A new dynamical approach of Emden-Fowler equations and systems

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Abstract

We give a new approach on general systems of the form

$$(G)\begin{cases} -\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \varepsilon_1 |x|^a u^s v^\delta, \\ -\Delta_q v = -\operatorname{div}(|\nabla v|^{q-2}\nabla u) = \varepsilon_2 |x|^b u^\mu v^m, \end{cases}$$

where $Q, p, q, \delta, \mu, s, m,\ a, b$ are real parameters, $Q, p, q \neq 1$, and $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$. In the radial case we reduce the problem to a quadratic system of four coupled first order autonomous equations, of Kolmogorov type. It allows to obtain new local and global existence or nonexistence results. We consider in particular the case $\varepsilon_1 = \varepsilon_2 = 1$. We describe the behaviour of the ground states in two cases where the system is variational. We give a result of existence of ground states for a nonvariational system with $p = q = 2$ and $s = m > 0$, that improves the former ones. It is obtained by introducing a new type of energy function. In the nonradial case we solve a conjecture of nonexistence of ground states for the system with $p = q = 2$, $\delta = m + 1$ and $\mu = s + 1$.

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1 Introduction

In this paper we consider the nonnegative solutions of Emden-Fowler equations or systems in $\mathbb{R}^N (N \geqq 1)$,

$$-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \varepsilon_1 |x|^a u^Q, \tag{1.1}$$

$$(G) \begin{cases} -\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \varepsilon_1 |x|^a u^s v^\delta, \\ -\Delta_q v = -\operatorname{div}(|\nabla v|^{q-2} \nabla u) = \varepsilon_2 |x|^b u^\mu v^m, \end{cases} \tag{1.2}$$

where $Q, p, q, \delta, \mu, s, m, a, b$ are real parameters, $Q, p, q \neq 1$, and $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$. These problems are the subject of a very rich litterature, either in the case of source terms ($\varepsilon_1 = \varepsilon_2 = 1$) or absorption terms ($\varepsilon_1 = \varepsilon_2 = 1$) or mixed terms ($\varepsilon_1 = -\varepsilon_2$). In the sequel we are concerned by the radial solutions, except at Section 9 where the solutions may be nonradial.

In this article we we give a <u>*new way of studying the radial solutions*</u>. In Section 2 we reduce system (G) to a quadratic autonomous system:

$$(M) \begin{cases} X_t = X \left[X - \frac{N-p}{p-1} + \frac{Z}{p-1} \right], \\ Y_t = Y \left[Y - \frac{N-q}{q-1} + \frac{W}{q-1} \right], \\ Z_t = Z \left[N + a - sX - \delta Y - Z \right], \\ W_t = W \left[N + b - \mu X - mY - W \right], \end{cases}$$

where $t = \ln r$, and

$$X(t) = -\frac{ru'}{u}, \quad Y(t) = -\frac{rv'}{v}, \quad Z(t) = -\varepsilon_1 r^{1+a} u^s v^\delta \frac{u'}{|u'|^p}, \quad W(t) = -\varepsilon_2 r^{1+b} u^\mu v^m \frac{v'}{|v'|^q}. \tag{1.3}$$

This system is of Kolmogorov type. The reduction is valid for equations and systems with source terms , absorption terms , or mixed terms . It is remarkable that in the new system, p and q appear only as <u>*simple coefficients*</u>, which allows to treat any value of the parameters, even p or $q < 1$, and s, m, δ or $\mu < 0$.

In Section 3 we revisit the well-known scalar case (1.1), where (G) becomes two-dimensional. We show that the phase plane of the system gives <u>*at the same time*</u> the behaviour of the two equations

$$-\Delta_p u = |x|^a u^Q \text{ and } -\Delta_p u = -|x|^a u^Q,$$

which is a kind of <u>*unification of the two problems*</u>, with source terms or absorption terms. For the case of source term ($\varepsilon_1 = 1$), we find again the results of [2], [19], showing that the new dynamical approach is simple and does not need regularity results or energy functions. Moreover it gives a model for the study of system (G). Indeed if $p = q$, $a = b$ and $\delta + s = \mu + m$, system (G) admits solutions of the form (u, u), where u is a solution of (1.1) with $Q = \delta + s$.

In the sequel of the article we study the case of source terms, i.e. $(G) = (S)$, where

$$(S) \begin{cases} -\Delta_p u = |x|^a u^s v^\delta, \\ -\Delta_q v = |x|^b u^\mu v^m. \end{cases} \tag{1.4}$$

This system has been studied by many authors, in particular the Hamiltonian problem $s = m = 0$, in the linear case $p = q = 2$, see for example [20], [31], [29], [9], [33], [14], and the potential system where $\delta = m+1$, $\mu = s+1$ and $a = b$, see [7], [34], [35]; the problem with general powers has been studied in [3], [39], [40], [41] in the linear case and [6], [12], [42] in the quasilinear case, see also [1], [10], [13].

Here we suppose that $\delta, \mu > 0$, so that the system is always coupled, $s, m \geqq 0$, and we assume for simplicity

$$1 < p, q < N, \qquad \min(p+a, q+b) > 0, \qquad D = \delta\mu - (p-1-s)(q-1-m) > 0. \tag{1.5}$$

We say that a positive solution (u, v) in $(0, R)$ is <u>regular</u> at 0 if $u, v \in C^2(0, R) \cap C([0, R))$. Condition $\min(p+a, q+b) > 0$ guaranties the existence of local regular solutions. Then $u, v \in C^1([0, R))$. when $a, b > -1$, and $u'(0) = v'(0) = 0$. The assumption $D > 0$ is a classical condition of superlinearity for the system.

We are interessed in the existence or nonexistence of <u>*ground states*</u>, called G.S., that means global positive (u, v) in $(0, \infty)$ and regular at 0. We exclude the case of "trivial" solutions, $(u, v) = (0, C)$ or $(C, 0)$, where C is a constant, which can exist when $s > 0$ or $m > 0$.

In Section 4 we give a series of <u>*local existence or nonexistence*</u> results concerning system (S), which complete the nonexistence results found in the litterature. They are not based on the fixed point method, quite hard in general, see for example [19], [27]. We make a dynamical analysis of the linearization of system (M) near each fixed point, which appears to be performant, even for the regular solutions. For a better exposition, the proofs are given at Section 10.

In Section 5 we study the <u>*global existence*</u> of G.S. This problem has been often compared with the nonexistence of positive solutions of the Dirichlet problem in a ball, see [29], [30], [12], [13]. Here we use a shooting method adapted to system (M), which allows to avoid questions of regularity of system (S). We give a new way of comparison, and improve the former results:

Theorem 1.1 *(i) Assume* $s < \frac{N(p-1)+p+pa}{N-p}$ *and* $m < \frac{N(q-1)+q+qb}{N-q}$. *If system* (S) *has no G.S., then*

(i) there exist regular radial solutions such that $X(T) = \frac{N-p}{p-1}$ *and* $Y(T) = \frac{N-q}{q-1}$ *for some* $T > 0$, *with* $0 < X < \frac{N-p}{p-1}$ *and* $0 < Y < \frac{N-q}{q-1}$ *on* $(-\infty, T)$.

(ii) there exists a positive radial solution (u, v) *of the Dirichlet problem in a ball* $B(0, R)$.

This result is a key tool in the next Sections for proving the existence of a G.S. It gives also new existence results for the Dirichlet problem, see Corollary 5.3. We also give a complementary result:

Proposition 1.2 *Assume* $s \geqq \frac{N(p-1)+p+pa}{N-p}$ *and* $m \geqq \frac{N(q-1)+q+qb}{N-q}$. *Then all the regular radial solutions are G.S.*

In Section 6 we study the <u>*radial*</u> solutions of the well known Hamiltonian system

$$(SH) \begin{cases} -\Delta u = |x|^a \, v^\delta, \\ -\Delta v = |x|^b \, u^\mu, \end{cases}$$

corresponding to $p = q = 2 < N$, $s = m = 0$, $a > -2$, which is variational. In the case $a = b = 0$, a main conjecture was made in [32]:

Conjecture 1.3 *System (SH) with $a = b = 0$ admits no (radial or nonradial) G.S. if and only if (δ, μ) is under the hyperbola of equation*

$$\frac{N}{\delta+1} + \frac{N}{\mu+1} = N - 2.$$

The question is still open; it was solved in the radial case in [26], [29], then partially in [31], [9], and up to the dimension $N = 4$ in [33], see references therein. Here we find again and extend to the case $a, b \neq 0$ some results of [20] relative to the G.S., with a shorter proof. We also give an existence result for the Dirichlet problem improving a result of [14].

Theorem 1.4 *Let $\mathcal{H}_0$ be the critical hyperbola in the plane (δ, μ) defined by*

$$\frac{N+a}{\delta+1} + \frac{N+b}{\mu+1} = N - 2. \tag{1.6}$$

Then

(i) System (SH) admits a (unique) radial G.S. if and only if (δ, μ) is above $\mathcal{H}_0$ or on $\mathcal{H}_0$.

(ii) The radial Dirichlet problem in a ball has a solution if and only if (δ, μ) is under $\mathcal{H}_0$.

(iii) On $\mathcal{H}_0$ the G.S. has the following behaviour at ∞ : assuming for example $\delta > \frac{N+a}{N-2}$, then $\lim_{r\to\infty} r^{N-2}u(r) = \alpha > 0$, and

$$\begin{aligned}
\lim_{r\to\infty} r^{(N-2)\mu-(2+b)}v = \beta > 0 \qquad & \text{if } \mu < \frac{N+b}{N-2}, \\
\lim_{r\to\infty} r^{N-2}v = \beta > 0 \qquad & \text{if } \mu > \frac{N+b}{N-2}, \\
\lim_{r\to\infty} r^{N-2}\left|\ln r\right|^{-1} v = \beta > 0 \qquad & \text{if } \mu = \frac{N+b}{N-2}.
\end{aligned}$$

Our proofs use a Pohozaev type function; in terms of the new variables X, Y, Z, W, it contains a quadratic factor

$$\begin{aligned}
\mathcal{E}_H(r) &= r^N \left[u'v' + r^b \frac{|u|^{\mu+1}}{\mu+1} + r^a \frac{|v|^{\delta+1}}{\delta+1} + \frac{N+a}{\delta+1}\frac{vu'}{r} + \frac{N+b}{\mu+1}\frac{uv'}{r} \right] \\
&= r^{N-2}uv \left[XY - \frac{Y(N+b-W)}{\mu+1} - \frac{(N+a-Z)X}{\delta+1} \right].
\end{aligned} \tag{1.7}$$

As observed in ([20]) the G.S. can present a non-symmetric behaviour. This non-symmetry phenomena has to be taken in account for solving conjecture (1.3).

In Section 7 we consider the radial solutions of a <u>*nonvariational system*</u>:

$$(SN) \begin{cases} -\Delta u = |x|^a \, u^s v^\delta, \\ -\Delta v = |x|^a \, u^\mu v^s, \end{cases}$$

where $p = q = 2 < N$, $a = b > -2$ and $m = s > 0$. For small s it appears as a perturbation of system (SH). In the litterature very few results are known for such nonvariational systems. Our main result in this Section is a new result of <u>*existence of G.S.*</u> valid for any s:

Theorem 1.5 *Consider the system* (SN), *with* $N > 2$, $a > -2$. *We define a curve* $\mathcal{C}_s$ *in the plane* (δ, μ) *by*

$$\frac{N+a}{\mu+1} + \frac{N+a}{\delta+1} = N - 2 + \frac{(N-2)s}{2} \min(\frac{1}{\mu+1}, \frac{1}{\delta+1}), \tag{1.8}$$

located under the hyperbola defined by (1.6). If (δ, μ) *is above* $\mathcal{C}_s$, *system* (SN) *admits a G.S.*

This result is obtained by constructing a new type of energy function which contains two terms in X^2, Y^2 :

$$\begin{aligned}\Phi(r) &= r^N \left[u'v' + r^b \frac{u^{\mu+1}v^s}{\mu+1} + r^a \frac{u^s v^{\delta+1}}{\delta+1} + \frac{N+a}{\delta+1}\frac{vu'}{r} + \frac{N+b}{\mu+1}\frac{uv'}{r} + \frac{s}{2(\delta+1)}\frac{vu'^2}{u} + \frac{s}{2(\mu+1)}\frac{uv'^2}{v}\right] \\ &= r^{N-2}uv\left[XY - \frac{Y(N+b-W)}{\mu+1} - \frac{(N+a-Z)X}{\delta+1} + \frac{s}{2(\delta+1)}X^2 + \frac{s}{2(\mu+1)}Y^2\right]. \end{aligned} \tag{1.9}$$

In Section 8 we consider the <u>*radial*</u> solutions of the potential system

$$(SP)\begin{cases} -\Delta_p u = |x|^a \, u^s v^{m+1}, \\ -\Delta_q v = |x|^a \, u^{s+1} v^m, \end{cases}$$

where $\delta = m+1, \mu = s+1$ and $a = b$, which is variational, see [34], [35]. Using system (M) we deduce new results of existence:

Theorem 1.6 *Let* $\mathcal{D}$ *be the critical line in the plane* (m, s) *defined by*

$$N + a = (m+1)\frac{N-q}{q} + (s+1)\,\frac{N-p}{p}.$$

Then

(i) System (SP) *admits a radial G.S. if and only if* (m, s) *is above or on* $\mathcal{D}$.

(ii) On $\mathcal{D}$ *the G.S. has the following behaviour: suppose for example* $q \leqq p$. *Let* $\lambda^* = N + a - (s+1)\frac{N-p}{p-1} - m\frac{N-q}{q-1}$. *Then* $\lim_{r\to\infty} r^{\frac{N-p}{p-1}} u(r) = \alpha > 0$, *and*

$$\lim_{r\to\infty} r^{\frac{N-q}{q-1}} v(r) = \beta > 0 \qquad \text{if } \lambda^* < 0, \tag{1.10}$$

$$\lim_{r\to\infty} r^{\frac{\frac{N-p}{p-1}\mu-(q+b)}{q-1-m}} v(r) = \beta > 0 \qquad \text{if } \lambda^* > 0, \tag{1.11}$$

$$\lim_{r\to\infty} r^{\frac{N-q}{q-1}} |\ln r|^{-\frac{1}{q-1-m}} v(r) = \beta > 0 \qquad \text{if } \lambda^* = 0. \tag{1.12}$$

In particular (1.10) holds if $p = q$, *or* $q \leqq m+1$.

(iii) The radial Dirichlet problem in a ball has a solution if and only if (m, s) *is under* $\mathcal{D}$.

In that case we use the following energy function, which deserves to be compared with the one of Section 6 , since it has also a quadratic factor:

$$\mathcal{E}_P(r) = r^N \left[(s+1)(\frac{|u'|^p}{p'} + \frac{N-p}{p}\frac{u\,|u'|^{p-2}\,u'}{r}) + (m+1)(\frac{|v'|^q}{q'} + \frac{N-q}{q}\frac{v\,|v'|^{q-2}\,v'}{r}) + r^a u^{s+1} v^{m+1} \right]$$

$$= r^{N-2-a}\frac{|u'|^{p-1}\,|v'|^{q-1}}{u^s v^m}\left[ZW - \frac{(s+1)W(N-p-(p-1)X)}{p} - \frac{(m+1)Z(N-q-(q-1)Y)}{q} \right]. \tag{1.13}$$

Finally in Section 9 we deduce a <u>*nonradial*</u> result for the potential system in the case of two Laplacians:

$$(SL) \left\{ \begin{array}{l} -\Delta u = |x|^a\, u^s v^{m+1}, \\ -\Delta v = |x|^a\, u^{s+1} v^m. \end{array} \right.$$

Our result proves a conjecture proposed in [7], showing that in the subcritical case there exists no G.S.:

Theorem 1.7 *Assume $a > -2$ and $s, m \geqq 0$. If*

$$s + m + 1 < \min(\frac{N+2}{N-2}, \frac{N+2+2a}{N-2}), \tag{1.14}$$

then system (SL) admits no (radial or nonradial) G.S.

Our proof uses the estimates of [7], which up to now are the only extensions of the results of [18] to systems. It is based on the construction of a nonradial Pohozaev function extending the radial one given at (1.13) for $p = q = 2$, different from the energy function used in [7].

The case of the system (G) with absorption terms ($\varepsilon_1 = \varepsilon_2 = -1$) or mixed terms ($\varepsilon_1 = -\varepsilon_2 = 1$), studied in [4], [5], will be the subject of a second article. Our approach also extends to a system with <u>gradient terms</u> and <u>doubly singular</u>:

$$\left\{ \begin{array}{l} -\operatorname{div}(|x|^c\, u^\rho\, |\nabla u|^{p-2}\,\nabla u) = \varepsilon_1\, |x|^a\, u^s v^\delta\, |\nabla u|^\eta\, |\nabla v|^\ell\,, \\ -\operatorname{div}(|x|^d\, v^\lambda\, |\nabla v|^{p-2}\,\nabla v) = \varepsilon_2\, |x|^b\, u^\mu v^m\, |\nabla u|^\nu\, |\nabla v|^\kappa\,, \end{array} \right. \tag{1.15}$$

which will be studied in another work.

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2 Reduction to a quadratic system

2.1 The change of unknowns

Here we consider the radial positive solutions $r \mapsto (u(r), v(r))$ of system (G) on any interval (R_1, R_2), that means

$$\left\{ \begin{array}{l} \left(|u'|^{p-2}\,u'\right)' + \frac{N-1}{r}\,|u'|^{p-2}\,u' = r^{1-N}\left(r^{N-1}\,|u'|^{p-2}\,u'\right)' = -\varepsilon_1 r^a u^s v^\delta, \\ \left(|v'|^{q-2}\,v'\right)' + \frac{N-1}{r}\,|v'|^{q-2}\,v' = r^{1-N}\left(r^{N-1}\,|v'|^{p-2}\,v'\right)' = -\varepsilon_2 r^b u^\mu v^m. \end{array} \right.$$

Near any point r where $u(r) \neq 0, u'(r) \neq 0$ and $v(r) \neq 0$, $v'(r) \neq 0$ we define

$$X(t) = -\frac{ru'}{u}, \quad Y(t) = -\frac{rv'}{v}, \quad Z(t) = -\varepsilon_1 r^{1+a} u^s v^\delta \left|u'\right|^{-p} u', \quad W(t) = -\varepsilon_2 r^{1+b} u^\mu v^m \left|v'\right|^{-q} v', \tag{2.1}$$

where $t = \ln r$. Then we find the system

$$(M) \begin{cases} X_t = X\left[X - \frac{N-p}{p-1} + \frac{Z}{p-1}\right], \\ Y_t = Y\left[Y - \frac{N-q}{q-1} + \frac{W}{q-1}\right], \\ Z_t = Z\left[N + a - sX - \delta Y - Z\right], \\ W_t = W\left[N + b - \mu X - mY - W\right]. \end{cases}$$

This sytem is *quadratic*, and moreover a very simple one, of *Kolmogorov type*: it admits four invariant hyperplanes: $X = 0, Y = 0, Z = 0, W = 0$. As a first consequence all the fixed points of the system are explicite. The trajectories located on these hyperplanes do not correspond to a solution of system (G); they will be called *nonadmissible*.

