L^∞ estimates and uniqueness results for nonlinear parabolic equations with gradient absorption terms

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Abstract

Here we study the nonnegative solutions of the viscous Hamilton-Jacobi problem

$$\begin{cases} u_t - \nu\Delta u + |\nabla u|^q = 0, \\ u(0) = u_0, \end{cases}$$

in $Q_{\Omega,T} = \Omega \times (0,T)$, where $q > 1, \nu \geqq 0, T \in (0,\infty]$, and $\Omega = \mathbb{R}^N$ or Ω is a smooth bounded domain, and $u_0 \in L^r(\Omega), r \geqq 1$, or $u_0 \in \mathcal{M}_b(\Omega)$. We show L^∞ decay estimates, valid for *any weak solution*, *without any conditions* as $|x| \to \infty$, and *without uniqueness assumptions*. As a consequence we obtain new uniqueness results, when $u_0 \in \mathcal{M}_b(\Omega)$ and $q < (N+2)/(N+1)$, or $u_0 \in L^r(\Omega)$ and $q < (N+2r)/(N+r)$. We also extend some decay properties to quasilinear equations of the model type

$$u_t - \Delta_p u + |u|^{\lambda-1} u|\nabla u|^q = 0$$

where $p > 1, \lambda \geqq 0$, and u is a signed solution.

Keywords Viscous Hamilton-Jacobi equation; quasilinear parabolic equations with gradient terms; regularity; decay estimates; regularizing effects; uniqueness results.

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1 Introduction

Here we study some parabolic equations with eventual gradient absorption terms. We are mainly concerned by the nonnegative solutions of the well known viscous parabolic Hamilton-Jacobi equation

$$u_t - \nu\Delta u + |\nabla u|^q = 0 \tag{1.1}$$

in $Q_{\Omega,T} = \Omega \times (0,T)$, $T \leqq \infty$, where $q > 1, \nu \geqq 0$, and $\Omega = \mathbb{R}^N$, or Ω is a smooth bounded domain of $\mathbb{R}^N$ and $u = 0$ on $\partial\Omega \times (0,T)$. We also consider the (signed) solutions of equations of the type

$$u_t - \Delta_p u + |u|^{\lambda-1} u|\nabla u|^q = 0 \tag{1.2}$$

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where $p > 1$ and Δ_p is the p-Laplacian, or more generally involving a quasilinear operator, nonnecessarily monotone,

$$u_t - \operatorname{div}(\mathrm{A}(x,t,u,\nabla u)) + g(x,u,\nabla u) = 0 \tag{1.3}$$

with natural growth conditions on A, and nonnegativity conditions

$$\mathrm{A}(x,t,u,\eta).\eta \geqq \nu\,|\eta|^p\,, \qquad g(x,u,\eta)u \geqq \gamma\,|u|^{\lambda+1}\,|\nabla u|^q \qquad \gamma \geqq 0, \nu \geqq 0, \tag{1.4}$$

where $\lambda \geqq 0$.

We denote by$\mathcal{M}_b(\Omega)$ the set of bounded Radon measures in Ω, and $\mathcal{M}_b^+(\Omega)$ the subset of nonnegative ones. We set $Q_{\Omega,s,\tau} = \Omega \times (s,\tau)\,,$ for any $0 \leqq s < \tau \leqq \infty$, thus $Q_{\Omega,T} = Q_{\Omega,0,T}$.

We study the Cauchy problem with rough initial data

$$u(.,0) = u_0, \qquad u_0 \in L^r(\Omega), r \geqq 1, \quad \text{or } u_0 \in \mathcal{M}_b(\Omega).$$

Our purpose is to give some decay estimates, and a regularizing effect L^∞ estimates, for the solutions, in terms of initial data u_0, and universal estimates when Ω is bounded, *under very few assumptions on the solutions*. In this problem two regularizing effects can occur, the first one due to the gradient term $|\nabla u|^q$, when $\gamma > 0$, the second to the operator when $\nu > 0$. A part of these estimates are well known when the solutions can be approximated by smooth solutions, and satisfy some conditions as $|x| \to \infty$ when $\Omega = \mathbb{R}^N$, of boundedness or integrability, for example semi-group solutions. Our approach is different: our results are valid for all the solutions of the equation *in a weak sense*: in the sense of distributions for the case of the Laplacian, in the renormalized sense in the case of a general operator; and we make *no assumption of existence or uniqueness*. Moreover in the case of the Hamilton-Jacobi equation in $\mathbb{R}^N$, we make *no assumption as* $|x| \to \infty$*, all our assumptions are local.* As a consequence we deduce *new uniqueness results* for equation (1.1) in $\mathbb{R}^N$ or in bounded Ω.

In order to get regularizing properties, we give at Section 2 an iteration lemma based of Moser's method, inspired by the results of [39], and we compare it to results of [34] obtained from Stampacchia's method. The Moser's method, based on the choice of test functions of the form $|u|^{\alpha-1}u$, $\alpha > 0$, appears to be well adapted to equations in a L^1 context. Since such functions are not always admissible, we combine the method with a *regularization* in case of equation (1.1) in $\mathbb{R}^N$, and a *truncature* in the case of the Dirichlet problem, for the same equation, and for the general equation (1.3).

In Section 3 we study the case of Hamilton-Jacobi equation (1.1) in $\mathbb{R}^N$, for which there is a huge literature. Among them we quote only some significative contributions and refer to the references therein: [1], [11], [6], [14], [37], see also [6], [13], [29]. One of our main results reads as follows:

Theorem 1.1 *Let $u \in L^1_{loc}(Q_{\mathbb{R}^N,T})$, with $|\nabla u| \in L^q_{loc}(Q_{\mathbb{R}^N,T})$, be any nonnegative solution of equation (1.1) in $\mathcal{D}'(Q_{\mathbb{R}^N,T})$.*

(i) Let $u_0 \in L^r(\mathbb{R}^N), r \geqq 1$. Assume that $u \in C([0,T)\,;L^r_{loc}(\mathbb{R}^N))$ and $u(.,0) = u_0$. Then $u \in C([0,T)\,;L^r(\mathbb{R}^N))$; and for any $t \in (0,T)$, $u(.,t) \in L^\infty(\mathbb{R}^N)$ and

$$\|u(.,t)\|_{L^r(\mathbb{R}^N)} \leqq \|u_0\|_{L^r(\mathbb{R}^N)}, \tag{1.5}$$

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq Ct^{-\sigma_{r,q,N}} \|u_0\|_{L^r(\mathbb{R}^N)}^{\varpi_{r,q,N}}, \qquad C = C(N,q,r),$$

where σ, ϖ are given for $q < N$ by $\sigma_{r,q,N} = 1/(rq/N + q - 1) = N\varpi_{r,q,N}/rq$; and if $\nu > 0$, $N > 2$,

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq Ct^{-\frac{N}{2r}} \|u_0\|_{L^r(\mathbb{R}^N)}, \qquad C = C(N,q,r,\nu).$$

*(ii) Let $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$ and assume that $u(.,t)$ converges weakly * to u_0 as $t \to 0$. Then $u \in C((0,T); L^1(\mathbb{R}^N))$, and for any $t \in (0,T)$, $\|u(.,t)\|_{L^r(\mathbb{R}^N)} \leqq \int_{\mathbb{R}^N} du_0$,*

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq Ct^{-\sigma_{1,q,N}} (\int_{\mathbb{R}^N} du_0)^{\varpi_{1,q,N}}, \qquad C = C(N,q),$$
$$\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq Ct^{-\frac{N}{2}} \int_{\mathbb{R}^N} du_0, \qquad C = C(N,q,\nu), \qquad \text{if } \nu > 0.$$

For any $q \leq 2$, we deduce estimates of the gradient, obtained from Bernstein technique. As a consequence we improve some uniqueness results of [11] and [14]:

Theorem 1.2 *(i) Let $1 < q < (N+2)/(N+1)$, and $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$. Then there exists a unique nonnegative function $u \in L^1_{loc}(Q_{\mathbb{R}^N,T})$, such that $|\nabla u| \in L^q_{loc}(Q_{\mathbb{R}^N,T})$, solution of equation (1.1) in $\mathcal{D}'(Q_{\mathbb{R}^N,T})$ such that*

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(.,t)\psi dx = \int_{\mathbb{R}^N} \psi du_0, \qquad \forall \psi \in C_c(\mathbb{R}^N).$$

(ii) Let $u_0 \in L^r(\mathbb{R}^N)$, $r \geqq 1$ and $1 < q < (N+2r)/(N+r)$. Then there exists a unique nonnegative solution u as above, such that $u \in C\left([0,T); L^r_{loc}(\mathbb{R}^N)\right)$ and $u(.,0) = u_0$.

We also find again the existence result of [14, Theorem 4.1] for any $u_0 \in L^r(\mathbb{R}^N)$, $r \geqq 1$, see Proposition 3.28. Finally we improve the estimate (1.5) when $q < (N+2r)/(N+r)$, see Theorem 3.30.

In Section 4 we study the Dirichlet problem in a bounded domain Ω:

$$\begin{cases} u_t - \nu\Delta u + |\nabla u|^q = 0, & \text{in } Q_{\Omega,T}, \\ u = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0 \geqq 0, \end{cases} \tag{1.6}$$

Here also the problem is the object of many works, such as [23], [7], [38], [8], [34]. We give decay properties and regularizing effects valid for *any weak solution* of the problem, in particular the universal estimate

$$\|u(.,t)\|_{L^\infty(\Omega)} \leqq Ct^{-\frac{1}{q-1}} \quad \text{in } (0,T),$$

where $C = C(N,q)$, see Theorem 4.12. And we improve the uniqueness results of [7]:

Theorem 1.3 *Assume that Ω is bounded.*

(i) Let $1 < q < (N+2)/(N+1)$, and $u_0 \in \mathcal{M}_b^+(\Omega)$. Then there exists a unique nonnegative function $u \in C((0,T); L^1(\Omega)) \cap L^1_{loc}((0,T); W_0^{1,1}(\Omega))$, such that $|\nabla u|^q \in L^1_{loc}((0,T); L^1(\Omega))$ solution of equation (1.1) in $\mathcal{D}'(Q_{\Omega,T})$ such that

$$\lim_{t\to 0} \int_\Omega u(.,t)\psi dx = \int_\Omega \psi du_0, \qquad \forall \psi \in C_b(\Omega).$$

(ii) Let $u_0 \in L^r(\Omega)$, $r \geqq 1$, and $1 < q < (N+2r)/(N+r)$. Then there exists a unique nonnegative solution u as above, such that $u \in C([0,T); L^r(\Omega))$ and $u(.,0) = u_0$.

And we show the existence of solutions for any $u_0 \in L^r(\Omega)$, $r \geqq 1$, see Proposition 4.17.

In Section 5 we extend some results of section 4 to the case of the quasilinear equations (1.3), with initial data $u_0 \in L^r(\Omega)$ or u_0 measure, and u may be a signed solution. In the case of equation

$$u_t - \Delta_p u = 0,$$

with rough initial data, several local or global L^∞ estimates and Harnack properties have been obtained in the last decades, see for example the pioneer works of [39], [25], [26], [31], and [24], [20] and references therein. Regularizing properties for equation (1.2) are given in [33] in an hilbertian context in case $g = 0$ or $p = 2$.

For this kind of problems, we combine our iteration method of Section 2 with a notion of *renormalized solution*, developped by many authors [18], [33],[36], well adapted to our context of rough initial data: we do not require that $u(.,t) \in L^2(\Omega)$, but we only assume that the truncates $T_k(u)$ of u by $k > 0$ lie in $L^p((0,T); W^{1,p}(\Omega))$. We prove decay and L^∞ estimates of the following type: if $\gamma > 0$, for any $r \geqq 1$, $p > 1$ and for example $q \in (1,N)$, then

$$\|u(.,t)\|_{L^\infty(\Omega)} \leqq Ct^{-\sigma} \|u_0\|_{L^r(\Omega)}^{\varpi}, \qquad \sigma = \frac{1}{\frac{rq}{N} + \lambda + q - 1} = \frac{N}{rq}\varpi, \tag{1.7}$$

and we deduce a universal estimate as before. If $\nu > 0$, then for any $r \geqq 1$, and any $p \in (1,N)$ such that $p > 2N/(N+2)$,

$$\|u(.,t)\|_{L^\infty(\Omega)} \leqq Ct^{-\tilde{\sigma}} \|u_0\|_{L^r(\Omega)}^{\tilde{\varpi}}, \qquad \tilde{\sigma} = \frac{1}{\frac{rp}{N} + p - 2} = \frac{N}{rp}\tilde{\varpi}. \tag{1.8}$$

Such methods can also be extended to porous media equations, and doubly nonlinear equations involving operators of the form $u \mapsto -\Delta_p(|u|^{m-1} u)$.

2 A Moser's type iteration lemma

We begin by a simple bootstrap property, used for example in [39]: We recall the proof for simplicity:

Lemma 2.1 *Let $\omega \in (0,1)$ and $\sigma > 0$, and $K > 0$. Let y be any positive function on $(0,T)$ such that for any $0 < s < t < T$*

$$y(t) \leqq K(t-s)^{-\sigma} y^{\omega}(s)$$

and $y(t) \leqq Mt^{-\sigma}$ for some $M > 0$. Then y satisfies an estimate independent of M : for any $t \in (0,T)$

$$y(t) \leqq 2^{\sigma(1-\omega)^{-2}} (Kt^{-\sigma})^{(1-\omega)^{-1}}$$

Proof. We get by induction

$$\begin{cases} y(t) \leqq K2^{\sigma} t^{-\sigma} y^{\omega}(t/2), \\ y^{\omega}(t/2) \leqq K^{\omega} 2^{2\sigma\omega} t^{-\sigma\omega} y^{\omega^2}(t/2^2), \dots \\ y^{\omega^{n-1}}(t/2^{n-1}) \leqq K^{\omega^{n-1}} 2^{n\sigma\omega^{n-1}} t^{-\sigma\omega^{n-1}} y^{\omega^n}(t/2^n), \\ y^{\omega^n}(t/2^n) \leqq 2^{n\sigma\omega^n} t^{-\sigma\omega^n} M^{\omega^n}. \end{cases}$$

Then

$$y(t) \leqq K^{\sum_{k=0}^{n-1} \varpi^k} t^{-\sigma \sum_{k=0}^{n} \varpi^k} 2^{\sigma \sum_{k=0}^{n} (k+1)\varpi^k)} M^{\omega^{n+1}}$$

and going to the limit as $n \to \infty$, we get the conclusion, since $\lim M^{\omega^{n+1}} = 1$. ■

In the sequel we use the following iteration property:

Lemma 2.2 *Let $m > 1$, $\theta > 1$ and $\lambda \in \mathbb{R}$ and $C_0 > 0$. Let $v \in C([0,T); L^1_{loc}(\Omega))$ be nonnegative, and $v_0 = v(x,0) \in L^r(\Omega)$ for some $r \geqq 1$ such that*

$$r > \frac{N}{m}(1-m-\lambda); \tag{2.1}$$

If $r > 1$ we assume that for any $0 \leqq s < t < T$ and any $\alpha \geqq r-1$ there holds

$$\frac{1}{\alpha+1} \int_{\Omega} v^{\alpha+1}(.,t)dx + \frac{C_0}{\beta^q} \int_s^t (\int_{\Omega} v^{\beta m\theta}(.,\tau)dx)^{\frac{1}{\theta}} d\tau \leqq \frac{1}{\alpha+1} \int_{\Omega} v^{\alpha+1}(.,s)dx \leqq \infty \tag{2.2}$$

where

$$\beta = \beta(\alpha) = 1 + \frac{\alpha+\lambda}{m}.$$

If $r = 1$ we make one of the two following assumptions:

(H_1) (2.2) holds for any $\alpha \geqq 0$,

(H_2) $\int_{\Omega} v(.,t)dx \leqq \int_{\Omega} v_0 dx$ for any $t \in (0,T)$, and $v_0 \in L^{\rho}(\Omega)$ for some $\rho > 1$, and (2.2) holds for any $\alpha \geqq \rho - 1$.

(i) Then there exists $C > 0$, depending on N, m, r, λ, C_0, and eventually ρ, such that for any $t \in (0,T)$,

$$\|v(.,t)\|_{L^{\infty}(\Omega)} \leqq Ct^{-\sigma_{r,m,\lambda,\theta}} \|v_0\|_{L^r(\Omega)}^{\varpi_{r,m,\lambda,\theta}}, \tag{2.3}$$

where

$$\sigma_{r,m,\lambda,\theta} = \frac{1}{\frac{r}{\theta'} + \lambda + m - 1} = \frac{\theta'}{r} \varpi_{r,m,\lambda,\theta}. \tag{2.4}$$

(ii) Moreover if $\lambda + m - 1 > 0$, and Ω is bounded, then a universal estimate holds, with a constant C depending on $N, m, r, \lambda, C_0, |\Omega|$ and eventually ρ : for any $t \in (0,T)$,

$$\|v(.,t)\|_{L^{\infty}(\Omega)} \leqq Ct^{-\frac{1}{m-1+\lambda}}.$$

Proof. (i) • **Case** $r>1$. Let $\alpha \geqq r-1$. From (2.2), It implies the decay: $\int_\Omega v^{\alpha+1}(t)dx$ is decreasing for $t>s$. And $\int_\Omega v^{\beta m\theta}(.,\xi)dx$ is finite for almost any $\xi \in (s,t)$. From assumption (2.1), and $\alpha \geqq r-1$, there holds $\beta m\theta > \alpha+1$. Replacing α by $\beta m\theta - 1$, taking $\xi_n \to s$ we have $\int_\Omega v^{\beta m\theta}(.,\xi)dx \leqq \int_\Omega u^{\beta m\theta}(.,\xi_n)dx$ for any $\xi > \xi_n$, then

$$(t-\xi_n)(\int_\Omega v^{\beta m\theta}(.,\xi_n)dx)^{\frac{1}{\theta}} \leqq \int_{\xi_n}^t (\int_\Omega v^{\beta m\theta}(.,\xi)dx)^{\frac{1}{\theta}} dt$$

and also

$$\int_\Omega v^{\alpha+1}(.,t)dx + \frac{C_0(\alpha+1)}{\beta^q}(\int_{\xi_n}^t (\int_\Omega v^{\beta m\theta}(.,\xi)dx)^{\frac{1}{\theta}} dt \leqq \int_\Omega v^{\alpha+1}(.,\xi_n)dx \leqq \int_\Omega v^{\alpha+1}(.,s)dx.$$

thus

$$\int_\Omega u^{\alpha+1}(.,t)dx + \frac{C_0(\alpha+1)}{\beta^q}(t-\xi_n)(\int_\Omega v^{\beta m\theta}(.,\xi_n)dx)^{\frac{1}{\theta}} \leqq \int_\Omega v^{\alpha+1}(.,s)dx.$$

Then going to the limit as $n\to\infty$, since $v \in C([0,T]\,;L^1_{loc}(\Omega))$ when $\xi_n \to s$, $v(.,\xi_n) \to v(.,s)$ in $L^1(\Omega)$, and after extraction, a.e. in Ω. Then from the Fatou lemma,

$$\int_\Omega v^{\alpha+1}(.,t)dx + \frac{C_0(\alpha+1)}{\beta^q}(t-s)(\int_\Omega v^{\beta m\theta}(.,s)dx)^{\frac{1}{\theta}} \leqq \int_\Omega v^{\alpha+1}(.,s)dx.$$

Hence

$$\|v(t)\|^{\beta m\theta}_{L^{\beta m\theta}(\Omega)} \leqq \left(\frac{\beta^m}{C_0(\alpha+1)}\frac{1}{t-s}\|v(s)\|^{\alpha+1}_{L^{\alpha+1}(\Omega)}\right)^\theta, \tag{2.5}$$

We start from $s=0$, we have $v_0 \in L^r(\Omega)$.

