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ScienceDirect
J. Differential Equations 244 (2008) 274–308
Journal of Differential Equations
www.elsevier.com/locate/jde

Separable solutions of some quasilinear equations with source reaction

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Received 12 June 2006; revised 27 July 2007

Available online 26 November 2007

Abstract

We study the existence of singular solutions to the equation $-\operatorname{div}(|Du|^{p-2}Du) = |u|^{q-1}u$ under the form $u(r,\theta) = r^{-\beta}\omega(\theta)$, $r > 0$, $\theta \in S^{N-1}$. We prove the existence of an exponent q below which no positive solutions can exist. If the dimension is 2 we use a dynamical system approach to construct solutions.
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MSC: 35K60; 34C05; 34C35

Keywords: p-Laplacian; Singularities; Phase-plane analysis; Poincaré map; Painlevé integral

1. Introduction

The study of isolated singularities of solutions of quasilinear equations started with the celebrated works of Serrin [20,21] dealing with expressions such as

$$\operatorname{div} A\big((x,u,Du)\big) + B(x,u,Du) = 0 \tag{1.1}$$

where A and B are respectively vector-valued and real-valued Carathéodory functions satisfying the same power p-growth with $p \geqslant 1$. One of the main results of these works stated that the type

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1 Supported by a grant from the Lebanese University.

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doi:10.1016/j.jde.2007.10.024

of singularities is dictated by the diffusion operator A. Later on the particular cases of superlinear semilinear elliptic equations was considered, either with an absorption

$$-\Delta u + |u|^{q-1}u = 0 \tag{1.2}$$

[5,24], or with a source reaction

$$\Delta u + u^q = 0 \tag{1.3}$$

[2,10,17], and in all cases $q > 1$. One of the main facts of these studies relied in the existence of critical thresholds where the interaction of the diffusion and the reaction terms could create unexpected phenomena. As a natural generalisation, the same analysis was carried on for

$$-\operatorname{div}\bigl(|Du|^{p-2}Du\bigr) + |u|^{q-1}u = 0 \tag{1.4}$$

[9], and

$$\operatorname{div}\bigl(|Du|^{p-2}Du\bigr) + u^q = 0 \tag{1.5}$$

[22], in the range $0 < p - 1 < q$. In all these works, the radial explicit solutions, whenever they exist, played a key role.

Similarly, the study of the boundary behaviour of solutions of quasilinear equations has a natural starting point in the description of their isolated singularities on the boundary. Besides the historical results of Fatou, Herglotz and Doob on the boundary trace of positive harmonic and super harmonic functions, equations of types (1.2), (1.3) and (1.4) have already been considered [4,6,11,12,26,27]. In the present article we consider equations of type (1.5). The problem can be stated under the following form: Assume Ω is an open subset of $\mathbb{R}^N$, $a \in \partial\Omega$ and $u \in C(\overline{\Omega} \setminus \{a\}) \cap C^1(\Omega)$ is a solution of one of the above equations which vanishes on $\partial\Omega \setminus \{a\}$, what is the behaviour of $u(x)$ when $x \to a$. The simplest configuration corresponds to $\Omega = \mathbb{R}^N_+$, and $a = 0$ (or more generally, if Ω is a cone and the singular point a its vertex 0). For such geometry, the key-stone element for describing the behaviour of u near 0 is played by *separable solutions*, whenever they exist. These solutions, which have the form

$$u(x) = u(r, \sigma) = r^{-\beta}\omega(\sigma), \quad r > 0,\ \sigma \in S^{N-1}, \tag{1.6}$$

have already proved their importance for (1.2), (1.3) and (1.4). It is expected that such will be the case for (1.5), even if the full theory will be much more difficult to develop because of the absence of comparison principle and a priori estimates near $x = 0$. It is straightforward that, if u is a separable solution of (1.5) in $\mathbb{R}^N$,

$$\beta = \frac{p}{q + 1 - p} := \beta_q, \tag{1.7}$$

which is positive since $q > p - 1$. Furthermore ω is a solution of

$$-\nabla'.\bigl(\bigl(\beta_q^2\omega^2 + |\nabla'\omega|^2\bigr)^{p/2-1}\nabla'\omega\bigr) - |\omega|^{q-1}\omega = \lambda_{q,p}\bigl(\beta_q^2\omega^2 + |\nabla'\omega|^2\bigr)^{p/2-1}\omega, \tag{1.8}$$

in S_+^{N-1}, where ∇' is the covariant gradient on S^{N-1}, $\nabla'.$ the divergence operator acting on vector fields on S^{N-1} and

$$\lambda_{q,p} = \beta_q(q\beta_q - N).$$

When $p = 2$, $\beta_q = 2/(q-1)$ and (1.8) becomes

$$-\Delta'\omega - |\omega|^{q-1}\omega = \lambda_{q,2}\omega, \tag{1.9}$$

where Δ' is the Laplace–Beltrami operator on S^{N-1} and

$$\lambda_{q,2} = \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right).$$