We suppose that the discriminant of the system

$$D = \delta\mu - (p-1-s)(q-1-m) \neq 0. \tag{2.2}$$

Then one can express u, v in terms of the new variables:

$$u = r^{-\gamma}(\left|X\right|^{p-1}\left|Z\right|)^{(q-1-m)/D}(\left|Y\right|^{q-1}\left|W\right|)^{\delta/D}, \qquad v = r^{-\xi}(\left|X\right|^{p-1}\left|Z\right|)^{\mu/D}(\left|Y\right|^{q-1}\left|W\right|)^{(p-1-s)/D}, \tag{2.3}$$

where γ and ξ are defined by

$$\gamma = \frac{(p+a)(q-1-m) + (q+b)\delta}{D}, \qquad \xi = \frac{(q+b)(p-1-s) + (p+a)\mu}{D}, \tag{2.4}$$

or equivalently by

$$(p-1-s)\gamma + p + a = \delta\xi, \qquad (q-1-m)\xi + q + b = \mu\gamma. \tag{2.5}$$

Since system (M) is autonomous, each admissible trajectory $\mathcal{T}$ in the phase space corresponds to a solution (u, v) of system (G) unique up to a scaling: if (u, v) is a solution, then for any $\theta > 0$, $r \mapsto (\theta^\gamma u(\theta r), \theta^\xi v(\theta r))$ is also a solution.

2.2 Fixed points of system (M)

System (M) has at most 16 fixed points. The main fixed point is

$$M_0 = (X_0, Y_0, Z_0, W_0) = (\gamma, \xi, N - p - (p-1)\gamma, N - q - (q-1)\xi), \tag{2.6}$$

corresponding to the particular solutions

$$u_0(r) = Ar^{-\gamma}, v_0(r) = Br^{-\xi}, \quad A, B > 0, \tag{2.7}$$

when they exist, depending on $\varepsilon_1, \varepsilon_2$. The values of A and B are given by

$$
\begin{aligned}
A^D &= \left(\varepsilon_1 \gamma^{p-1}(N-p-\gamma(p-1))\right)^{q-1-m} \left(\varepsilon_2 \xi^{q-1}(N-q-(q-1)\xi)\right)^{\delta}, \\
B^D &= \left(\varepsilon_2 \xi^{q-1}(N-q-(q-1)\xi)\right)^{p-1-s} \left(\varepsilon_1 \gamma^{p-1}(N-p-(p-1)\gamma)\right)^{\mu}.
\end{aligned}
$$

The other fixed points are

$$
\begin{aligned}
&0 = (0,0,0,0), \quad N_0 = (0,0,N+a,N+b), \quad A_0 = (\frac{N-p}{p-1}, \frac{N-q}{q-1}, 0, 0), \\
&I_0 = (\frac{N-p}{p-1}, 0,0,0), \quad J_0 = (0, \frac{N-q}{q-1}, 0, 0), \quad K_0 = (0,0,N+a,0), \quad L_0 = (0,0,0,N+b), \\
&G_0 = (\frac{N-p}{p-1}, 0, 0, N+b-\frac{N-p}{p-1}\mu), \quad H_0 = (0, \frac{N-q}{q-1}, N+a-\frac{N-q}{q-1}\delta, 0),
\end{aligned}
$$

and if $m \neq q-1$,

$$
\begin{aligned}
P_0 &= (\frac{N-p}{p-1}, \frac{\frac{N-p}{p-1}\mu-(q+b)}{q-1-m}, 0, \frac{(q-1)(N+b-\frac{N-p}{p-1}\mu)-m(N-q)}{q-1-m}), \\
C_0 &= \left(0, -\frac{q+b}{q-1-m}, 0, \frac{(N+b)(q-1)-m(N-q)}{q-1-m}\right), \\
R_0 &= \left(0, -\frac{q+b}{q-1-m}, N+a+\delta\frac{b+q}{q-1-m}, \frac{(N+b)(q-1)-m(N-q)}{q-1-m}\right),
\end{aligned}
$$

and by symmetry, if $s \neq p-1$,

$$
\begin{aligned}
Q_0 &= (\frac{\frac{N-q}{q-1}\delta-(p+a)}{p-1-s}, \frac{N-q}{q-1}, \frac{(p-1)(N+a-\frac{N-q}{q-1}\delta)-s(N-p)}{p-1-s}, 0), \\
D_0 &= \left(-\frac{p+a}{p-1-s}, 0, \frac{(N+a)(p-1)-s(N-p)}{p-1-s}, 0\right), \\
S_0 &= \left(-\frac{p+a}{p-1-s}, 0, \frac{(N+a)(p-1)-s(N-p)}{p-1-s}, N+b+\mu\frac{a+p}{p-1-s}\right).
\end{aligned}
$$

2.3 First comments

Remark 2.1 *This formulation allows to treat more general systems with signed solutions by reducing the study on intervals where u and v are nonzero. Consider for example the problem*

$$
-\Delta_p u = \varepsilon_1 |x|^a |u|^s |v|^{\delta-1} v, \quad -\Delta_q v = \varepsilon_2 |x|^b |v|^m |u|^{\mu-1} u.
$$

On any interval where $uv > 0$, the couple $(|u|, |v|)$ is a solution of (G). On any interval where $u > 0 > v$, the couple $(u, |v|)$ satisfies (G) with $(\varepsilon_1, \varepsilon_2)$ replaced by $(-\varepsilon_1, -\varepsilon_2)$.

Remark 2.2 *There is another way for reducing the system to an autonomous form: setting*

$$
\mathrm{U}(t) = r^{\gamma} u, \quad \mathrm{V}(t) = r^{\xi} v, \quad \mathrm{H}(t) = -r^{(\gamma+1)(p-1)} |u'|^{p-2} u', \quad \mathrm{K}(t) = -r^{(\xi+1)(q-1)} |v'|^{q-2} v',
$$

with $t = \ln r$, we find

$$\left\{ \begin{array}{ll} \mathrm{U}_t = \gamma \mathrm{U} - |\mathrm{H}|^{(2-p)/(p-1)}\,\mathrm{H}, & \mathrm{V}_t = \zeta \mathrm{U} - |\mathrm{K}|^{(2-q)/(q-1)}\,\mathrm{K}, \\ \mathrm{H}_t = (\gamma(p-1)+p-N)\mathrm{H} + \varepsilon_1 \mathrm{U}^s \mathrm{V}^\delta, & \mathrm{K}_t = (\zeta(q-1)+q-N)\mathrm{K} + \varepsilon_2 \mathrm{U}^\mu \mathrm{V}^m. \end{array} \right. \tag{2.8}$$

It extends the well-known transformation of Emden-Fowler in the scalar case when $p = 2$, used also in [2] for general p, see Section 3. When $p = q = 2$ we obtain

$$\left\{ \begin{array}{l} \mathrm{U}_{tt} + (N-2-2\gamma)\mathrm{U}_t - \gamma(N-2-\gamma)\mathrm{U} + \varepsilon_1 \mathrm{U}^s \mathrm{V}^\delta = 0, \\ \mathrm{V}_{tt} + (N-2-2\xi)\mathrm{V}_t - \xi(N-2-\xi)\mathrm{V} + \varepsilon_2 \mathrm{U}^\mu \mathrm{V}^m = 0, \end{array} \right. \tag{2.9}$$

which was extended to the nonradial case and used for Hamiltonian systems ($s = m = 0$), with source terms in [9] ($\varepsilon_1 = \varepsilon_2 = 1$) and absorption terms in [4] ($\varepsilon_1 = \varepsilon_2 = -1$). Our system is more adequated for finding the possible behaviours: unlike system (2.8)it has no singularity, since it is polynomial, also its fixed points at ∞ are not concerned when we deal with solutions $u, v > 0$.

Remark 2.3 *In the specific case $p = q = 2$, setting*

$$z = XZ = \varepsilon_1 r^{2+a} |u|^{s-2} u\, |v|^{\delta-1} v, \qquad w = YW = \varepsilon_2 r^{2+b} |u|^{\mu-1} u\, |v|^{m-2} v,$$

we get the following system

$$\left\{ \begin{array}{ll} X_t = X^2 - (N-2)X + z, & Y_t = Y^2 - (N-2)Y + w, \\ z_t = z\left[2 + a + (1-s)X - \delta Y\right], & w_t = w\left[2 + b - \mu X + (1-m)Y\right]. \end{array} \right.$$

It has been used in [20] for studying the Hamiltonian system (SH). Even in that case we will show at Section 6 that system (M) is more performant, because it is of Kolmogorov type.

Remark 2.4 *Assume $p = q$ and $a = b$. Setting $t = k\hat{t}$ and $\left(\hat{X}, \hat{Y}, \hat{Z}, \hat{W}\right) = k(X, Y, Z, W)$, we obtain a system of the same type with N, a replaced by $\hat{N}, \hat{a}$, with*

$$\frac{\hat{N} - p}{N - p} = k = \frac{\hat{N} + \hat{a}}{N + a}.$$

It corresponds to the change of unknowns

$$r = \hat{r}^k, \qquad \hat{u}(\hat{r}) = C_1 u(r), \quad \hat{v}(\hat{r}) = C_2 u(r), \qquad C_1 = k^{p(p-1-m+\delta)/D}, \quad C_2 = k^{(p(p-1-s+\mu))/D}.$$

From (2.3) and (2.4), we get $\hat{\gamma}/\gamma = \hat{\xi}/\xi = k = \frac{p+\hat{a}}{p+a}$. There is one free parameter. In particular
1) we get a system without power ($\hat{a} = 0$), by taking

$$\hat{N} = \frac{p(N+a)}{p+a}, \qquad k = \frac{p}{p+a};$$

2) we get a system in dimension $\hat{N} = 1$, by taking

$$k = -\frac{p-1}{N-p} < 0,, \qquad \hat{a} = \frac{p + a - (N+a)p}{N-p}.$$

3 The scalar case

We first study the signed solutions of two scalar equations with source or absorption:

$$-\Delta_p u = -r^{1-N}\left(r^{N-1}\left|u'\right|^{p-2}u'\right)' = \varepsilon\,|x|^a\,|u|^{Q-1}\,u, \tag{3.1}$$

with $\varepsilon = \pm 1$, $1 < p < N$, $Q \neq p-1$ and $p + a > 0$.

We cannot quote all the huge litterature concerning its solutions, supersolutions or subsolutions, from the first studies of Emden and Fowler for $p = 2$, recalled in [16]; see for example [2] and [37], for any $p > 1$, and references therein. We set

$$Q_1 = \frac{(N+a)(p-1)}{N-p}, \qquad Q_2 = \frac{N(p-1)+p+pa}{N-p}, \qquad \gamma = \frac{p+a}{Q+1-p}.$$

From Remark 2.4 we could reduce the system to the case $a = 0$, in dimension $\hat{N} = p(N+a)/(p+a)$. However we do not make the reduction, because we are motivated by the study of system (G), and also by the nonradial case.

3.1 A common phase plane for the two equations

Near any point r where $u(r) \neq 0$ (positive or negative), and $u'(r) \neq 0$ setting

$$X(t) = -\frac{ru'}{u}, \qquad Z(t) = -\varepsilon r^{1+a}\,|u|^{Q-1}\,u\left|u'\right|^{-p}u', \tag{3.2}$$

with $t = \ln r$, we get a 2-dimensional system

$$(M_{scal})\begin{cases} X_t = X\left[X - \frac{N-p}{p-1} + \frac{Z}{p-1}\right], \\ Z_t = Z\left[N + a - QX - Z\right]. \end{cases}$$

and then $|u| = r^{-\gamma}(\,|Z|\,|X|^{p-1})^{1/(Q+1-p)}$. This change of unknown was mentioned in [11] in the case $p = 2, \varepsilon = 1$ and $N = 3$. It is remarkable that system (M_{scal}) is the same for the *two cases* $\varepsilon = \pm 1$, the only difference is that $X(t)Z(t)$ has the sign of ε :

The equation with source ($\varepsilon = 1$) is associated to the 1^{st} and 3^{rd} quadrant. It is well known that any local solution has a unique extension on $(0, \infty)$. The 1^{st} quadrant corresponds to the intervals where $|u|$ is decreasing, which can be of the following types $(0,\infty)$, $(0, R_2)$, (R_1, ∞), (R_1, R_2), $0 < R_1 < R_2 < \infty$. The 3^{rd} quadrant corresponds to the intervals (R_1, R_2) where $|u|$ is increasing.

The equation with absorption ($\varepsilon = -1$) is associated to the 2^{nd} and 4^{th} quadrant. It is known that the solutions have at most one zero, and their maximal interval of existence can be $(0, R_2)$, (R_1, ∞), (R_1, R_2) or $(0, \infty)$. The 2^{nd} quadrant corresponds to the intervals (R_1, R_2) where $|u|$ is increasing. The 4^{th} quadrant corresponds to the intervals $(0, R_2)$ or (R_1, ∞) where $|u|$ is decreasing.

The fixed points of (M_{scal}) are

$$M_0 = (X_0,, Z_0) = (\gamma, N-p-(p-1)\gamma), \quad (0,0), \quad N_0 = (0, N+a), \quad A_0 = (\frac{N-p}{p-1}, 0).$$

In particular M_0 is in the 1^{st} quadrant whenever $\gamma < \frac{N-p}{p-1}$, equivalently $Q > Q_1$, see fig. 1, and in the 4^{th} quadrant whenever $Q < Q_1$, see fig. 2. It corresponds to the solution

$$u(r) = Ar^{-\gamma}, \quad \text{for } \varepsilon = 1, Q > Q_1, \quad \text{or } \varepsilon = -1, Q < Q_1,$$

where $A = \left(\varepsilon\gamma^{p-1}(N-p-\gamma(p-1))\right)^{1/(Q-p+1)}$.

3.2 Local study

We examine the fixed points, where for simplicity we suppose $Q \neq Q_1$, and we deduce local results for the two equations:

• Point $(0,0)$: it is a saddle point, and the only trajectories that converge to $(0,0)$ are the separatrix, contained in the lines $X = 0, Y = 0$, they are not admissible.

• Point N_0 : it is a saddle point: the eigenvalues of the linearized system are $\frac{p}{p-1}$ and $-N$. the trajectories ending at N_0 at ∞ are located on the set $Z = 0$, then there exists a unique trajectory starting from $-\infty$ at N_0; it corresponds to the local existence and uniqueness of regular solutions, which we obtain easily.

• Point A_0 : the eigenvalues of the linearized system are $\frac{N-p}{p-1}$ and $\frac{N-p}{p-1}(Q_1 - Q)$. If $Q < Q_1$, A_0 is an unstable node. There is an infinity of trajectories starting from A_0 at $-\infty$; then $X(t)$ converges exponentially to $\frac{N-p}{p-1}$, thus $\lim_{r\to 0} r^{\frac{N-p}{p-1}} u = \alpha > 0$. The corresponding solutions u satisfy the equation with a Dirac mass at 0. There exists no solution converging to A_0 at ∞. If $Q > Q_1$, A_0 is a saddle point; the trajectories starting from A_0 at $-\infty$ are not admissible; there is a trajectory converging at ∞, and then $\lim_{r\to\infty} r^{\frac{N-p}{p-1}} u = \alpha > 0$.

• Point M_0 : the eigenvalues λ_1, λ_2 of the linearized system are the roots of equation

$$\lambda^2 + (Z_0 - X_0)\lambda + \frac{Q-p+1}{p-1} X_0 Z_0 = 0.$$

For $\varepsilon = 1$, M_0 is defined for $Q > Q_1$; the eigenvalues are imaginary when $X_0 = Z_0$, equivalently $\gamma = (N-p)/p$, $Q = Q_2$. When $Q < Q_2$, M_0 is a source, there exists an infinity of trajectories such that $\lim_{r\to 0} r^\gamma u = A$. When $Q > Q_2$, M_0 is a sink, and there exists an infinity of trajectories such that $\lim_{r\to\infty} r^\gamma u = A$. When $Q = Q_2$, M_0 is a center, from [2], see fig. 1. For $\varepsilon = -1$, M_0 is defined for $Q < Q_1$, it is a saddle-point, see fig. 2. There exist two trajectories $\mathcal{T}_1, \mathcal{T}_1'$ converging at ∞, such that $\lim_{r\to\infty} r^\gamma u = A$ and two trajectories $\mathcal{T}_2$, $\mathcal{T}_2'$, converging at 0, such that $\lim_{r\to 0} r^\gamma u = A$.

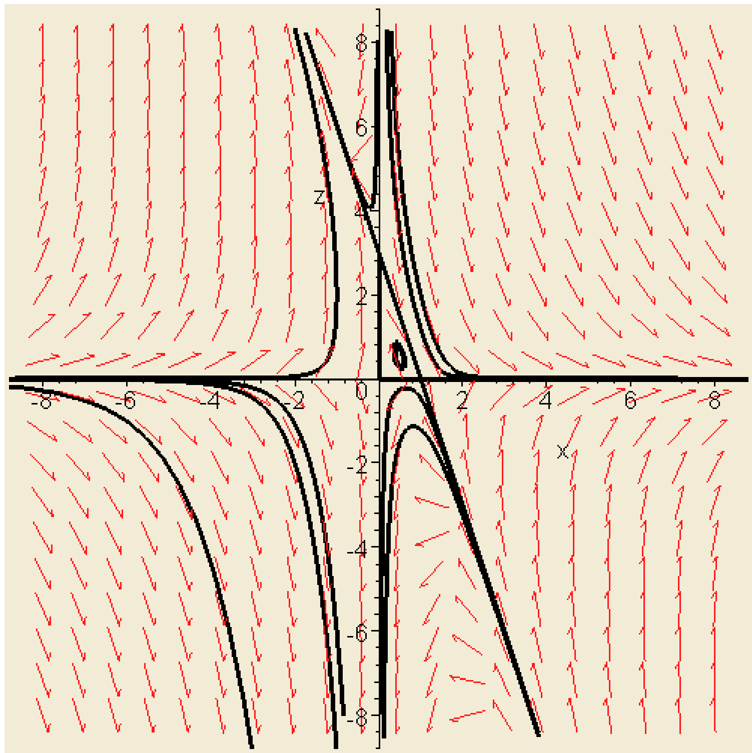

Figure 1: Case $Q = Q_2 : N = 3, Q = 5$

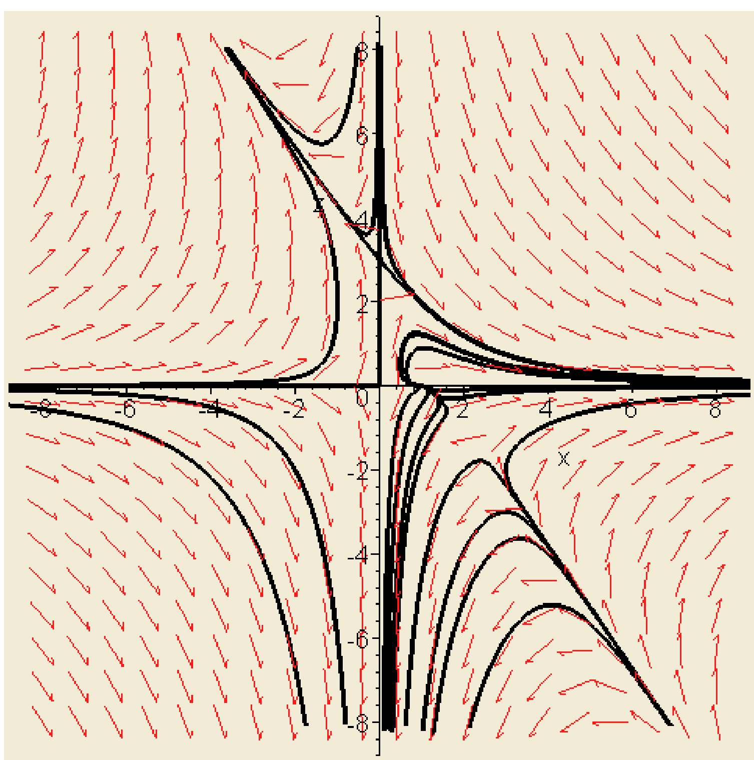

Figure 2: Case $Q < Q_1 : N = 3, Q = 2$

3.3 Global study

Remark 3.1 *System* (M_{scal}) *has no limit cycle for* $Q \neq Q_2$. *It is evident when* $\varepsilon = -1$. *When* $\varepsilon = 1$, *as noticed in [19], it comes from the Dulac's theorem: setting* $X_t = f(X,Z)$, $\quad Z_t = g(X,Z)$, *and*

$$B(X,Z) = X^{pQ/(Q+1-p)-2} Z^{(p/(Q+1-p)-1}, \quad M = B_X X_t + B_Z Z_t + B(f_X + g_Z),$$

then $M = KB$ *with* $K = (Q_2 - Q)\gamma(N-p)/p$, *thus* M *has no zero for* $Q \neq Q_2$.