We take $\alpha_0 = r-1$, thus $\int_\Omega v^{\alpha_0+1}(t)dx$ is finite, and set $\beta_0 = 1 + (r-1+\lambda)/m$. We define increasing sequences $(t_n), (\alpha_n), (r_n), (\beta_n)$, by $t_0 = 0$, $r_0 = r$ and for any $n \geqq 1$,

$$t_n = t(1-\frac{1}{2^n}), \qquad r_n = \alpha_n + 1, \qquad \beta_n = 1 + \frac{\alpha_n+\lambda}{m}, \qquad r_{n+1} = \beta_n m\theta = (r_n+\lambda+m-1)\theta.$$

In (2.5), we replace $s, t, \alpha, \beta m\theta$, by $t_n, t_{n+1} r_n, r_{n+1}$, and get

$$\|v(t_{n+1})\|_{L^{r_{n+1}}(\Omega)} \leqq \left(\frac{1}{C_0(m\theta)^m}\frac{r^m_{n+1}}{r_n}\frac{1}{t_{n+1}-t_n}\right)^{\frac{\theta}{r_{n+1}}} \|v(t_n)\|^{\frac{\theta.r_n}{r_{n+1}}}_{L^{r_n}(\Omega)}. \tag{2.6}$$

It follows that

$$\|v(t_{n+1})\|_{L^{r_{n+1}}(\Omega)} \leqq I_n J_n L_n \|u_0\|^{\frac{\theta^{n+1}.r}{r_{n+1}}}_{L^r(\Omega)} \tag{2.7}$$

where

$$I_n = \prod_{k=1}^{n+1}(\frac{r_k^m}{r_{k-1}})^{\frac{\theta^{n+2-k}}{r_{n+1}}}, \quad J_n = \prod_{k=1}^{n+1}\left(\frac{1}{t_k-t_{k-1}}\right)^{\frac{\theta^{n+2-k}}{r^{n+1}}}, \quad L_n = (C_0(m\theta)^q)^{-\sum_{k=1}^{n+1}\frac{\theta^{n+2-k}}{r_{n+1}}}.$$

Since $r_n = \theta^n(r + (\lambda+m-1)\theta'(1-\theta^{-n}))$, it is clear that

$$\lim \frac{\theta^{n+1}r}{r_{n+1}} = \varpi_{r,m,\lambda,\theta}, \qquad \lim \frac{1}{r_{n+1}}\sum_{k=1}^{n+1}\theta^{n+2-k} = \sigma_{r,m,\lambda}, \qquad \lim \sum_{k=1}^{n+1} k\theta^{1-k} = \theta'^2 \tag{2.8}$$

Thus, it follows

$$\lim J_n = 2^{-\frac{\varpi_{r,m,\lambda,\theta}}{r}\theta'^2}, \qquad \lim L_n = (C_0(m\theta)^q)^{-\sigma_{r,m,\lambda,\theta}}. \tag{2.9}$$

And I_n has a finite limit $\ell = \ell(N, m, r, \lambda)$ as $n \to \infty$. Indeed,

$$\ln I_n = \frac{m}{r_{n+1}} \sum_{k=1}^{n+1} \theta^{n+2-k} \ln r_k - \frac{1}{r_{n+1}} \sum_{k=0}^{n} \theta^{n+1-k} \ln r_k = \frac{\theta^{n+1}}{r_{n+1}} (m\theta \sum_{k=1}^{n+1} \theta^{-k} \ln r_k - \sum_{k=0}^{n} \theta^{-k} \ln r_k)$$

and the sum $S = \sum_{k=0}^{n} \theta^{-k} \ln r_k$ is finite, since $r_k \leqq \theta^k(r + |\lambda + m - 1|\,\theta')$. Then I_n has a finite limit $\ell = \ell(N, m, r, \lambda, \theta) = \exp(r^{-1}\varpi_{r,m,\lambda}((m\theta - 1)S - m\theta \ln r))$. Thus we can go to the limit in (2.7), and the conclusion follows.

• **Case** $r = 1$. If (H$_1$) holds we can take $\alpha_0 = r - 1 = 0$ and the proof is done. Next assume (H$_2$) Then we obtain, for any $0 \leqq s < t < T$, and a constant C as before,

$$\begin{aligned}
\|v(.,t)\|_{L^\infty(\Omega)} &\leqq C(t-s)^{-\sigma_{\rho,m,\lambda,\theta}} \|v(.,s)\|_{L^\rho(\Omega)}^{\varpi_{\rho,m,\lambda,\theta}} \\
&\leqq C(t-s)^{-\sigma_{\rho,m,\lambda}} \|v(.,s)\|_{L^\infty(\Omega)}^{\varpi_{\rho,m,\lambda,\theta}(\rho-1)/\rho} \|v(.,s)\|_{L^1(\Omega)}^{\varpi_{\rho,m,\lambda,\theta}/\rho} \\
&\leqq C\|v_0\|_{L^1(\Omega)}^{\varpi_{\rho,m,\lambda,\theta}/\rho} (t-s)^{-\sigma_{\rho,m,\lambda}} \|v(.,s)\|_{L^\infty(\Omega)}^{\varpi_{\rho,m,\lambda,\theta}(\rho-1)/\rho}
\end{aligned}$$

Let $y(t) = \|v(.,t)\|_{L^\infty(\Omega)}$. We can apply Lemma 2.1 to y, with

$$\sigma = \sigma_{\rho,m,\lambda,\theta}, \qquad \omega = \frac{\varpi_{\rho,m,\lambda,\theta}}{\rho'}, \qquad K = C\|v_0\|_{L^1(\Omega)}^{\varpi_{\rho,m,\lambda,\theta}/\rho}, \qquad M = C\|v_0\|_{L^\rho(\Omega)}^{\varpi_{\rho,m,\lambda,\theta}/\rho}.$$

Indeed $\omega < 1$ from assumption (2.1) with $r = 1$. Then there holds

$$\|v(.,t)\|_{L^\infty(\Omega)} \leqq 2^{\sigma(1-\omega)^{-2}} (Kt^{-\sigma})^{(1-\omega)^{-1}} = 2^{\sigma(1-\omega)^{-2}} C^{(1-\omega)^{-1}} t^{-\sigma(1-\omega)^{-1}} \|v_0\|_{L^1(\Omega)}^{\varpi_{\rho,q,\lambda,\theta}/\rho((1-\omega))}.$$

And we observe that $\sigma(1-\omega)^{-1} = \sigma_{1,m,\lambda,\theta}$ and $\varpi_{\rho,m,\lambda,\theta}/\rho((1-\omega)) = \varpi_{1,m,\lambda,\theta}$, then with a new constant C, now depending on ρ,

$$\|v(.,t)\|_{L^\infty(\Omega)} \leqq Ct^{-\sigma_{1,m,\lambda,\theta}} \|v_0\|_{L^1(\Omega)}^{\varpi_{1,m,\lambda,\theta}}. \tag{2.10}$$

(ii) Assume that Ω is bounded, thus $L^r(\Omega) \subset L^1(\Omega)$. We use the result with $r = 1$ and obtain (2.10), and, for any $0 < s < t < T$,

$$\|v(t)\|_{L^\infty(\Omega)} \leqq C(t-s)^{-\sigma_{1,m,\lambda,\theta}} \|v(s)\|_{L^1(\Omega)}^{\varpi_{1,m,\lambda}} \leqq C(t-s)^{-\sigma_{1,m,\lambda}} |\Omega|^{\varpi_{1,m,\lambda,\theta}} \|v(s)\|_{L^\infty(\Omega)}^{\varpi_{1,m,\lambda,\theta}}$$

where $C = C(N, m, \lambda, C_0)$ (or $C = C(N, m, \lambda, C_0, \rho)$. And $\varpi_{1,m,\lambda,\theta} < 1$, because $\lambda + m - 1 > 0$. Then we can apply Lemma 2.1, and we get

$$\|v(\tau)\|_{L^\infty(\Omega)} \leqq 2^{\sigma_{1,m,\lambda}(1-\varpi_{1,m,\lambda,\theta})^{-2}} (C\,|\Omega|^{\varpi_{1,m,\lambda,\theta}}\, t^{-\sigma_{1,m,\lambda,\theta}})^{(1-\varpi_{1,m,\lambda,\theta})^{-1}} = M\tau^{-\frac{1}{m-1+\lambda}}, \forall \tau \in (0, T).$$

with $M = M(N, m, \lambda, C_0, |\Omega|)$ (or $M = M(N, m, \lambda, C_0, |\Omega|\,, \rho)$). ■

Remark 2.3 *This lemma can be compared with the result of [34, Theorem 2.1] obtained by the Stampacchia's method. In order to obtain decay estimates for the solutions u of a parabolic equation such as (1.1) or (3.18), the Moser's method consists to take as test functions powers $|u|^{\alpha-1}u$ of u; the Stampacchia's method uses test functions of the form $(u-k)^{+}\mathrm{sign}u$. If one applies to sufficiently smooth solutions, both techniques leed to decay estimates of the same type. In the case of weaker solutions, the second method supposes that the functions $(u-k)^{+}$ are admissible in the equation, which leads to assume that $u(.,t) \in W^{1,2}(\Omega)$, see [34]. In the sequel we combine Moser's method with regularization or truncature of u, in order to admit powers as test functions.*

3 The Hamilton-Jacobi equation in $\mathbb{R}^N$

3.1 Different notions of solution

The Hamilton-Jacobi equation was first studied with smooth initial data. Let us recall the main results:

• For any nonnegative $u_0 \in C_b^2\left(\mathbb{R}^N\right)$, from [1] there a unique global solution in $C^{2,1}(\mathbb{R}^N\times[0,\infty))$ such that

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq \|u_0\|_{L^\infty(\mathbb{R}^N)}, \qquad \|\nabla u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq \|\nabla u_0\|_{L^\infty(\mathbb{R}^N)}.$$

Some estimates of the gradient, independant of ν, have been obtained **for this solution**, by using the Bersnstein technique, which consists in derivating the equation, and computing the equation satisfied by $|\nabla u|^2$: first from [32]

$$\|\nabla u(.,t)\|^q_{L^\infty(\mathbb{R}^N)} \leqq \frac{\|u_0\|_{L^\infty(\mathbb{R}^N)}}{t},$$

then from [11],

$$\|\nabla(u^{\frac{q-1}{q}})(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq C_q t^{-1/2}\|u_0\|^{\frac{q-1}{q}}_{L^\infty(\mathbb{R}^N)}, \tag{3.1}$$

$$\left\|\nabla(u^{(q-1)/q}(.,t)\right\|_{L^\infty(\mathbb{R}^N)} \leqq \frac{q-1}{q}\frac{1}{t^{1/q}}, \qquad \text{equivalently} \qquad |\nabla u(.,t)|^q \leqq \frac{u(.,t)}{t} \quad a.e.\text{in } \mathbb{R}^N \tag{3.2}$$

• For any nonnegative $u_0 \in C_b\left(\mathbb{R}^N\right)$, from [30] there exists a unique solution such that $u \in C^{2,1}(Q_{\mathbb{R}^N,\infty})$ and $u \in C(\mathbb{R}^N\times[0,\infty)\cap L^\infty(\mathbb{R}^N\times(0,\infty))$, and from [6] estimates (3.1) and (3.2) are still valid.

In case of rough initial data $u_0 \in \mathcal{M}_b(\mathbb{R}^N)$ or $u \in L^r(\mathbb{R}^N)$, $r \geqq 1$, existence results have been obtained in [11], [14] at section by using different formulations involving the semi-group of heat equation. Here we consider the solutions in a weaker sense, which does not use this formulation.

Definition 3.1 *We say that a nonnegative function u is a **weak solution** (resp. subsolution) of equation of (1.1) in $Q_{\mathbb{R}^N,T}$, if $u \in L^1_{loc}(Q_{\mathbb{R}^N,T})$, and $|\nabla u| \in L^q_{loc}(Q_{\mathbb{R}^N,T})$, and*

$$\int_0^T\int_\Omega(-u\varphi_t - u\Delta\varphi + |\nabla u|^q\varphi)dxdt = 0, \quad (resp. \leqq), \quad \forall\varphi \in \mathcal{D}^+(Q_{\mathbb{R}^N,T}). \tag{3.3}$$

Remark 3.2 *Recall that from [16], any weak solution satisfies*

$$u \in L^\infty_{loc}(Q_{\mathbb{R}^N,T}), \qquad \nabla u \in L^2_{loc}(Q_{\mathbb{R}^N,T}), \qquad u \in C((0,T); L^\rho_{loc}(\mathbb{R}^N)) \quad \forall \rho \geqq 1. \tag{3.4}$$

Hence (3.3) is equivalent to:

$$\int_0^T \int_\Omega (-u\varphi_t + \nabla u.\nabla\varphi + |\nabla u|^q \varphi) dxdt = 0, \forall \varphi \in \mathcal{D}(Q_{\mathbb{R}^N,T}), \tag{3.5}$$

and we have and there holds for any $s, \tau \in (0,T)$,

$$\int_{\mathbb{R}^N} u(.,\tau)\varphi(.,\theta)dx - \int_{\mathbb{R}^N} u(.,s)\varphi(.,s)dx + \int_s^\tau \int_{\mathbb{R}^N} (-u\varphi_t + \nabla u.\nabla\varphi + |\nabla u|^q\varphi)dxdt = 0 \tag{3.6}$$

and then for any $\psi \in C^2_c\left(\mathbb{R}^N\right)$,

$$\int_{\mathbb{R}^N} u(.,\tau)\psi dx - \int_{\mathbb{R}^N} u(.,s)\psi dx + \int_s^\tau \int_{\mathbb{R}^N} (\nabla u.\nabla\psi + |\nabla u|^q \psi dxdt = 0 \tag{3.7}$$

In this section we study the Cauchy problem

$$\left\{ \begin{array}{l} u_t - \Delta u + |\nabla u|^q = 0, \quad \text{in } Q_{\mathbb{R}^N,T}, \\ u(x,0) = u_0 \geqq 0 \quad \text{in } \mathbb{R}^N, \end{array} \right. \tag{3.8}$$

Definition 3.3 *Let $u_0 \in L^r_{loc}\left(\mathbb{R}^N\right), r \geqq 1$.*

*We say that u is a **weak L^r_{loc} solution** if u is a weak solution of (1.1) and the extension of u by u_0 at time 0 satisfies $u \in C\left([0,T); L^r_{loc}(\mathbb{R}^N)\right)$.*

*We say that u is a **weak r solution** of problem (3.8) if it is a weak solution of equation (1.1) such that*

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u^r(.,t)\psi dx = \int_{\mathbb{R}^N} u_0^r \psi dx, \qquad \forall \psi \in C_c(\mathbb{R}^N). \tag{3.9}$$

Definition 3.4 *Let u_0 be any nonnegative Radon measure in $\mathbb{R}^N$, we say that u is a **weak $\mathcal{M}_{loc}$ solution** of problem (3.8) if it is a weak solution of (1.1) such that*

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(.,t)\psi dx = \int_{\mathbb{R}^N} \psi du_0, \qquad \forall \psi \in C_c(\mathbb{R}^N), \tag{3.10}$$

Remark 3.5 *Obviously, any weak L^r_{loc} solution is a weak r solution. When $r = 1$, the notions of weak 1-solution and weak $\mathcal{M}_{loc}$ solution coincide. When $r > 1$, u is a weak L^r_{loc} solution if and only if it is a weak r solution and*

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(.,t)\psi dx = \int_{\mathbb{R}^N} u_0 \psi dx, \qquad \forall \psi \in C_c(\mathbb{R}^N). \tag{3.11}$$

Indeed if $u(.,t)$ converges to u_0 in $L^r_{loc}(\mathbb{R}^N)$ as $t \to 0$, then it satisfies (3.9) and (3.11). The converse is true. Indeed let u satisfy (3.9) and (3.11). Then $u(.,t)$ is bounded in L^r_{loc}, there exists $t_n \to 0$ such that $u(.,t) \to v$ in $\mathcal{D}'(\mathbb{R}^N)$ with $v \in L^r_{loc}$. And $u(.,t_n) \to u_0$ in $\mathcal{D}'(\mathbb{R}^N)$, then $v = u_0$, hence it is true for any $t \to 0$. Then for any nonnegative $\psi \in C_c(\mathbb{R}^N)$, $u(.,t)\psi \to u_0\psi$ weakly and in norm, thus strongly in $L^r(\mathbb{R}^N)$, thus $u(.,t)$ converges to u_0 in $L^r_{loc}(\mathbb{R}^N)$.

Other types of solutions using the semigroup of the heat equation have been introduced in ([14]):

Definition 3.6 *Let $u_0 \in L^r\left(\mathbb{R}^N\right)$. A function u is called* ***mild L^r solution*** *of problem (3.8) if $u \in C([0,T)\,; L^r\left(\mathbb{R}^N\right))$, and $|\nabla u|^q \in L^1_{loc}([0,T)\,; L^r\left(\mathbb{R}^N\right))$ and*

$$u(.,t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}|\nabla u(.,s)|^q ds \qquad \text{in } L^r(\mathbb{R}^N).$$

Here $e^{t\Delta}$ is the semi-group of the heat equation acting on $L^r\left(\mathbb{R}^N\right)$.

Definition 3.7 *Let $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$. A function u is called* ***mild $\mathcal{M}$ solution*** *of (3.8) if $u \in C_b((0,T); L^1\left(\mathbb{R}^N\right))$ and $|\nabla u|^q \in L^1_{loc}([0,T)\,; L^1\left(\mathbb{R}^N\right))$ and for any $0 < t < T$,*

$$u(.,t) = e^{t\Delta}u_0(.) - \int_0^t e^{(t-s)\Delta}|\nabla u(.,s)|^q ds \qquad \text{in } L^1(\mathbb{R}^N), \tag{3.12}$$

where $e^{t\Delta}$ is defined on $\mathcal{M}_b^+(\mathbb{R}^N)$ as the adjoint of the operator $e^{t\Delta}$ on $C_0(\mathbb{R}^N)$, the space of continuous functions on $\mathbb{R}^N$ which tend to 0 as $|x| \to \infty$.

Remark 3.8 *Every mild L^r solution is a weak L^r_{loc} solution.*

Remark 3.9 *Any mild $\mathcal{M}$ solution is a weak $\mathcal{M}_{loc}$ solution. Indeed for any $0 < \epsilon < t < T$, we find*

$$u(.,t) = e^{(t-\epsilon)\Delta}u(.,\epsilon) - \int_\epsilon^t e^{(t-s)\Delta}|\nabla u(.,s)|^q ds \qquad \text{in } L^1(\mathbb{R}^N),$$

*and $u(.,\epsilon) \in L^1(\mathbb{R}^N)$, then u is a weak solution on (ϵ, T), then on $(0,T)$. As $t \to 0$, $u(.,t) - e^{t\Delta}u_0(.)$ converges to 0 in $L^1(\mathbb{R}^N)$, then weakly *, and $e^{t\Delta}u_0(.) \to u_0$ weakly *, then (3.10) holds.*

Another definition of solution with initial data measure was given in ([11]):

Definition 3.10 *Let $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$. A function u is called* ***weak semi-group solution*** *if $u \in C((0,T); L^1\left(\mathbb{R}^N\right))$ and $|\nabla u|^q \in L^1_{loc}([0,T)\,; L^1\left(\mathbb{R}^N\right))$ and for any $0 < \epsilon < t < T$,*

$$u(.,t) = e^{(t-\epsilon)\Delta}u(.,\epsilon) - \int_\epsilon^t e^{(t-s)\Delta}|\nabla u(.,s)|^q ds \qquad \text{in } L^1(\mathbb{R}^N), \tag{3.13}$$

$$\lim_{t\to 0}\int_{\mathbb{R}^N} u(.,t)\varphi dx = \int_{\mathbb{R}^N}\varphi du_0, \qquad \forall \varphi \in C_b(\mathbb{R}^N), \tag{3.14a}$$

We first prove that the two definitions coincide:

Lemma 3.11 *Let $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$. Then*

$$u \text{ is a mild } \mathcal{M} \text{ solution of (3.8)} \Longleftrightarrow u \text{ is a weak semi-group solution of (3.8).}$$

Proof. (i) Let u be a mild $\mathcal{M}$ solution. Then clearly (3.13) holds. Moreover for any $\psi \in C_0\left(\mathbb{R}^N\right)$, from the assumption on the gradient,

$$< e^{t\Delta}\mu_0, \psi >=< \mu_0, e^{t\Delta}\psi >= \int_{\mathbb{R}^n} e^{t\Delta}\psi d\mu_0 = \int_{\mathbb{R}^N} (u(.,t) + \int_0^t e^{(t-s)\Delta}|\nabla u(.,s)|^q ds)\psi dx$$

By approximation the relation extends to any $\varphi \in C_b\left(\mathbb{R}^N\right)$:

$$\begin{aligned}\int_{\mathbb{R}^N} e^{t\Delta}\varphi d\mu_0 &= \int_{\mathbb{R}^N} u(.,t)\varphi dx + \int_{\mathbb{R}^N} (\int_0^t e^{(t-s)\Delta}|\nabla u(.,s)|^q ds)\varphi dx \\ &= \int_{\mathbb{R}^N} u(.,t)\varphi dx + \int_0^t \int_{\mathbb{R}^N} |\nabla u|^q \varphi dx ds\end{aligned}$$

since the measure is bounded. And from the integrability of the gradient and the Lebesgue theorem in $L^1(\mathbb{R}^N, d\mu_0)$, we deduce

$$\lim_{t\to 0} \int_0^t \int_{\mathbb{R}^N} |\nabla u|^q \varphi dx ds = 0, \qquad \lim_{t\to 0} \int_{\mathbb{R}^N} e^{t\Delta}\varphi d\mu_0 = \int_{\mathbb{R}^N} \varphi d\mu_0,$$

since $\left\|e^{t\Delta}\varphi\right\|_{L^\infty(\mathbb{R}^N)} \leq \|\varphi\|_{L^\infty(\mathbb{R}^N)}$ and $e^{t\Delta}\varphi$ converges to φ everywhere as $t \to 0$; thus (3.14a) holds.