If S is a subdomain of S^{N-1}, Eq. (1.9), considered in S, is the Euler–Lagrange variation of the functional

$$I(\psi) = \int_S \left(\frac{1}{2}|\nabla\psi|^2 + \frac{\lambda_{q,2}}{2}\psi^2 - \frac{1}{q+1}|\psi|^{q+1}\right) d\sigma. \tag{1.10}$$

For any $1 < q < (N+1)/(N-3)$ (any $q > 1$ if $N = 2$ or 3) this functional satisfies the Palais–Smale condition. Furthermore, if $\lambda_{q,2} < \lambda_{S,2}$ ($\lambda_{S,2}$ is the first eigenvalue of $-\Delta'$ in $W_0^{1,2}(S)$), Ambrosetti–Rabinowitz theorem [1] or Pohozaev fibration method [18,19] apply and yield to the existence of nontrivial positive solutions to (1.9) in S vanishing on ∂S; while if $\lambda_{q,2} \geqslant \lambda_{S,2}$ no such solution exists.

When $p \neq 2$, Eq. (1.8) cannot be associated to any functional defined on S^{N-1}, except if $q = q_c = (N(p-1)+p)/(N-p)$ (the critical Sobolev exponent for $W^{1,p}$, when $N > p$); therefore, finding functions satisfying it is not straightforward. Besides the constant solutions which exist as soon as $q\beta_q < N$, it is not easy to prove the existence of nonconstant solutions. As in the case $p = 2$, it is remarkable to see that existence, or nonexistence, of solutions of (1.8) is associated to some spectral problem, although this problem is not standard at all: if one looks for the existence of a positive p-harmonic function v in the cone $C_S = \{(r,\sigma)\colon\ r > 0,\ \sigma \in S\}$ vanishing on ∂S, under the form $v(r,\sigma) = r^{-\beta}\phi(\sigma)$, one finds that ϕ is a positive solution of the so-called *spherical p-harmonic spectral equation on S*, namely

$$\begin{cases} -\nabla'.\left(\left(\beta^2\phi^2 + |\nabla'\phi|^2\right)^{p/2-1}\nabla'\phi\right) = \lambda\left(\beta^2\phi^2 + |\nabla'\phi|^2\right)^{p/2-1}\phi & \text{in } S, \\ \phi = 0 & \text{in } \partial S, \end{cases} \tag{1.11}$$

and $\lambda = \beta(\beta(p-1) + p - N)$. The difficulty of this problem is two-fold since β is unknown and (1.11) is not the Euler–Lagrange equation of any functional. However, given a smooth subdomain $S \subset S^{N-1}$, it is proved in [25], following a shooting method due to Tolksdorff [23], that there exists a couple $(\beta,\phi) = (\beta_S,\phi_S)$, where $\beta_S > 0$ is unique and ϕ_S is defined up to a homothety, such that (1.11) holds. Denoting

$$\lambda_S = \beta_S\left(\beta_S(p-1) + p - N\right),$$

the couple (ϕ_S, λ_S) is the natural generalization of the first eigenfunction and eigenvalue of the Laplace–Beltrami operator in $W_0^{1,2}(S)$ since $\lambda_S = \lambda_{S,2}$ when $p = 2$. The case $N = 2$ is treated in [14,15] by ODE methods. Our first theorem is a nonexistence which extends the one already mentioned in the case $p = 2$.