Then from the Poincaré-Bendixson theorem, any trajectory bounded near $\pm\infty$ converges to one of the fixed points. Thus we find again global results:

• Equation with source ($\varepsilon = 1$). If $Q < Q_1$, there is no G.S.: the regular trajectory $\mathcal{T}$ issued from N_0 cannot converge to a fixed point. Then X tends to ∞ and the regular solutions u are changing sign, there is no G.S., see fig. 2.

If $Q_1 < Q < Q_2$, the regular trajectory $\mathcal{T}$ cannot converge to M_0; if it converges to A_0, it is the unique trajectory converging to A_0; the set delimitated by $\mathcal{T}$ and $X = 0, Z = 0$ is invariant, thus it contains M_0; and the trajectories issued from M_0 cannot converge to a fixed point, which is contradictory. then again X tends to ∞ on $\mathcal{T}$ and the regular solutions u are changing sign.. The trajectory ending at A_0 converges to M_0 at $-\infty$; then there exist solutions $u > 0$ such that $\lim_{r\to 0} r^\gamma u = A$ and $\lim_{r\to 0} r^{\frac{N-p}{p-1}} u = \alpha > 0$.

If $Q > Q_2$, the only singular solution at 0 is u_0, and the regular solutions are G.S., with $\lim_{r\to\infty} r^\gamma u = A$. Indeed M_0 is a sink; the trajectory ending at A_0 cannot converge to N_0 at $-\infty$, thus X converges to 0, and Z converges to ∞, then u cannot be positive on $(0,\infty)$.The trajectory issued from N_0 converges to M_0.

• Equation with absorption ($\varepsilon = -1$). If $Q > Q_1$, all the solutions u defined near 0 are regular; indeed the trajectories cannot converge to a fixed point.

If $Q < Q_1$, see fig. 2, we find again easily a well known result: there exists a positive solution u_1, unique up to a scaling, such that $\lim_{r\to 0} r^{\frac{N-p}{p-1}} u_1 = \alpha > 0$, and $\lim_{r\to\infty} r^\gamma u_1 = A$. Indeed the eigenvalues at M_0 satisfy $\lambda_1 < 0 < \lambda_2$. There are two trajectories $\mathcal{T}_1, \mathcal{T}_1'$ associated to λ_1, and the eigenvector $(X_0 + |\lambda_1|, -\frac{X_0}{p-1})$. The trajectory $\mathcal{T}_1$ satisfies $X_t > 0 > Z_t$ near ∞, and $X > \frac{N-p}{p-1}$, since $Z_0 < 0$, and X cannot take the value $\frac{N-p}{p-1}$ because at such a point $X_t < 0$; then $\frac{N-p}{p-1} < X < X_0$ and $X_t > 0$ as long as it is defined; similarly $Z_0 < Z < 0$ and $Z_t < 0$; then $\mathcal{T}_1$ converge to a fixed point, necessarily A_0, showing the existence of u_1. The trajectory $\mathcal{T}_1'$ corresponds to solutions u such that $\lim_{r\to\infty} r^\gamma u = A$ and $\lim_{r\to R} u = \infty$ for some $R > 0$. There are two trajectories $\mathcal{T}_2$, $\mathcal{T}_2'$, associated to λ_2, defining solutions u such that $\lim_{r\to 0} r^\gamma u = A$ and changing sign, or with a minimum point and $\lim_{r\to R} u = \infty$ for some $R > 0$. The regular trajectory starts from N_0 in the 2^{nd} quadrant, it cannot converge to a fixed point, then $\lim_{r\to R} u = \infty$ for some $R > 0$.

• Critical case $Q = Q_2$: it is remarkable that system (M_{scal}) admits <u>another invariant line</u>, namely $A_0 N_0$, given by

$$\frac{X}{p'} + \frac{Z}{Q_2+1} - \frac{N-p}{p} = 0, \tag{3.3}$$

see fig. 1. It precisely corresponds to well-known solutions of the two equations

$$u = c(K^2 + r^{(p+a)/(p-1)})^{(p-N)/(p+a)}, \text{for } \varepsilon = 1; \quad u = c\left|K^2 - r^{(p+a)/(p-1)}\right|^{(p-N)/(p+a)}, \text{for } \varepsilon = -1,$$

where $K^2 = c^{Q-p+1}(N+a)^{-1}\left((N-p)/(p-1)\right)^{1-p}.$

Remark 3.2 *The global results have been obtained without using energy functions. The study of [2] was based on a reduction of type of Remark 2.2, using an energy function linked to the new unknown. Other energy functions are well-known, of Pohozaev type:*

$$\mathcal{F}_\sigma(r) = r^N\left[\frac{|u'|^p}{p'} + \varepsilon r^a \frac{|u|^{Q+1}}{Q+1} + \sigma\frac{u\,|u'|^{p-2}\,u'}{r}\right] = r^{N-p}\,|u|^p\,|X|^{p-2}\,X\left[\frac{X}{p'} + \frac{Z}{Q+1} - \sigma\right],$$

with $\sigma = \frac{N-p}{p}$, satisfying $\mathcal{F}_\sigma'(r) = r^{N-1+a}\left(\frac{N+a}{Q+1} - \frac{N-p}{p}\right)|u|^{Q+1}$, or with $\sigma = \frac{N+a}{Q+1}$, leading to $\mathcal{F}_\sigma'(r) = r^{N-1}\left(\frac{N+a}{Q+1} - \frac{N-p}{p}\right)|u'|^p$. In the critical case $Q = Q_2$, all these functions coincide and they are constant, in other words system (M_{scal}) has a first integral. We find again the line (3.3): the G.S. are the functions of energy 0.

4 Local study of system (S)

In all the sequel we study the system with source terms: $(G) = (S)$. Assumption (1.5) is the most interesting case for studying the existence of the G.S.

We first study the local behaviour of nonnegative solutions (u,v) defined near 0 or near ∞. It is well known that any solution (u,v) positive on some interval $(0,R)$ satisfies $u',v' < 0$ on $(0,R)$. Any solution (u,v) positive on (R,∞), satisfies $u',v' < 0$ near ∞. We are reduced to study the system in the region $\mathcal{R}$ where $X,Y,Z,W > 0$, and consider the fixed points in $\bar{\mathcal{R}}$. Then

$$X(t) = -\frac{ru'}{u}, \quad Y(t) = -\frac{rv'}{v}, \quad Z(t) = \frac{r^{1+a}u^s v^\delta}{|u'|^{p-1}}, \quad W(t) = \frac{r^{1+b}v^m u^\mu}{|v'|^{q-1}}; \tag{4.1}$$

and (X,Y,Z,W) is a solution of system (M) in $\mathcal{R}$ if and only if (u,v) defined by

$$u = r^{-\gamma}(ZX^{p-1})^{(q-1-m)/D}(WY^{q-1})^{\delta/D}, \qquad v = r^{-\xi}(WY^{q-1})^{(p-1-s)/D}(ZX^{p-1})^{\mu/D} \tag{4.2}$$

is a positive solution with $u',v' < 0$. Among the fixed points, the point M_0 defined at (2.6) lies in $\mathcal{R}$ if and only if

$$0 < \gamma < \frac{N-p}{p-1} \quad \text{and} \quad 0 < \xi < \frac{N-q}{q-1}. \tag{4.3}$$

The local study of the system near M_0 appears to be tricky, see Remark 4.2. A main difference with the scalar case is that there always exist a trajectory converging to M_0 at $\pm\infty$:

Proposition 4.1 *(Point M_0) Assume that (4.3) holds. Then there exist trajectories converging to M_0 as $r \to \infty$, and then solutions (u,v) being defined near ∞, such that*

$$\lim_{r\to\infty} r^\gamma u = \alpha > 0, \quad \lim_{r\to\infty} r^\xi v = \beta > 0. \tag{4.4}$$

There exist trajectories converging to M_0 as $r \to 0$, and thus solutions (u,v) being defined near 0 such that

$$\lim_{r\to 0} r^\gamma u = \alpha > 0, \quad \lim_{r\to 0} r^\xi v = \beta > 0. \tag{4.5}$$

Proof. Here $M_0 \in \mathcal{R}$; setting $X = X_0 + \tilde{X}, Y = Y_0 + \tilde{Y}, Z = Z_0 + \tilde{Z}, W = W_0 + \tilde{W}$, the linearized system is

$$\begin{cases} \tilde{X}_t = X_0(\tilde{X} + \frac{1}{p-1}\tilde{Z}), \\ \tilde{Y}_t = Y_0(\tilde{Y} + \frac{1}{q-1}\tilde{W}), \\ \tilde{Z}_t = Z_0(-s\tilde{X} - \delta\tilde{Y} - \tilde{Z}), \\ \tilde{W}_t = W_0(-\mu\tilde{X} - m\tilde{Y} - \tilde{W}). \end{cases}$$

The eigenvalues are the roots $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, of equation

$$\left[(\lambda - X_0)(\lambda + Z_0) + \frac{s}{p-1}X_0Z_0\right]\left[(\lambda - Y_0)(\lambda + W_0) + \frac{m}{q-1}Y_0W_0\right] - \frac{\delta\mu}{(p-1)(q-1)}X_0Y_0Z_0W_0 = 0. \tag{4.6}$$

This equation is of the form

$$f(\lambda) = \lambda^4 + E\lambda^3 + F\lambda^2 + G\lambda - H = 0,$$

with

$$\begin{cases} E = Z_0 - X_0 + W_0 - Y_0, \\ F = (Z_0 - X_0)(W_0 - Y_0) - \frac{s+p-1}{p-1}X_0Z_0 - \frac{m+q-1}{q-1}Y_0W_0, \\ G = -\frac{q-1-m}{q-1}Y_0W_0(Z_0 - X_0) - \frac{p-1-s}{p-1}X_0Z_0(W_0 - Y_0), \\ H = \frac{D}{(p-1)(q-1)}X_0Y_0Z_0W_0. \end{cases}$$

From (1.5) we have $H > 0$, then $\lambda_1\lambda_2\lambda_3\lambda_4 < 0$. There exist two real roots $\lambda_3 < 0 < \lambda_4$, and two roots λ_1, λ_2, real with $\lambda_1\lambda_2 > 0$, or complex. Therefore there exists at least one trajectory converging to M_0 at ∞ and another one at $-\infty$. Then (4.4) and (4.5) follow from (4.2). Moreover the convergence is monotone for X, Y, Z, W. ∎

Remark 4.2 *There exist imaginary roots, namely* $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 = 0$, *if and only if there exists* $c > 0$ *such that* $f(ci) = 0$, *that means* $Ec^2 - G = 0$, *and* $c^4 - Fc^2 - H = 0$, *equivalently*

$$E = G = 0, \quad \text{or } EG > 0 \text{ and } G^2 - EFG - E^2H = 0.$$

Condition $E = G = 0$ *means that*

(i) either $Z_0 = X_0$ *and* $W_0 = Y_0$, *i.e.*

$$(\gamma, \xi) = (\frac{N-p}{p}, \frac{N-q}{q}), \tag{4.7}$$

in other words $(\delta, \mu) = (\frac{q(N(p-1-s)+p(1-s+a))}{p(N-q)}, \frac{p(N(q-1-m)+q(1-m+b))}{q(N-p)})$.

(ii) or $(p-1-s)(q-1-m) > 0$ *and* (γ, ξ) *satisfies*

$$\begin{cases} 2N - p - q = p\gamma + q\xi, \\ (1 - \frac{m}{q-1})\xi(N - q - (q-1)\xi) = (1 - \frac{s}{p-1})\gamma(N - p - (p-1)\gamma). \end{cases} \tag{4.8}$$

This gives in general 0,1 or 2 values of (γ, ξ). *For example, in the case* $\frac{m}{q-1} = \frac{s}{p-1} \neq 1$, *and* $(p-2)(q-2) > 0$ *and* $N > \frac{pq-p-q}{p+q-2}$ *we find another value, different from the one of (4.7) for* $p \neq q$:

$$(\gamma, \xi) = (N\frac{q-2}{pq-p-q} - 1, N\frac{p-2}{pq-p-q} - 1). \tag{4.9}$$

Moreover the computation shows that it can exist imaginary roots with $E, G \neq 0$.

In the case $p = q = 2$ and $s = m$ the situation is interesting:

Proposition 4.3 *Assume $p = q = 2$ and $s = m < \frac{N}{N-2}$, with $\delta + 1 - s > 0, \mu + 1 - s > 0$. In the plane (δ, μ), let $\mathcal{H}_s$ be the hyperbola of equation*

$$\frac{1}{\delta + 1 - s} + \frac{1}{\mu + 1 - s} = \frac{N-2}{N - (N-2)s}, \tag{4.10}$$

equivalently $\gamma + \xi = N - 2$. Then $\mathcal{H}_s$ is contained in the set of points (δ, μ) for which the linearized system at M_0 has imaginary roots, and equal when $s \leqq 1$.

Proof. The assumption $D > 0$ imply $\delta + 1 - s > 0$ and $\mu + 1 - s > 0$; condition $E = G = 0$ implies $s < N/(N-2)$ and reduces to condition (4.10). Moreover if $s \leqq 1$, all the cases are covered. Indeed $2G = (s-1)E\left[Y_0 Z_0 + X_0 W_0\right]$, hence $GE \leqq 0$. ∎

Next we give a summary of the local existence results obtained by linearization around the other fixed points of system (M) proved in Section 10. Recall that $t \to -\infty$ as $r \to 0$ and $t \to \infty$ as $r \to \infty$.

Proposition 4.4 *(Point N_0) A solution (u, v) is regular if and only if the corresponding trajectory converges to N_0 when $r \to 0$. For any $u_0, v_0 > 0$, there exists a unique local regular solution (u, v) with initial data (u_0, v_0).*

Proposition 4.5 *(Point A_0) If $s\frac{N-p}{p-1} + \delta\frac{N-q}{q-1} > N + a$ and $\mu\frac{N-p}{p-1} + m\frac{N-q}{q-1} > N + b$, there exist (admissible) trajectories converging to A_0 when $r \to \infty$. If $s\frac{N-p}{p-1} + \delta\frac{N-q}{q-1} < N + a$ and $\mu\frac{N-p}{p-1} + m\frac{N-q}{q-1} < N + b$, the same happens when $r \to 0$. In any case*

$$\lim r^{\frac{N-p}{p-1}} u = \alpha > 0, \quad \lim r^{\frac{N-q}{q-1}} v = \beta > 0. \tag{4.11}$$

If $s\frac{N-p}{p-1} + \delta\frac{N-q}{q-1} < N + a$ or $\mu\frac{N-p}{p-1} + m\frac{N-q}{q-1} < N + b$, there exists no trajectory converging when $r \to \infty$; if $s\frac{N-p}{p-1} + \delta\frac{N-q}{q-1} > N + a$ or $\mu\frac{N-p}{p-1} + m\frac{N-q}{q-1} > N + b$, there exists no trajectory converging when $r \to 0$.

Proposition 4.6 *(Point P_0) 1) Assume that $q > m + 1$ and $q + b < \frac{N-p}{p-1}\mu < N + b - m\frac{N-q}{q-1}$. If $\gamma < \frac{N-p}{p-1}$ there exist trajectories converging to P_0 when $r \to \infty$ (and not when $r \to 0$). If $\gamma > \frac{N-p}{p-1}$ the same happens when $r \to 0$ (and not when $r \to \infty$).*

2) Assume that $q < m + 1$ and $q + b > \frac{N-p}{p-1}\mu > N + b - m\frac{N-q}{q-1}$ and $q\frac{N-p}{p-1}\mu + m(N-q) \neq N(q-1) + (b+1)q$. If $\gamma < \frac{N-p}{p-1}$ there exist trajectories converging to P_0 when $r \to 0$ (and not when $r \to \infty$). If $\gamma > \frac{N-p}{p-1}$ there exist trajectories converging when $r \to \infty$ (and not when when $r \to 0$).

In any case, setting $\kappa = \frac{1}{q-1-m}(\frac{N-p}{p-1}\mu - (q+b))$, there holds

$$\lim r^{\frac{N-p}{p-1}} u = \alpha > 0, \quad \lim r^{\kappa} v = \beta > 0. \tag{4.12}$$

Remark 4.7 *This result improves the results of existence obtained by the fixed point theorem in [27] in the case of system* (RP) *with* $p = q = 2, a = 0, N = 3,\ 2s + m \neq 3$. *The proof is quite simpler..*

Proposition 4.8 *(Point* I_0*) If* $\frac{N-p}{p-1}s > N+a$ *and* $\frac{N-q}{q-1}\mu > N+b$, *there exist trajectories converging to* I_0 *when* $r \to \infty$, *and then*

$$\lim_{r\to\infty} r^{\frac{N-p}{p-1}} u = \beta, \quad \lim_{r\to\infty} v = \alpha > 0. \tag{4.13}$$

For any $s, m \geqq 0$, *there is no trajectory converging when* $r \to 0$.

Proposition 4.9 *(Point* G_0*) Suppose* $\frac{N-p}{p-1}\mu < N + b$. *If* $q + b < \frac{N-p}{p-1}\mu$ *and* $N + a < \frac{N-p}{p-1}s$, *there exist trajectories converging to* G_0 *when* $r \to \infty$. *If* $\frac{N-p}{p-1}\mu < q + b$ *and* $\frac{N-p}{p-1}s < N + a$, *the same happens when* $r \to 0$. *In any case*

$$\lim r^{\frac{N-p}{p-1}} u = \beta, \quad \lim v = \alpha > 0. \tag{4.14}$$

Proposition 4.10 *(Point* C_0*) Suppose* $N + b < \frac{N-q}{q-1}m$ *(hence* $q < m+1$*) with* $m \neq \frac{N(q-1)+(b+1)q}{N-q}$, *and* $\delta > \frac{(N+a)(m+1-q)}{q+b}$. *Then there exist trajectories converging to* C_0 *when* $r \to \infty$ *(and not when* $r \to 0$*), and then*

$$\lim u = \alpha > 0, \quad \lim r^k v = \beta, \tag{4.15}$$

where $k = \frac{q+b}{m+1-q}$.

Proposition 4.11 *(Point* R_0*) Assume that* $N + b < \frac{N-q}{q-1}m$ *(hence* $q < m + 1$*) with* $m \neq \frac{N(q-1)+b+bq}{N-q}$, *and* $\delta < \frac{(N+a)(m+1-q)}{q+b}$. *If* $\frac{(p+a)(m+1-q)}{q+b} < \delta$, *there exist trajectories converging to* R_0 *when* $r \to \infty$ *(and not when* $r \to 0$*). If* $\delta < \frac{(p+a)(m+1-q)}{q+b}$, *there exist trajectories converging when* $r \to 0$ *(and not when* $r \to \infty$*), and then (4.15) holds again.*

We obtain similar results of convergence to the points Q_0, J_0, H_0, D_0, S_0 by exchanging p, δ, s, a and q, μ, m, b. There is no admissible trajectory converginf to $0, K_0, L_0$, see Remark 10.1.