(ii) Let u be a weak semi-group solution. Then obviously $u \in C_b((0,T); L^1\left(\mathbb{R}^N\right))$. As $\epsilon \to 0$, we have

$$\lim_{\epsilon\to 0} \int_\epsilon^t e^{(t-s)\Delta}|\nabla u(.,s)|^q ds = \int_0^t e^{(t-s)\Delta}|\nabla u(.,s)|^q ds \qquad \text{in } L^1(\mathbb{R}^N).$$

Then

$$\lim_{\epsilon\to 0} e^{(t-\epsilon)\Delta}u(.,\epsilon) = u(.,t) + \int_0^t e^{(t-s)\Delta}|\nabla u(.,s)|^q ds \qquad \text{in } L^1(\mathbb{R}^N).$$

Moreover (3.14a) entails that that $u(.,\epsilon) \to u_0$ in $\mathcal{S}'(\mathbb{R}^N)$ and

$$\lim_{\epsilon\to 0} e^{(t-\epsilon)\Delta}u(.,\epsilon) = e^{t\Delta}u_0 \qquad \text{in } \mathcal{S}'(\mathbb{R}^N); \tag{3.15}$$

indeed for any $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$\begin{aligned}\left|< e^{(t-\epsilon)\Delta}u(.,\epsilon) - e^{t\Delta}u_0, \varphi >\right| &\leq \left|< e^{t\Delta}(u(.,\epsilon) - u_0(.), \varphi >\right| + \left|\int_{\mathbb{R}^N} (u(x,\epsilon)((e^{(t-\epsilon)\Delta} - e^{t\Delta})\varphi)(x)dx\right| \\ &\leq \left|< e^{t\Delta}(u(.,\epsilon) - u_0(.), \varphi >\right| \\ &\quad + \|u(.,\epsilon)\|_{L^1(\mathbb{R}^N)} \left\|(e^{(t-\epsilon)\Delta} - e^{t\Delta})\varphi\right\|_{L^\infty(\mathbb{R}^N)})\end{aligned}$$

and $e^{t\Delta} is$ continuous on $\mathcal{S}(\mathbb{R}^N)$. Then for any $\varphi \in \mathcal{S}(\mathbb{R}^N)$, we have

$$< e^{t\Delta}u_0, \varphi >= \int_{\mathbb{R}^n} u(.,t)\varphi dx + \int_{\mathbb{R}^n} (\int_0^t e^{(t-s)\Delta}|\nabla u(.,s)|^q ds)\varphi dx$$

which extends to any $\varphi \in C_0(\mathbb{R}^N)$ by density. Thus (3.12) follows. ■

Let us recall the main existence results using semi-groups:

• If $1 < q < (N+2)/(N+1)$, for any $u_0 \in \mathcal{M}_b(\mathbb{R}^N)$, from [11], there exists a weak semi-group solution u of problem (3.8), obtained by approximation, and $u \in C^{2,1}(Q_{\mathbb{R}^N,\infty})$. The existence of a mild $\mathcal{M}$ solution is also proved in [14] from the Banach fixed point theorem, and the notions are equivalent from Lemma 3.11. In any case uniqueness results are obtained *under additional conditions of punctual or integral conditions on the gradient.*

• If $u_0 \in L^r(\mathbb{R}^N)$, $r \geqq 1$, and $r > N(q-1)/(2-q)$, which means $q < (N+2r)/(N+r)$, there exists a mild L^r solution of (3.8), and uniqueness holds in the class of pointwise mild solutions such that $u \in C([0,T); L^r(\mathbb{R}^N)) \cap C((0,T); W^{1,qr}(\mathbb{R}^N))$, from [14, Theorem 2.1]. Moreover if $q \leq 2$, there exists a pointwise mildsolution of (3.8) for any $r \geqq 1$ but uniqueness is not known for $q < 2$, see [14, Theorem 4.1]. For $q > 2$, existence holds under the restriction that u_0 is a limit of a monotone sequence of continuous functions, and is not known in the general case.

Remark 3.12 *All the definitions of semi-group solutions assume an integrability property of $|\nabla u|^q$, global in space. Observe also that (3.14a) is assumed for any $\varphi \in C_b(\mathbb{R}^N)$. On the contrary, our definitions of weak solutions are local in space, they do not require such global properties.*

Finally we mention another weaker form of semi-group solutions, given in ([14]), which will be used in the sequel:

Definition 3.13 *Let $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$. Then u is a pointwise mild solution of (3.8) if $u \in L^1_{loc}(Q_{\mathbb{R}^N,T})$, and $|\nabla u|^q \in L^1_{loc}(Q_{\mathbb{R}^N,T})$, and*

$$u(x,t) = (e^{t\Delta}u_0)(x) - \int_0^t \int_{\mathbb{R}^N} g(x-y,t-s)|\nabla u(y,s)|^q dyds \qquad \text{for a.e. } (x,t) \in Q_{\mathbb{R}^N,T},$$

where g is the heat kernel.

Remark 3.14 *For $r \geqq 1$, it is clear that every mild L^r solution is a pointwise mild solution. If $u_0 \in L^1(\mathbb{R}^N)$ every pointwise mild solution is a mild L^1 solution; if $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$, every pointwise mild solution, is a mild $\mathcal{M}$ solution. see [14, Proposition 1.1 and Remark 1.2].*

3.2 Decay of the norms

Next we show a *decay result* for the solutions of Hamilton Jacobi equations, which is valid for any $q > 1$, and for *all the weak solutions*, *with no condition of boundedness at infinity*.

When $q \leqq 2$, any weak solution u of equation (1.1) is smooth: $u \in C^{2,1}(Q_{\mathbb{R}^N,T})$, from [16, Theorem 2.15]. Since it may be false for $q > 2$, we regularize u by convolution, setting

$$u_\varepsilon = u * \varrho_\varepsilon,$$

where $(\varrho_\varepsilon)_{\varepsilon>0}$ is a sequence of mollifiers. We recall that for given $0 < s < \tau < T$, and ε small enough, u_ε is a *subsolution* of equation (1.1), see [16]:

$$(u_\varepsilon)_t - \nu\Delta u_\varepsilon + |\nabla u_\varepsilon|^q \leqq 0, \qquad \text{in } Q_{\mathbb{R}^N,s,\tau} \tag{3.16}$$

Theorem 3.15 *Assume $q > 1$. Let $r \geqq 1$. Let $u_0 \in L^r(\mathbb{R}^N)$ be nonnegative. Let u be any nonnegative weak r solution of problem (3.8).*

(i) Then $u(.,t) \in L^r\left(\mathbb{R}^N\right)$ for any $t \in (0,T)$, and

$$\int_{\mathbb{R}^N} u^r(.,t)dx \leqq \int_{\mathbb{R}^N} u_0^r dx. \tag{3.17}$$

(ii) Moreover $u^{r-1}|\nabla u|^q \in L^1_{loc}([0,T);L^1\left(\mathbb{R}^N\right))$; and $u^{r-2}|\nabla u|^2 \in L^1_{loc}([0,T);L^1\left(\mathbb{R}^N\right))$ if $r > 1$ and $\nu > 0$; and for any $t \in (0,T)$,

$$\int_{\mathbb{R}^N} u^r(.,t)dx + r\int_0^t\int_{\mathbb{R}^N} u^{r-1}|\nabla u|^q dxdt + r(r-1)\nu\int_0^t\int_{\mathbb{R}^N} u^{r-2}|\nabla u|^2 dxdt = \int_{\mathbb{R}^N} u_0^r dx, \qquad \text{if } r > 1, \tag{3.18}$$

$$\int_{\mathbb{R}^N} u(.,t)dx + \int_0^t\int_{\mathbb{R}^N} |\nabla u|^q dxdt = \int_{\mathbb{R}^N} u_0 dx, \qquad \text{if } r = 1, \tag{3.19}$$

$$\lim_{t\to 0}\int_{\mathbb{R}^N} u^r(.,t)dx = \int_{\mathbb{R}^N} u_0^r dx. \tag{3.20}$$

(iii) $u^{q-1+r} \in L^1_{loc}(([0,T);W^{1,1}\left(\mathbb{R}^N\right))$, and if $\nu > 0$, then $u^{r/2} \in L^2_{loc}([0,T);W^{1,2}\left(\mathbb{R}^N\right))$.
(iv) If u is a weak L^r_{loc} solution, then $u \in C([0,T);L^r\left(\mathbb{R}^N\right))$.

Proof. (i) **First step: case** $q' > N/r$. That means $r \geqq N$ or q is small enough: $1 < q < N/(N-r)$.

Let $0 < s < \tau < T$ be fixed and $\varepsilon > 0$ small enough. Let $\delta > 0$, and $u_{\varepsilon,\delta} = u_\varepsilon + \delta$. For any $R > 0$, we consider $\xi(x) = \xi_R(x) = \psi(x/R)$,where $\psi(x) \in [0,1]$, $\psi(x) = 1$ for $|x| \leqq 1, \psi(x) = 0$ for $|x| \geqq 2$. Then multiplying (3.16) by $u_{\varepsilon,\delta}^{r-1}\xi^\lambda$ where $\lambda > 0$, we get

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{r}\int_{\mathbb{R}^N} u_{\varepsilon,\delta}^r\xi^\lambda dx\right) + (r-1)\nu\int_{\mathbb{R}^N} u_{\varepsilon,\delta}^{r-2}\left|\nabla u_{\varepsilon,\delta}\right|^2\xi^\lambda dx + \int_{\mathbb{R}^N}|\nabla u_{\varepsilon,\delta}|^q u_{\varepsilon,\delta}^{r-1}\xi^\lambda dx \\ &\leqq -\lambda\int_{\mathbb{R}^N} u_{\varepsilon,\delta}^{r-1}\xi^{\lambda-1}\nabla u_{\varepsilon,\delta}.\nabla\xi dx,\end{aligned}$$

and from the Hölder inequality

$$\lambda\int_{\mathbb{R}^N} u_{\varepsilon,\delta}^{r-1}\xi^{\lambda-1}\left|\nabla u_{\varepsilon,\delta}\right|\left|\nabla\xi\right|dx \leqq \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u_{\varepsilon,\delta}|^q u_{\varepsilon,\delta}^{r-1}\xi^\lambda dx + C(q,\lambda)\int_{\mathbb{R}^N} u_{\varepsilon,\delta}^{r-1}\xi^{\lambda-q'}\left|\nabla\xi\right|^{q'}dx,$$

$$\int_{\mathbb{R}^N} u_{\varepsilon,\delta}^{r-1}\xi^{\lambda-q'}\left|\nabla\xi\right|^{q'}dx \leqq \left(\int_{\mathbb{R}^N} u_{\varepsilon,\delta}^r\xi^\lambda dx\right)^{1/r'}\left(\int_{\mathbb{R}^N}\xi^{\frac{\lambda}{r}-q'}\left|\nabla\xi\right|^{rq'}dx\right)^{1/r}.$$

Choosing $\lambda = rq'$ we deduce

$$\frac{d}{dt}\left(\left(\int_{\mathbb{R}^N} u_{\varepsilon,\delta}^r\xi^\lambda dx\right)^{1/r}\right) \leqq C(q,\lambda)\left(\int_{\mathbb{R}^N}\left|\nabla\xi\right|^{rq'}dx\right)^{1/r} \leqq CR^{\frac{N}{r}-q'}$$

where $C = C(N,q,\alpha,\psi)$. By integration, for any $0 < s \leqq \sigma < t \leqq \tau$,

$$\left(\int_{\mathbb{R}^N} u_{\varepsilon,\delta}^r(.,t)\xi^\lambda dx\right)^{1/r} \leqq \left(\int_{\mathbb{R}^N} u_{\varepsilon,\delta}^r(.,\sigma)\xi^\lambda dx\right)^{1/r} + C\tau R^{\frac{N}{r}-q'}.$$

with a new constant C as above. Let $R_0 > 0$ be fixed and take $R > R_0$, thus

$$\left(\int_{B_{R_0}} u^r_{\varepsilon,\delta}(.,t)dx\right)^{1/r} \leqq \left(\int_{B_{2R}} u^r_{\varepsilon,\delta}(.,\sigma)\xi^\lambda dx\right)^{1/r} + C\tau R^{\frac{N}{r}-q'}$$

Then we make successively $\delta \to 0$, and then $\varepsilon \to 0$. From (3.9), we deduce that

$$\left(\int_{B_{R_0}} u(.,t)^r dx\right)^{1/r} \leqq \left(\int_{\mathbb{R}^N} u(.,\sigma)^r\xi^\lambda dx\right)^{1/r} + C\tau R^{\frac{N}{r}-q'} \tag{3.21}$$

and then from (3.9) we can make $\sigma \to 0$ and obtain

$$\left(\int_{B_{R_0}} u(.,t)^r dx\right)^{1/r} \leqq \left(\int_{\mathbb{R}^N} u(.,\sigma)^r\xi^\lambda dx\right)^{1/r} + C\tau R^{\frac{N}{r}-q'} \leqq \left(\int_{\mathbb{R}^N} u_0^r dx\right)^{1/r} + C\tau R^{\frac{N}{r}-q'}$$

and finally we make $R \to \infty$ and then $R_0 \to \infty$.

Second step: case $q' < N/r$. Then $r < N$ and $q \geqq N/(N-r) > 1$. Then we fix some $k \in (1, N/(N-r))$. For any $\eta \in (0,1)$, we have $\eta|\nabla u|^k \leqq \eta + |\nabla u|^q$, hence the function

$$w_\eta = \eta^{1/(k-1)}(u - \eta t)$$

satisfies

$$(w_\eta)_t - \Delta w_\eta + |\nabla w_\eta|^k \leqq 0$$

in the weak sense. Thanks to Kato's inequality, see [21, Lemma 1], [4], we deduce that

$$(w_\eta^+)_t - \Delta w_\eta^+ + |\nabla w_\eta^+|^k \leqq 0, \tag{3.22}$$

in $\mathcal{D}'(Q_{\mathbb{R}^N,T})$. And w_η^+ has the same regularity as u, and moreover it satisfies an analogous property to (3.9):

$$\lim_{t\to 0}\int_{\mathbb{R}^N} (w_\eta^+)^r(.,t)\psi dx = \int_{\mathbb{R}^N} (\eta^{1/(k-1)}u_0)^r\psi dx, \qquad \forall\psi \in C_c(\mathbb{R}^N), \tag{3.23}$$

Indeed

$$\begin{aligned}\left|\int_{\mathbb{R}^N} ((u-\eta t)^+)^r - u^r(.,t))\psi dx\right| &\leqq \int_{\{u\geqq\eta t\}} |u(.,t)-\eta t)^r - u^r(.,t)|\,\psi dx + \int_{\{u\leqq\eta t\}} u^r(.,t))\psi dx \\ &\leqq r\eta t\int_{\mathbb{R}^N} u^{r-1}(.,t)\psi dx + Ct^r \\ &\leqq r\eta t(\int_{\mathbb{R}^N} u^r(.,t)dx)^{1/r'}(\int_{\mathbb{R}^N} \psi^r dx)^{1/r} + Ct^r\end{aligned}$$

then

$$\lim_{t\to 0}\int_{\mathbb{R}^N} ((u-\eta t)^+)^r - u^r(.,t))\psi dx = 0$$

and (3.23) follows from (3.9) applied to $\eta^{1/(k-1)}u$. From the first step we deduce that $w_\eta^+(t) \in L^r\left(\mathbb{R}^N\right)$ and

$$\int_{\mathbb{R}^N} (w_\eta^+)^r(.,t)dx \leqq \eta^{r/(k-1)}\int_{\mathbb{R}^N} u_0^r dx.$$

Then $\|(u-\eta t)^+\|_{L^r(\mathbb{R}^N)} \leqq \|u_0\|_{L^r(\mathbb{R}^N)}$. Then for any $R>0$, since $u \leqq \eta t + (u-\eta t)^+$,

$$\|u(.,t)\|_{L^r(B_R)} \leqq \|u_0\|_{L^r(\mathbb{R}^N)} + \eta t\,|B_R|^r$$

Going to the limit as $\eta \to 0$ for fixed R, we get $\|u(.,t)\|_{L^r(B_R)} \leqq \|u_0\|_{L^r(\mathbb{R}^N)}$, then going to the limit as $R \to \infty$ we deduce that $u(.,t) \in L^r\left(\mathbb{R}^N\right)$ and (3.17) holds.

(ii) Considering again $u_{\varepsilon,\delta}$ as above, and setting $F_\varepsilon = |\nabla u|^q * \varrho_\varepsilon$, there holds

$$(u_{\varepsilon,\delta})_t - \nu\Delta u_{\varepsilon,\delta} + F_\varepsilon = 0,$$

then

$$\begin{aligned}&\frac{d}{dt}\left(\int_{\mathbb{R}^N} u_{\varepsilon,\delta}^r \xi^\lambda dx\right) + r(r-1)\nu\int_{\mathbb{R}^N} u_{\varepsilon,\delta}^{\alpha-1}\left|\nabla u_{\varepsilon,\delta}\right|^2 \xi^\lambda dx + r\int_{\mathbb{R}^N} F_\varepsilon u_{\varepsilon,\delta}^\alpha \xi^\lambda dx\\ &= -r\int_{\mathbb{R}^N} u_{\varepsilon,\delta}^{r-1}\nabla u_{\varepsilon,\delta}.\nabla(\xi^\lambda)dx = \int_{\mathbb{R}^N} u_{\varepsilon,\delta}^r \Delta(\xi^\lambda)dx\end{aligned}$$

then for any $0<\sigma<t<T$,

$$\begin{aligned}&\int_{\mathbb{R}^N} u_{\varepsilon,\delta}^r(.,t)\xi^\lambda dx + r\int_\sigma^t\int_{\mathbb{R}^N} u_{\varepsilon,\delta}^\alpha F_\varepsilon \xi^\lambda dxdt\\ &+r(r-1)\nu\int_{\mathbb{R}^N} u_{\varepsilon,\delta}^{\alpha-1}\left|\nabla u_{\varepsilon,\delta}\right|^2\xi^\lambda dx = \int_{\mathbb{R}^N} u_{\varepsilon,\delta}^r(.,\sigma)\xi^\lambda dx + \int_\sigma^t\int_{\mathbb{R}^N} u_{\varepsilon,\delta}^r\Delta(\xi^\lambda)dx\end{aligned}$$

First we can go to the limit as $\varepsilon \to 0$, because $u \in L^\infty_{loc}(Q_{\mathbb{R}^N,T})$, and $|\nabla u|^2 \in L^1_{loc}(Q_{\mathbb{R}^N,T})$, and $F_\varepsilon \to |\nabla u|^q$ in $L^1_{loc}(Q_{\mathbb{R}^N,T})$. Setting $v_\delta = u+\delta$, we obtain for almost any σ, t, and in fact for any σ, t by the continuity,

$$\begin{aligned}&\int_{\mathbb{R}^N} v_\delta^r(.,t)\xi^\lambda dx + r\int_\sigma^t\int_{\mathbb{R}^N} v_\delta^\alpha\,|\nabla u|^q\,\psi dxdt\\ &+r(r-1)\nu\int_{\mathbb{R}^N} v_\delta^{\alpha-1}\,|\nabla u|^2\,\xi^\lambda dx = \int_{\mathbb{R}^N} v_\delta^r(.,\sigma)\xi^\lambda dx + \int_\sigma^t\int_{\mathbb{R}^N} v_\delta^r\Delta(\xi^\lambda)dx\end{aligned}$$

Next we go to the limit as $\delta \to 0$: from the Fatou Lemma we deduce that $\int_\sigma^t\int_{\mathbb{R}^N} u^{r-1}\,|\nabla u|^q\,\psi dxdt$ and $(r-1)\nu\int_{\mathbb{R}^N} u^{\alpha-1}\,|\nabla u|^2\,\xi^\lambda dx$ are finite, and then from Lebesgue we obtain the equality

$$\begin{aligned}&\int_{\mathbb{R}^N} u^r(.,t)\xi^\lambda dx + r\int_\sigma^t\int_{\mathbb{R}^N} u^\alpha\,|\nabla u|^q\,\xi^\lambda dxdt\\ &+r(r-1)\nu\int_{\mathbb{R}^N} u^{\alpha-1}\,|\nabla u|^2\,\xi^\lambda dx = \int_{\mathbb{R}^N} u^r(.,\sigma)\xi^\lambda dx + \int_\sigma^t\int_{\mathbb{R}^N} u^r\Delta(\xi^\lambda)dx.\end{aligned}$$

Next we go to the limit as $\sigma \to 0$, from (3.9). In the same way we deduce that

$$\begin{aligned}&\int_{\mathbb{R}^N} u^r(.,t)\xi^\lambda dx + r\int_0^t\int_{\mathbb{R}^N} u^\alpha\,|\nabla u|^q\,\xi^\lambda dxdt\\ &+\alpha r\int_0^t\int_{\mathbb{R}^N} u^{\alpha-1}\,|\nabla u|^2\,\xi^\lambda dx = \int_{\mathbb{R}^N} u_0^r(.,\sigma)\xi^\lambda dx + \int_\sigma^t\int_{\mathbb{R}^N} u^r\Delta(\xi^\lambda)dx\end{aligned}$$

Now $u(.,t) \in L^r\left(\mathbb{R}^N\right)$ for any $t \in [s,\tau]$, and

$$\int_\sigma^t \int_{\mathbb{R}^N} u^r \Delta(\xi^\lambda) dx \leqq \frac{C}{R^2}\tau \int_{\mathbb{R}^N} u^r(\sigma) dx$$

and we can make $R \to \infty$. We get that

$$\int_0^\tau \int_{\mathbb{R}^N} u^{r-1} |\nabla u|^q dxdt + (r-1)\nu \int_0^\tau \int_{\mathbb{R}^N} u^{r-2} |\nabla u|^2 dxdt < \infty$$

and, from the Lebesque theorem, we deduce

$$\int_{\mathbb{R}^N} u^r(.,t)dx + r\int_0^t \int_{\mathbb{R}^N} u^\alpha |\nabla u|^q dxdt + r(r-1)\nu \int_0^t \int_{\mathbb{R}^N} u^{\alpha-1} |\nabla u|^2 dx = \int_{\mathbb{R}^N} u_0^r dx \qquad (3.26)$$

Hence (3.18) follows, which implies directly (3.20).

