have

$$\liminf \int_\Omega |\nabla u_n|^p \, \mathrm{d}x \geqslant \int_\Omega |\nabla u|^p \, \mathrm{d}x. \tag{4.17}$$

At this stage in the proof, once again, let us pick up an integer number m. It is easy to see that

$$\begin{aligned} F_V(u_n) &= \int_\Omega |\nabla u_n|^p \, \mathrm{d}x + \int_\Omega V(x)|u_n|^p \, \mathrm{d}x \\ &\geqslant \int_\Omega |\nabla u_n|^p \, \mathrm{d}x + \int_{U_m} V(x)|u_n|^p \, \mathrm{d}x. \end{aligned} \tag{4.18}$$

We take the inferior limit in the inequality. Since V is bounded on U_m,

$$\lambda_1 \geqslant \int_\Omega |\nabla u|^p \, \mathrm{d}x + \int_{U_m} V(x)|u|^p \, \mathrm{d}x. \tag{4.19}$$

m is arbitrary, hence

$$\lambda_1 \geqslant \int_\Omega |\nabla u|^p \, \mathrm{d}x + \int_\Omega V(x)|u|^p \, \mathrm{d}x. \tag{4.20}$$

This implies that u belongs to $W^{1,p,V}(\Omega)$. Moreover, from the definition of λ_1, we get the converse inequality. □

Remark 4.3. V has a well in each U_m. $a_m \to +\infty$ replaces $h \to 0$ for the concentration of the minimizing sequence. Moreover, assumptions of this theorem are close to those of the Cwickel-Lieb-Rosenblyum formula.

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