5 Global results for system (S)

We are concerned by the existence of global positive solutions. First we find again easily some known results by using our dynamical approach.

Proposition 5.1 *Assume that system* (S) *admits a positive solution* (u, v) *in* $(0, \infty)$. *Then the corresponding solution* (X, Y, Z, W) *of system* (M) *stays in the box*

$$\mathcal{A} = \left(0, \frac{N-p}{p-1}\right) \times \left(0, \frac{N-q}{q-1}\right) \times (0, N+a) \times (0, N+b), \tag{5.1}$$

in other words

$$ru'+\frac{N-p}{p-1}u>0, \quad rv'+\frac{N-q}{q-1}v>0, \quad r^{1+a}u^{s}v^{\delta}<(N+a)\left|u'\right|^{p-1}, \quad r^{1+b}u^{\mu}v^{m}<(N+b)\left|v'\right|^{q-1}. \tag{5.2}$$

and then

$$u^{s-p+1}v^{\delta} \leqq C_1 r^{-(p+a)}, \quad u^{\mu}v^{m-q+1} \leqq C_2 r^{-(q+b)}, \quad \text{in } (0,\infty), \tag{5.3}$$

where $C_1=(N+a)(\frac{N-p}{p-1})^{p-1}, C_2=(N+b)(\frac{N-q}{q-1})^{q-1}$, *and*

$$\lim_{r\to 0} r^{\frac{N-p}{p-1}}u=c_1 \geqq 0, \quad \lim_{r\to 0} r^{\frac{N-q}{q-1}}v(r)=c_2 \geqq 0, \quad \lim\inf_{r\to\infty} r^{\frac{N-p}{p-1}}u>0, \quad \lim\inf_{r\to\infty} r^{\frac{N-q}{q-1}}v>0. \tag{5.4}$$

As a consequence if $s \leqq p-1$ *or* $m \leqq q-1$, *we have*

$$u \leqq K_1 r^{-\gamma}, \quad v \leqq K_2 r^{-\xi}, \quad \text{in } (0,\infty), \tag{5.5}$$

with $K_1=C_1^{(q-1-m)/D}C_2^{\delta/D}, K_2=C_1^{\mu/D}C_2^{(p-1-s)/D}$.

Proof. The solution of system (M) in $\mathcal{R}$ defined on $\mathbb{R}$. On the hyperplane $X=\frac{N-p}{p-1}$ we have $X_t>0$, the field is going out. If at some time t_0, $X(t_0)=\frac{N-p}{p-1}$, then $X(t)>\frac{N-p}{p-1}$ for $t>t_0$, in turn $X_t \geqq X\left[X-\frac{N-p}{p-1}\right]>0$, since since $Z>0$, thus $X(t)>2\frac{N-p}{p-1}$ for $t>t_1>t_0$, then $X_t \geqq X^2/2$, which implies that X blows up in finite time; thus $X(t)<\frac{N-p}{p-1}$ on $\mathbb{R}$; in the same way $Y(t)<\frac{N-q}{q-1}$. On the hyperplane $Z=N+a$ we have $Z_t<0$, the field is entering. If at some time t_0, $Z(t_0)=N+a$ then $Z(t)>N+a$ for $t<t_0$, then $Z_t \leqq Z(N+a-Z)$, since $sX+\delta Y>0$, and Z blows up in finite time as above; thus $Z(t)<N+a$ on $\mathbb{R}$, in the same way $W(t)<N+b$. Then (5.2),(5.3) and (5.5) follows. By integration it implies that $r^{(N-p)/(p-1)}u(r)$ is nondecreasing near 0 or ∞, hence (5.4) holds. ∎

Next we prove Theorem 1.1.

Proof of Theorem 1.1. (i) The trajectories of the regular solutions start from $N_0=(0,0,N+a,N+b)$, from Proposition 4.4, and the unstable variety $\mathcal{V}_u$ has dimension 2, from (10.1), (10.2). It is given locally by $Z=\varphi(X,Y), W=\psi(X,Y)$ for $(X,Y)\in B(0,\rho)\setminus\{0\}\subset\mathbb{R}^2$.

To any $(x,y)\in B(0,\rho)\setminus\{0\}$ we associate the unique trajectory $\mathcal{T}_{x,y}$ in $\mathcal{V}_u$ going through this point. If T^* is the maximal interval of existence of a solution on $\mathcal{T}_{x,y}$, then $\lim_{t\to T^*}(X(t)+Y(t))=\infty$. Indeed Z, and W satisfy $0<Z<N+a$, $0<W<N+b$ as long as the solution exists, because at a time T where $Z(T)=N+a$, we have $Z_t<0$. If there exists a first time T such that $X(T)=\frac{N-p}{p-1}$ or $Y(T)=\frac{N-q}{q-1}$, then $T<T^*$. We consider the open rectangle $\mathcal{N}$ of submits

$$(0,0), \quad \varpi_1=\left(\frac{N-p}{p-1},0\right), \quad \varpi_2=\left(0,\frac{N-q}{q-1}\right), \quad \varpi=\left(\frac{N-p}{p-1},\frac{N-q}{q-1}\right).$$

Let $\mathcal{U}=\{(x,y)\in B(0,\rho):x,y>0\}$; then $\mathcal{U}=\mathcal{S}_1\cup\mathcal{S}_2\cup\mathcal{S}_3\cup\mathcal{S}$, where

$$\begin{cases} \mathcal{S}_i=\{(x,y)\in\mathcal{U}:\mathcal{T}_{x,y} \text{ leaves } \mathcal{N} \text{ on } (\varpi_i,\varpi)\}, \quad i=1,2, \\ \mathcal{S}_3=\{(x,y)\in\mathcal{U}: \ \mathcal{T}_{x,y} \text{ leaves } \mathcal{N} \text{ at } \varpi\}, \quad \mathcal{S}=\{(x,y)\in\mathcal{U}: \ \mathcal{T}_{x,y} \text{ stays in } \mathcal{N}\}. \end{cases}$$

Any element of $\mathcal{S}$ defines a G.S. Assume $s < \frac{N(p-1)+p+pa}{N-p}$. Let us show that $\mathcal{S}_1$ is nonempty. Consider the trajectory $\mathcal{T}_{\bar{x},0}$ on $\mathcal{V}_u$ associated to $(\bar{x},0)$, with $\bar{x} \in (0,\rho)$, going through $\bar{M} = (\bar{x}, 0, \varphi(\bar{x},0), \psi(\bar{x},0))$; it is <u>*not admissible*</u> for our problem, since it is in the hyperplane $Y = 0$: it satisfies the system

$$\begin{cases} X_t = X\left[X - \frac{N-p}{p-1} + \frac{Z}{p-1}\right], \\ Z_t = Z\left[N + a - sX - Z\right], \\ W_t = W\left[N + b - \mu X - W\right], \end{cases}$$

which is <u>*not completely coupled*</u>. The two equations in X, Z corresponds to the equation

$$-\Delta_p U = r^a U^s. \tag{5.6}$$

The regular solutions of (5.6) are changing sign, since s is subcritical, see Section 3. Consider the solution $(\bar{X}, \bar{Y}, \bar{Z}, \bar{W})$ of system (M), of trajectory $\mathcal{T}_{\bar{x},0}$, going through $\bar{M}$ at time 0; it satisfies $\bar{Y} = 0$, and $\bar{X}(t) > 0$, $\bar{Z}(t) > 0$ tend to ∞ in finite time T^*, then for any given $C \geqq \frac{N-p}{p-1}$, there exist a first time $T < T^*$ such that $\bar{X}(T) = C$, and $\bar{Y}(T) = 0$. We have $\lim_{t\to-\infty} \bar{W} = N + b$, and necessarily $0 < \bar{W} < N + b$, in particular $0 < \bar{W}(T) < N + b$; and $\bar{W}_t$ is bounded on $(-\infty, T^*)$, then $\bar{W}$ has a finite limit at T^*. The field at time T is transverse to the hyperplane $X = \frac{N-p}{p-1}$: we have $\bar{X}_t \geqq C\frac{Z(T)}{p-1} > 0$, since $\bar{Z}(T) > 0$. From the continuous dependance of the initial data at time 0, for any $\varepsilon > 0$, there exists $\eta > 0$ such that for any $(x,y) \in B((\bar{x},0),\eta)$ and for any (X,Y,Z,W) on $\mathcal{T}_{x,y}$, there exists a first time T_ε such that $X(T_\varepsilon) = C$, and $|Y(t)| \leqq \varepsilon$ for any $t \leqq T_\varepsilon$, in particular for any $(x,y) \in B((\bar{x},0),\eta)$ with $y > 0$, and then $0 < Y(t) \leqq \varepsilon$ for any $t \leqq T_\varepsilon$. Let us take $C = \frac{N-p}{p-1}$. Then $(x,y) \in \mathcal{S}_1$. The same arguments imply that $\mathcal{S}_1$ is open. Similarly assuming $m < \frac{N(q-1)+q+qb}{N-q}$ implies that $\mathcal{S}_2$ is nonempty and open. By connexity $\mathcal{S}$ is empty if and only if $\mathcal{S}_3$ is nonempty.

(ii) Here the difficulty is due to the fact that the zeros of u, v correspond to infinite limits for X, Y, and then the argument of continuous dependance is no more available. We can write $\mathcal{U} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{S}$, where

$$\begin{cases} \mathcal{M}_1 = \{(x,y) \in \mathcal{U} \text{ and } \mathcal{T}_{x,y} \text{ has an infinite branch in } X \text{ with } Y \text{ bounded}\}, \\ \mathcal{M}_2 = \{(x,y) \in \mathcal{U} : \ \mathcal{T}_{x,y} \text{ has an infinite branch in } Y \text{ with } X \text{ bounded}\}, \\ \mathcal{M}_3 = \{(x,y) \in \mathcal{U} : \ \mathcal{T}_{x,y} \text{ has an infinite branch in } (X,Y)\}. \end{cases}$$

In other words, $\mathcal{M}_1$ is the set of $(x,y) \in \mathcal{U}$ such that for any (X,Y,Z,W) on $\mathcal{T}_{x,y}$, there exists a T^* such that $\lim_{t\to T^*} X(t)) = \infty$, and $Y(t)$ stays bounded on $(-\infty, T^*)$, that means the set of $(x,y) \in \mathcal{U}$ such that for any solution (u,v) corresponding to $\mathcal{T}_{x,y}$, u vanishes before v; similarly for $\mathcal{M}_2$. Otherwise $\mathcal{M}_3$ is the set of $(x,y) \in \mathcal{U}$ such that there exists a T^* such that $\lim_{t\to T^*} X(t) = \lim_{t\to T^*} Y(t) = \infty$, that means (u,v) vanish at the same $R^* = e^{T^*}$. In that case, from the Höpf Lemma, $\lim_{r\to R} \frac{u'}{(r-R)u} = 1$, then $\lim_{t\to T^*} \frac{X}{Y} = 1$.

We are lead to show that $\mathcal{M}_1$ is nonempty and open for $s < \frac{N(p-1)+p+pa}{N-p}$. We consider again the trajectory $\bar{\mathcal{T}}$ and take C large enough: $C = 2(\frac{N-p}{p-1} + \frac{N+|b|}{q-1})$. Let $\varepsilon \in \left(0, \frac{C}{2}\right)$. For any $(x,y) \in B((\bar{x},0),\eta)$ with $y > 0$, and any (X,Y,Z,W) on $\mathcal{T}_{x,y}$, there is a first time T_ε such that $X(T_\varepsilon) = C$, and $0 < Y(t) \leqq \varepsilon$ for any $t \leqq T_\varepsilon$. And X is increasing and $X_t \geqq X(X - C)$, thus there exists T^* such that $\lim_{t\to T^*} X(t) = \infty$. Setting $\varphi = X/Y$, we find

$$\frac{\varphi_t}{\varphi} = X - Y + \frac{Z}{p-1} - \frac{W}{q-1} + \frac{N-q}{q-1} - \frac{N-p}{p-1} \geqq X - Y - \frac{C}{2}$$

then $\varphi_t(T_\varepsilon) > 0$. Let $\theta = \sup\{t > T_\varepsilon : \varphi_t > 0\}$; suppose that θ is finite; then $\varphi(\theta) > \varphi(T_\varepsilon) = C/\varepsilon > 2$ and $X(\theta) \leqq Y(\theta) + C < X(\theta)/2 + C$, which is contradictory. Then φ is increasing up to T^*; if $\lim_{t\to T^*} Y(t) = \infty$, then $\lim_{t\to T^*} \varphi = 1$, which is impossible. Then $(x, y) \in \mathcal{M}_1$, thus $\mathcal{M}_1$ is nonempty. In the same way $\mathcal{M}_1$ is open. Indeed for any $(\bar{x}, \bar{y}) \in \mathcal{M}_1$ there exists $M > 0$ such that $0 < \bar{Y}(t) \leqq M/2$ on $\mathcal{T}_{\bar{x},\bar{y}}$. To conclude we argue as above, with $(\bar{x}, 0)$ replaced by $(\bar{x}, \bar{y})$, and C replaced by $C + M$. ■

Proof of Proposition 1.2. Assume $s \geqq \frac{N(p-1)+p+pa}{N-p}$. Consider the Pohozaev type function

$$\mathcal{F}(r) = r^N \left[\frac{|u'|^p}{p'} + \frac{r^a u^{s+1}}{s+1} v^\delta + \frac{N-p}{p} \frac{u\,|u'|^{p-2}\,u'}{r} \right] = r^{N-p} u^p \left[\frac{X}{p'} + \frac{1}{s+1} Z - \frac{N-p}{p} \right]. \quad (5.7)$$

We find $\mathcal{F}(0) = 0$ and

$$\begin{aligned} \mathcal{F}'(r) &= r^{N-1+a} \left[\left(\frac{N+a}{s+1} - \frac{N-p}{p} \right) v^\delta u^{s+1} + \frac{\delta}{s+1} r u^{s+1} v^{\delta-1} v' \right] \\ &= r^{N-1+a} v^\delta u^{s+1} \left[\frac{N+a}{s+1} - \frac{N-p}{p} - \frac{\delta Y}{s+1} \right] \end{aligned} \quad (5.8)$$

From our assumption, $\mathcal{F}$ is decreasing, and $Z > 0$, thus $X < \frac{N-p}{p-1}$. Then $\mathcal{S}_1, \mathcal{S}_3$ are empty. If moreover $m \geqq \frac{N(q-1)+q+qb}{N-q}$ then $\mathcal{S}_2$ is empty, therefore $\mathcal{S} = \mathcal{U}$. ■

Remark 5.2 *Let us only assume that $s \geqq \frac{N(p-1)+p+pa}{N-p}$. If one function has a first zero, it is v. Indeed if there exists a first value R where $u(R) = 0$, and $v(r) > 0$ on $[0, R)$, then $\mathcal{F}(R) = \frac{R^N}{p'} |u'(R)|^p > 0$.*

As a first consequence we obtain existence results for the Dirichlet problem. It solves an open problem in the case $s > p - 1$ or $m > q - 1$, and extends some former results of [12] and [42]. Our proof, based on the shooting method differs from the proof of [12], based on degree theory and blow-up technique. Our results extend the ones of [3, Theorem 2.2] relative to the case $p = q = 2$, obtained by studying the equation satisfied by a suitable function of u, v.

Corollary 5.3 *system (S) admits no G.S. and then there is a radial solution of the Dirichlet problem in a ball in any of the following cases:*

(i) $p < s+1, q < m+1$, *and* $\min(s\frac{N-p}{p-1} + \frac{N-q}{q-1}\delta - (N+a), \frac{N-p}{p-1}\mu + m\frac{N-q}{q-1} - (N+b)) \leqq 0$;

(ii) $p < s+1$, $q > m+1$ *and* $s\frac{N-p}{p-1} + \frac{N-q}{q-1}\delta - (N+a) \leqq 0$ *or* $\gamma - \frac{N-p}{p-1} > 0$;

(iii) $p > s+1, q > m+1$ *and* $\max(\gamma - \frac{N-p}{p-1}, \xi - \frac{N-q}{q-1}) \geqq 0$;

(iv) $p \geqq s+1, q \geqq m+1$ *and* $\max(\gamma - \frac{N-p}{p-1}, \xi - \frac{N-q}{q-1}) > 0$.

Proof. From Theorem 1.1, we are reduced to prove the nonexistence of G.S.

(i) Assume $p < s+1$, and $s\frac{N-p}{p-1} + \frac{N-q}{q-1}\delta - (N+a) < 0$. We have $-\Delta_p u \geqq C r^{a - \frac{N-q}{q-1}\delta} u^s$ for large r. From [6, Theorem 3.1], we find $u = O(r^{-(p+a-\frac{N-q}{q-1}\delta)/(s+1-p)})$, and then $s\frac{N-p}{p-1} + \frac{N-q}{q-1}\delta - (N+a) \geqq 0$, from (5.4), which contradicts our assumption. In case of equality, we find $-\Delta_p u \geqq Cr^{-N}$ for large r, which is impossible. Then there exists no G.S. This improves ythe result of [12] where the minimum is replaced by a maximum.

(ii) Assume $p < s+1$, $q > m+1$ and $\gamma - \frac{N-p}{p-1} > 0$; then $u = O(r^{-\gamma})$, which contradicts (5.4). If $\gamma - \frac{N-p}{p-1} = 0$, then $\lim r^{\frac{N-p}{p-1}} u = \alpha > 0$, and $\xi > \frac{N-q}{q-1}$. Hence $-\Delta_q v \geqq Cr^{b - \frac{N-p}{p-1}\mu} v^m$ for large r, then $v \geqq Cr^{(q+b-\frac{N-p}{p-1}\delta)/(q-1-m)} = Cr^{-\xi}$. There exists $C_1 > 0$ such that $C_1 \leqq r^{\xi} v \leqq 2C_1$ for large r, from [6, Theorem 3.1] and (5.5), then $-\Delta_p u \geqq Cr^{-N}$ for some $C > 0$, which is again contradictory.

(iii) (iv) The nonexistence of G.S is obtained by extension of the proof of [12] to the case $a, b \neq 0$. Moreover (iii) implies the nonexistence of positive solution (u, v), radial or not, in any exterior domain $(R, \infty) \times (R, \infty), R > 0$ from [6]. ■

Corollary 5.4 *Assume (4.3) with $p = q = 2$. If $\delta + s \geqq \frac{N+2+2a}{N-2}$ and $\mu + m \geqq \frac{N+2+2b}{N-2}$, then system (S) admits a G.S.*

Proof. It was shown in [28], [41] by the moving spheres method that the Dirichlet problem has no radial or nonradial solution. Then Theorem 1.1 applies again. ■

We aso extend and improve a result of nonexistence of [10] for the case $p = q = 2, a = 0, s > 1$:

Proposition 5.5 *Assume $s + 1 > p$ or $\gamma > \frac{N-p}{p}$, and*

$$s + \frac{p(N-q)}{(q-1)(N-p)}\delta < \frac{N(p-1) + pa + p}{N-p} \tag{5.9}$$

Then system (S) admits no G.S. and then there is a solution of the Dirichlet problem. The same happens by exchanging p, s, δ, a, γ with q, m, μ, b, ξ.