(iii) Setting $v = u^m$ with $m = (q-1+r)/q < r$, we have $|\nabla v|^q \in L^1_{loc}([0,T)\,; L^1(\mathbb{R}^N))$, and $v \in L^\infty((0,T); L^{\frac{r}{m}}\left(\mathbb{R}^N\right))$. From the Gagliardo-Nirenberg inequality, we deduce that

$$\|v(.,t)\|_{L^q(\mathbb{R}^N)} \leqq \|v(.,t)\|^{1-k}_{L^{\frac{r}{m}}(\mathbb{R}^N)} \|\nabla v(.,t)\|^k_{L^q(\mathbb{R}^N)}, \qquad \frac{1}{k} = 1 + \frac{rq'}{N}. \qquad (3.27)$$

Then by integration, for any $0 < \tau < T$, using Hölder inequality,

$$\int_0^\tau \int_{\mathbb{R}^N} v^q(.,t)dxdt = \int_0^\tau \int_{\mathbb{R}^N} u^{q-1+r}(.,t)dxdt \leqq C(\tau)\, \|v\|^{(1-k)q}_{L^\infty((0,\tau);L^{\frac{r}{m}}(\mathbb{R}^N))} (\int_0^\tau \int_{\mathbb{R}^N} |\nabla v|^q\, dxdt)^k$$

Then $u \in L^{q-1+r}(Q_{\mathbb{R}^N,\tau})$, and $v^q = u^{q-1+r} \in L^1((0,\tau); W^{1,1}\left(\mathbb{R}^N\right))$, $v \in L^q((0,\tau); W^{1,q}\left(\mathbb{R}^N\right))$. If $\nu > 0$, we also have $u^{r-2}\,|\nabla u|^2 = \left|\nabla(u^{r/2})\right|^2 \in L^1(Q_{\mathbb{R}^N,\tau})$, and $u^{r/2} \in L^2(Q_{\mathbb{R}^N,\tau})$, then $u^{r/2} \in L^2((0,\tau); W^{1,2}\left(\mathbb{R}^N\right))$.

(iv) Here we assume that $u \in C([0,T)\,; L^r_{loc}\left(\mathbb{R}^N\right))$. First assume $r > 1$. Then from a diagonal procedure, there exists $t_n \to 0$ such that $u(.,t_n) \to u_0$ a.e. in $\mathbb{R}^N$, and $\|u(.,t_n)\|_{L^r(\mathbb{R}^N)} \to \|u_0\|_{L^r(\mathbb{R}^N)}$, and $u(.,t_n) \to u_0$ weakly in $L^r\left(\mathbb{R}^N\right)$. Then it holds from any sequence, and $u \in C([0,T)\,; L^r\left(\mathbb{R}^N\right))$. Next assume $r = 1$; let $t_n \to t \in [0,T)$. We have for any $p > 0$,

$$\begin{aligned}
\int_{\mathbb{R}^N} |u(t_n) - u_0|\, dx &\leqq \int_{B_p} |u(t_n) - u_0|\, dx + \int_{\mathbb{R}^N \backslash B_p} |u(t_n) - u_0|\, dx \\
&\leqq \int_{B_p} |u(t_n) - u_0|\, dx + \int_{\mathbb{R}^N \backslash B_p} u(t_n) dx + \int_{\mathbb{R}^N \backslash B_p} u_0 dx \\
&= \int_{B_p} |u(t_n) - u_0|\, dx + \int_{\mathbb{R}^N} u(t_n) dx - \int_{B_p} u_0 dx + \int_{B_p} (u(t) - u_0) dx + \int_{\mathbb{R}^N \backslash B_p} u_0 dx \\
&\leqq 2\int_{B_p} |u(t_n) - u_0|\, dx + \int_{\mathbb{R}^N} u(t_n) dx - \int_{\mathbb{R}^N} u_0 dx + 2\int_{\mathbb{R}^N \backslash B_p} u_0 dx
\end{aligned}$$

And the result follows because $\int_{\mathbb{R}^N \backslash B_p} u_0 dx \to 0$ as $p \to \infty$, since $u_0 \in L^1\left(\mathbb{R}^N\right)$. ■

The decay result is also available for initial data measures, where we do not assume that $q < (N+2)/(N+1)$:

Theorem 3.16 *Assume $q > 1$. Let $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$ and u be any non-negative weak $\mathcal{M}_{loc}$ solution of equation (3.8) in $Q_{\mathbb{R}^N,T}$. Then $u(.,t) \in L^1\left(\mathbb{R}^N\right)$ for any $t > 0$, and*

$$\int_{\mathbb{R}^N} u(.,t)dx \leqq \int_{\mathbb{R}^N} du_0. \tag{3.28}$$

Moreover $u \in C((0,T); L^1\left(\mathbb{R}^N\right))$, $|\nabla u|^q \in L^1_{loc}([0,T); L^1\left(\mathbb{R}^N\right))$ and

$$\int_{\mathbb{R}^N} u(.,t)dx + \int_0^t \int_{\mathbb{R}^N} |\nabla u|^q dxdt = \int_{\mathbb{R}^N} du_0, \tag{3.29}$$

and

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(.,t)\varphi dx = \int_{\mathbb{R}^N} \varphi du_0, \qquad \forall \varphi \in C_b(\mathbb{R}^N) \tag{3.30}$$

Proof. We obtain in the same way, as in (3.21),

$$\int_{B_{R_0}} u(.,t)dx \leqq \int_{\mathbb{R}^N} u(.,t)\xi^\lambda dx \leqq \int_{\mathbb{R}^N} u(.,\sigma)\xi^\lambda dx + C\tau R^{N-q'}$$

and we can go to the limit as $\sigma \to 0$ from (3.10), then

$$\int_{B_{R_0}} u(.,t)dx \leqq \int_{\mathbb{R}^N} \xi^\lambda du_0 + C\tau R^{N-q'} \leqq \int_{\mathbb{R}^N} du_0 + C\tau R^{N-q'}$$

Then going to the limit as $R \to \infty$, and then as $R_0 \to \infty$, we deduce that (3.28) holds, and we still obtain (3.29) holds. And $u \in C((0,T); L^1\left(\mathbb{R}^N\right))$, from the Lebesgue theorem, because $u \in C((0,T); L^1_{loc}\left(\mathbb{R}^N\right))$, and $u \in L^\infty((0,T); L^1\left(\mathbb{R}^N\right))$

Let us show (3.30): let $\varphi \in C_b(\mathbb{R}^N)$ be nonnegative, we can assume that φ takes its values in $[0,1]$. Let $t_n \to 0$. We know that $\lim \int_{\mathbb{R}^N} u(.,t_n)dx = \int_{\mathbb{R}^N} du_0$. Let $\psi_p \in \mathcal{D}(\mathbb{R}^N)$ with values in $[0,1]$, $\psi_p(x) = 1$ if $|x| \leqq p$, 0 if $|x| \geqq 2p$. Then for fixed p, $\lim \int_{\mathbb{R}^N} u(.,t_n)\varphi\psi_p dx = \int_{\mathbb{R}^N} \varphi\psi_p du_0$. Let $\eta > 0$.

$$\begin{aligned} &\left|\int_{\mathbb{R}^N} u(.,t_n)\varphi dx - \int_{\mathbb{R}^N} \varphi du_0\right| \\ &\leqq \left|\int_{\mathbb{R}^N} u(.,t_n)\varphi\psi_p dx - \int_{\mathbb{R}^N} \varphi\psi_p du_0\right| + \int_{\mathbb{R}^N} \varphi(1-\psi_p)du_0 + \int_{\mathbb{R}^N} u(.,t_n)\varphi(1-\psi_p)dx \end{aligned}$$

and $\int_{\mathbb{R}^N} (1-\psi_p)du_0 \to 0$ as $p \to \infty$ from the Lebesgue Theorem, then for some p_η we have $\int_{\mathbb{R}^N} (1-\psi_p)du_0 \leqq \eta$. As $n \to \infty$,

$$\int_{\mathbb{R}^N} u(.,t_n)(1-\psi_{p_\eta})dx \to \int_{\mathbb{R}^N} du_0 - \int_{\mathbb{R}^N} \psi_{p_\eta} du_0 = \int_{\mathbb{R}^N} (1-\psi_{p_\eta})du_0$$

Then $\int_{\mathbb{R}^N} u(.,t_n)\varphi(1-\psi_{p_\eta})dx \leqq 2p_\eta$ for large n, and $\left[\int_{\mathbb{R}^N} u(.,t_n)\varphi\psi_{p_\eta} dx - \int_{\mathbb{R}^N} \varphi\psi_{p_\eta} du_0\right] \leqq p_\eta$ for large n, hence we majorizate by 4η, hence the result. ∎

3.3 Regularizing effects

Here we deduce of the decay estimates a regularizing effect *without any condition at* ∞, achieving the proof of Theorem 1.1.

Theorem 3.17 *Let $q > 1$. Let $r \geqq 1$ and $u_0 \in L^r(\mathbb{R}^N)$. Let u be any non-negative weak L^r_{loc} solution of problem (3.8) in $Q_{\mathbb{R}^N,T}$ (3.9). Then $u(.,t) \in L^\infty(\mathbb{R}^N)$ for any $t \in (0,T)$ and*

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq Ct^{-\sigma_{rq,N}} \|u_0\|^{\varpi_{r,q,N}}_{L^r(\mathbb{R}^N)}, \tag{3.31}$$

where $C = C(N,q,r)$ and $\sigma_{r,q,N}, \varpi_{r,q,N}$ are given by

$$\sigma_{r,q,N} = \frac{1}{\frac{rq}{N} + q - 1}, \qquad \varpi_{r,q,N} = \frac{rq}{N}\sigma_{r,q}, \qquad \text{if } q < N, \tag{3.32}$$

and

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq Ct^{-\frac{N}{2r}} \|u_0\|_{L^r(\mathbb{R}^N)}, \qquad \text{if } \nu > 0,\ 2 < N. \tag{3.33}$$

where $C = C(N,q,r,\nu)$.

Proof. Since u is a weak L^r_{loc} solution, then $u \in C([0,T);L^r(\mathbb{R}^N))$, from Theorem 3.15, thus for any $0 \leqq s < T$, u is a weak r solution in $Q_{\mathbb{R}^N,s,T}$; and $\int_{\mathbb{R}^N} u^r(s)dx < \infty$ with $r \geqq 1$; for any $s \leqq t < T$, and for any $\alpha \geqq 0$ such that $\int_{\mathbb{R}^N} u^{\alpha+1}(s)dx < \infty$, applying Theorem 3.15 to u starting at point s, denoting $\beta = 1 + \alpha/q$, we have

$$\frac{1}{\alpha+1}\int_{\mathbb{R}^N} u^{\alpha+1}(.,t)dx + \int_s^t \int_{\mathbb{R}^N} |\nabla(u^\beta)|^q dxdt \leqq \frac{1}{\alpha+1}\int_{\mathbb{R}^N} u^{\alpha+1}(.,s)dx \tag{3.34}$$

and $u^\beta(.,t) \in L^q(Q_{\mathbb{R}^N,s,\tau})$ for a.e.t.

(i) Proof of (3.31). First suppose $q < N$. Then from the Sobolev injection of $W^{1,q}(\mathbb{R}^N)$ into $L^{q^*}(\mathbb{R}^N)$,

$$\frac{1}{\alpha+1}\int_{\mathbb{R}^N} u^{\alpha+1}(.,t)dx + \frac{C_{N,q}}{\beta^q}\int_s^t (\int_{\mathbb{R}^N} u^{\beta q^*}(.,t)dx)^{\frac{q}{q^*}})dt \leqq \frac{1}{\alpha+1}\int_{\mathbb{R}^N} u^{\alpha+1}(.,s)dx$$

so that we can apply Lemma 2.2 with $m = q$ and $\theta = N/(N-q)$ and deduce (3.31). If $q \geqq N$ we still obtain (3.31), with $\sigma_{r,q,N} = 1/(q+r-1) = \varpi_{r,q,N}/r$ if $q > N$, and $\sigma_{r,q,N} = 1/(N(1-\delta)+r-1) = \varpi_{r,q,N}/r(1-\delta)$ if $q = N$, where $\delta \in (0,1)$ is arbitrary. Indeed, if $q = N$,then $W^{1,q}(\mathbb{R}^N) \subset L^{q\theta}(\mathbb{R}^N)$ for any $\theta > 0$, and Lemma 2.2 applies. If $q > N$, $W^{1,q}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$, and then $t \mapsto \|u(.,t)\|^{\beta q}_{L^\infty(\mathbb{R}^N)} = \|u(.,t)\|^{q+r-1}_{L^\infty(\mathbb{R}^N)}$ is nonincreasing, thus for any $r \geqq 1$, from (3.34),

$$\int_{\mathbb{R}^N} u^r(.,t)dx + C_{N,N}rt\,\|u(.,t)\|^{q+r-1}_{L^\infty(\mathbb{R}^N)} \leqq \int_{\mathbb{R}^N} u_0^r dx.$$

(ii) Proof of (3.33). Assume $\nu > 0, N > 2$. For any $\alpha > 0$ such that $\int_{\mathbb{R}^N} u^{\alpha+1}(s)dx < \infty$

$$\frac{1}{\alpha+1}\int_{\mathbb{R}^N} u^{\alpha+1}(t)dx + \frac{\alpha}{\tilde{\beta}^2}\nu\int_\sigma^t \int_{\mathbb{R}^N} \left|\nabla(u^{\tilde{\beta}})\right|^2 dx \leqq \frac{1}{\alpha+1}\int_{\mathbb{R}^N} u^{\alpha+1}(\sigma)dx$$

where $\tilde{\beta} = (\alpha+1)/2$; and $u^{\tilde{\beta}} \in L^2_{loc}((0,\tau); W^{1,2}(\mathbb{R}^N))$. From the Sobolev injection of $W^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$, we get

$$\frac{1}{\alpha+1}\int_{\mathbb{R}^N} u^{\alpha+1}(t)dx + \frac{\alpha C_N}{\tilde{\beta}^2}\nu\int_s^t (\int_{\mathbb{R}^N} u^{\tilde{\beta}2^*})^{2^*/2}dx \leqq \frac{1}{\alpha+1}\int_{\mathbb{R}^N} u^{\alpha+1}(s)dx.$$

First suppose $r>1$. Then we can apply Lemma 2.2 with $C_0 = (r-1)C_N\nu$, $q=2, \theta = N/(N-2)$ and $\lambda = -1$, since $\tilde{\beta} = 1+(\alpha-1)/2$, and $r > N(1-2+1)/2$, and obtain (3.33). Next assume $r=1$. Then $u \in C([0,T); L^1(\mathbb{R}^N)) \cap L^\infty_{loc}((0,T); L^\infty(\mathbb{R}^N))$ *because of estimate* (3.31), then $C([0,T); L^\rho(\mathbb{R}^N))$ for any $\rho > 1$, for example with $\rho = 2$, and $\|u(.,t)\|_{L^1(\mathbb{R}^N)}$ is nonincreasing, from Theorem 3.15, hence we can still apply Lemma 2.2 on (ϵ, t) for $0<\epsilon<t<T$

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq C(t-\epsilon)^{-\frac{N}{2}}\|u(.,\epsilon)\|_{L^1(\mathbb{R}^N)} \leqq C(t-\epsilon)^{-\frac{N}{2}}\|u_0\|_{L^1(\mathbb{R}^N)}$$

Then we still obtain (3.33) with $C = C(N,q,r,\nu)$. ■

Remark 3.18 *If $N \leqq 2$ we obtain similarly that $\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq Ct^{-\tilde{\sigma}}\|u_0\|^{\tilde{\varpi}}_{L^r(\mathbb{R}^N)}$ with $\tilde{\sigma} = 1/r = \tilde{\varpi}/r$ if $N=1$, and $\tilde{\sigma} = 1/(r-2\delta)$=$\tilde{\varpi}/r(1-\delta)$ if $N=2$.*

Remark 3.19 *As a consequence, for any $k \geqq 1, q<N$,*

$$\|u(.,t)\|_{L^{kr}(\mathbb{R}^N)} \leqq Ct^{-\frac{\sigma_{r,q}}{k'}}\|u_0\|^{\frac{\varpi_{r,q}}{k'}+\frac{1}{k}}_{L^r(\mathbb{R}^N)}, \tag{3.35}$$

$$\|u(.,t)\|_{L^{kr}(\mathbb{R}^N)} \leqq Ct^{-\frac{N}{2rk'}}\|u_0\|_{L^r(\mathbb{R}^N)}, \qquad \text{if } \nu > 0. \tag{3.36}$$

Indeed it follows from (3.17) and (3.31), (3.33) by interpolation

$$\|u(.,t)\|_{L^{kr}(\mathbb{R}^N)} \leqq \|u(.,t)\|^{1/k'}_{L^\infty(\mathbb{R}^N)}\|u(.,t)\|^{1/k}_{L^r(\mathbb{R}^N)}.$$

Remark 3.20 *If $q \leqq 2$, then $u \in C^{2,1}(Q_{\mathbb{R}^N,T})$, from the regularity result of [16, Theorem 2.12]. In this case* ***we do not need to introduce the regularization by*** *u_ε; we only need to introduce $u+\delta$, when $r>1$ and make $\delta \to 0$.*

In case of initial data measures, we obtain in the same way:

Theorem 3.21 *Assume $q>1$. Let $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$ and u be any non-negative weak $\mathcal{M}_{loc}$ solution of equation (3.8) in $Q_{\mathbb{R}^N,T}$. Then*

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq Ct^{-\sigma_{1,q}}(\int_{\mathbb{R}^N} du_0)^{\varpi_{1,q}},$$

where $\sigma_{1,q}, \varpi_{1,q}$ are given at (3.32), and $C = C(N,q)$. Moreover if $\nu>0$, with $C = C(N,q,\nu)$,

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq Ct^{-\frac{N}{2}}\int_{\mathbb{R}^N} du_0. \tag{3.37}$$

Proof. Taking $\epsilon > 0$, we have for any $t \geqq \epsilon$,

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq C(t-\epsilon)^{-\sigma_1}\|u(.,t)\|_{L^1(\mathbb{R}^N)}^{\sigma_2} \leqq C(t-\epsilon)^{-\sigma_1}(\int_{\mathbb{R}^N} du_0)^{\sigma_2}$$

and then we make $\varepsilon \to 0$ and deduce the estimate. If $\nu > 0$, we also obtain for $t > s > 0$

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq C(t-s)^{-\frac{N}{2}} \int_{\mathbb{R}^N} u(.,s)dx$$

and going to the limit as $s \to 0$, we deduce (3.37). ∎

Remark 3.22 *Up to now, the decay estimate (3.17) and the L^∞ estimate (3.31) of u were proved in case $u_0 \in C_b\left(\mathbb{R}^N\right) \cap L^r\left(\mathbb{R}^N\right)$ and for the unique bounded solution u of problem (3.8), and based on the estimate (3.2) given in [14, Theorem 5.6]; indeed (3.31) follows from the Gagliardo-Nirenberg estimate:*

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq \|\nabla u(.,t)\|_{L^\infty(\mathbb{R}^N)}^{\frac{N}{N+r}}\|u(.,t)\|_{L^r(\mathbb{R}^N)}^{\frac{r}{N+r}} \leqq C(q,r)\|u(.,t)\|_{L^\infty(\mathbb{R}^N)}^{\frac{N}{q(N+r)}}\|u_0\|_{L^r(\mathbb{R}^N)}^{\frac{r}{N+r}}.$$

3.4 Further estimates and convergence results for $q \leqq 2$.

Here we consider the case $1 < q \leqq 2$. From the L^∞ estimates above, and the interior regularity of u, we deduce new local estimates and convergence results:

Corollary 3.23 *Assume $1 < q \leqq 2$.*

(i) Any nonnegative weak L^r_{loc} solution (resp. $\mathcal{M}_{loc}$ solution) u of problem (3.8) with initial data $u_0 \in L^r(\mathbb{R}^N)$, $r \geqq 1$ (resp. $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$) satisfies $u \in C^{2,1}(Q_{\mathbb{R}^N,T}) \cap L^\infty_{loc}((0,T)\,; C_b(\mathbb{R}^N))$.

(ii) Let $(u_{0,n})$ be any bounded sequence in $L^r(\mathbb{R}^N)$, $r \geqq 1$ (resp. in $\mathcal{M}_b^+(\mathbb{R}^N)$). For any $n \in \mathbb{N}$, let u_n be any nonnegative weak L^r_{loc} solution (resp. $\mathcal{M}_{loc}$ solution) of problem (3.8) with initial data $u_{0,n}$. Then one can extract a subsequence converging in $C^{2,1}_{loc}(Q_{\mathbb{R}^N,T})$ to a weak solution u of (1.1) in $Q_{\mathbb{R}^N,T}$.