Proof. Consider the function $\mathcal{F}$ defined at (5.7). Suppose that there exists a G.S. Then from (5.1) and (5.9) we find

$$\frac{N+a}{s+1} - \frac{N-p}{p} - \frac{\delta Y}{s+1} > \frac{N+a}{s+1} - \frac{N-p}{p} - \frac{\delta}{s+1}\frac{N-q}{q-1} \geqq 0.$$

From (5.8), we deduce that $\mathcal{F}$ is nondecreasing. First suppose $s + 1 > p$. From (5.3) and (5.4),it follows that $u = O(r^{-k})$ at ∞, with $k = (p+a-\delta\frac{N-q}{q-1})/(s-p+1)$. In turn $r^{N-p}u^p = O(r^{(N-p)-kp}) = o(1)$ from (5.9), then $\mathcal{F}(r) = o(1)$ near ∞. Next assume $s + 1 \leqq p$ and $\gamma > \frac{N-p}{p}$. Then $r^{N-p}u^p = O(r^{N-p-\gamma p})$, hence $\mathcal{F}(r) = o(1)$ near ∞. In any case we get a contradiction. ■

6 The Hamiltonian system

Here we consider the nonnegative solutions of the variational Hamiltonian problem (SH) in $\Omega \subset \mathbb{R}^N$

$$(SH) \begin{cases} -\Delta u = |x|^a v^{\delta}, \\ -\Delta v = |x|^b u^{\mu}, \end{cases}$$

where $p = q = 2 < N$, $s = m = 0$, $a > b > -2$, and $D = \delta\mu - 1 > 0$. For this case we find

$$\gamma = \frac{(2+a)+(2+b)\delta}{D}, \quad \xi = \frac{2+b+(2+a)\mu}{D}, \quad \gamma + 2 + a = \delta\xi, \quad \xi + 2 + b = \mu\gamma.$$

The particular solution $(u_0(r), v_0(r)) = (Ar^{-\gamma}, Br^{-\xi})$ exists for $0 < \gamma < N-2$, $0 < \xi < N-2$. Here X, Y, Z, W are defined by

$$X(t) = \frac{r\,|u'|}{u}, \qquad Y(t) = \frac{r\,|v'|}{v}, \qquad Z(t) = \frac{r^{1+a}v^{\delta}}{|u'|}, \qquad W(t) = \frac{r^{1+b}u^{\mu}}{|v'|},$$

with $t = \ln r$, and system (M) becomes

$$(MH)\begin{cases} X_t = X\left[X - (N-2) + Z\right], \\ Y_t = Y\left[Y - (N-2) + W\right], \\ Z_t = Z\left[N + a - \delta Y - Z\right], \\ W_t = W\left[N + b - \mu X - W\right] \end{cases}$$

This system has a <u>*Pohozaev type*</u> function, well known at least in the case $a = b = 0$, given at (1.7):

$$\begin{aligned}\mathcal{E}_H(r) &= r^N\left[u'v' + r^b\frac{|u|^{\mu+1}}{\mu+1} + r^a\frac{|v|^{\delta+1}}{\delta+1} + \frac{N+a}{\delta+1}\frac{vu'}{r} + \frac{N+b}{\mu+1}\frac{uv'}{r}\right] \\ &= r^{N-2}uv\left[XY - \frac{Y(N+b-W)}{\mu+1} - \frac{(N+a-Z)X}{\delta+1}\right] \\ &= r^{N-2-\gamma-\xi}(ZX)^{(\mu+1)/D}(WY)^{(\delta+1)/D}\left[XY - \frac{Y(N+b-W)}{\mu+1} - \frac{(N+a-Z)X}{\delta+1}\right].\end{aligned}$$

It can also be found by a direct computation, and $\mathcal{E}_H$ satisfies

$$\mathcal{E}_H'(r) = r^{N-1}u'v'\left(\frac{N+a}{\delta+1} + \frac{N+b}{\mu+1} - (N-2)\right).$$

We define the <u>*critical case*</u> as the case where (δ, μ) lie on the hyperbola $\mathcal{H}_0$ given by

$$\frac{N+a}{\delta+1} + \frac{N+b}{\mu+1} = N-2, \text{ equivalently } \gamma + \xi = N-2. \tag{6.1}$$

In this case $\gamma = \frac{N+b}{\mu+1}, \xi = \frac{N+a}{\delta+1}$, and $\mathcal{E}_H'(r) \equiv 0$. It corresponds to the existence of a first integral of system (M), which can also be expressed in the variables U$= r^{\gamma}u$,V$= r^{\xi}v$ of Remark 2.2:

$$\mathcal{E}_H(r) = \mathrm{U}_t\mathrm{V}_t - \gamma\xi\mathrm{U}\mathrm{V} + \frac{\mathrm{U}^{\mu+1}}{\mu+1} + \frac{\mathrm{V}^{\delta+1}}{\delta+1} = C.$$

The supercritical case is defined as the case where (δ, μ) is above $\mathcal{H}$, equivalently $\gamma + \xi < N-2$ and the subcritical case corresponds to (δ, μ) under $\mathcal{H}$.

Remark 6.1 *The energy $\mathcal{E}_{H,0}$ of the particular solution associated to M_0 is always negative, given by $\mathcal{E}_{H,0} = -\frac{D}{(\mu+1)(\delta+1)}r^{N-2-\gamma-\xi}X_0Y_0(Z_0X_0)^{(\mu+1)/D}(W_0Y_0)^{(\delta+1)/D}$.*

Remark 6.2 *In the case $a=b=0$, it is known that there exists a solution of the Dirichlet problem in any bounded regular domain Ω of $\mathbb{R}^N$, see for example [15], [20]; for general a,b, some restrictions on the coefficients appear, see [23] and [14].*

Next consider the critical and supercritical cases. When $a=b=0$, there exists no solution if Ω is starshaped, see [36]. Here we show the existence of G.S. for general a,b. The existence in the critical case with $a=b=0$ was first obtained in [22], then in the supercritical case in [29], and uniqueness was proved in [20], [29]. The proofs of [29] are quite long due to regularity problems, when δ or $\mu<1$, which play no role in our quadratic system.

Remark 6.3 *The particular case $\delta=\mu$ and $a=b$ is easy to treat. Indeed in that case $u=v$ is a solution of the scalar equation $\Delta u+|x|^a|u|^{\delta-1}u=0$, for which the critical case is given by $\delta=(N+2+2a)/(N-2)$. Moreover if system (SH) admits a G.S., or a solution of the Dirichlet problem in a ball, it satisfies $u=v$, from [3]. Then we are completely reduced to the scalar case. In particular, in the critical case, the G.S. are given explicitely by: $u=v=c(K+r^{(2+a)})^{(2-N)/(2+a)}$, where $K=c^{\delta-1}/(N+a)(N-2)$; in other words they satisfy (3.3) with $X=Y$ and $Z=W$, i.e.*

$$\frac{X(t)}{N-2}+\frac{Z(t)}{N+a}-1=0.$$

Near ∞, the G.S. is (obviously) symmetrical: it joins the points N_0 and A_0.

Remark 6.4 *Consider the case $\delta=1$, $a=b=0$, which is the case of the biharmonic equation*

$$\Delta^2 u=u^\mu.$$

Recall that it is the only case where the conjecture (1.3) was completely proved by Lin in [21]. In the critical case $\mu=(N+4)/(N-4)$, the G.S. are also given explicitely, see [20]:

$$u(r)=c(K+r^2)^{(4-N)/2},\quad K=c^{\mu-1}/(N-4)(N-2)N(N+2).$$

They satisfy the relation $XY=\frac{N-Z}{2}X+\frac{N-4}{2N}(N-W)Y$, and moreover we find that they are on an hyperplane, of equation

$$\frac{(N-2)X(t)}{N(N-4)}+\frac{Z(t)}{N}-1=0.$$

Observe also that the G.S. is not symmetrical near ∞: u behaves like r^{4-N} and v behaves like r^{2-N}. The trajectory in the phase space joins the points N_0 and $Q_0=(N-4,N-2,2,0)$.

Proof of Theorem 1.4. 1) *Existence or nonexistence results:*

• In the supercritical or critical case we apply any of the two conditions of Theorem 1.1: Here $\mathcal{E}_H(0)=0$, and $\mathcal{E}_H$ is nonincreasing; there does not exist solutions of (M) such that at some time T, $X(T)=Y(T)=N-2$, because at the time T,

$$XY-\frac{Y(N+b-W)}{\mu+1}-\frac{(N+a-Z)X}{\delta+1}=(N-2)\left[N-2-\frac{N+a}{\delta+1}-\frac{N+b}{\mu+1}+\frac{W}{\mu+1}+\frac{Z}{\delta+1}\right]>0$$

since $W > 0, Z > 0$, thus $\mathcal{E}_H(e^T) > 0$, which is impossible. Otherwise there exists no solution of the Dirichlet problem in a ball $B(0,R)$, because $\mathcal{E}_H(R) = R^N u'(R)v'(R) > 0$ from the Höpf Lemma. Then there exists a G.S. The uniqueness is proved in [20].

• In the subcritical case there is no radial G.S.: it would satisfy $\mathcal{E}_H(0) = 0$, and $\mathcal{E}_H$ is nondecreasing, $\mathcal{E}_H(r) \leqq Cr^{N-2-\gamma-\xi}$ from (5.1), and $\gamma + \xi > (N-2)$, then $\lim_{r\to\infty} \mathcal{E}_H(r) = 0$. From Theorem 1.1, there exists a solution of the Dirichlet problem.

2) *Behaviour of the G.S. in the critical case.*

It is easy to see that the condition (1.6) implies $\mu > \frac{2+b}{N-2}$ and $\delta > \frac{2+a}{N-2}$, and that $\delta \leqq \frac{N+a}{N-2}$ and $\mu \leqq \frac{N+b}{N-2}$ cannot hold simultaneously. One can suppose that $\delta > \frac{N+a}{N-2}$. Let $\mathcal{T}$ be the unique trajectory of the G.S.. Then $\mathcal{E}_H(0) = 0$, thus $\mathcal{T}$ lies on the variety $\mathcal{V}$ of energy 0, defined by

$$\frac{X(N+a-Z)}{\delta+1} + \frac{Y(N+b-W)}{\mu+1} = XY. \tag{6.2}$$

From (5.2) $\mathcal{T}$ starts from the point N_0, and from (5.1) $\mathcal{T}$ stays in

$$\mathcal{A} = \left\{(X,Y,Z,W) \in \mathbb{R}^4 : 0 < X < N-2, \quad 0 < Y < N-2, \quad 0 < Z < N+a, \quad 0 < W < N+b\right\}.$$

(i) Suppose that $\mathcal{T}$ converges to a fixed point of the system in $\bar{\mathcal{R}}$. Then the only possible points are A_0, P_0, Q_0 which are effectively on $\mathcal{V}$. Indeed $I_0,\ J_0, G_0, H_0 \notin \mathcal{V}$. But $Q_0 = ((N-2)\delta - (2+a), N-2, N+a-(N-2)\delta, 0) \notin \bar{\mathcal{R}}$, since $\delta > \frac{N+a}{N-2}$. And $P_0 \in \bar{\mathcal{R}}$ if and only if $\mu \leqq \frac{N+b}{N-2}$.

If $\mu > \frac{N+b}{N-2}$, then $\mathcal{T}$ converges to A_0. If $\mu < \frac{N+b}{N-2}$, no trajectory converges to A_0, from Proposition 4.5, thus $\mathcal{T}$ converges to P_0. If $\mu \neq \frac{N+b}{N-2}$ the convergence is exponential, thus the behaviour of u, v follows. If $\mu = \frac{N+b}{N-2}$, then $\mathcal{T}$ converges converges to $A_{0,} = P_0$; the eigenvalues given by (10.3) satisfy $\lambda_1 = \lambda_2 = N-2$, $\lambda_3 = N+a-\delta(N-2) < 0$ and $\lambda_4 = 0$; the projection of the trajectory on the hyperplane $Y = N-2$ satisfies the system

$$X_t = X\left[X - (N-2) + Z\right], \qquad Z_t = Z\left[N+a-\delta(N-2) - Z\right]$$

which presents a saddle point at $(N-2, 0)$, thus the convergence of X and Z is exponential, in particular we deduce the behaviour of u. The trajectory enters by the central variety of dimension 1, and by computation we deduce that $Y - (N-2) = -t^{-1} + O(t^{-2+\varepsilon})$ near ∞, and the behaviour of v follows.

(ii) Let us show that $\mathcal{T}$ converges to a fixed point. We eliminate W from (6.2) and we get a still quadratic system in (X,Y,Z) :

$$\left\{\begin{array}{c} X_t = X\left[X - (N-2) + Z\right], \\ Y_t = Y\left[Y + b + 2 - (\mu+1)X\right] + \frac{\mu+1}{\delta+1}X(N+a-Z), \\ Z_t = Z\left[N+a-\delta Y - Z\right]. \end{array}\right. \tag{6.3}$$

We have $X_t \geqq 0$, and $Y_t \geqq 0$ near $-\infty$. Suppose that X has a maximum at t_0 followed by a minimum at t_1. At these times $X_{tt} = XZ_t$, thus we find $Z_t(t_0) < 0 < Z_t(t_1)$. There exists $t_2 \in (t_0, t_1)$ such that $Z_t(t_2) = 0$, and t_2 is a minimum. At this time $Z(t_2) = N+a-\delta Y(t_2)$, $Z_{tt}(t_2) = -\delta(ZY_t)(t_2)$ hence

$$Y_t(t_2) = Y(t_2)\left[Y(t_2) + b + 2 - \frac{\mu+1}{\delta+1}X(t_2)\right] < 0$$

and $X_t(t_2) < 0$, hence $(X+Z)(t_2) < N-2$, and

$$N-2-X(t_2) > Z(t_2) > N+a-\delta(\frac{\mu+1}{\delta+1}X(t_2)-b-2)$$

$$(a+2)+\delta(b+2) < (\delta\frac{\mu+1}{\delta+1}-1)X(t_2) = \frac{\delta(2+b)+(2+a)}{(N-2)\delta-(2+a)}X(t_2)$$

but $X(t_2) < X(t_0) < \delta(N-2)-(2+a)$, which is contradictory. Then X has at most one extremum, which is a maximum, and then it has a limit in $(0, N-2]$ at ∞. In the same way, by symmetry, Y has at most one extremum, which is a maximum, and has a limit in $(0, N-2]$ at ∞. Then Z has at most one extremum, which is a minimum. Indeed at the points where $Z_t = 0$, $-Z_{tt}$ has the sign of Y_t. Thus Z has a limit in $[0, N+a)$, similarly W has a limit in $[0, N+b)$. ■

Open problems: 1) For the case $\delta = \mu$, in the critical case it is well known that there exist solutions (u,v) of system (SH) of the form (u,u), such that $r^\gamma u$ is periodic in $t = \ln r$. They correspond to a periodic trajectory for the scalar system (M_{scal}) with $p=2$, and it admits an infinity of such trajectories. If $\delta \neq \mu$, does there exist solutions (u,v) such that $(r^\gamma u, r^\xi v)$ is periodic in t, in other words a periodic trajectory for system (MH)?

2) In the supercritical case, we cannot prove that the regular trajectory $\mathcal{T}$ converges to M_0, that means $\lim_{r\to\infty} r^\gamma u = A$, $\lim_{r\to\infty} r^\xi v = B$. Here $\mathcal{E}_H(0) = 0$, $\mathcal{E}_H$ is nonincreasing, then $\mathcal{E}_H$ is negative. The only fixed points of negative energy are M_0, G_0, H_0, but a G.S. satisfies (5.5), then it tends to $(0,0)$ at ∞, hence $\mathcal{T}$ cannot converge to G_0 or H_0 from Proposition 4.9; but we cannot prove that $\mathcal{T}$ converges to some fixed point.

7 A nonvariational system

Here we consider system (S) with $p=q=2, a=b$ and $s=m\neq 0$.

$$(SN)\left\{\begin{array}{l} -\Delta u = |x|^a\, u^s v^\delta, \\ -\Delta v = |x|^a\, u^\mu v^s, \end{array}\right.$$

where $D = \delta\mu-(1-s)^2 > 0$. In order to prove Theorem we can reduce the system to the case $a=0$, by changing N into $\hat{N} = \frac{2(N+a)}{2+a}$, from Remark 2.4; thus we assume $a=0$ in this Section. Here

$$X = -\frac{ru'}{u} \qquad Y = -\frac{rv'}{v}, \qquad Z(t) = -\frac{ru^s v^\delta}{u'}, \qquad W(t) = -\frac{ru^\mu v^s}{v'},$$

and system (M) becomes

$$(MN)\left\{\begin{array}{r} X_t = X\left[X-(N-2)+Z\right], \\ Y_t = Y\left[Y-(N-2)+W\right], \\ Z_t = Z\left[N-sX-\delta Y-Z\right], \\ W_t = W\left[N-\mu X-sY-W\right]. \end{array}\right.$$

We have chosen this system because it is not variational, and different hyperbolas in the plane (δ,μ), see fig. 3:

- the hyperbola $\mathcal{H}_s$ for which the linearized system at M_0 has two imaginary roots, given by

$$(\mathcal{H}_s) \qquad \frac{1}{\delta+1-s}+\frac{1}{\mu+1-s}=\frac{N-2}{N-(N-2)s}$$

whenever $s<\frac{N}{N-2}$, and $\delta+1-s>0$, $\mu+1-s>0$, from Proposition 4.3;

- the hyperbola $\mathcal{H}_0$ defined by

$$(\mathcal{H}_0) \qquad \frac{1}{\delta+1}+\frac{1}{\mu+1}=\frac{N-2}{N}; \tag{7.1}$$

it was shown in [26] that above $\mathcal{H}_0$ there exists no solution of the Dirichlet problem;

- an hyperbola $\mathcal{Z}_s$ introduced in [38] in case $s<\frac{N}{N-2}$, and $\min(\delta,\mu)>|s-1|$:

$$(\mathcal{Z}_s) \qquad \frac{1}{\delta+1}+\frac{1}{\mu+1}=\frac{N-2}{N-(N-2)s}, \tag{7.2}$$

- we introduce the new curve $\mathcal{C}_s$ defined <u>*for any* $s>0$</u> by

$$(\mathcal{C}_s) \qquad \frac{N}{\mu+1}+\frac{N}{\delta+1}=N-2+\frac{(N-2)s}{2}\min\left(\frac{1}{\mu+1},\frac{1}{\delta+1}\right),$$

We first extend and complete the results of [38] and [26]:

Proposition 7.1 *(i) Assume* $s<\frac{N}{N-2}$*, and* $\delta+1-s>0$, $\mu+1-s>0$*. Under the hyperbola* $\mathcal{Z}_s$*, system* (SN) *admits no G.S., and then there is a solution of the Dirichlet problem in a ball.*

(ii) Above $\mathcal{H}_0$ *there exists no solution of the Dirichlet problem. Thus there exists a G.S.*

Proof. (i) We consider an energy function with parameters $\alpha,\beta,\sigma,\theta$:

$$\mathcal{E}_N(r)=r^N\left[u'v'+\alpha u^{\mu+1}v^s+\beta v^{\delta+1}u^s+\frac{\sigma}{r}vu'+\frac{\theta}{r}uv'\right] \tag{7.3}$$

$$=r^{N-2}uv\Psi_0=r^{N-2-\gamma-\xi}(ZX)^{\xi/2}(WZ)^{\gamma/2}\Psi_0, \tag{7.4}$$

from (4.2), where

$$\Psi_0(X,Y,Z,W)=XY+\alpha WY+\beta ZX-\sigma X-\theta Y. \tag{7.5}$$

We get

$$\begin{aligned} r^{1-N}(uv)^{-1}\mathcal{E}_N'(r)&=(\sigma+\theta-(N-2))XY+(N\alpha-\theta)YW+(N\beta-\sigma)XZ\\ &\quad-(\alpha(\mu+1)-1)XYW-(\beta(\delta+1)-1)XYZ-\alpha sY^2W-\beta sX^2Z. \end{aligned}$$

Taking $\alpha=\frac{1}{\mu+1},\beta=\frac{1}{\delta+1}$, we find

$$r^{3-N}(uv)^{-1}\mathcal{E}_N'(r)=(\sigma+\theta-(N-2))XY+(N\alpha-\theta-\alpha sY)YW+(N\beta-\sigma-\beta sX)XZ. \tag{7.6}$$

If there exists a G.S., from (5.1) it satisfies $X,Y<N-2$, hence

$$r^{3-N}(uv)^{-1}\mathcal{E}_N'(r)>(\sigma+\theta-(N-2))XY+((N-(N-2)s)\alpha-\theta)YW+((N-(N-2)s)\beta-\sigma)XZ. \tag{7.7}$$

Taking $\theta = \frac{N-(N-2)s}{\mu+1}, \sigma = \frac{N-(N-2)s}{\delta+1}$, we deduce that $\mathcal{E}_N' > 0$ under $\mathcal{Z}_s$. Moreover $\mathcal{Z}_s$ is under $\mathcal{H}_s$, thus $\gamma + \xi > N-2$. Then $\mathcal{E}_N(r) = O(r^{N-2-\gamma-\xi})$ tends to 0 at ∞, which is contradictory.

(ii) Taking $\alpha = \frac{1}{\mu+1} = \frac{\theta}{N}, \beta = \frac{1}{\delta+1} = \frac{\sigma}{N}$, it comes from (7.6)

$$r^{3-N}(uv)^{-1}\mathcal{E}_N'(r) = (\frac{N}{\delta+1} + \frac{N}{\mu+1} - (N-2))XY - \alpha s Y^2 W - \beta s X^2 Z$$

hence $\mathcal{E}_N' < 0$ when (7.1) holds. At the value R where $u(R) = v(R) = 0$, we find $\mathcal{E}_N(R) = R^N u'(R) v'(R) > 0$, which is a contradiction. ■

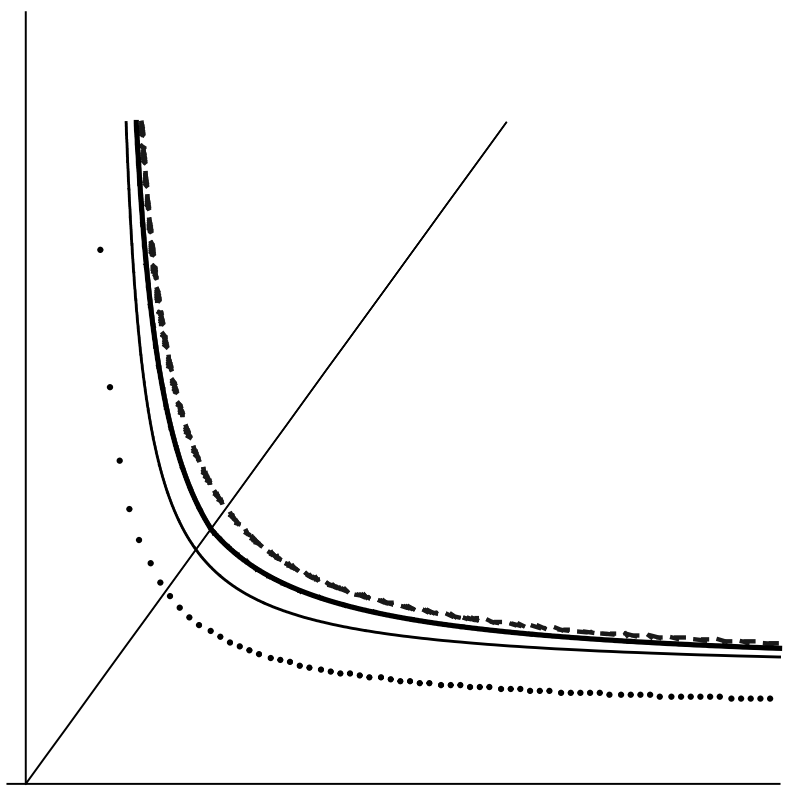

Figure 3 : The level curves for system (SN) — — — curve $\mathcal{H}_0$ —— curve $\mathcal{H}_s$ —— curve $\mathcal{C}_s$ ...... curve $\mathcal{Z}_s$

Remark 7.2 *(i) When the four curves are simultaneously defined, they are in the following order, from below to above: $\mathcal{Z}_s, \mathcal{H}_s, \mathcal{C}_s, \mathcal{H}_0$. They intersect the diagonal $\delta = \mu$ repectively for*

$$\delta = \frac{N+2}{N-2} - 2s, \quad \delta = \frac{N+2}{N-2} - s, \quad \delta = \frac{N+2}{N-2} - \frac{s}{2}, \quad \delta = \frac{N+2}{N-2}.$$

(ii) For $\delta = \mu$, system (SN) has a G.S. for $\delta \geqq \frac{N+2}{N-2} - s$. Indeed it admits solutions of the form (U,U), where U is a solution of equation $-\Delta U = U^{s+\delta}$. Suppose moreover $s \leqq \delta$. If $1 - s < \delta < \frac{N+2}{N-2} - s$, then there exists no G.S; indeed all such solutions satisfy $u = v$, from [3, Remark 3.3].

Then the point $P_s = \left(\frac{N+2}{N-2} - s, \frac{N+2}{N-2} - s\right)$ *appears to be the separation point on the diagonal; notice that* $P_s \in \mathcal{H}_s$.

Next we prove our main existence result of existence of a G.S. valid without restrictions on s. The main idea is to introduce a *new energy function* Φ *by adding two terms in* X^2 *and* Y^2 to the energy $\mathcal{E}_N$ defined at (7.3). It is constructed in order that Φ' does not contain Y and Z. Then we consider the set of couples (X, Y) such that Φ' has a sign, which is bounded by a cubic curve. When (δ, μ) is above $\mathcal{C}_s$, the cubic curve is exterior to the square

$$K = [0, N-2] \times [0, N-2], \tag{7.8}$$

and then we can apply Theorem 1.1.

Proof of Theorem 1.5. From Theorem 1.1, if $s \geqq \frac{N+2}{N-2}$, all the regular solutions are G.S.. Thus we can assume $s < \frac{N+2}{N-2}$. Let $j, k \in \mathbb{R}$ be parameters, and

$$\begin{aligned}
\Phi(r) &= \mathcal{E}_N(r) + r^N \left[k\frac{s}{2}\frac{vu'^2}{u} + j\frac{s}{2}\frac{uv'^2}{v}\right] \\
&= r^N \left[u'v' + \alpha u^{\mu+1}v^s + \beta v^{\delta+1}u^s + \frac{\sigma}{r}vu' + \frac{\theta}{r}uv' + k\frac{s}{2}\frac{vu'^2}{u} + j\frac{s}{2}\frac{uv'^2}{v}\right] \\
&= r^{N-2}uv\Psi = r^{N-2-\gamma-\xi}(ZX)^{\xi/2}(WY)^{\gamma/2}\Psi,
\end{aligned}$$

where

$$\Psi(X, Y, Z, W) = XY + \alpha WY + \beta ZX - \sigma X - \theta Y + k\frac{s}{2}X^2 + j\frac{s}{2}Y^2.$$

Then

$$\begin{aligned}
r^{3-N}(uv)^{-1}\Phi'(r) &= (\sigma + \theta - (N-2))XY + (N\alpha - \theta)YW + (N\beta - \sigma)XZ \\
&\quad - (\alpha(\mu+1) - 1)XYW - (\beta(\delta+1) - 1)XYZ + (j - \alpha)sY^2W + (k - \beta)sX^2Z \\
&\quad + ksX^2[X - (N-2)] + jsY^2[Y - (N-2)] + (N - 2 - X - Y)(k\frac{s}{2}X^2 + j\frac{s}{2}Y^2).
\end{aligned}$$

We eliminate the terms in Z, W by taking $j = \alpha = \frac{1}{\mu+1}$, $k = \beta = \frac{1}{\delta+1}$, $\theta = N\alpha$, $\sigma = N\beta$. Then we get the function Φ defined at (1.9). Computing its derivative, we obtain after reduction

$$\begin{aligned}
\mathcal{B}(X, Y) &:= -\frac{2}{s}r^{3-N}(uv)^{-1}\Phi'(r) \\
&= \beta X^2(N - 2 - X) + \alpha Y^2(N - 2 - Y) + XY\left[\beta X + \alpha Y + \frac{2}{s}(N - 2 - N\alpha - N\beta)\right].
\end{aligned}$$

From Proposition 7.1 we can assume that $N(\alpha + \beta) - (N - 2) > 0$. We determine the sign of $\mathcal{B}$ on the boundary ∂K of the square K defined at (7.8). We have $\mathcal{B}(0, Y) = \alpha Y^2(N - 2 - Y) \geqq 0$ and $\mathcal{B}(X, 0) = \beta X^2(N - 2 - X) \geqq 0$. In particular $\mathcal{B}(0, 0) = 0$. Otherwise $\mathcal{B}(N - 2, Y) = Y\Theta(Y)$ with

$$\Theta(Y) = \alpha Y[2(N-2) - Y] + (N-2)((N-2)\beta + \frac{2}{s}(N - 2 - N\alpha - N\beta).$$

On the interval $[0, N-2]$, there holds $\Theta(Y) > \Theta(0)$. By hypothesis, (δ, μ) is above $\mathcal{C}_s$, or equivalently

$$(\alpha+\beta)\frac{N}{N-2} - 1 \leqq \frac{s}{2}\min(\alpha,\beta); \tag{7.9}$$

consequently $\mathcal{B}(N-2, Y) \geqq 0$ and similarly $\mathcal{B}(X, N-2) \geqq 0$. Then $\mathcal{B}$ is nonnegative on ∂K and is zero at $(0,0),(0,N-2),(N-2,0)$. The curve $\mathcal{B}(X,Y) = 0$ is a cubic with a double point at $(0,0)$, which is isolated under the condition (7.9): $\mathcal{B}(X,Y) > 0$ near $(0,0)$, except at this point. Then $\mathcal{B}(X,Y) > 0$ on the interior of K.

Suppose that there exists a regular solution such that $X(T) = Y(T) = N-2$ at the same time T. Indeed up to this time (X,Y) stays in K, thus the function Φ is decreasing. We have $\Phi(0) = 0$, and at the value $R = e^T$, we find

$$\Phi(R) = R^{N-2-\gamma-\xi}(N-2)^{\xi+\gamma+2}\left[\frac{\alpha W + \beta Z}{N-2} + 1 - (\beta+\alpha)(\frac{N}{N-2} - \frac{s}{2})\right]$$

then $\Phi(R) > 0$, since $\min(\alpha,\beta) < \alpha+\beta$. Therefore from Theorem 1.1, there exists a G.S. ∎

Remark 7.3 *We wonder if the limit curve for existence of G.S. would be $\mathcal{H}_s$, or another curve $\mathcal{L}_s$ defined by*

$$(\mathcal{L}_s) \qquad \frac{1}{\delta+1} + \frac{1}{\mu+1} = \frac{N-2}{N - \frac{(N-2)s}{2}},$$

which ensures that $\Phi(R) > 0$, and also $\mathcal{B}(N-2, N-2) > 0$. This curve cuts the diagonal at the same point $P_s = \left(\frac{N+2}{N-2} - s, \frac{N+2}{N-2} - s\right)$ as $\mathcal{H}_s$. Notice that $\mathcal{L}_s$ is under $\mathcal{H}_s$.

8 The radial potential system

Here we study the nonnegative radial solutions of system (SP) :

$$(SP)\begin{cases} -\Delta_p u = |x|^a\, u^s v^{m+1}, \\ -\Delta_q v = |x|^a\, u^{s+1} v^m, \end{cases}$$

with $a = b, \delta = m+1, \mu = s+1$, and we assume (1.5). System (M) becomes

$$(MP)\begin{cases} X_t = X\left[X - \frac{N-p}{p-1} + \frac{Z}{p-1}\right], \\ Y_t = Y\left[Y - \frac{N-q}{q-1} + \frac{W}{q-1}\right], \\ Z_t = Z\left[N + a - sX - (m+1)Y - Z\right], \\ W_t = W\left[N + b - (s+1)X - mY - W\right]. \end{cases}$$

For this system D, γ and ξ are defined by

$$D = p(1+m) + q(1+s) - pq, \qquad (p-1-s)\gamma + p + a = (m+1)\xi, \qquad (q-1-m)\xi + q + b = (s+1)\gamma,$$

thus γ and ξ are linked independtly of s, m by the relation

$$p(\gamma+1) = q(\xi+1) = \frac{pq(m+s+2+a)}{D}. \tag{8.1}$$

The system is *variational*. It admits an energy function, given at (1.13), which can also can be obtained by a direct computation in terms of X, Y, Z, W:

$$\mathcal{E}_P(r) = \psi\left[ZW - \frac{s+1}{p}W((N-p)-(p-1)X) - \frac{m+1}{q}Z((N-q)-(q-1)Y)\right], \tag{8.2}$$

where

$$\psi = \frac{r^{N-2-a}\left|u'\right|^{p-1}\left|v'\right|^{q-1}}{u^s v^m} = r^{N-(\gamma+1)p}\left[X^{q(s+1)(p-1)}Y^{p(m+1)(q-1)}Z^{p(q-m-1)}W^{q(p-s-1)}\right]^{1/D}.$$

Then we find

$$\mathcal{E}_P'(r) = (N+a-(s+1)\frac{N-p}{p} - (m+1)\frac{N-q}{q})r^{N-1+a}u^{s+1}v^{m+1}.$$

Thus we define a critical line $\mathcal{D}$ as the set of $(\delta,\mu) = (m+1, s+1)$ such that

$$N+a = (m+1)\frac{N-q}{q} + (s+1)\,\frac{N-p}{p}, \tag{8.3}$$

equivalent to $pq(m+s+2+a) = ND$, or $N+a = (m+1)\xi + (s+1)\gamma$, or

$$(\gamma,\xi) = (\frac{N-p}{p}, \frac{N-q}{q})$$

The subcritical case is given by the set of points under $\mathcal{D}$, equivalently $\gamma > \frac{N-p}{p}$, $\xi > \frac{N-q}{q}$ or $(s+1)\,\gamma + (m+1)\xi > N+a$. The supercritical case is the set of points above $\mathcal{D}$.

Remark 8.1 *The energy* $\left(\mathcal{E}_{\mathcal{P}}\right)_0$ *of the particular solution associated to* M_0 *is still negative:* $\left(\mathcal{E}_P\right)_0 = -\frac{D}{pq}r^{N+a-(\gamma+1)p}\left[X_0^{q(p-1)}Y_0^{p(q-1)}Z_0^{q(s+1)}W_0^{p(m+1)}\right]^{1/D}$.

Remark 8.2 *When* $p = q = 2$, *another energy function can be associated to the transformation given at Remark 2.2: the system (2.9) relative to* $u(r) = r^{-\gamma}\,U(t), \quad v(r) = r^{-\xi}\,V(t)$ *is*

$$\begin{cases} \mathrm{U}_{tt} + (N-2-2\gamma)\mathrm{U}_t - \gamma(N-2-\gamma)\mathrm{U} + \mathrm{U}^s\mathrm{V}^{m+1} = 0 \\ \mathrm{V}_{tt} + (N-2-2\gamma)\mathrm{V}_t - \gamma(N-2-\gamma)\mathrm{V} + \mathrm{U}^{s+1}\mathrm{V}^m = 0 \end{cases} \tag{8.4}$$

and the function

$$E_P(t) = \frac{s+1}{2}(\mathrm{U}_t^2 - \gamma(N-2-\gamma)\mathrm{U}^2) + \frac{m+1}{2}(\mathrm{V}_t^2 - \gamma(N-2-\gamma)\mathrm{V}^2 + \mathrm{U}^{s+1\mathrm{V}m+1} \tag{8.5}$$

satisfies

$$(E_P)_t = -(N-2-2\gamma)\left[(s+1)\mathrm{U}_t^2 + (m+1)\mathrm{V}_t^2\right]$$

It differs from $\mathcal{E}_P$, *even in the critical case. This point is crucial for Section 9.*

It has been proved in [34], [35], that in the subcritical case with $a = 0$, there exists a solution of the Dirichlet problem in any bounded regular domain Ω of $\mathbb{R}^N$; and in the supercritical case there exists no solution if Ω is starshaped. Here we prove two results of existence or nonexistence of G.S. which seem to be new:

Proof of Theorem 1.6. 1) *Existence or nonexistence results.*

• In the supercritical or critical case there exists a G.S. From Theorem 1.1, if it were not true, then there would exist regular positive solutions of (MP) such that $X(T) = \frac{N-p}{p-1}$ and $Y(T) = \frac{N-q}{q-1}$. It would satisfy $\mathcal{E}_P \leqq 0$. Then at time T, we find $\mathcal{E}_P(R) > 0$, from (8.2), since $W > 0, Z > 0$, which is impossible.

• In the subcritical case, there exists no G.S. Suppose that there exists one. Now $\mathcal{E}_P$ is nondecreasing, hence $\mathcal{E}_P \geqq 0$. Its trajectory stays in the box $\mathcal{A}$ defined by (5.1), thus it is bounded. If $q \geqq m+1$ and $p \geqq s+1$, we deduce that , $\mathcal{E}_P(r) = O(r^{N-(\gamma+1)p})$ from (8.2), then $\mathcal{E}_P$ tends to 0 at ∞, which is contradictory. Next consider the general case. We have

$$\begin{aligned} \mathcal{E}_P(r) &\leqq r^{N-(\gamma+1)p} \left[X^{q(p-1)} Y^{p(q-1)} Z^{p(q-m-1)} W^{q(p-s-1)} \right]^{1/D} ZW \\ &= r^{N-(\gamma+1)p} \left[X^{q(p-1)} Y^{p(q-1)} Z^{q(1+s)} W^{p(1+m)} \right]^{1/D}, \end{aligned}$$

then the same result holds. Consequently, from Theorem 1.1, there exists a solution of the Dirichlet problem

2) *Behaviour of the G.S. in the critical case.*

Let $\mathcal{T}$ be the trajectory of a G.S.; then $\mathcal{E}_P(0) = 0$, thus $\mathcal{T}$ lies on the variety $\mathcal{V}$ of energy 0, also defined by

$$qW\left[(s+1)((p-1)X - (N-p)) + pZ\right] = p(m+1)Z\left[(N-q) - (q-1)Y)\right] \tag{8.6}$$

and $Y < \frac{N-q}{q-1}$, hence $(s+1)((p-1)X - (N-p)) + pZ > 0$. From (5.2), $\mathcal{T}$ starts from $N_0 = (0, 0, N+a, N+b)$ and stays in $\mathcal{A}$. Eliminating W in system (M), we find a system of three equations

$$\begin{cases} X_t = X\left[X - \frac{N-p}{p-1} + \frac{Z}{p-1}\right], \\ Y_t = YF, \\ Z_t = Z\left[N + a - sX - (m+1)Y - Z\right], \end{cases}$$

where

$$F(X,Y,Z) = \frac{1}{q}\left[\frac{N-q}{q-1} - Y\right] \frac{p(m+1-q)Z + q(s+1)((N-p) - (p-1)X)}{(s+1)((p-1)X - (N-p)) + pZ}.$$

(i) If $\mathcal{T}$ converges to a fixed point of the system in $\bar{\mathcal{R}}$, the possible points on $\mathcal{V}$ are $A_0, I_0, J_0, P_0, Q_0, G_0, H_0, R_0, S_0$. The eigenvalues of the linearized problem at A_0, given by (10.3) satisfy

$$\lambda_1, \lambda_2 > 0, \lambda_3 = N + a - s\frac{N-p}{p-1} - (m+1)\frac{N-q}{q-1} \leqq \lambda_4 = \lambda^* = N + a - (s+1)\frac{N-p}{p-1} - m\frac{N-q}{q-1},$$

since $q \leqq p$, and $\lambda_3 < \lambda^*$ for $q \neq p$, and $\lambda_3 = \lambda^* < 0$ for $q = p$, from (8.3). Then A_0 can be attained only when $\lambda^* \leqq 0$, from Proposition 4.5. And P_0 can be attained only if

$$q > m+1, \; \lambda^* \geqq 0 \; \text{ and } \; q + a < (s+1)\frac{N-p}{p-1}, \tag{8.7}$$

from Proposition 4.6, because $\gamma = \frac{N-p}{p} < \frac{N-p}{p-1}$. We observe that the condition $\lambda^* \geqq 0$ joint to (8.3) implies $m+1 < q < p$ and is equivalent to (8.7). Indeed it implies

$$\frac{N-p}{p-1}(s+1) \leqq N + a - m\frac{N-q}{q-1} = N + a - \frac{q}{q-1}(N + a - \frac{N-q}{q} - (s+1)\,\frac{N-p}{p});$$

then

$$(s+1)\,\frac{N-p}{p-1}\frac{q-p}{p} \leqq -(a+q),$$

thus $q < p$. From (8.3) we obtain

$$(N-q)(\frac{m+1}{q} - 1) = q + a - (s+1)\frac{N-p}{p} \leqq (s+1)\,\frac{N-p}{p}(\frac{p-q}{p-1} - 1) < 0,$$

hence $m+1 < q$ and (8.7) follows. By symmetry, Q_0 cannot be attained since $q \leqq p$. Then A_0 and P_0 are incompatible, unless $A_0 = P_0$, and P_0 is not attained when $p = q$.