Proof. From [16, Theorem 2.16] there there exists $\gamma \in (0,1)$ such that for any nonnegative weak solution of equation (1.1) u in $Q_{\mathbb{R}^N,T}$ and any ball $B_R \subset \mathbb{R}^N$, and $0 < s < \tau < T$,

$$\|u\|_{C^{2+\gamma,1+\gamma/2}(Q_{B_R,s,\tau})} \leqq C\Phi(\|u\|_{L^\infty(Q_{B_{2R},s/2,\tau})}).$$

where $C = C(N,q,R,s,\tau)$ and Φ is a continuous increasing function. From estimates (3.31), (3.37), we deduce that $u \in L^\infty_{loc}((0,T)\,; C_b(\mathbb{R}^N))$ and

$$\|u\|_{C^{2+\gamma,1+\gamma/2}(Q_{B_R,s,\tau})} \leqq C\Phi(\|u_0\|_{L^r(\mathbb{R}^N)}), \qquad (\text{resp. } \|u\|_{C^{2+\gamma,1+\gamma/2}(Q_{B_R,s,\tau})} \leqq C\Phi(\int_{\mathbb{R}^N} du_0) \quad (3.38)$$

and the conclusions follow. ∎

We also deduce global gradient estimates in $\mathbb{R}^N$:

Corollary 3.24 *Assume $1 < q \leqq 2$ (i) Let $u_0 \in L^r(\mathbb{R}^N)$, $r \geqq 1$. Then any weak L^r_{loc} solution u of problem (3.8) satisfies*

$$\|\nabla u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq Ct^{-\vartheta_{r,q}} \|u_0\|_{L^r(\mathbb{R}^N)}^{\varkappa_{r,q}}, \tag{3.39}$$

$$\vartheta_{r,q} = \frac{N+r}{rq+N(q-1)}, \qquad \varkappa_{r,q} = \frac{r}{rq+N(q-1)}$$

and $|\nabla u|^q \in L^\infty_{loc}((0,T); L^r(\mathbb{R}^N))$, and

$$\int_{\mathbb{R}^N} |\nabla u(.,t)|^{qr}\, dx \leqq C_q t^{-r(\frac{q}{2}+\sigma_{r,q}(q-1))} \|u_0\|_{L^r(\mathbb{R}^N)}^{(1+\varpi_{r,q}(q-1))r} \tag{3.40}$$

And for $\nu > 0$,

$$\|\nabla u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq Ct^{-\frac{1}{q}(\frac{N}{2r}+1)} \|u_0\|_{L^r(\mathbb{R}^N)}^{\frac{1}{q}}; \tag{3.41}$$

$$\int_{\mathbb{R}^N} |\nabla u(.,t)|^{qr}\, dx \leqq C_q t^{-r(\frac{q}{2}+\frac{N}{2r}(q-1))} \|u_0\|_{L^r(\mathbb{R}^N)}^{qr}, \tag{3.42}$$

moreover if $q < 2$, u is a pointwise mild solution.

(ii) Let $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$. Then any weak $\mathcal{M}_{loc}$ solution of (3.8) satisfies

$$\|\nabla u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq Ct^{-\vartheta_{1,q}} (\int_{\mathbb{R}^N} du_0)^{\varkappa_{1,q}},$$

and $|\nabla u| \in L^\infty_{loc}((0,T); L^q(\mathbb{R}^N)$; and for $\nu > 0$,

$$\|\nabla u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq Ct^{-\frac{1}{q}(\frac{N}{2}+1)} (\int_{\mathbb{R}^N} du_0)^{1/q}.$$

As a consequence, in any case u is defined on $(0,\infty)$.

Proof. (i) Let $u_0 \in L^r(\mathbb{R}^N)$, $r \geqq 1$. Then for any $\epsilon > 0$, $u(.,\epsilon) \in C_b(\mathbb{R}^N)$, from Corollary 3.23. From [30], u is the unique solution v such that $v \in C^{2,1}\left(\mathbb{R}^N \times (\epsilon,T)\right) \cap C_b\left(\mathbb{R}^N \times [\epsilon,T)\right)$, and $v(.,\epsilon) = u(.,\epsilon)$; since $v \in C_b^2\left(\mathbb{R}^N \times (\epsilon,T)\right)$, we deduce that $u \in C_b^2\left(\mathbb{R}^N \times (0,T)\right)$; and for any $\epsilon \leqq t < T$,

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq \|u(.,\epsilon)\|_{L^\infty(\mathbb{R}^N)}, \qquad \|\nabla u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq \|\nabla u(.,\epsilon)\|_{L^\infty(\mathbb{R}^N)},$$

and

$$|\nabla u(.,t)|^q \leqq C_q \frac{u(.,t)}{t-\epsilon}, \qquad \text{a.e. in } \mathbb{R}^N. \tag{3.43}$$

From the decay estimates, we also have $\|u(.,\epsilon)\|_{L^r(\mathbb{R}^N)} \leqq \|u_0\|_{L^r(\mathbb{R}^N)}$. And $u(.,\epsilon) \in L^{\tilde{r}}(\mathbb{R}^N)$ for any $\tilde{r} \in [r,\infty]$, and $u \in C([\epsilon,T)\,; L^{\tilde{r}}(\mathbb{R}^N))$. Going to the limit in (3.43) as $\epsilon \to 0$, we deduce (3.39) from (3.31), and (3.41) from (3.33). Moreover $|\nabla u|^q \in L^\infty_{loc}((0,T); L^r(\mathbb{R}^N))$, since

$$\|\nabla u(.,t)\|_{L^{qr}(\mathbb{R}^N)} \leqq Ct^{-1/q} \|u_0\|_{L^r(\mathbb{R}^N)}^{1/q}.$$

More precisely we get from estimate (3.1),

$$\|\nabla(u^{\frac{q-1}{q}}(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq C_q(t-\epsilon)^{-1/2}\|u(.,\epsilon)\|_{L^\infty(\mathbb{R}^N)}^{\frac{q-1}{q}}$$

then from estimate (3.33), for any $t \in (0,T)$, with other constants C_q,

$$\|\nabla(u^{\frac{q-1}{q}}(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq C_q t^{-1/2}\|u(.,\frac{t}{2})\|_{L^\infty(\mathbb{R}^N)}^{\frac{q-1}{q}}$$

$$|\nabla u(.,t)|^q \leqq C_q t^{-q/2}\|u(.,\frac{t}{2})\|_{L^\infty(\mathbb{R}^N)}^{q-1} u(.,t),$$

then from estimate (3.31) we get

$$\int_{\mathbb{R}^N} |\nabla u(.,t)|^{qr}\, dx \leqq C_q \|u_0\|_{L^r(\mathbb{R}^N)}^{\varpi_{r,q}(q-1)r} t^{-r(\frac{q}{2}+\sigma_{r,q}(q-1))} \int_{\mathbb{R}^N} u(.,t)^r dx$$

then (3.40) follows.

Assume that $\nu > 0$; then (3.42) follows from (3.33). Moreover, from [30, Theorem 6], $u(.,t) \in C_b^2(\mathbb{R}^N)$ for any $t \in (\epsilon,T)$, in particular $u(.,2\epsilon) \in C_b^2(\mathbb{R}^N)$, then for any $t \geqq \epsilon$, and any $x \in \mathbb{R}^N$,

$$u(x,t) = e^{(t-2\epsilon)\Delta}u(x,2\epsilon) - \int_{2\epsilon}^t \int_{\mathbb{R}^N} g(x-y,t-s)|\nabla u(y,s)|^q dyds, \tag{3.44}$$

see for example [6, Proposition 4.2]. But $u(x,2\epsilon)$ converges to u_0 in $L^r(\mathbb{R}^N)$, and then $e^{(t-2\epsilon)\Delta}u(.,\epsilon)$ converges to $e^{t\Delta}u_0$ in $L^r(\mathbb{R}^N)$. Then we can go to the limit as $\epsilon \to 0$ in (3.44), for a.e. $x \in \mathbb{R}^N$: the integral is convergent, then u is a pointwise mild solution.

(ii) For Theorem 3.16, we have $u(.,t) \in L^1(\mathbb{R}^N)$ for $t \geqq \epsilon > 0$, which gives from (i)

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq C(t-\varepsilon)^{-\sigma_{1,q}}\|u(.,\epsilon)\|_{L^1(\mathbb{R}^N)}^{\varpi_{1,q}} \leqq C(t-\varepsilon)^{-\sigma_{1,q}}(\int_{\mathbb{R}^N} du_0)^{\varpi_{1,q}}$$

and then we go to the limit as $\epsilon \to 0$. And $|\nabla u| \in L^\infty_{loc}((0,T);L^q(\mathbb{R}^N)$, since

$$\|\nabla u(.,t)\|_{L^q(\mathbb{R}^N)} \leqq Ct^{-1/q}(\int_{\mathbb{R}^N} du_0)^{1/q}.$$

And the estimates (3.40) and (3.42) hold with $r=1$ and $\|u_0\|_{L^1(\mathbb{R}^N)}$ replaced by $\int_{\mathbb{R}^N} du_0$. ∎

3.5 Existence and uniqueness results for $q \leq 2$

Let $u_0 \in L^r(\mathbb{R}^N)$, $r \geqq 1$. We first consider the subcritical case $q < (N+2r)/(N+r)$, equivalently $q < 2$ and $r > N(q-1)/(2-q)$.

Theorem 3.25 *Let $u_0 \in L^r(\mathbb{R}^N)$, $r \geqq 1$. Suppose $1 < q < (N+2r)/(N+r)$. Then any weak L^r_{loc} solution u of problem (3.8) satisfies*

$$|\nabla u|^q \in L^1_{loc}([0,T)\,;L^r(\mathbb{R}^N)). \tag{3.45}$$

And

u is a weak L^r_{loc} solution $\Longleftrightarrow$ u is a mild L^r solution.

Proof. Let u be any weak L^r_{loc} solution. Then from (3.40),

$$\int_0^\tau \|\nabla u(.,t)\|^q_{L^{qr}(\mathbb{R}^N)}\,dt = \int_0^\tau (\int_{\mathbb{R}^N} |\nabla u(.,t)|^{qr}\,dx)^{\frac{1}{r}}dt \leqq C\int_0^\tau t^{-(\frac{q}{2}+\sigma_{r,q}(q-1))}dt$$

with $C = C_q\|u_0\|^{(1+\varpi_{r,q}(q-1))r}_{L^r(\mathbb{R}^N)}$, and $q/2+\sigma_{r,q}(q-1) < 1$ is equivalent to $q < (N+2r)/(N+r)$; if $\nu > 0$, the estimate (3.42) leads to the same conclusion, since $q/2+(q-1)N/2r < 1$ is still equivalent to $q < (N+2r)/(N+r)$. Then (3.45) holds. Moreover from Corollary 3.24, u is a mild pointwise solution:

$$u(.,t) = e^{t\Delta}u_0(.) - \int_0^t \int_{\mathbb{R}^N} g(x-y,t-s)|\nabla u(y,s)|^q dyds; \tag{3.46}$$

and $u \in C([0,T)\,; L^r\left(\mathbb{R}^N\right))$ from Theorem 3.15, and $f = |\nabla u|^q \in L^1_{loc}([0,T)\,; L^r(\mathbb{R}^N))$, thus the relation (3.46) holds in $L^r(\mathbb{R}^N)$,

$$u(.,t) = (e^{t\Delta}u_0) - \int_0^t e^{(t-s)\Delta}\,|\nabla u(.,s)|^q\,(s)ds \qquad \text{in } L^r(\mathbb{R}^N), \tag{3.47}$$

that means u is a mild L^rsolution. Conversely it is clear that any mild L^rsolution is a weak L^r_{loc} solution. ■

Next we deduce the uniqueness results of Theorem 1.2.

Theorem 3.26 *Let $u_0 \in L^r(\mathbb{R}^N)$. Assume $1 < q < (N+2r)/(N+r)$, or $q = 2$. Then there exists a unique weak L^r_{loc} solution u of problem (3.8). In the first case, $u \in C((0,T); W^{1,qr}(\mathbb{R}^N))$.*

Proof. (i) Case $1 < q < (N+2r)/(N+r)$. From [14, Theorem 2.1], there exists a mild L^r solution, then it is a L^r_{loc} solution. Let us show the uniqueness. Let u be any weak L^r_{loc} solution, thus u is a mild L^r solution, from Theorem 3.25. And $u \in L^\infty((0,T); L^r(\mathbb{R}^N))$ from Theorem 3.15, and $u \in L^\infty_{loc}((0,T); W^{1,qr}\left(\mathbb{R}^N\right))$, since $|\nabla u| \in L^\infty_{loc}((0,T); L^{qr}\left(\mathbb{R}^N\right))$ from Theorem 3.25 and $u \in L^\infty_{loc}((0,T); L^{qr}\left(\mathbb{R}^N\right))$ by interpolation. . Then we enter in the class of uniqueness $u \in L^\infty_{loc}((0,T); W^{1,qr}\left(\mathbb{R}^N\right))$ required in [14, Lemma 2.2 and Remark 2.5]. Thus u is unique, and satisfies $u \in C((0,T); W^{1,qr}(\mathbb{R}^N))$, from [14, Theorem 2.1].

(ii) Case $q = 2$. From [14, Theorem 4.2] there exists a unique solution u such that $u \in C([0,T)\,; L^r\left(\mathbb{R}^N\right)) \cap u \in C^{2,1}(\left(Q_{\mathbb{R}^N,\infty}\right)$ solution of (1.1) at each point. Then it is a weak L^r_{loc} solution. Reciprocally any weak L^r_{loc} solution u satisfies the conditions above, from Theorem 3.15 and [16, Theorem 2.16]. ■

Theorem 3.27 *Assume that $1 < q < (N+2)/(N+1)$. Let $u_0 \in \mathcal{M}^+_b(\mathbb{R}^N)$. Then there exists a unique weak $\mathcal{M}_{loc}$ solution of problem (3.8).*

Proof. The existence of a weak semi-group solution was obtained in [11] by approximation. The existence of a mild $\mathcal{M}$ solution was proved in [14, Theorem 2.2], and the two notions are equivalent from Lemma 3.11. In any case the solution is a weak $\mathcal{M}_{loc}$ solution. Next consider any solution $\mathcal{M}_{loc}$ solution u. Then $u(.,t) \in L^\infty(\mathbb{R}^N)$ for any $t \in (\epsilon,T)$ by applying theorem 3.17 from $\epsilon/2$. Then again we deduce $u(.,\epsilon) \in C_b(\mathbb{R}^N)$, then (3.43). From Theorem 3.15 we obtain again that $u \in L^\infty_{loc}((0,T); W^{1,q}(\mathbb{R}^N))$. And moreover from the uniqueness after ϵ, we have $u \in C((\epsilon, T); W^{1,q}(\mathbb{R}^N))$ from Theorem 3.26. Then $u \in C((0,T); W^{1,q}(\mathbb{R}^N))$. And u satisfies (3.30) from Theorem 3.16. Then u is a weak semi-group solution, then a mild $\mathcal{M}$ solution from Lemma 3.11. Then we enter the class of uniqueness of [14, Theorem 2.2]. We can also prove the uniqueness directly: if u_1, u_2 are two solutions, since they are mild $\mathcal{M}$ solutions, we have

$$(u_1 - u_2)(.,t) = \int_0^t e^{(t-s)\Delta}(|\nabla u_1(.,s)|^q - |\nabla u_2(.,s)|^q)ds$$

and we know that $|\nabla u_j|^q \in C((0,T); L^r(\mathbb{R}^N))$, hence

$$\begin{aligned}
\|\nabla(u_1 - u_2)(.,t)\|_{L^{qr}(\mathbb{R}^N)} &\leqq \int_0^t \left\|\nabla(e^{(t-s)\Delta})\right\|_{L^1(\mathbb{R}^N)} \||\nabla u_1(.,s)|^q - |\nabla u_2(.,s)|^q\|_{L^{qr}(\mathbb{R}^N)} ds \\
&\leqq C\int_0^t (t-s)^{-1/2} \max_{j=1,2} \|\nabla u_j(.,s)\|^{q-1}_{L^\infty(\mathbb{R}^N)} \|\nabla(u_1 - u_2)(.,s)\|_{L^{qr}(\mathbb{R}^N)} ds \\
&\leqq C\int_0^t (t-s)^{-1/2} s^{-(q-1)\vartheta_{1,q}} \|\nabla(u_1 - u_2)(.,s)\|_{L^{qr}(\mathbb{R}^N)} ds
\end{aligned}$$

and we can apply the singular Gronwall lemma when $2 < (q-1)\vartheta_{1,q}$, which means precisely $q < \frac{N+2}{N+1}$. Then $\nabla(u_1 - u_2)(.,t) = 0$ in $L^{qr}(\mathbb{R}^N)$, hence $u_1 = u_2$. ■

Finally we give a short proof of the existence result of [14, Theorem 4.1].

Proposition 3.28 *Let $1 < q < 2$. For any nonnegative $u_0 \in L^r(\mathbb{R}^N), r \geq 1$, there exists a mild pointwise solution u of problem (3.8), and $u \in C([0,T); L^r(\mathbb{R}^N))$.*

Proof. Let $u_{0,n} = \min(u_0, n)$. Then $u_{0,n} \in L^\rho(\mathbb{R}^N)$ for any $\rho \geq r$. Choosing $\rho > N(q-1)/(2-q)$, that means $q < q_\rho$, from [14, Theorem 2.1], there exists a mild L^ρ solution u_n with initial data $u_{0,n}$, and $u_n \in C((0,T); C^2_b(\mathbb{R}^N)) \cap C^{2,1}(Q_{\mathbb{R}^N,T})$. Then (u_n) is nondecreasing from the comparison principle, and $u_n(.,t) \leq e^{t\Delta}u_0 \leq Ct^{-N/2r}\|u_0\|_{L^r(\mathbb{R}^N)}$. From Corollary 3.23, (u_n) converges in $C^{2,1}_{loc}(Q_{\mathbb{R}^N,T})$ to a weak solution u of (1.1) in $Q_{\mathbb{R}^N,T}$, and $u(.,t) \leq e^{t\Delta}u_0$. Moreoever $(|\nabla u_n|^q)$ is bounded in $L^1_{loc}([0,T); L^1_{loc}(\mathbb{R}^N))$: indeed for any $\xi \in \mathcal{D}^+(\mathbb{R}^N)$, with values in $[0,1]$, and any $0 < s < t < T$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} u_n(t,.)\xi^{q'}dx + \int_s^t\int_{\mathbb{R}^N} |\nabla u_n|^q\xi^{q'}dx \leqq -q'\int_s^t\int_{\mathbb{R}^N} \xi^{1/(q-1)}\nabla u_n.\nabla\xi dx + \int_{\mathbb{R}^N} u_n(s,.)\xi^{q'}dx \\
&\leqq \frac{1}{2}\int_s^t\int_{\mathbb{R}^N} |\nabla u_n|^q\xi^{q'}dx + Ct\int_{\mathbb{R}^N} |\nabla\xi|^{q'}dx + \int_{\mathbb{R}^N} u_n(s,.)\xi^{q'}dx
\end{aligned}$$

and $u_n \in C([0,T); L^\rho(\mathbb{R}^N))$, thus we can go to the limit as $s \to 0$:

$$\int_{\mathbb{R}^N} u_n(t,.)\xi^{q'}dx + \frac{1}{2}\int_s^t\int_{\mathbb{R}^N} |\nabla u_n|^q\xi^{q'}dx \leqq Ct\int_{\mathbb{R}^N} |\nabla\xi|^{q'}dx + \int_{\mathbb{R}^N} u_0\xi^{q'}dx.$$

Thus $|\nabla u|^q \in L^1_{loc}([0,T); L^1_{loc}(\mathbb{R}^N))$, hence, from [16, Proposition 2.11], u admits a trace as $t \to 0$: there exists a Radon measure μ_0 in $\mathbb{R}^N$, such that $u(.,t)$ converges weakly* to μ_0. And $e^{t\Delta}u_0$ converges to u_0 in $L^r(\mathbb{R}^N)$, thus $\mu_0 \in L^1_{loc}(\mathbb{R}^N)$ and $0 \leq \mu_0 \leq u_0$; and $u_n \leq u$, thus

$u_{0,n} \leq \mu_0$, hence $\mu_0 = u_0$. Also there exists a function $g \in L^r(\mathbb{R}^N)$ such that $u(.,t) \leqq g$ for small t. Then the nonnegative function $e^{t\Delta}u_0 - u(.,t)$ converges weakly* to 0, and then in $L^1_{loc}(\mathbb{R}^N)$. Hence $u(.,t)$ converges to u_0 in $L^1_{loc}(\mathbb{R}^N)$, then in $L^r(\Omega)$ from the Lebesgue theorem. Thus $u \in C([0,T)\,;L^r\left(\mathbb{R}^N\right))$. In particular u is a weak L^r_{loc} solution, then a pointwise mild solution, from Corollary 3.24. ■

Remark 3.29 *The uniqueness of the solution is still an open problem when $u_0 \in L^r(\mathbb{R}^N)$ and $q \geqq (N+2r)/(N+r)$.*

3.6 More decay estimates for $q < (N+2r)/(N+r)$

Here, we exploit theorem 3.15 to obtain a better decay estimate of the L^r norm when $u_0 \in L^r(\mathbb{R}^N)$ and $q < (N+2r)/(N+r)$, which appears to be new for $r > 1$. In case $r = 1$ we find again the result of [2], proved *under the assumption that the energy relation (3.29) holds.*

Theorem 3.30 *Let $r \geqq 1$ and $1 < q < (N+2r)/(N+r)$. Let u be any non-negative weak r solution of problem (3.8) with $u_0 \in L^r(\mathbb{R}^N)$. Then there exists $C = C(N,q,r)$ such that for any $t \in (0,T)$*

$$\int_{\mathbb{R}^N} u^r(.,t)dx \leq C(\int_{\{|x|>\sqrt{t}\}} u_0^r(x)dx + t^{-\frac{ar-N}{2}}), \qquad a = \frac{2-q}{q-1}. \tag{3.48}$$

As a consequence, $\lim_{t\longrightarrow\infty} \|u(t)\|_{L^r(\mathbb{R}^N)} = 0$ *and*

$$r\int_0^\infty \int_{\mathbb{R}^N} u^{r-1}|\nabla u|^q dxdt + r(r-1)\nu \int_0^\infty \int_{\mathbb{R}^N} u^{r-2}|\nabla u|^2 dxdt = \int_{\mathbb{R}^N} u_0^r dx.$$

Proof. We still consider $v = u^m$ with $m = (q-1+r)/q < r$. Let $E(s) = \int_{\mathbb{R}^N} u^r(.,s)dx$, thus from the energy relation (3.18) of theorem 3.15, $E \in W^{1,1}((0,T))$ and for almost any $s \in (0,T)$,

$$E'(s) = -r(r-1)\int_{\mathbb{R}^N} |\nabla u|^2 u^{r-2}(.,s)dx - \int_{\mathbb{R}^N} |\nabla u|^q u^{r-1}(.,s)dx \leq 0.$$