(ii) Next we show that $\mathcal{T}$ converges to A_0 or to P_0. If t is an extremum value of Y, then

$$(\frac{m+1}{q} - 1)Z(t) + \frac{s+1}{p}((N-p) - (p-1)X(t)) = 0. \tag{8.8}$$

This relation implies $q > m+1$ and

$$X_t(t) = \frac{X(t)Z(t)}{p-1}\left[1 + \frac{p(m+1-q)}{q(s+1)}\right] = \frac{DX(t)Z(t)}{(p-1)q(s+1)} > 0.$$

In the same way, if t is an extremum value of X, then $p > s+1$ and $Y_t(t) > 0$. Near $-\infty$, there holds $X_t, Y_t \geqq 0$, and $Z_t, W_t \leqq 0$, from the linearization near N_0. Suppose that X has a maximum at t_0 followed by a minimum at t_1. Then $p > s+1$, and Y is increasing on $[t_0, t_1]$. At time t_0 we have $(p-1)X(t_0) + Z(t_0) = N - p$ and $X_{tt}(t_0) \leqq 0$, thus $Z_t(t_0) \leqq 0$; eliminating Z we deduce $p + a + (p-1-s)X(t_0) \leqq (m+1)Y(t_0)$ and similarly $(m+1)Y(t_1) \leqq p + a + (p-1-s)X(t_1)$; hence $Y(t_1) < Y(t_0)$, which is a contradiction. Thus X and Y can have at most one maximum, and in turn they have no maximum point. Therefore X and Y are increasing, and they are bounded, hence X has a limit in $\left(0, \frac{N-p}{p-1}\right]$ and Y has a limit in $\left(0, \frac{N-q}{q-1}\right]$. Then Z, W are decreasing; indeed at each time where $Z_t = 0$, we have $Z_{tt} = Z(-sX_t - (m+1)Y_t) < 0$, thus it is a maximum, which is impossible.

Then $\mathcal{T}$ converges to a fixed point of the system. Moreover, since X and Y are increasing, it cannot be one of the points $I_0, J_0, G_0, H_0, R_0, S_0$. It is necessarily A_0 or P_0. We distinguish two cases:

- Case $q \leqq m+1$. Then $\mathcal{T}$ converges to A_0, and $\lambda_3, \lambda^* < 0$, then (1.10) follows.

• Case $q > m+1$. Then $\mathcal{T}$ converges to A_0 (resp. P_0) when $\lambda^* \leqq 0$ (resp. $\lambda^* \geqq 0$). If the inequalities are strict, we deduce the convergence of u and v from Propositions 4.5 and 4.6, and (1.11) follows. If $\lambda^* = 0$, then $P_0 = A_0$, and $\lambda_3 = \frac{(N-1)(q-p)}{(p-1)(q-1)} < 0$. The projection of the trajectory $\mathcal{T}$ in $\mathbb{R}^3$ on the plane $Y = \frac{N-q}{q-1}$ satisfies the system

$$X_t = X\left[X - \frac{N-p}{p-1} + \frac{Z}{p-1}\right], \qquad Z_t = Z\left[N + a - sX - (m+1)\frac{N-q}{q-1} - Z\right]$$

which presents a saddle point at $(\frac{N-p}{p-1}, 0)$, thus the convergence of X and Z is exponential, in particular we deduce the behaviour of u. The trajectory enters by the central variety of dimension 1, and by computation we deduce that $Y = \frac{N-q}{q-1} - \frac{1}{q-1-m}t^{-1} + O(t^{-2+\varepsilon})$, then (1.12) follows. ■

9 The nonradial potential system of Laplacians

Here we study the possibly <u>*nonradial*</u> solutions of the system of the preceeding Section when $p = q = 2$:

$$(SL)\begin{cases} -\Delta u = |x|^a u^s v^{m+1}, \\ -\Delta v = |x|^a u^{s+1} v^m, \end{cases}$$

with $D = s + m$. We solve an open problem of [7]: the nonexistence of (radial or nonradial) G.S. under condition (1.14).

It was shown in [7] in the case $N + a \geqq 4$. The problem was open when $N + a < 4$, and $m+s+1 > (N+a)/(N-2)$, which implies $N < 6$. Indeed in the case $m+s+1 \leqq (N+a)/(N-2)$, there are no solutions of the exterior problem, see [6, Theorem 5.3]. Recall that the main result of [7] is the obtention of apriori estimates near 0 or ∞, by using the Bernstein technique introduced in [18] and improved in [8]. Then the behaviour of the solutions is obtained by using the change of unknown

$$u(r,\theta) = r^{-\gamma}\mathrm{U}(t,\theta), \qquad v(r,\theta) = r^{-\gamma}\mathrm{V}(t,\theta), \qquad t = \ln r,$$

extending the transformation of Remark 8.2 to the nonradial case (in fact here t is $-t$ in [7]); it leads to the system

$$\begin{aligned} &\mathrm{U}_{tt} + (N-2-2\gamma)\mathrm{U}_t + \Delta_S \mathrm{U} - \gamma(N-2-\gamma)\mathrm{U} + \mathrm{U}^s\mathrm{V}^{m+1} = 0, \\ &\mathrm{V}_{tt} + (N-2-2\gamma)\mathrm{V}_t + \Delta_S \mathrm{V} - \gamma(N-2-\gamma)\mathrm{V} + \mathrm{U}^{s+1}\mathrm{V}^m = 0, \end{aligned}$$

where Δ_S is the Laplace-Beltrami operator on S_{N-1}. A corresponding energy is introduced in [7]:

$$\begin{aligned} E_L(t) = {}& \frac{s+1}{2}\int_{S^{N-1}} (\mathrm{U}_t^2 - |\nabla_S \mathrm{U}|^2 - \gamma(N-2-\gamma)\mathrm{U}^2)d\theta \\ &+ \frac{m+1}{2}\int_{S^{N-1}} (\mathrm{V}_t^2 - |\nabla_S \mathrm{V}|^2 - \xi(N-2-\xi)\mathrm{V}^2)d\theta + \int_{S^{N-1}} \mathrm{U}^{s+1}\mathrm{V}^{m+1}d\theta, \end{aligned}$$

extending (8.5) to the nonradial case; it satisfies

$$(E_L)_t = -(N-2-2\gamma)\int_{S^{N-1}} \left[(s+1)\mathrm{U}_t^2 + (m+1)\mathrm{V}_t^2\right] d\theta$$

Here we construct <u>*another energy function*</u>, extending the Pohozaev function defined at (1.13) to the nonradial case.

Lemma 9.1 *Consider the function $\mathcal{E}_L(r)$ defined by*

$$
\begin{aligned}
r^{-N}\mathcal{E}_L(r) &= \frac{s+1}{2}\int\limits_{S^{N-1}}\left[u_r^2 - r^{-2}\left|\nabla_S u\right|^2 + (N-2)\frac{uu_r}{r}\right]d\theta \\
&\quad + \frac{m+1}{2}\int\limits_{S^{N-1}}\left[(\frac{\partial v}{\partial \nu})^2 - r^{-2}\left|\nabla_S v\right|^2 + (N-2)\frac{vv_r}{r}\right]d\theta + r^a\int\limits_{S^{N-1}} u^{s+1}v^{m+1}d\theta.
\end{aligned}
$$

Then the following relation holds:

$$
r^{1-N}\mathcal{E}_L'(r) = (N+a-(s+1)\frac{N-2}{2} - (m+1)\frac{N-2}{2})r^a\int\limits_{S^{N-1}} u^{s+1}v^{m+1}d\theta.
$$

Proof. In terms of t, we find

$$
\begin{aligned}
&\mathcal{E}_L(t) = \mathcal{E}_{L,1}(t) + \mathcal{E}_{L,2}(t) + \mathcal{E}_{L,3}(t), \text{ with} \\
&\mathcal{E}_{L,1}(t) = \frac{s+1}{2}e^{(N-2)t}\int\limits_{S^{N-1}}\left[u_t^2 - \left|\nabla_S u\right|^2 + (N-2)uu_t\right]d\theta, \\
&\mathcal{E}_{L,2}(t) = \frac{m+1}{2}e^{(N-2)t}\int\limits_{S^{N-1}}\left[v_t^2 - \left|\nabla_S v\right|^2 + (N-2)vv_t\right]d\theta, \quad \mathcal{E}_{L,3}(t) = e^{(N+a)t}\int\limits_{S^{N-1}} u^{s+1}v^{m+1}d\theta,
\end{aligned}
$$

and u satisfies the equations

$$
u_{tt} + (N-2)u_t + \Delta_S u + e^{(2+a)t}u^s v^{m+1} = 0, \tag{9.1}
$$

$$
(e^{(N-2)t}u_t)_t + e^{(N-2)t}\Delta_S u + e^{(N+a)t}u^s v^{m+1} = 0, \tag{9.2}
$$

and v satisfies symmetrical equations. Multiplying (9.2) by u and (9.1) by $(s+1)e^{(N-2)t}u_t$, we obtain

$$
\begin{aligned}
0 &= \int\limits_{S^{N-1}} u(e^{(N-2)t}u_t)_t + e^{(N-2)t}\int\limits_{S^{N-1}} u\Delta_S u + e^{(N+a)t}\int\limits_{S^{N-1}} u^{s+1}v^{m+1} \\
&= \frac{d}{dt}\int\limits_{S^{N-1}} ue^{(N-2)t}u_t - e^{(N-2)t}\int\limits_{S^{N-1}} (u_t^2 + |\nabla_S u|^2) + e^{(N+a)t}\int\limits_{S^{N-1}} u^{s+1}v^{m+1}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d}{dt}\int\limits_{S^{N-1}} \frac{s+1}{2}(N-2)ue^{(N-2)t}u_t - \frac{s+1}{2}(N-2)e^{(N-2)t}\int\limits_{S^{N-1}} (u_t^2 + |\nabla_S u|^2) \\
&= -\frac{s+1}{2}(N-2)e^{(N+a)t}\int\limits_{S^{N-1}} u^{s+1}v^{m+1},
\end{aligned}
$$

and symmetrically for v, and adding the equalities we deduce

$$
\begin{aligned}
0 &= (s+1)(e^{(N-2)t}\frac{d}{dt}\int\limits_{S^{N-1}}(\frac{u_t^2-|\nabla_S u|^2}{2}+(N-2)e^{(N-2)t}\int\limits_{S^{N-1}}u_t^2 \\
&+(m+1)(e^{(N-2)t}\frac{d}{dt}\int\limits_{S^{N-1}}\frac{v_t^2-|\nabla_S v|^2}{2}+(N-2)e^{(N-2)t}\int\limits_{S^{N-1}}v_t^2 \\
&+\frac{d}{dt}(e^{(N+a)t}\int\limits_{S^{N-1}}u^{s+1}v^{m+1})-(N+a)e^{(N+a)t}\int\limits_{S^{N-1}}u^{s+1}v^{m+1}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{e^{(N-2)t}}{2}\int\limits_{S^{N-1}}((s+1)(u_t^2-|\nabla_S u|^2)+(m+1)(v_t^2-|\nabla_S v|^2))+e^{(N+a)t}\int\limits_{S^{N-1}}u^{s+1}v^{m+1}\right] \\
&+\frac{N-2}{2}e^{(N-2)t}\int\limits_{S^{N-1}}((s+1)(u_t^2+|\nabla_S u|^2)+(m+1)(v_t^2+|\nabla_S v|^2)) \\
&=(N+a)e^{(N+a)t}\int\limits_{S^{N-1}}u^{s+1}v^{m+1},
\end{aligned}
$$

hence

$$
(\mathcal{E}_L)_t(t)=(N+a-(s+1)\frac{N-2}{2}-(m+1)\frac{N-2}{2})e^{(N+a)t}\int\limits_{S^{N-1}}u^{s+1}v^{m+1}d\theta.
$$

■

Proof of Theorem 1.7. Suppose that there exists a G.S. Since $s+m+1<(N+2+2a)/(N-2)$ we deduce that E_L and $\mathcal{E}_L$ are increasing and start from 0, then they stay positive. From [7, Corollary 6.4], since $s+m+1<(N+2)/(N-2)$, three eventualities can hold. The first one is that (u,v) behaves like the particular solution (u_0,v_0); it cannot hold because E_L has a negative limit, see [7, Remark 6.3]. The second one is that (u,v) is regular at ∞, that means $\lim_{|x|\to\infty}|x|^{N-2}u=\alpha>0$, $\lim_{|x|\to\infty}|x|^{N-2}v=\beta>0$; it cannot hold because $\lim_{t\to\infty}E_L(t)=0$. It remains a third eventuality: when for example $m>(N+a)/(N-2)$, and (u,v) has the following behaviour at ∞ :

$$
\lim_{r\to\infty}u=\alpha>0,\ \text{and}\ \lim_{|x|\to\infty}|x|^k v=\beta>0 \text{ or } 0,\qquad \text{with } k=(2+a)/(m-1). \tag{9.3}
$$

The condition on m implies that $N<4-a$ from assumption (1.14). In that case $\lim_{t\to\infty}E_L(t)=\infty$, which gives no contradiction. Here we show that a contradiction holds by using the new energy function $\mathcal{E}_L$.

First recall the proof of (9.3). Making the substitution

$$
u(r,\theta)=u(t,\theta),\qquad v(r,\theta)=r^{-k}\mathbf{V}(t,\theta),\qquad t=\ln r,\theta\in S_{N-1},
$$

we get

$$
\left\{
\begin{array}{c}
u_{tt}+(N-2)u_t+\Delta_S u+e^{-2kt}u^s\mathbf{V}^{m+1}=0, \\
\mathbf{V}_{tt}+(N-2-2k)\mathbf{V}_t+\Delta_S\mathbf{V}-k(N-2-k)\mathbf{V}+u^{s+1}\mathbf{V}^m=0.
\end{array}
\right. \tag{9.4}
$$

Then $u, \mathbf{V}$ are bounded near ∞, and from [7, Proposition 4.1] u converges exponentially to the constant α, more precisely

$$\| |u - \alpha| + |u_t| + |\nabla_S u| \|_{C^0(S^{N-1})} = O(e^{-(N-2)t}), \tag{9.5}$$

because $k \neq (N-2)/2$ and all the derivatives of $\mathbf{V}$ up to the order 2 are bounded. The equation in $\mathbf{V}$ takes the form

$$\mathbf{V}_{tt} + (N-2-2k)\mathbf{V}_t + \Delta_S \mathbf{V} - k(N-2-k)\mathbf{V} + \alpha^{s+1}\mathbf{V}^m + \varphi = 0$$

where φ and its derivatives up to the order 2 are $O(e^{-(N-2)t})$. From [7, Theorem 4.1], the function $\mathbf{V}$ converges to β or to 0 in $C^2(S^{N-1})$.

Next we define

$$f(t) = e^{(N-2)t} \int_{S^{N-1}} u_t d\theta = r^{N-1} \int_{S^{N-1}} u_r d\theta.$$

Then

$$\mathcal{E}_{L,1}(t) = (N-2)\frac{s+1}{2}\alpha f(t) + O((e^{-(N-2)t}))$$

from (9.5). Moreover from (9.4),

$$f_t(t) = -e^{(N-2-2k)t} \int_{S^{N-1}} u^s \mathbf{V}^{m+1} d\theta < 0.$$

Since u is regular at 0, $f(t) = 0(e^{(N-1)t})$ at $-\infty$, in particular $\lim_{t \to -\infty} f(t) = 0$. And $f_t(t) = O(e^{(N-2-2k)t}) = O(e^{-t})$ at ∞, then $f(t)$ has a finite negative limit $-\ell^2$; and

$$\lim_{t \to \infty} \mathcal{E}_{L,1}(t) = -(N-2)\frac{s+1}{2}\alpha \ell^2.$$

Moreover $v = e^{-kt}\mathbf{V}$, and $\mathbf{V}$ and its derivatives up to the order 2 are bounded, thus

$$\mathcal{E}_{L,2}(t) = O(e^{(N-2-2k)t}) = O(e^{-t})$$

Finally

$$\mathcal{E}_{L,3}(t) = O(e^{(N+a-k(m+1))t})$$

and $N + a - k(m+1) < \frac{2-N}{m-1} < 0$. Then $\mathcal{E}_L$ has a finite limit $\theta < 0$ at ∞, which is contradictory. ■

10 Analysis of the fixed points

Here we make the local analysis around the fixed points.

Proof of Proposition 4.4. (i) Consider a regular solution (u, v) with initial data (u_0, v_0). When when $r \to 0$, we have

$$(-r^{N-1} |u'|^{p-2} u')' = r^{N-1+a} u_0^s v_0^\delta (1 + o(1)), \qquad -|u'|^{p-2} u' = \frac{1}{N+a} r^{1+a} u_0^s v_0^\delta (1 + o(1)),$$

thus from (2.1), when $t \to -\infty$

$$X(t) = (\frac{1}{N+a} u_0^{s+1-p} v_0^{\delta})^{1/(p-1)} e^{(p+a)t/(p-1)}(1+o(1)),$$
$$Y(t) = (\frac{1}{N+b} u_0^{\mu} v_0^{m+1-q})^{1/(q-1)} e^{(q+b)t/(q-1)}(1+o(1)),$$

and $\lim_{t\to-\infty} Z = N+a$, $\lim_{t\to-\infty} W = (N+b)$. In particular the trajectory tends to $N_0 = (0,0,N+a,N+b)$.

(ii) Reciprocally, consider a trajectory converging to N_0. Setting $Z = N+a+\tilde{Z}, W = N+b+\tilde{W}$, the linearized system is

$$X_t = \frac{p+a}{p-1}X, \quad Y_t = \frac{q+b}{q-1}Y, \quad \tilde{Z}_t = (N{+}a)\left[-sX - \delta Y - \tilde{Z}\right], \quad \tilde{W}_t = (N{+}b)\left[-\mu X - mY - \tilde{W}\right]. \tag{10.1}$$

The eigenvalues are

$$\lambda_1 = \frac{p+a}{p-1} > 0, \quad \lambda_2 = \frac{q+b}{q-1} > 0, \quad \lambda_3 = -(N+a) < 0, \quad \lambda_4 = -(N+b) < 0. \tag{10.2}$$

The unstable variety $\mathcal{V}_u$ and the stable variety $\mathcal{V}_s$ have dimension 2. Notice that $\mathcal{V}_s$ is contained in the set $X = Y = 0$, thus no admissible trajectory converges to N_0 when $r \to \infty$, and there exists an infinity of admissible trajectories in $\mathcal{R}$, converging to N_0 when $r \to 0$. Moreover we get $\lim_{t\to-\infty} e^{-(p+a)/(p-1)t}X(t) = \kappa > 0$ and $\lim_{t\to-\infty} e^{-(q+b)/(q-1)t}Y(t) = \ell > 0$. Thus (u,v) have a positive limit $(u_0,v_0) = ((N+a)\kappa^{p-1})^{(q-1-m)/D}((N+b)\ell^{q-1})^{\delta/D}$ from (4.2), (4.1), hence (u,v) is a regular solution.

Next we show that *for any* $\kappa > 0, \ell > 0$ there exists a unique local solution such that $\lim_{t\to-\infty} e^{-(p+a)t/(p-1)}X(t) = \kappa$ and $\lim_{t\to-\infty} e^{-(q+b)/(q-1)t}Y = \ell$. On $\mathcal{V}_u$, we get a system of two equations of the form

$$X_t = X(\lambda_1 + F(X,Y)), \quad Y_t = Y(\lambda_2 + G(X,Y)),$$

where $F = AX + BY + f(X,Y)$, where f is a smooth function with $f_X(0,0) = f_Y(0,0) = 0$, similarly for G. Setting $X = e^{\lambda_1 t}(\kappa + x)$, $Y = e^{\lambda_2 t}(\ell + y)$, and assuming $\lambda_2 \geqq \lambda_1$ and setting $\rho = e^{\lambda_1 t}$ we obtain

$$x_\rho = \frac{1}{\rho}(\kappa + x)F(\rho(\kappa+x), \rho^{\lambda_2/\lambda_1}(\ell+y)), \qquad y_\rho = (\ell+y)G(\rho(\kappa+x), \rho^{\lambda_2/\lambda_1}(\ell+y)),$$

with $x(0) = y(0) = 0$. Then we get local existence and uniqueness. Hence for any $u_0, v_0 > 0$ there exists a regular solution (u,v) with initial data (u_0,v_0). Moreover $u,v \in C^1([0,R))$ when $a,b > -1$.

■

Proof of Proposition 4.5. The linearization at $A_0 = \left(\frac{N-p}{p-1}, \frac{N-q}{q-1}, 0, 0\right)$ gives, with $X = \frac{N-p}{p-1} + \tilde{X}, Y = \frac{N-q}{q-1} + \tilde{Y}$,

$$\tilde{X}_t = \frac{N-p}{p-1}\left[\tilde{X} + \frac{Z}{p-1}\right], \quad \tilde{Y}_t = \frac{N-q}{q-1}\left[\tilde{Y} + \frac{W}{q-1}\right], \quad Z_t = \lambda_3 Z, \quad W_t = \lambda_4 W.$$

The eigenvalues are

$$\lambda_1 = \frac{N-p}{p-1} > 0,\ \lambda_2 = \frac{N-q}{q-1} > 0,\ \lambda_3 = N+a-s\frac{N-p}{p-1}-\delta\frac{N-q}{q-1},\ \lambda_4 = N+b-\mu\frac{N-p}{p-1}-m\frac{N-q}{q-1}. \tag{10.3}$$

• Convergence when $r \to \infty$: If $\lambda_3 > 0$, or $\lambda_4 > 0$, then the stable variety$\mathcal{V}_s$ has at most dimension 1, it satisfies $W = 0$ or $Z = 0$, hence there is no admissible trajectory converging to A_0 at ∞. If $\lambda_3 < 0$, and $\lambda_4 < 0$, then $\mathcal{V}_s$ has dimension 2. Moreover $\mathcal{V}_s \cap \{Z = 0\}$ has dimension 1 : the corresponding system in X, Y, W has the eigenvalues $\lambda_1, \lambda_2, \lambda_4$; similarly $\mathcal{V}_s \cap \{W = 0\}$ has dimension 1. Then there exist trajectories in $\mathcal{V}_s$ such that $Z > 0$ and $W > 0$, included in $\mathcal{R}$ and thus admissible. They satisfy $\lim e^{-\lambda_3 t} Z = C_3 > 0, \lim e^{-\lambda_4 t} W = C_4 > 0$, then (4.11) follows from (4.2).

• Convergence when $r \to 0$: If $\lambda_3 < 0$, or $\lambda_4 < 0$, the unstable variety $\mathcal{V}_u$ has at most dimension 3, and it satisfies $W = 0$ or $Z = 0$. Therefore there is no admissible trajectory converging at $-\infty$. If $\lambda_3, \lambda_4 > 0$, then $\mathcal{V}_u$ has dimension 4; in that case there exist admissible trajectories, and (4.11) follows as above. ∎

Proof of Proposition 4.6. We set $P_0 = \left(\frac{N-p}{p-1}, Y_*, 0, W_*\right)$, with

$$Y_* = \frac{\frac{N-p}{p-1}\mu - (q+b)}{q-1-m}, \qquad W_* = \frac{(q-1)(N+b-\frac{N-p}{p-1}\mu) - m(N-q)}{q-1-m},$$

for $m+1 \neq q$. The linearization at P_0 gives, with $X = \frac{N-p}{p-1} + \tilde{X}, Y = Y_* + \tilde{Y}$, $W = W_* + \tilde{W}$,

$$\tilde{X}_t = \frac{N-p}{p-1}\left[\tilde{X} + \frac{Z}{p-1}\right], \quad \tilde{Y}_t = Y_*\left[\tilde{Y} + \frac{\tilde{W}}{q-1}\right], \quad Z_t = \lambda_3 Z, \quad \tilde{W}_t = W_*\left[-\mu\tilde{X} - m\tilde{Y} - \tilde{W}\right]$$

The eigenvalues are

$$\lambda_1 = \frac{N-p}{p-1} > 0, \quad \lambda_3 = N + a - s\frac{N-p}{p-1} - \delta Y_* = \frac{D}{q-1-m}(\gamma - \frac{N-p}{p-1}),$$

and the roots λ_2, λ_4 of equation

$$\lambda^2 - (Y_* - W_*)\lambda + \frac{m+1-q}{q-1}Y_*W_* = 0$$

Then if $\lambda_3 < 0$ (resp. $\lambda_3 > 0$) there is no admissible trajectory converging when $r \to 0$ (resp. $r \to \infty$). Indeed $\mathcal{V}_u = \mathcal{V}_u \cap \{Z = 0\}$ (resp. $\mathcal{V}_s = \mathcal{V}_s \cap \{Z = 0\}$).

1) Suppose that $q > m+1$. Since $q + b < \frac{N-p}{p-1}\mu < N + b - m\frac{N-q}{q-1}$, we have $P_0 \in \mathcal{R}$, and $\lambda_2\lambda_4 < 0$. First assume $\lambda_3 < 0$, that means $\gamma < \frac{N-p}{p-1}$. Then $\mathcal{V}_s$ has dimension 2, and $\mathcal{V}_s \cap \{Z = 0\}$ has dimension 1, there exists trajectories with $Z > 0$, which are admissible, converging when $r \to \infty$. Next assume $\lambda_3 > 0$. Then $\mathcal{V}_u$ has dimension 3, and $\mathcal{V}_u \cap \{Z = 0\}$ has dimension 2. Thus there exist admissible trajectories converging when $t \to -\infty$.

2) Suppose that $q < m+1$. Since $q+b > \frac{N-p}{p-1}\mu > N+b-m\frac{N-q}{q-1}$, we have $P_0 \in \mathcal{R}$, and $\lambda_2\lambda_4 > 0$. We assume $q\frac{N-p}{p-1}\mu + m(N-q) \neq N(q-1) + (b+1)q$, that means $Y_* \neq W_*$. First suppose $\lambda_3 > 0$, that means $\gamma < \frac{N-p}{p-1}$. If $\mathrm{Re}\lambda_2 > 0$, then $\mathcal{V}_u$ has dimension 4, or $\mathrm{Re}\lambda_2 < 0$ then $\mathcal{V}_u$ has dimension

2 and $\mathcal{V}_u \cap \{Z=0\}$ has dimension 1. In any case, there exist admissible trajectories converging when $r \to 0$. Next assume $\lambda_3 < 0$. If $\mathrm{Re}\lambda_2 > 0$, then $\mathcal{V}_s$ has dimension 1, and $\mathcal{V}_s \cap \{Z=0\} = \emptyset$. If $\mathrm{Re}\lambda_2 < 0$, then $\mathcal{V}_s$ has dimension 3. In any case $\mathcal{V}_s$ contains trajectories with $Z>0$, which are admissible, converging when $r \to \infty$.

Those trajectories satisfy $\lim e^{-\lambda_3 t} Z = C_3 > 0$, $\lim X = \frac{N-p}{p-1}$, $\lim Y = Y_*$ and $\lim W = W_*$, thus (4.12) follows from (4.2) and (2.5). ■

Proof of Proposition 4.8. The linearization at $I_0 = (\frac{N-p}{p-1}, 0, 0, 0)$ gives, with $X = \frac{N-p}{p-1} + \tilde{X}$,

$$\tilde{X}_t = \frac{N-p}{p-1}(\tilde{X} + \frac{Z}{p-1}), \quad Y_t = -\frac{N-q}{q-1}Y, \quad Z_t = (N+a-s\frac{N-p}{p-1})Z, \quad W_t = (N+b-\mu\frac{N-p}{p-1})W.$$

The eigenvalues are

$$\lambda_1 = \frac{N-p}{p-1} > 0, \quad \lambda_2 = -\frac{N-q}{q-1} < 0, \quad \lambda_3 = N+a-s\frac{N-p}{p-1}, \quad \lambda_4 = N+b-\mu\frac{N-p}{p-1}.$$

• Convergence when $r \to \infty$: If $\lambda_3 > 0$ or $\lambda_4 > 0$, then $\mathcal{V}_s = \mathcal{V}_s \cap \{Z=0\}$ or $\mathcal{V}_s = \mathcal{V}_s \cap \{W=0\}$. There is no admissible trajectory converging at ∞. Next suppose that $\lambda_3, \lambda_4 < 0$. Then $\mathcal{V}_s$ has dimension 3; it contains trajectories with $Y, Z, W > 0$, which are admissible. They satisfy $\lim X = \frac{N-p}{p-1}$, $\lim e^{-\lambda_2 t} Y = C_2 > 0$, $\lim e^{-\lambda_3 t} Z = C_3 > 0$, $\lim e^{-\lambda_4 t} W = C_4 > 0$, then (4.13) follows from (4.2) and (2.4).

• Convergence when $r \to 0$: Since $\lambda_2 < 0$ we have $\mathcal{V}_u = \mathcal{V}_u \cap \{Y=0\}$, hence there is no admissible trajectory converging when $r \to 0$. ■

Proof of Proposition 4.9. The point $G_0 = (\frac{N-p}{p-1}, 0, 0, N+b-\frac{N-p}{p-1}\mu) \in \mathcal{R}$ since $\frac{N-p}{p-1}\mu < N+b$. The linearization at G_0 gives, with $X = \frac{N-p}{p-1} + \tilde{X}, W = N+b-\frac{N-p}{p-1}\mu + \tilde{W}$,

$$\begin{aligned} \tilde{X}_t &= \frac{N-p}{p-1}\left[\tilde{X} + \frac{Z}{p-1}\right], \quad Y_t = \frac{Y}{q-1}(q+b-\frac{N-p}{p-1}\mu), \\ Z_t &= (N+a-s\frac{N-p}{p-1})Z, \quad W_t = (N+b-\frac{N-p}{p-1}\mu)\left[-\mu\tilde{X} - mY - \tilde{W}\right] \end{aligned}$$

The eigenvalues are

$$\lambda_1 = \frac{N-p}{p-1} > 0, \ \lambda_2 = \frac{1}{q-1}(q+b-\frac{N-p}{p-1}\mu), \ \lambda_3 = N+a-s\frac{N-p}{p-1}, \ \lambda_4 = \frac{N-p}{p-1}\mu - N - b < 0.$$

• Convergence when $r \to \infty$: If $\lambda_2 > 0$, or $\lambda_3 > 0$, then $\mathcal{V}_s = \mathcal{V}_s \cap \{Y=0\}$ or $\mathcal{V}_s = \mathcal{V}_s \cap \{Z=0\}$, there is no admissible trajectory converging at ∞. Assume $\lambda_2, \lambda_3 < 0$, then $\mathcal{V}_s$ has dimension 3, it contains trajectories with $Y, Z > 0$, which are admissible.

• Convergence when $r \to 0$: If $\lambda_3 < 0$, or $\lambda_2 < 0$ there is no admissible trajectory. If $\lambda_2, \lambda_3 > 0$ then $\mathcal{V}_s$ has dimension 3, it contains admissible trajectories.

In any case $\lim X = \frac{N-p}{p-1}$, $\lim e^{-\lambda_2 t} Y = C_2 > 0$, $\lim e^{-\lambda_3 t} Z = C_3 > 0$, $\lim W = N+b-\frac{N-p}{p-1}\mu$, hence (4.13) still follows from (4.2) and (2.4). ■

Proof of Proposition 4.10. We set $C_0 = \left(0, \bar{Y}, 0, \bar{W}\right)$, with

$$\bar{Y} = \frac{q+b}{m+1-q}, \qquad \bar{W} = \frac{m(N-q)-(N+b)(q-1)}{m+1-q}. \tag{10.4}$$

Then $C_0 \in \mathcal{R}$ if $\frac{N-q}{q-1}m > N+b$, implying $q < m+1$. The linearization at C_0 gives, with $Y = \bar{Y} + \tilde{Y}$ and $W = \bar{W} + \tilde{W}$

$$X_t = -\frac{N-p}{p-1}X, \quad \tilde{Y}_t = \bar{Y}\left[\tilde{Y} + \frac{\tilde{W}}{q-1}\right], \quad Z_t = \lambda_3 Z, \quad W_t = \bar{W}\left[-\mu X - m\tilde{Y} - \tilde{W}\right].$$

The eigenvalues are

$$\lambda_1 = -\frac{N-p}{p-1}, \quad \lambda_3 = N + a - \delta\bar{Y},$$

and the roots λ_2, λ_4 of equation

$$\lambda^2 - (\bar{Y} - \bar{W})\lambda + \frac{m+1-q}{q-1}\bar{Y}\bar{W} = 0 \tag{10.5}$$

then $\lambda_2\lambda_4 > 0$. We assume $m \neq \frac{N(q-1)+(b+1)q}{N-q}$, that means $\bar{Y} \neq \bar{W}$.

• Convergence when $r \to \infty$: if $\lambda_3 > 0$ we have $\mathcal{V}_s = \mathcal{V}_s \cap \{Z = 0\}$, hence there is no admissible trajectory. Next assume that $\lambda_3 < 0$, that means $\delta > (N+a)\frac{m+1-q}{q+b}$.If $\operatorname{Re}\lambda_2 < 0$ (resp. > 0) then $\mathcal{V}_s$ has dimension 4 (resp. 2) and $\mathcal{V}_s \cap \{X = 0\}$ and $\mathcal{V}_s \cap \{Z = 0\}$ have dimension 3 (resp. 1) then there exist trajectories with $X, Z > 0$, which are admissible.

In any case $\lim e^{-\lambda_1 t}X = C_1 > 0$, $\lim Y = \bar{Y}$, $\lim e^{-\lambda_3 t}Z = C_3 > 0$, $\lim W = \bar{W}$, then (4.15) follows.

• Convergence when $r \to 0$: Since $\lambda_1 < 0$ we have $\mathcal{V}_u = \mathcal{V}_u \cap \{X = 0\}$, hence there is no admissible trajectory. ■

Proof of Proposition 4.11. We set $R_0 = \left(0, \bar{Y}, \bar{Z}, \bar{W}\right)$, where $\bar{Y}, \bar{W}$ are defined at (10.4), and $\bar{Z} = N+a-\delta\frac{b+q}{m+1-q}$.Under our assumptions it lies in $\mathcal{R}$. Setting $Y = \bar{Y}+\tilde{Y}, Z = \bar{Z}+\tilde{Z}, W = \bar{W}+\tilde{W}$, the linearization at R_0 gives

$$X_t = \lambda_1 X, \quad \tilde{Y}_t = \bar{Y}\left[\tilde{Y} + \frac{\tilde{W}}{q-1}\right], \quad Z_t = \bar{Z}\left[-sX - \delta\tilde{Y} - \tilde{Z}\right], \quad W_t = \bar{W}\left[-\mu X - m\tilde{Y} - \tilde{W}\right];$$

the eigenvalues are

$$\lambda_1 = \frac{1}{p-1}(p + a - \delta\frac{b+q}{m+1-q}), \quad \lambda_3 = -\bar{Z} < 0;$$

and the roots λ_2, λ_4 of equation of equation (10.5).

• Convergence when $r \to \infty$: If $\lambda_1 > 0$, that means $(p+a)\frac{m+1-q}{q+b} < \delta$, then $\mathcal{V}_s = \mathcal{V}_s \cap \{X = 0\}$, hence there is no admissible trajectory. Next assume $\lambda_1 < 0$; if $\operatorname{Re}\lambda_2 < 0$ (resp. > 0) then $\mathcal{V}_s$ has dimension 4(resp. 2) and $\mathcal{V}_s \cap \{X = 0\}$ has dimension 3 (resp. 1) then there exist admissible trajectories.

• Convergence when $r \to 0$: If $\lambda_1 < 0$, then $\mathcal{V}_u = \mathcal{V}_u \cap \{X = 0\}$, hence there is no admissible trajectory. Next assume $\lambda_1 > 0$. If $\operatorname{Re}\lambda_2 = \operatorname{Re}\lambda_4 < 0$ (resp. > 0) then $\mathcal{V}_s$ has dimension 4 (resp. 2) and $\mathcal{V}_s \cap \{X = 0\}$ has dimension 3 (resp. 1) then there exist admissible trajectories.

In any case $\lim e^{-\lambda_1 t}X = C_1 > 0$, $\lim Y = \bar{Y}$, $\lim Z = \bar{Z}$, $\lim W = \bar{W}$, then (4.15) holds again.■

Remark 10.1 *Finally there is no admissible trajectory converging to* $0 = (0,0,0,0)$*, or* $K_0 = (0,0,N+a,0)$*, or* $L_0 = (0,0,0,N+b)$*. Indeed the linearization at* 0 *gives*

$$X_t = -\frac{N-p}{p-1}X, \quad Y_t = -\frac{N-q}{q-1}Y, \quad Z_t = (N+a)Z, \quad W_t = (N+b)W$$

Then $\mathcal{V}_s$ *and* $\mathcal{V}_u$ *have dimension* 2*, hence* $\mathcal{V}_s$ *is contained in* $\{Z = W = 0\}$*, and* $\mathcal{V}_u$ *in* $\{X = Y = 0\}$*. The linearization at* K_0 *gives, with* $Z = N + a + \tilde{Z}$*,*

$$X_t = \frac{p+a}{p-1}X, \quad Y_t = -\frac{N-q}{q-1}Y, \quad Z_t = (N+a)\left[-sX - \delta Y - \tilde{Z}\right], \quad W_t = (N+b)W.$$

The eigenvalues are $\frac{p+a}{p-1}, -\frac{N-q}{q-1}, -(N+a)$*,* $N+b$*. Then* $\mathcal{V}_s$ *and* $\mathcal{V}_u$ *have dimension* 2*, hence* $\mathcal{V}_s$ *is contained in* $\{Z = W = 0\}$*, and* $\mathcal{V}_u$ *in* $\{Y = 0\}$*. The case of* L_0 *follows by symmetry.*

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