Next, we set $E = E_1 + E_2$ with $E_1(s) = \int_{\{|x|<2R\}} u^r(x,s)dx$, $E_2(s) = \int_{\{|x|\geq 2R\}} u^r(x,s)dx$. From the Gagliardo-Nirenberg inequality (3.27), we obtain successively

$$\begin{aligned} E_1(s) = \int_{\{|x|<2R\}} v^{\frac{r}{m}}(x,s)dx &\leq \left(\int_{\{|x|<2R\}} v^q(x,s)dx\right)^{\frac{r}{mq}} (2R)^{1-\frac{r}{mq}} \\ &\leqq C\|\nabla v(s)\|^{\frac{kr}{m}}_{L^q(\mathbb{R}^N)} \|v(s)\|^{\frac{(1-k)r}{m}}_{L^{r/m}(\mathbb{R}^N)} R^{N(1-\frac{r}{mq})} \\ &\leqq \frac{1}{2}\|v(s)\|^{\frac{r}{m}}_{L^{r/m}(\mathbb{R}^N)} + C(N,q,r)\|\nabla v(s)\|^{\frac{kr}{m}}_{L^q(\mathbb{R}^N)} R^{\frac{N}{k}(1-\frac{r}{mq})}, \end{aligned}$$

thus

$$E(s) \leqq C(\|\nabla v(s)\|^{\frac{r}{m}}_{L^q(\mathbb{R}^N)} R^{\frac{N}{k}(1-\frac{r}{mq})} + 2E_2(s)). \tag{3.49}$$

Consider two smooth cut-off functions φ, η with values in $[0,1]$, such that $\varphi = 1$ in B_1, with support in $\overline{B_2}$, and $\eta = 1 - \varphi$, and put $\varphi_l(x) = \varphi(\frac{x}{l})$, $\eta_R(x) = \eta(\frac{x}{R})$. As in the first step of theorem 3.15, we obtain for any $0 < \sigma < s < t < T$, and $l > 2R$,

$$\left(\int_{\mathbb{R}^N} u^r(.,s)\varphi_l^\lambda \eta_R^\lambda dx\right)^{\frac{1}{r}} \leq \left(\int_{\mathbb{R}^N} u^r(.,\sigma)\varphi_l^\lambda \eta_R^\lambda dx\right)^{\frac{1}{r}} + C(s-\sigma)(R^{\frac{N}{r}-q'} + l^{\frac{N}{r}-q'}), \tag{3.50}$$

with $\lambda = rq'$ Noting that our assumption on q implies $N < rq'$. As $\sigma \to 0$ and $l \to \infty$. we deduce

$$\left(\int_{\mathbb{R}^N} u^r(x,s)\eta_R dx \right)^{\frac{1}{r}} \leq \left(\int_{\mathbb{R}^N} u_0^r(x)\eta_R dx \right)^{\frac{1}{r}} + CsR^{\frac{N}{r}-q'},$$

hence, taking $R = \sqrt{t}$, and setting

$$\rho = r + \frac{N - rq'}{2} = \frac{(N+2r) - q(N+r)}{2(q-1)} = \frac{ar - N}{2},$$

we find

$$E_2(s) \leq A(t) = C \left(\int_{\{|x| > \sqrt{t}\}} u_0^r(x)dx + t^{-\rho} \right),$$

where with a new constant C. Next, we set $F(s) = E(s) - 2A(t)$. Either there exists $t_0 \in (0,t)$ such that $F(t_0) \leq 0$, then $F(s) \leq 0$, $\forall s \in (t_0, t)$, thus by continuity, $E(t) \leq 2A(t)$, hence (3.48) holds. Or $F(s) > 0$, $\forall s \in (0,t)$. Since

$$-F'(s) \geq \int_{\mathbb{R}^N} |\nabla u|^q u^{r-1}(x,s)dx = \int_{\mathbb{R}^N} |\nabla v(x,s)|^q dx \tag{3.51}$$

it follows from (3.49) that $F(s) \leqq C(-F'(s))^{\frac{r}{mq}} t^{\frac{N}{2k}(1-\frac{r}{mq})}$. Thus by integration

$$C(t-s)t^{-\frac{N}{2k}(1-\frac{r}{mq})} \leqq F(t)^{-\frac{q-1}{r}} - F(s)^{\frac{q-1}{r}}.$$

Then as $s \longrightarrow 0$ we get $F(t) \leqq Ct^{-\rho}$, since $\rho = r/(q-1) - N/2k$, and (3.48) still holds. ∎

Remark 3.31 *The case $r = 1$ has been the object of many works, assuming that $u_0 \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$. There holds $\lim_{t \longrightarrow \infty} \|u(t)\|_{L^1(\mathbb{R}^N)} = 0$ if and only if $q \leq (N+2)/(N+1)$, see [1], [11], [3], [29]. When $q < (N+2)/(N+1)$, the absorption plays a role in the asymptotics. From [9], if $lim_{|x| \to \infty} |x|^a \ u_0(x) = 0$, then $u(.,t)$ converges as $t \to \infty$ to the very singular solution constructed in [35], [12]. In that case $\int_{\mathbb{R}^N} u(.,t)dx$ behaves like $t^{-(a-N)/2}$ for large t, and estimate (3.48) is sharp. If $q > (N+2)/(N+1)$, and $u_0 \in L^1(\mathbb{R}^N)$, then $u(.,t)$ behaves as the fundamental solution of heat equation, see [9].*

Our result is new when $u_0 \in L^r(\mathbb{R}^N)$, $r > 1$ and $u_0 \notin L^1(\mathbb{R}^N)$. When $q > (N+2)/(N+1)$, and u_0 is bounded and behaves like $|x|^{-b}$ as $|x| \to \infty$ with $b \in (a, N)$, it has been shown that $u(.,t)$ behaves as the selfsimilar solution of the heat equation with initial data $|x|^{-b}$, see [17]. In that case $u_0 \in L^r(\mathbb{R}^N)$ for any $r > N/b$ and $\int_{\mathbb{R}^N} u^r(.,t)dx$ behaves like $t^{-(br-N)/2}$. Thus (3.48) is sharp as $b \to a$.

4 The Dirichlet problem in $Q_{\Omega,T}$

Here we study equation (1.1) in case of a regular bounded domain Ω, with Dirichlet conditions on $\partial\Omega \times (0,T)$, with $\nu = 1$:

$$(D_{\Omega,T}) \left\{ \begin{array}{l} u_t - \Delta u + |\nabla u|^q = 0, \quad \text{in } Q_{\Omega,T}, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \end{array} \right. \tag{4.1}$$

Let us recall some well-known results in case of smooth initial data. For any nonnegative $u_0 \in C_0^1\left(\overline{\Omega}\right)$, there exists a unique solution $u \in C^{2,1}\left(\Omega \times (0,\infty)\right) \cap C\left(\overline{\Omega} \times [0,\infty)\right)$, such that $|\nabla u| \in C\left(\overline{\Omega} \times [0,\infty)\right)$. Universal a priori estimates are given in [23]: there exists a constant $C > 0$ and a function $D \in C((0,\infty)$ such that

$$u(.,t) \leqq C(1+t^{-1/(q-1)})d(x,\partial\Omega), \qquad |\nabla u(.,t)| \leqq D(t). \tag{4.2}$$

The estimate on u is based on the construction of supersolutions, and the estimate of the gradient is deduced from the first one by the Bernstein technique.

As in section 3, we study the problem with rough initial data, and introduce different notions of solutions.

4.1 Solutions of the heat equation with L^1 data

In the following, since the regularization used at Section does not provide estimates up to the boundary, thus we use another argument: the notion of *entropy solution,* introduce din [36], for the problem

$$\begin{cases} u_t - \Delta u = f, \quad \text{in } Q_{\Omega,s,\tau}, \\ u = 0 \quad \text{on } \partial\Omega \times (s,\tau), \\ u(.,s) = u_s \geqq 0 \end{cases} \tag{4.3}$$

when f and u_s are integrable, that we recall now. For any $k > 0$ and $\theta \in \mathbb{R}$, we define as usual the trucation fonction T_k and a primitive Θ_k by

$$T_k(\theta) = \max(-k, \min(k,\theta)), \qquad \Theta_k(s) = \int_0^r T_k(\theta)d\theta. \tag{4.4}$$

Definition 4.1 *Let $s,\tau \in \mathbb{R}$ with $s < \tau$, and $f \in L^1(Q_{\Omega,s,\tau})$ and $u_s \in L^1(\Omega)$. A function $u \in C([s,\tau]\,; L^1(\Omega))$ is an entropy solution of the problem (4.3) if and for any $k > 0, T_k(u) \in L^2((s,\tau)\,; W_0^{1,2}(\Omega))$ and*

$$\begin{array}{c} \int_\Omega \Theta_k(u-\varphi)(.,\tau)dx - \int_\Omega \Theta_k(u_s - \varphi(.,s)dx + \int_s^\tau \langle \varphi_t, T_k(u-\varphi)\rangle dt \\ + \int_s^\tau \int_\Omega (\nabla u.\nabla T_k(u-\varphi) - fT_k(u-\varphi)dxdt \leq 0 \end{array} \tag{4.5}$$

for any $\varphi \in L^2((s,\tau); W^{1,2}(\Omega)) \cap L^\infty\left(Q_{\Omega,\tau}\right)$ such that $\varphi_t \in L^2((s,\tau); W^{-1,2}(\Omega))$.

Other notions of solutions have been used for this problem, see [7], recalled below. In fact they are equivalent: here $e^{t\Delta}$ denotes the semi-group of the heat equation with Dirichlet conditions acting on $L^1\left(\Omega\right)$,

Lemma 4.2 *Let -$\infty < s < \tau < \infty$, $f \in L^1(Q_{\Omega,s,\tau})$, $u_s \in L^1(\Omega)$ and $u \in C([s,\tau]\,; L^1(\Omega))$, $u(.,s) = u_s$. Then denoting the three properties are equivalent:*

(i) u is a weak solution of problem (4.3) in $Q_{\Omega,s,\tau}$, that means $u \in L^1_{loc}((s,\tau); W_0^{1,1}\left(\Omega\right))$ and

$$u_t - \Delta u = f, \quad \text{in } \mathcal{D}'(Q_{\Omega,s,\tau}); \tag{4.6}$$

(ii) u is a mild solution of (4.3), that means, for any $t \in [s,\tau]$,

$$u(.,t) = e^{(t-s)\Delta}u_s + \int_s^t e^{(t-\sigma)\Delta} f(\sigma) d\sigma \qquad in\ L^1(\Omega); \tag{4.7}$$

(iii) u is an entropy solution of (4.3).

Such a solution exists, is unique, and will be called weak solution of (4.3).

Proof. It follows from the existence and uniqueness of the solutions of (i) from [4, Lemma 3.4], as noticed in [7], and of the entropy solutions, see [18]. ■

As a consequence, when u is bounded, we can admit test functions of the form u^α :

Lemma 4.3 *Let $s, \tau \in \mathbb{R}$ with $s < \tau$, and $f \in L^1(Q_{\Omega,s,\tau})$ and u be any nonnegative* ***bounded*** *weak solution in $Q_{\Omega,s,\tau}$ of (4.3).*

Then for any $\alpha > 0$, we have $u^{\alpha-1}|\nabla u|^2 \in L^1(Q_{\Omega,s,\tau})$ and

$$\frac{1}{\alpha+1}\int_\Omega u^{\alpha+1}(.,\tau))dx + \alpha \int\int_{Q_{\Omega,s,\tau}} u^{\alpha-1}|\nabla u|^2 dxdt = \frac{1}{\alpha+1}\int_\Omega u^{\alpha+1}(.,s))dx + \int_s^\tau \int_\Omega f u^\alpha dxdt. \tag{4.8}$$

Proof. There holds $u \in L^2((s,\tau); W_0^{1,2}(\Omega)) \cap L^\infty(Q_{\Omega,s,\tau})$, and $u_t \in L^2((s,\tau); W^{-1,2}(\Omega)) + L^1(Q_{\Omega,s,\tau})$, then any function $\varphi \in L^2((s,\tau); W_0^{1,2}(\Omega)) \cap L^\infty(Q_{\Omega,s,\tau})$ is admissible in equation (4.6). In particular for any $\alpha > 0$, we can take $\varphi = M_{\alpha,\delta}(u) = (u+\delta)^\alpha - \delta^\alpha$, with $\delta > 0$. Integrating on $[s,\tau]$ we deduce that

$$\int_s^\tau < u_t, \varphi > + \alpha \int\int_{Q_{\Omega,s,\tau}} (u+\delta)^{\alpha-1}|\nabla u|^2 dxdt = \int_s^\tau \int_\Omega f M_{\alpha,\delta}(u) dxdt.$$

Let $k > 0$ such that $\sup_{Q_{\Omega,s,\tau}} u \leqq k$, thus $u = T_k(u)$. Moreover the function $\theta \mapsto M(\theta) = (T_k(\theta) + \delta)^\alpha - \delta^\alpha$ is continuous on $\mathbb{R}^+$and piecewise C^1 such that $M(0) = 0$ and M' has a compact support. Denoting $\mathcal{M}_{\alpha,\delta}(r) = (u+\delta)^{\alpha+1}/(\alpha+1) - \delta^\alpha u$, we can integrate by parts from [28, Lemma 7.1], and deduce that

$$\int_\Omega \mathcal{M}_{\alpha,\delta}(u)(.,\tau))dx - \int_\Omega \mathcal{M}_{\alpha,\delta}(u)(.,s))dx + \alpha \int\int_{Q_{\Omega,s,\tau}} (u+\delta)^{\alpha-1}|\nabla u|^2 dxdt = \int_s^\tau \int_\Omega f M_{\alpha,\delta}(u) dxdt$$

and then we go to the limit as $\delta \to 0$ from the Fatou Lemma and then from the Lebesgue theorem. Thus (4.8) holds for $\alpha > 0$. ■

Remark 4.4 *From [28], the notion of entropy solution of (4.3) is also equivalent to the notion of renormalized solution, that we develop in Section 5. Lemma 4.3 is a special case of a much more general property of the truncates when u is not necessarily bounded, see Lemma 5.4.*

4.2 Different notions of solutions of problem $(D_{\Omega,T})$

Definition 4.5 *We say thay u is a **weak solution** of the problem* $(D_{\Omega,T})$ *if* $u \in C((0,T); L^1(\Omega)) \cap L^1_{loc}((0,T); W_0^{1,1}(\Omega))$, *such that* $|\nabla u|^q \in L^1_{loc}((0,T); L^1(\Omega))$ *and u satisfies*

$$u_t - \Delta u + |\nabla u|^q = 0, \quad \text{in } \mathcal{D}'(Q_{\Omega,T}). \tag{4.9}$$

Next we study the Cauchy problem

$$\begin{cases} u_t - \Delta u + |\nabla u|^q = 0, & \text{in } Q_{\Omega,T}, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0 \geqq 0 \end{cases} \tag{4.10}$$

with $u_0 \in L^r(\Omega)$, $r \geqq 1$, or only $u_0 \in \mathcal{M}_b^+(\Omega)$. Here in any case $u_0 \in \mathcal{M}_b^+(\Omega)$.

Definition 4.6 *If* $u_0 \in L^r(\Omega), r \geqq 1$, *we say that u is a **weak** L^r **solution** of problem (4.10) if it is a weak solution of* $(D_{\Omega,T})$, *such that the extension of u by* u_0 *at time* 0 *satisfies* $u \in C([0,T); L^r(\Omega))$.

Definition 4.7 *For any* $u_0 \in \mathcal{M}_b^+(\Omega)$, *we say that u is a weak* $\mathcal{M}$ *solution of problem (4.10) if it is a weak solution of* $(D_{\Omega,T})$, *such that*

$$\lim_{t \to 0} \int_\Omega u(.,t)\psi dx = \int_\Omega \psi du_0, \qquad \forall \psi \in C_b(\Omega). \tag{4.11}$$

Some semi-group notions of solutions have been introduced in [7], for any nonnegative $u_0 \in \mathcal{M}_b^+(\Omega)$. Here $e^{t\Delta}u_0 = \int_\Omega g_\Omega(.,y,t)du_0(y)$, where g_Ω is the heat kernel with Dirichlet conditions on $\partial\Omega$.

Definition 4.8 *For any* $u_0 \in \mathcal{M}_b^+(\Omega)$, *a function u is a **mild solution** of problem (4.10) if* $u \in C((0,T); L^1(\Omega))$, *and* $|\nabla u|^q \in L^1_{loc}([0,T); L^1(\Omega))$ *and*

$$u(.,t) = e^{t\Delta}u_0(.) - \int_0^t e^{(t-s)\Delta}|\nabla u(.,s)|^q ds \qquad \text{in } L^1(\Omega), \tag{4.12}$$

Remark 4.9 *As it was shown in [7, p.1420], from Lemma 4.2, u is a mild solution if and only if u is a weak* $\mathcal{M}$ *solution such that* $|\nabla u|^q \in L^1_{loc}([0,T); L^1(\Omega))$; *and then* $u \in L^1_{loc}([0,T); W_0^{1,1}(\Omega))$.

Remark 4.10 *As in Remark 3.12, the definition of mild solution requires an integrability property of the gradient up to time* 0, *namely* $|\nabla u|^q \in L^1_{loc}([0,T); L^1(\Omega))$. *The definition of weak solution only assumes that* $|\nabla u|^q \in L^1_{loc}((0,T); L^1(\Omega))$.

4.3 Decay and regularizing effect

Here Ω is bounded, then the situation is simpler than in $\mathbb{R}^N$, because we take benefit of the regularizing effect of the semi-group $e^{t\Delta}$ associated with the first eigenvalue λ_1 of the Laplacian, and also since $L^r(\Omega) \subset L^1(\Omega)$.

Lemma 4.11 *Let $q > 1$, and $u_0 \in L^r(\Omega)$, $r \geqq 1$. 1) Let u be any non-negative weak L^r -solution of problem (4.10).*

(i) Then $u(.,t) \in L^\infty(\Omega)$ for any $t > 0$, and

$$\|u(.,t)\|_{L^r(\Omega)} \leqq Ce^{-\lambda_1 t}\|u_0\|_{L^r(\Omega)}, \qquad \|u(.,t)\|_{L^\infty(\Omega)} \leqq Ct^{-\frac{N}{2r}}e^{-\lambda_1 t}\|u_0\|_{L^r(\Omega)}. \tag{4.13}$$

(ii) Moreover $|\nabla u|^q \in L^1_{loc}([0,T);L^1(\Omega))$, and

$$\int_\Omega u(.,t)dx + \int_0^t \int_\Omega |\nabla u|^q dxdt \leqq \int_\Omega u_0 dx. \tag{4.14}$$

If $r > 1$, then $u^{r-1}|\nabla u|^q \in L^1_{loc}([0,T);L^1(\Omega))$; we have $u^{r-2}|\nabla u|^2 \in L^1_{loc}([0,T);L^1(\Omega))$ and

$$\frac{1}{r}\int_\Omega u^r(.,t)dx + \int_0^t \int_\Omega u^{r-1}|\nabla u|^q dxdt + (r-1)\int_0^t \int_\Omega u^{r-2}|\nabla u|^2 dxdt = \frac{1}{r}\int_\Omega u_0^r dx, \tag{4.15}$$

As a consequence, $u^{q-1+r} \in L^1_{loc}(([0,T);W_0^{1,1}(\Omega))$.

2) Let $u_0 \in \mathcal{M}_b^+(\Omega)$ and u be any non-negative weak $\mathcal{M}$ solution of problem (4.10). Then (4.13) and (4.14) still hold as in case $u_0 \in L^1(\Omega)$, where the norm $\|u_0\|_{L^1(\Omega)}$ is replaced by $\int_\Omega du_0$. In particular u is a mild solution.

Proof. 1) (i) Let $0 < \epsilon < \tau < T$. Since u is a weak solution of $(D_{\Omega,T})$, we can apply Lemma 4.2 with $f = -|\nabla u|^q$ in $Q_{\Omega,\epsilon,\tau}$. Thus u is a mild solution of the problem in $Q_{\Omega,\epsilon,\tau}$: for any $t \in [\epsilon,\tau]$,

$$u(.,t) = e^{(t-\epsilon)\Delta}u(.,\epsilon) - \int_\epsilon^t e^{(t-\sigma)\Delta}|\nabla u|^q d\sigma \qquad \text{in } L^1(\Omega).$$

thus $u(.,t) \leqq e^{(t-\epsilon)\Delta}u(.,\epsilon)$. From our assumptions $u \in C([0,T);L^r(\Omega))$, thus we deduce $u(.,t) \leqq e^{t\Delta}u_0$ as $\epsilon \to 0$. Then (4.13) follows.

(ii) The function u is bounded in $Q_{\Omega,s,\tau}$, thus from Lemma 4.3, for any $\rho > 1$,

$$\frac{1}{\rho}\int_\Omega u^\rho(.,t)dx + \int_\epsilon^t \int_\Omega u^{\rho-1}|\nabla u|^q dxdt + (\rho-1)\int_\epsilon^t \int_\Omega u^{\rho-2}|\nabla u|^2 dxdt = \frac{1}{\rho}\int_\Omega u^\rho(.,\epsilon)dx. \tag{4.16}$$

and we make $\rho \to 1$. From Fatou Lemma we deduce that $|\nabla u|^q \in L^1(Q_{\Omega,\epsilon,\tau})$ and

$$\int_\Omega u(.,t)dx + \int_\epsilon^t \int_\Omega |\nabla u|^q dxdt \leqq \int_\Omega u(.,\epsilon)dx.$$

As $\epsilon \to 0$ we deduce that $|\nabla u|^q \in L^1(Q_{\Omega,\tau})$ and (4.14) holds. If $r > 1$, we can take $\rho = r$ in (4.16) and obtain (4.15) as $\epsilon \to 0$. Then $u^{q-1+r} \in L^1_{loc}(([0,T);W_0^{1,1}(\Omega))$ as in the case of $\mathbb{R}^N$.

2) The same estimates hold because $\lim_{\epsilon\to 0}\|u(.,\epsilon)\|_{L^1(\Omega)} = \int_\Omega du_0$. ∎

Theorem 4.12 *Let $q>1$ and $u_0 \in L^r(\Omega)$, $r \geqq 1$. 1) Let u be any non-negative weak L^r -solution of problem (4.10). Then*

$$\|u(.,t)\|_{L^\infty(\Omega)} \leqq Ct^{-\sigma_{r,q}} \|u_0\|^{\varpi_{r,q}}_{L^r(\Omega)}, \tag{4.17}$$

where $\sigma_{r,q}, \varpi_{r,q}$ are given at (3.32).

2) Any non-negative weak solution u of $(D_{\Omega,T})$ satisfies the universal estimate, where $C=C(N,q)>0$,

$$\|u(.,t)\|_{L^\infty(\Omega)} \leqq Ct^{-\frac{1}{q-1}}. \tag{4.18}$$

Proof. 1) For any $\alpha>0$, setting $\rho=1+\alpha$, and $0<\epsilon \leqq s<t<T$, setting $\beta=1+\alpha/q$, we have from (4.16),

$$\frac{1}{\alpha+1}\int_\Omega u^{\alpha+1}(.,t)dx + \int_s^t \int_\Omega \left|\nabla(u^\beta)\right|^q dxdt \leqq \frac{1}{\alpha+1}\int_\Omega u^{\alpha+1}(.,s)dx.$$

And $u^\beta(.,t) \in L^\infty(Q_{\Omega,s,\tau})$ for a.e. $t>0$, then $u^\beta(.,t) \in W^{1,q}(\Omega)$; and $u(.,t) \in W_0^{1,1}(\Omega))$ hence $u^\beta(.,t) \in W_0^{1,q}(\Omega)$, then from the Sobolev injection of $W_0^{1,q}(\Omega)$ into $L^{q^*}(\Omega)$, for any $s<t$,

$$\frac{1}{\alpha+1}\int_\Omega u^{\alpha+1}(.,t)dx + \frac{C_{N,q}}{\beta^q}\int_s^t (\int_\Omega u^{\beta q^*}(.,\sigma)dx)^{\frac{q}{q^*}} dt \leqq \frac{1}{\alpha+1}\int_\Omega u^{\alpha+1}(.,s)dx.$$

Then we can apply Lemma 2.2 on $[\epsilon,T)$, and deduce estimates for $\epsilon<t<T$,

$$\|u(.,t)\|_{L^\infty(\Omega)} \leqq C(t-\epsilon)^{-\sigma_{r,q}} \|u(.,\epsilon)\|^{\varpi_{r,q}}_{L^r(\Omega)},$$

$$\|u(.,t)\|_{L^\infty(\Omega)} \leqq C(t-\epsilon)^{-\frac{1}{q-1}}.$$

and we deduce (4.17) and (4.18) as $\epsilon \to 0$.

2) Let u be any weak solution of $(D_{\Omega,T})$. Let $\epsilon>0$. Since $u \in C([\epsilon,T);L^1(\Omega))$ we find, for any $t \in [\epsilon,T)$,

$$\|u(.,t)\|_{L^\infty(\Omega)} \leqq C(t-\epsilon)^{-\frac{1}{q-1}}$$

with $C=C(N,q)$, and deduce (4.18) for any $t \in (0,T)$ by letting ϵ tend to 0. ■

Remark 4.13 *The same decay estimates where shown in [34] in case $q<2$, for any weak L^r solution u such that $u \in C((0,T);L^2(\Omega)) \cap L^2((0,T);W_0^{1,2}(\Omega))$, and $(u-k)^+$ is admissible as a test function in the equation; this implies integrability properties of $u|\nabla u|^q$. Our result is valid without any of these conditions.*

4.4 Existence and uniqueness results for $q \leqq 2$

Here we consider the case $1<q\leqq 2$. From the universal a priori estimate (4.18), we deduce new convergence results:

Corollary 4.14 *Assume $1<q\leqq 2$. Then*

(i) any weak solution u of problem $(D_{\Omega,T})$ satisfies $u \in C^{2,1}\left(Q_{\mathbb{R}^N,T}\right) \cap C^{1,0}\left(\overline{\Omega}\times(0,T)\right)$;

(ii) for any sequence of weak solutions (u_n) of $(D_{\Omega,T})$, one can extract a subsequence converging in $C^{2,1}_{loc}(Q_{\mathbb{R}^N,T}) \cap C^{1,0}\left(\overline{\Omega}\times(0,T)\right)$ to a weak solution u of $(D_{\Omega,T})$.

Proof. (i) From [16, Theorem 2.17], any weak solution u of $(D_{\Omega,T})$ such that $u \in L^\infty_{loc}((0,T)\,;L^\infty(\Omega))$ satisfies $u \in C^{2,1}\left(Q_{\mathbb{R}^N,T}\right) \cap C^{1,0}\left(\overline{\Omega} \times (0,T)\right)$. And any weak solution $u \in L^\infty_{loc}((0,T)\,;L^\infty(\Omega))$, from Theorem 4.12,3.

(ii) Moreover (u_n) is uniformly bounded in $L^\infty_{loc}\left(0,T); L^\infty\left(\Omega\right)\right)$. From [16, Theorem 2.13], there exists $\upsilon \in (0,1)$ such that, for any $0 < s < \tau < T$,

$$\|u_n\|_{C(\overline{\Omega}\times[s,\tau])} + \|\nabla u_n\|_{C^{\upsilon,\upsilon/2}(\overline{\Omega}\times[s,\tau])} \leqq C\Phi(\|u_n\|_{L^\infty(Q_{\Omega,s/2,\tau})}) \tag{4.19}$$

where $C = C((N,q,\Omega,s,\tau,\upsilon)$, and Φ is an increasing function. The conclusion follows. ■

Theorem 4.15 *Suppose $1 < q < (N+2)/(N+1)$. For any $u_0 \in \mathcal{M}_b^+(\Omega)$, problem (4.10) admits a unique weak $\mathcal{M}$ solution.*

Proof. From Lemma 4.11, u is a mild $\mathcal{M}$ solution, and then it is the unique mild $\mathcal{M}$ solution, from [7, Theorem 3.2]. ■

Next assume that $u_0 \in L^r(\Omega)$ and $q < (N+2r)/(N+r)$. In [7, Theorem 3.3], it is proved that there exists a weak L^r solution such that $u \in L^q_{loc}([0,T)\,;W_0^{1,qr}(\Omega))$, and *it is unique in this space.* The local existence in an interval $(0,T_1)$ is obtained by the Banach fixed point theorem in a ball of radius K_1 of the space

$$X_{K_1}(T_1) = \left\{ u \in C((0,T_1]\,,W_0^{1,qr}(\Omega)) : \sup_{(0,t_1]} t^\theta(\|u(.,t)\|_{L^{qr}(\Omega)} + t^{\frac{1}{2}}\|\nabla u(.,t)\|_{L^{qr}(\Omega)}) < \infty \right\}$$

where $\theta = N/2rq'$, under the condition

$$\|u_0\|_{L^r(\Omega)} + K_1^q T_1^\gamma \leqq CK_1, \qquad \text{where } \gamma = 1 - q(\theta + 1/2) \quad \text{and } C = C(N,q,r,\Omega). \tag{4.20}$$

We prove the uniqueness *with no condition of integrability*:

Theorem 4.16 *Assume that $u_0 \in L^r(\Omega)$ and $1 < q < (N+2r)/(N+r)$. Then problem (4.10) admits a unique weak L^r solution.*

Proof. Let $\epsilon > 0$. From Theorem 4.12, u is bounded on (ϵ,T) for any $\epsilon \in (0,T)$. Then $u \in C^{2,1}(Q_{\Omega,T}) \cap C^{1,0}(\overline{\Omega} \times (0,T))$ because $q < 2$, from [16, Theorem 2.16]. From (4.2), there exists a function $D \in C((0,\infty)$ such that for any $\epsilon > 0$ and for $t \geqq \epsilon$

$$\|\nabla u(.,t)\|_{L^\infty(\Omega)} \leqq D(t-\epsilon).$$

Then $|\nabla u|$ is bounded in $Q_{\epsilon,T,\Omega}$ for any $\epsilon > 0$. Thus $u \in C((0,T), W_0^{1,qr}(\Omega))$. The problem with initial data $u(.,\epsilon)$ at time 0 has a unique solution v_ϵ such that $v_\epsilon \in C((0,T-\epsilon), W_0^{1,qr}(\Omega))$, then $v_\epsilon(.,t) = u(.,t+\epsilon)$. Let K_1 and T_1 such that (4.20) holds. Since $\|u(.,\epsilon)\|_{L^r(\Omega)} \leqq \|u_0\|_{L^r(\Omega)}$, we also have $\|v_\epsilon(0)\|_{L^r(\Omega)} + K_1^q T_1^\gamma \leqq CK_1$, thus for any $t \in (0,T_1)$

$$t^\theta(\|v_\epsilon(.,t)\|_{L^{qr}(\Omega)} + t^{\frac{1}{2}}\|\nabla v_\epsilon(.,t)\|_{L^{qr}(\Omega)} \leqq K_1.$$

Going to the limit as $\epsilon \to 0$ from the Fatou Lemma, we obtain

$$t^\theta(\|u(.,t)\|_{L^{qr}(\Omega)} + t^{\frac{1}{2}}\|\nabla u(.,t)\|_{L^{qr}(\Omega)} \leqq K_1.$$

Hence we enter the class of uniqueness. Then u is the unique solution constructed in [7]. ■

Finally we give existence results for any $u_0 \in L^r(\Omega), r \geq 1$, extending the results of [7, Theorem 3.4] for $u_0 \in L^1(\Omega)$, also proved for more general operators in [33]. We proceed as in Proposition 3.28.

Proposition 4.17 *Let $1 < q \leq 2$. For any nonnegative $u_0 \in L^r(\Omega), r \geq 1$, there exists a weak L^r solution of problem (4.10). And it is unique if $q = 2$.*

Proof. (i) Case $q < 2$. Let $u_{0,n} = \min(u_0, n)$. Then for $\rho > N(q-1)/(2-q)$, from [7, Theorem 3.3], there exists a mild solution u_n with initial data $u_{0,n}$, and $u_n \in C([0,T); L^\rho(\Omega)) \cap L^q((0,T); W_0^{1,q\rho}(\Omega) \cap C^{2,1}(Q_{\Omega,T})$. Then $u_n(.,t) \leq e^{t\Delta}u_0$, and (u_n) is nondecreasing and $|\nabla u_n|^q$ is bounded in $L^1_{loc}\left([0,T); L^1(\Omega)\right)$ from (4.14). From Corollary 3.23, (u_n) converges in $C^{2,1}_{loc}(Q_{\Omega,T})$ to a weak solution u of (1.1) in $Q_{\Omega,T}$, and then $u(.,t) \leq e^{t\Delta}u_0$ and $|\nabla u|^q \in L^1_{loc}\left([0,T); L^1(\Omega)\right)$. Thus from [16, Proposition 2.11], $u(.,t)$ converges weakly* to some Radon measure μ_0 on Ω. And $e^{t\Delta}u_0$ converges to u_0 in $L^r(\Omega)$, thus $\mu_0 \in L^1_{loc}(\Omega)$ and $0 \leq \mu_0 \leq u_0$. Since $u_n \leq u$, there holds $u_{0,n} \leq \mu_0$, hence $\mu_0 = u_0 \in L^r(\Omega)$. Also there exists a function $g \in L^r(\Omega)$ such that $u(.,t) \leqq g$ for small t. Then the nonnegative function $e^{t\Delta}u_0 - u(.,t)$ converges weakly* to 0, and then in $L^1_{loc}(\Omega)$. Hence $u(.,t)$ converges to u_0 in $L^1_{loc}(\Omega)$, then in $L^r(\Omega)$ from the Lebesgue theorem. Thus $u \in C([0,T); L^r(\Omega))$.

(ii) Case $q = 2$. As in [14, Theorem 4.2], using the classical transformation $v = 1 - e^{-u}$, it can be shown that there exists a unique solution u such that $u \in C([0,T); L^r(\Omega)) \cap C^{2,1}(Q_{\Omega,T}) \cap C^1\left(\overline{\Omega} \times (0,T)\right)$. Then it is a weak L^r solution. Reciprocally any weak L^r solution u satisfies the conditions above, from Corollary 4.14 and [16, Theorem 2.17]. ■

5 Regularizing effects for quasilinear Dirichlet problems

Next we extend some results of section 4 to a general quasilinear problem, where u may be a signed solution. In this section, we suppose Ω is a smooth bounded domain in $\mathbb{R}^N$. Let $p, q > 1$. Let A be a Caratheodory function on $Q_{\Omega,\infty} \times \mathbb{R} \times \mathbb{R}^N$ such that for any $(u, \eta) \in \mathbb{R} \times \mathbb{R}^N$, and a.e. $(x,t) \in Q_{\Omega,\infty}$,

$$|\mathrm{A}(x,t,u,\eta)| \leqq C(|\eta|^{p-1} + b(x,t)), \qquad C > 0, \quad b \in L^{p'}(Q_{\Omega,\infty}), \tag{5.1}$$

and A is nonnegative operator:

$$\mathrm{A}(x,t,u,\eta).\eta \geqq \nu \, |\eta|^p \qquad \nu \geqq 0, \tag{5.2}$$

with no monotonicity assumption.

Let g be a Caratheodory function on $Q_{\Omega,\infty} \times \mathbb{R}^+ \times \mathbb{R}^N$, such that

$$g(x,t,u,\eta)u \geqq \gamma \, |u|^{\lambda+1} \, |\eta|^q, \qquad \lambda \geqq 0, \quad \gamma \geqq 0. \tag{5.3}$$

Definition 5.1 *We say that A is coercive if (5.2) holds with $\nu > 0$, and g is coercive if (5.3) holds with $\gamma > 0$.*

We consider the solutions of the Dirichlet problem

$$(P_{\Omega,T}) \begin{cases} u_t - \operatorname{div}(\mathrm{A}(x,t,u,\nabla u)) + g(x,t,u,\nabla u) = 0, & \text{in } Q_{\Omega,T}, \\ u = 0, \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0 \end{cases} \tag{5.4}$$

where $u_0 \in L^r(\Omega)$, $r \geqq 1$ or only $u_0 \in \mathcal{M}_b(\Omega)$.

5.1 Solutions of quasilinear heat equation with L^1 data

Here we consider the problem in $Q_{\Omega,s,\tau}$

$$\begin{cases} u_t - \operatorname{div}(\mathrm{A}(x,t,u,\nabla u)) = f, & \text{in } Q_{\Omega,s,\tau}, \\ u = 0, & \text{on } \partial\Omega \times (s,\tau), \\ u(x,s) = u_s \end{cases} \tag{5.5}$$

First we recall the notion of renormalized solution introduced in [18] for this problem with L^1 data:

Definition 5.2 *Let $s,\tau \in \mathbb{R}$ with $s < \tau$, and $f \in L^1(Q_{\Omega,s,\tau})$ and $u_s \in L^1(\Omega)$. A function $u \in L^\infty((s,\tau); L^1(\Omega))$ is a renormalized solution in $Q_{\Omega,s,\tau}$ of (5.5) if $T_k(u) \in L^p((s,\tau); W_0^{1,p}(\Omega))$ for any $k \geqq 0$, and for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' has a compact support,*

$$(S(u))_t - \operatorname{div}(\mathrm{A}(x,t,u,\nabla u)S'(u)) + S''(u)(\mathrm{A}(x,t,u,\nabla u).\nabla u - S'(u)f = 0 \quad \text{in } \mathcal{D}'(Q_{\Omega,s,\tau}), \tag{5.6}$$

and $u(s) = u_s$, and

$$\lim_{n\to\infty} \int\int_{Q_{\Omega,s,\tau}\cap\{n\leqq u\leqq n+1\}} |\nabla u|^p dxdt = 0, \tag{5.7}$$

Remark 5.3 *The initial condition takes sense from [18], because $S(u)$ lies in the set*

$$E = \left\{ \varphi \in L^p((0,T); W_0^{1,p}(\Omega)) : \varphi_t \in L^{p'}((0,T); W^{-1,p'}(\Omega)) + L^1(Q_{\Omega,T}) \right\} \tag{5.8}$$

and $E \subset C([0,T]; L^1(\Omega))$; and any function $\varphi \in L^p((0,T); W_0^{1,p}(\Omega)) \cap L^\infty(Q_{\Omega,T})$ is admissible in equation (5.6). Moreover from [28, Lemma 7.1], $v = S(u)$ satisfies for any $\psi \in C^\infty([s,\tau] \times \bar{\Omega})$ the integration formula

$$\int_s^\tau < v_t, M(v)\psi >= \int_\Omega \mathcal{M}(v(.,\tau))\psi(.,\tau)dx - \int_\Omega \mathcal{M}(v(.,s))\psi(.,s)dx - \int_s^\tau \int_\Omega \psi_t \mathcal{M}(v)dxdt, \tag{5.9}$$

for any function M continuous and piecewise C^1 such that $M(0) = 0$ and M' has a compact support, where $\mathcal{M}(r) = \int_0^r M(\theta)d\theta$.

A main point in the sequel is the choice of test functions: here we approximate $|u|^{\alpha-1}u$ for $\alpha > 0$ by truncation. In the following lemma, we solve some technical difficulties arising because the truncates are not smooth enough to apply the integration formula, and moreover we do not assume $\alpha \geqq 1$.

Lemma 5.4 *Let $s,\tau \in \mathbb{R}$ with $s < \tau$, and $f \in L^1(Q_{\Omega,s,\tau})$ and $u \in C([s,\tau]; L^1(\Omega))$ be any non-negative renormalized solution in $Q_{\Omega,s,\tau}$ of (5.5), with $u_s = u(.,s)$. For any $\alpha > 0$ and $k > 0$, we set*

$$\mathcal{T}_{k,\alpha}(r) = \int_0^r |T_k(\theta)|^{\alpha-1} T_k(\theta)d\theta.$$

Then $|T_k(u)|^{\alpha-1} \mathrm{A}(x,t,u,\nabla u).\nabla(T_k(u)) \in L^1(Q_{\Omega,s,\tau})$ and

$$\begin{aligned} &\int_\Omega \mathcal{T}_{k,\alpha}(u)(.,\tau))dx + \alpha \int\int_{Q_{\Omega,s,\tau}} |T_k(u)|^{\alpha-1} \mathrm{A}(x,t,u,\nabla u).\nabla(T_k(u))dxdt \\ &= \int_\Omega \mathcal{T}_{k,\alpha}(u)(.,s))dx + \int_s^\tau \int_\Omega f |T_k(u)|^{\alpha-1} T_k(u)dxdt. \end{aligned} \tag{5.10}$$

Proof. Let $\alpha > 0, k > 0$ be fixed, and for any $n \geqq 2$, and $\theta \in \mathbb{R}$,

$$S_n(\theta) = \int_0^\theta (1 - |T_1(s - T_n(s)|)ds, \qquad n \geqq 2.$$

This function, introduced in [18], is a smoothing of the truncate T_{n+1}, such that $0 \leqq S_n(\theta)\theta \leqq T_{n+1}(\theta)\theta$, supp $S_n' \subset [-(n+1), n+1]$, and $S_n(\theta) = S_n(T_k(\theta))$ for any $n > k$. Let $\delta \in (0,k)$, and $n > k$. Setting

$$T_{\delta,k,\alpha}(\theta) = ((T_k(|\theta|) + \delta))^\alpha - \delta^\alpha)\text{sign}\theta, \qquad \mathcal{T}_{\delta,k,\alpha}(r) = \int_0^r T_{\delta,k,\alpha}(\theta)d\theta$$

Then we can take in (5.6) $S = S_n$ and $\varphi = T_{\delta,k,\alpha}(u) = T_{\delta,k,\alpha}(S_n(u))$. We obtain

$$\begin{aligned}\int_s^t &< (S_n(u))_t, \varphi > + \int_s^t \int_\Omega S_n'(u)\mathrm{A}(x,t,u,\nabla u).\nabla\varphi dxdt \\ &= \int_s^t \int_\Omega S_n'(u) f\varphi dxdt - \int_s^t \int_\Omega S_n''(u)(\mathrm{A}(x,t,u,\nabla u).\nabla u)\varphi dxdt.\end{aligned}$$

then from (5.9), we deduce

$$\begin{aligned}&\int_\Omega \mathcal{T}_{\delta,k,\alpha}(S_n(u)(.,\tau))dx + \alpha\int\int_{Q_{\Omega,s,\tau}} (T_k(|u|) + \delta)^{\alpha-1}\mathrm{A}(x,t,u,\nabla u).\nabla(T_k(u))dxdt \\ &= \int_\Omega \mathcal{T}_{\delta,k,\alpha}(S_n(u)(.,s))dx + \int_s^\tau \int_\Omega S_n'(u)f\varphi dxdt - \int_s^t \int_\Omega S_n''(u)(\mathrm{A}(x,t,u,\nabla u).\nabla u)\varphi dxdt\end{aligned}$$

First we make $\delta \to 0$. We have $|\mathcal{T}_{\delta,k,\alpha}(\theta)| \leqq k^\alpha |\theta|$ for any $\theta \in \mathbb{R}$, and $S_n(u) \in C([0,T]\,; L^1(\Omega))$, and S_n' is bounded, thus we can go to the limit in the right hand side. In the left hand side, From the positivity of A, and the Fatou Lemma we deduce that $T_k(|u|)^{\alpha-1}\mathrm{A}(x,u,\nabla u).\nabla T_k(u) \in L^1(Q_{\Omega,s,\tau})$. Then we can apply Lebesgue theorem: indeed $\mathrm{A}(x,u,\nabla u).\nabla T_k(u) \in L^1(Q_{\Omega,s,\tau})$ from (5.1), since $T_k(u) \in L^p((s,\tau); W_0^{1,p}(\Omega))$, and $(T_k(|u|) + \delta)^{\alpha-1} \leqq \max(T_k^{\alpha-1}(|u|), (k+1)^{\alpha-1})$. Then the same relation holds with $\delta = 0$, with $T_{0,k,\alpha}(r) = T_k^{\alpha-1}(|u|)T_k(u)$:

$$\begin{aligned}&\int_\Omega \mathcal{T}_{k,\alpha}(S_n(u)(.,\tau))dx - \int_\Omega \mathcal{T}_{k,\alpha}(S_n(u)(.,s))dx + \alpha\int_s^t \int_\Omega T_k^{\alpha-1}(|u|)\mathrm{A}(x,u,\nabla u).\nabla(T_k(u))dxdt \\ &= \int_s^\tau \int_\Omega S_n'(u)fT_{0,k,,\alpha}(u)dxdt - \int_s^t \int_\Omega S_n''(u)(\mathrm{A}(x,t,u,\nabla u).\nabla u)T_{0,k,,\alpha}(u)dxdt.\end{aligned}$$

Then we make $n \to \infty$. Since $u \in C([0,T]\,; L^1(\Omega))$, for any $t \in [s,\tau]$

$$\lim_{n\to\infty} \int_\Omega \mathcal{T}_{k,\alpha}(S_n(u)(.,t))dx = \int_\Omega \mathcal{T}_{k,\alpha}(u(.,t))dx,$$

moreover

$$\lim_{n\to\infty} \int_s^t \int_\Omega S_n''(u)(\mathrm{A}(x,t,u,\nabla u).\nabla u)\ T_{0,k,\alpha}(u)dxdt = 0$$

from (5.7), (5.1), since $S_n'' = -1_{[n,n+1]} + 1_{[-n,-n-1]}$. Moreover

$$\lim_{n\to\infty} \int_s^\tau \int_\Omega S_n'(u)fT_{0,k,,\alpha}(u)dxdt = \int_s^\tau \int_\Omega fT_{0,k,,\alpha}(u)dxdt$$

since $S_n'(u) \to 1$ a.e. and is uniformly bounded. Then (5.10) follows. ■

5.2 Notion of solutions of problem $(P_{\Omega,T})$

Definition 5.5 *We say that u is a* ***renormalized solution*** *of problem $(P_{\Omega,T})$ if:*

(i) $u \in C((0,T);L^1(\Omega))$, $T_k(u) \in L^p_{loc}((0,T);W_0^{1,p}(\Omega))$ for any $k \geqq 0$, and $g(x,u,\nabla u) \in L^1_{loc}((0,T);L^1(\Omega))$,

(i) for any $0 < s < \tau < T$, u is a renormalized solution of problem

$$\begin{cases} u_t - \operatorname{div}(\mathrm{A}(x,t,u,\nabla u)) + g(x,t,u,\nabla u) = 0, & \text{in } Q_{\Omega,s,\tau}, \\ u = 0, & \text{on } \partial\Omega \times (0,T), \end{cases}$$

with initial data $u(.,s)$;

(ii) for $u_0 \in L^r(\Omega)$, the extension of u by u_0 at time 0 belongs to $C([0,T)\,;L^r(\Omega))$; for $u_0 \in \mathcal{M}_b(\Omega)$, there holds

$$\lim_{t\to 0}\int_\Omega u(.,t)\psi dx = \int_\Omega \psi du_0, \qquad \forall\psi \in C_b(\Omega). \tag{5.11}$$

Remark 5.6 *Recall that ∇u is defined by $\nabla u = \nabla(T_k(u))$ on the set $|u| \leq k$. The assumption on g means that, for any $0 < s < \tau < T$,*

$$\int_{Q_{\Omega,s,\tau}} |g(.,u,\nabla u)|\, dxdt = \sum_{k=1}^{\infty}\int_{Q_{\Omega,s,\tau}\cap\{k-1\leqq|u|\leqq k\}} |g(.,u,\nabla(T_k(u))|\, dxdt < \infty.$$

We first prove decay properties of the solutions.

Theorem 5.7 *Let $p,q > 1$, and Ω be a regular bounded domain of $\mathbb{R}^N$. Let A and g satisfy (5.1) (5.2) and (5.3).*

1) Let $u_0 \in L^r(\Omega), r \geqq 1$ and u be any renormalized solution of $(P_{\Omega,T})$. Then for any $t \in [0,T)$,

$$\int_\Omega |u|^r\,(.,t)dx \leqq \int_\Omega |u_0|^r\, dx. \tag{5.12}$$

Moreover if $r > 1$, or if g is coercive, then $\gamma\,|u|^{\lambda+r-1}\,|\nabla u|^q + \nu\,|u|^{r-2}\,|\nabla u|^p \in L^1_{loc}([0,T)\,;L^1(\Omega))$, and

$$\int_\Omega |u|^r\,(.,t)dx + r\gamma\int_0^t\int_\Omega |u|^{\lambda+r-1}\,|\nabla u|^q dxdt + r(r-1)\nu\int_0^t\int_\Omega |u|^{r-2}\,|\nabla u|^p dxdt \leqq \int_\Omega |u_0|^r\, dx. \tag{5.13}$$

2) Let $u_0 \in \mathcal{M}_b^+(\Omega)$ and u be any nonnegative renormalized solution of $(P_{\Omega,T})$ of problem (4.10). Then the same conclusions hold as in case $u_0 \in L^1(\Omega)$, where the norm $\|u_0\|_{L^1(\Omega)}$ is replaced by $\displaystyle\int_\Omega du_0$.

Proof. 1) Let $0 < s < t < T$, then we have for any $\alpha > 0$, any $k > 0$, from Lemma 5.4,

$$\int_\Omega \mathcal{T}_{k,\alpha}(u)(.,\tau))dx + \alpha \int_s^t \int_\Omega |T_k(u)|^{\alpha-1} \mathrm{A}(x,t,u,\nabla u).\nabla(T_k(u))dxdt$$
$$= \int_\Omega \mathcal{T}_{k,\alpha}(u)(.,s))dx - \int_s^\tau \int_\Omega |T_k(u)|^{\alpha-1} T_k(u)g(.,u,\nabla u)dxdt$$

And $|T_k(u)|^{\alpha-1} T_k(u)g(.,u,\nabla u) \geq \gamma |T_k(u)|^{\alpha+\lambda} |\nabla T_k(u)|^q$ from (5.3). Then $\int_\Omega \mathcal{T}_{k,\alpha}(u)(.,t))$ is decreasing for any $k, \alpha > 0$, and

$$\int_\Omega \mathcal{T}_{k,\alpha}(u)(.,\tau))dx + \gamma \int_s^t \int_\Omega |T_k(u)|^{\alpha+\lambda} |\nabla T_k(u)|^q \, dxdt + \alpha\nu \int_s^t \int_\Omega |T_k(u)|^{\alpha-1} |\nabla T_k(u)|^p \, dxdt$$
$$\leqq \int_\Omega \mathcal{T}_{k,\alpha}(u)(.,s))dx \tag{5.14}$$

If $r > 1$, we can take $\alpha = r - 1 > 0$ in (5.14) and get

$$\int_\Omega \mathcal{T}_{k,r-1}(u)(.,t))dx + \gamma \int_s^t \int_\Omega |T_k(u)|^{r-1+\lambda} |\nabla T_k(u)|^q \, dxdt + \alpha\nu \int_s^t \int_\Omega |T_k(u)|^{r-2} |\nabla T_k(u)|^p \, dxdt$$
$$\leqq \int_\Omega \mathcal{T}_{k,r-1}(u)(.,s))dx \leqq \frac{1}{r} \int_\Omega |u|^r (.,s)dx \tag{5.15}$$

Since $u \in C([0,T); L^r(\Omega))$ we can go to the limit as $k \to \infty$, and $s \to 0$, and deduce that $\gamma |u|^{r-1+\lambda} |\nabla u|^q$ and $\alpha\nu |u|^{r-2} |\nabla u|^p$ belong to $L^1_{loc}\left([0,T); L^1(\Omega)\right)$ and for any $t \in (0,T)$,

$$\int_\Omega |u|^r (.,t)dx + r\gamma \int_0^t \int_\Omega |u|^{r-1+\lambda} |\nabla u|^q \, dxdt + r(r-1)\nu \int_0^t \int_\Omega |u|^{r-2} |\nabla u|^p \, dxdt \leqq \int_\Omega |u_0|^r \, dx.$$

If $r = 1$, we take any $\alpha > 0$ in (5.14) and observe that for any $\theta > 0$,

$$\frac{|T_k(\theta)|^{\alpha+1}}{\alpha+1} \leqq \mathcal{T}_{k,\alpha}(\theta) \leqq k^\alpha |\theta| \tag{5.16}$$

Then

$$\int_\Omega |T_k(u)|^{\alpha+1} (.,t))dx + (\alpha+1)\gamma \int_s^t \int_\Omega |T_k(u)|^{\alpha+\lambda} |\nabla T_k(u)|^q \, dxdt \leqq (\alpha+1)k^\alpha \int_\Omega |u| (.,s)dx$$

Then we go to the limit as $\alpha \to 0$, we deduce

$$\int_\Omega |T_k(u)| (.,t))dx + \gamma \int_s^t \int_\Omega |T_k(u)|^\lambda |\nabla T_k(u)|^q \, dxdt \leqq \int_\Omega |u| (.,s)dx \tag{5.17}$$

and then as $s \to 0$ we find

$$\int_\Omega T_k(u)(.,t))dx + \gamma \int_s^t \int_\Omega |T_k(u)|^\lambda |\nabla T_k(u)|^q \, dxdt \leqq \int_\Omega |u_0| \, dx \tag{5.18}$$

and finally $k \to \infty$, and deduce that $\int_\Omega |u| (.,t)dx \leqq \int_\Omega |u_0| \, dx$. Moreover if $\gamma > 0$, we find

$$\int_\Omega |u| (.,t)dx + \gamma \int_0^t \int_\Omega |u|^\lambda |\nabla u|^q dxdt \leqq \int_\Omega |u_0| \, dx,$$

thus (5.13) still holds with $r=1$.

2) We still find (5.17). And $\lim_{s\to 0}\int_\Omega u(.,s)dx = \displaystyle\int_\Omega du_0$ from (5.11), hence the conclusion. ■

Next we deduce L^∞ estimates, in particular a universal one.

Theorem 5.8 *Let $p>1$, $1<q<N$, and Ω be a regular bounded domain of $\mathbb{R}^N$. Assume (5.1) (5.2) and (5.3). Let $u_0\in L^r(\Omega), r\geqq 1$, and u be any renormalized solution of $(P_{\Omega,T})$.*

(i) If g is coercive, there exists $C=C(N,q,\lambda,\gamma,\Omega)$ independent of A, *such that*

$$\|u(.,t)\|_{L^\infty(\Omega)} \leqq Ct^{-\sigma_{r,q,\lambda}}\|u_0\|_{L^r(\Omega)}^{\varpi_{r,q,\lambda}}, \tag{5.19}$$

where

$$\sigma_{r,q,\lambda}=\frac{1}{\frac{rq}{N}+\lambda+q-1}, \qquad \varpi_{r,q,\lambda}=\frac{rq}{N}\sigma_{r,q,\lambda}$$

and there exists $C=C(N,q,\lambda,|\Omega|)$ such that

$$\|u(.,t)\|_{L^\infty(\Omega)} \leqq Ct^{-\frac{1}{q-1+\lambda}}. \tag{5.20}$$

(ii) If A *is coercive and $r>(2-p)N/p$, (in particular if $p>2N/(N+1)$), and $p<N$, then*

$$\|u(.,t)\|_{L^\infty(\Omega)} \leqq Ct^{-\sigma_{r,p,-1}}\|u_0\|_{L^r(\Omega)}^{\varpi_{r,p,-1}}, \tag{5.21}$$

where

$$\sigma_{r,p,-1}=\frac{1}{\frac{rp}{N}+p-2}, \qquad \varpi_{r,p,-1}=\frac{1}{1+\frac{N}{rp}(p-2)}$$

and if $p>2$, there exists $C=C(N,p,|\Omega|)$ such that

$$\|u(.,t)\|_{L^\infty(\Omega)} \leqq Ct^{-\frac{1}{p-2}}. \tag{5.22}$$

(iii) The same conclusions hold if u is nonnegative and $u_0\in\mathcal{M}_b^+(\Omega)$, as in case $u_0\in L^1(\Omega)$, where the norm $\|u_0\|_{L^1(\Omega)}$ is replaced by $\displaystyle\int_\Omega du_0$.

Proof. (i) Let $0<s<t<T$. Since g is coercive, from Theorem 5.7, for any $\alpha\geqq 0$ such that $|u|^{\alpha+1}(.,s)\in L^1(\Omega)$, we get from (5.13)

$$\int_\Omega |u|^{\alpha+1}(.,t)dx+(\alpha+1)\gamma\int_s^t\int_\Omega |u|^{\lambda+\alpha}|\nabla u|^q dxdt \leqq \int_\Omega |u|^{\alpha+1}(.,s)dx,$$

and in particular

$$\int_\Omega |T_k(u)|^{\alpha+1}(.,t)dx+(\alpha+1)\gamma\int_s^t\int_\Omega (T_k(u))^{\lambda+\alpha}|\nabla T_k(u)|^q\, dxdt \leqq \int_\Omega |u|^{\alpha+1}(.,s)dx$$

And $|u|^{\lambda+\alpha}|\nabla u|^q = |\nabla(|u|^{\beta-1}u)|^q$ with $\beta=1+(\alpha+\lambda)/q\geqq 1$. Then $|\nabla((|u|^{\beta-1}u)(.,t))|$, and also $|\nabla((|T_k(u)|^{\beta-1}T_k(u))(.,t))|$ belong to $L^q(\Omega)$ for a.e. $t>0$. Since $|T_k(u)|^{\beta-1}T_k(u)(.,t)\in$

$L^{\infty}(\Omega)$ it follows that $|T_k(u)|^{\beta-1} T_k(u)(.,t) \in W^{1,q}(\Omega)$. Moreover $T_k(u)(.,t) \in W_0^{1,p}(\Omega))$, hence $|T_k(u)|^{\beta-1} T_k(u)(.,t) \in W_0^{1,q}(\Omega)$. Then from the Sobolev injection of $W_0^{1,q}(\Omega)$ into $L^{q^*}(\Omega)$,

$$\int_\Omega |T_k(u)|^{\alpha+1}(.,t)dx + \gamma \frac{C_0(\alpha+1)}{\beta^q}\int_s^t (\int_\Omega |T_k(u)|^{\beta q^*}(.,\sigma)dx)^{\frac{q}{q^*}} d\sigma \leqq \int_\Omega |u|^{\alpha+1}(.,s)dx.$$

Going to the limit as $k \to \infty$, we find

$$\int_\Omega |u|^{\alpha+1}(.,t)dx + \gamma \frac{C_0(\alpha+1)}{\beta^q}\int_s^t (\int_\Omega |u|^{\beta q^*}(.,\sigma)dx)^{\frac{q}{q^*}} d\sigma \leqq \frac{1}{\alpha+1}\int_\Omega |u|^{\alpha+1}(.,s)dx.$$

Then we can apply Lemma 2.2 on $[\epsilon,T)$, with $m=q$ and $\theta=N/(N-q)$; we deduce the estimate for $[\epsilon,T)$,

$$\|u(.,t)\|_{L^\infty(\Omega)} \leqq C(t-\epsilon)^{-\sigma_{r,q,\lambda}} \|u(.,\epsilon)\|_{L^r(\Omega)}^{\varpi_{r,q,\lambda}},$$

with $C=C(N,q,\lambda,\gamma,\Omega)$. Finally we go to the limit as $\epsilon \to 0$, and get (5.19) for $u_0 \in L^r(\Omega)$, and the analogous when u is nonnegative and $u_0 \in \mathcal{M}_b^+(\Omega)$, and also (5.20).

(ii) Assume that A is coercive. Then for any $\alpha>0$,

$$\int_\Omega \mathcal{T}_{k,\alpha}(u)(.,t))dx + \alpha\nu \int_s^t \int_\Omega |T_k(u)|^{\alpha-1} |\nabla T_k(u)|^p dxdt \leqq \int_\Omega \mathcal{T}_{k,\alpha}(u)(.,s))dx$$

from (5.14). From the Sobolev injection of $W_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$, since $p<N$, we deduce

$$\frac{1}{\alpha+1}\int_\Omega u^{\alpha+1}(.,t)dx + \alpha\frac{C_0}{k^p}\int_s^t (\int_\Omega u^{kp^*}(.,\sigma)dx)^{\frac{q}{q^*}} dt \leqq \frac{1}{\alpha+1}\int_\Omega u^{\alpha+1}(.,s)dx.$$

with $k=1+(\alpha-1)/p$.

• First suppose $r>1$; then we start from $\alpha_0=r-1>0$, and we can apply Lemma 2.2 with $m=p$, $\theta=N/(N-q)$ and $\lambda=-1$. The condition (2.1) is satisfied, since $r>N(2-p)/p$.

• Next suppose $r=1$. Then $1>(2-p)N/p$, thus $p-1+p/N>1$. For any $\alpha>0$,

$$\int_\Omega |T_k(u)|^{\alpha+1}(.,t))dx + \alpha(\alpha+1)\nu \int_s^t \int_\Omega |T_k(u)|^{\alpha-1} |\nabla T_k(u)|^p dxdt \leqq (\alpha+1)k^\alpha \int_\Omega |u|(.,s)dx$$

Taking $\alpha=1$, we get from (5.12),

$$\nu \int_s^t \int_\Omega |\nabla T_k(u)|^p dxdt \leqq k\int_\Omega |u|(.,s)dx \leqq k\int_\Omega |u_0| dx.$$

And from (5.12), $u \in L^\infty\left((s,T);L^1(\Omega)\right)$, then from standard estimates, there holds $u \in L^\rho(Q_{\Omega,s,t})$ for any $\rho \in (1,p-1+p/N)$, see [19]. Then $|u|^\rho(.,t) \in L^1(\Omega)$ for almost any $t \in (0,T)$, hence we can apply Lemma 2.2 on $[\epsilon,T)$ for $\epsilon>0$, with the same parameters, after fixing such a $\rho=\rho_{p,N}$. We obtain that

$$\|u(.,t)\|_{L^\infty(\Omega)} \leqq C(t-\epsilon)^{-\sigma_{1,p,-1}} \|u(.,\epsilon)\|_{L^1(\Omega)}^{\varpi_{1,p,-1}},$$

where $C=C(N,p,\lambda,\rho_{p,N})=C(N,p,\lambda)$; finally we go to the limit as $\epsilon \to 0$ because $u \in C([0,T];L^1(\Omega))$. Estimate (5.22) follows, since $-1+p-1>0$.

(iii) Similarly, if $u_0 \in \mathcal{M}_b^+(\Omega)$ and u is nonnegative, we are lead to the same conclusions, where $\|u_0\|_{L^1(\Omega)}$ is replaced by $\int_\Omega du_0$. In particular (5.21) holds for $p>2N/(N+1)$, and (5.22) for $p>2$.
■

Remark 5.9 *As at Remark 3.18, we still obtain L^∞ estimates for $q \geq N$ or $p \geq N$. If g is coercive, we get (5.19) with $\sigma_{r,q,N} = 1/(q+r-1+\lambda) = \varpi_{r,q,N}/r$ if $q > N$, and $\sigma_{r,N,N} = 1/(N(1-\delta)+r-1+\lambda) = \varpi_{r,N,N}/r(1-\delta)$ where $\delta \in (0,1)$ is arbitrary. If A is coercive we get (5.21) with $\sigma_{r,p,-1} = 1/(r+p-2) = \varpi_{r,p,-1}/r$ if $p > N$, and $\sigma_{r,N,-1} = 1/(N(1-\delta)+p-2) = \varpi_{r,N,-1}/r(1-\delta)$.*

Remark 5.10 *Our results apply in particular to the problem*

$$\begin{cases} u_t - \operatorname{div}(\mathrm{A}(x,t,u,\nabla u)) = 0, & \text{in } Q_{\Omega,T}, \\ u = 0, \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0 \end{cases}$$

Thus we find again the estimates of [34, Theorem 5.3], with less regularity on the solutions: those estimates were proved for solutions $u \in C([0,T); L^r(\Omega))$ such that $u \in L^p((0,T); W_0^{1,p}(\Omega)) \cap C([0,T); L^2(\Omega))$. The notion of renormalized solutions, equivalent to the notion of entropy solutions of [36] (see [28]), is weaker.

Remark 5.11 *The extension of results of section 3 to the case of equation of type (1.2) in the case $\Omega = \mathbb{R}^N$ will be treated a further article.*

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