Theorem 1. *Let $S \subset S^{N-1}$ be a smooth subdomain. If $\beta_q \geqslant \beta_S$ there exists no positive solution of* (1.8) *in S which vanishes on ∂S.*

Apart the case $p = 2$, the existence counterpart of this theorem is not known in arbitrary dimension, except if $q = q_c$ in which case (1.5) is the Euler–Lagrange equation of the functional

$$J(\psi) = \int_S \left(\frac{1}{p} \left(\beta_{q_c}^2 \psi^2 + |\nabla' \psi|^2 \right)^{p/2} - \frac{1}{q_c + 1} |\psi|^{q_c+1} \right) d\sigma, \tag{1.12}$$

and applications of the already mentioned variational methods lead to an existence result.

However, when $N = 2$ the problem of finding solutions of (1.5) under the form (1.6) can be completely solved using dynamical systems methods. In order to point out a richer class of phenomena, we shall imbed this problem into a more general class of quasilinear equations with a potential, authorizing even the value $p = 1$. This equation is the following,

$$\operatorname{div}\left(|Du|^{p-2} Du\right) + |u|^{q-1} u - \frac{c}{|x|^p} |u|^{p-2} u = 0 \tag{1.13}$$

in $\mathbb{R}^2 \setminus \{0\}$, with $q > p - 1 \geqslant 0$ and $c \in \mathbb{R}$. If u is a solution under the form (1.6), β is equal to β_q, while ω is any 2π-periodic solution of

$$\begin{aligned} &\frac{d}{d\sigma}\left[\left(\beta_q^2 \omega^2 + \left(\frac{d\omega}{d\sigma}\right)^2\right)^{p-2)/2} \frac{d\omega}{d\sigma}\right] + \lambda_q \left[\beta_q^2 \omega^2 + \left(\frac{d\omega}{d\sigma}\right)^2\right]^{(p-2)/2} \omega \\ &\quad + |\omega|^{q-1} \omega - c|\omega|^{p-2} \omega = 0, \end{aligned} \tag{1.14}$$

where

$$\lambda_q = \beta_q \left(q\beta_q - 2\right) = \beta_q \left(p - 2 + (p - 1)\beta_q\right). \tag{1.15}$$

If we set

$$c_q = \beta_q^{p-2} \lambda_q = p^{p-1} \frac{(p - 2)q + 2(p - 1)}{(q + 1 - p)^p}, \tag{1.16}$$

then, if $c \leqslant c_q$, the only constant solution is the zero function, while if $c > c_q$, there exist two other constant solutions $\pm(c - c_q)^{1/(q+1-p)}$. Let us denote by $\mathcal{E}^+$ the set of positive solutions of (1.14) on S^1, $\mathcal{E}$ the set of sign changing solutions and $\mathcal{F} = \pm\mathcal{E}^+ \cup \mathcal{E}$ the set of all nonzero solutions. Our main result which gives the structure of the sets $\mathcal{E}$ and $\mathcal{E}^+$ is the following:

Theorem 2. *Assume $p > 1$, $q > p - 1$. Then*:

(i)
$$\bigcup_{\substack{k\in\mathbb{N}\\ k=k_q}}^{\infty} \{\omega_k(.+\psi)\colon \psi \in S^1\}, \tag{1.17}$$

in which expression ω_k is a function with least period $2\pi/k$, and $k_q = 1$ if $c \geqslant c_q$, or k_q is the smallest positive integer such that $k_q > M_q$, where

$$M_q = \frac{\pi\beta_q^{1-p}}{2\int_0^{\pi/2} \frac{1+(p-1)\tan^2\theta}{\beta_q^p(p-1)\tan^2\theta + c_q - c\cos^{p-2}\theta}\,d\theta}, \tag{1.18}$$

if $c < c_q$.

(ii) *If $c \leqslant c_q$, $\mathcal{E}^+$ is empty. If $0 < c - c_q \leqslant \beta_q^{p-1}/p$, $\mathcal{E}^+$ is reduced to the constant function $(c-c_q)^{1/(q+1-p)}$. If $c - c_q > \beta_q^{p-1}/p$, $\mathcal{E}^+$ contains the constant function $(c-c_q)^{1/(q+1-p)}$ and the set*

$$\mathcal{E}_*^+ = \bigcup_{\substack{k\in\mathbb{N}\\ k=1}}^{k_q^+} \{\omega_k^+(.+\psi)\colon \psi \in S^1\}, \tag{1.19}$$

where ω_k^+ is a nonconstant positive function with least period $2\pi/k$, and k_q^+ is the largest integer smaller than $(p\beta_q^{1-p}(c-c_q))^{1/2}$.

Since separable solutions of (1.5) defined in a cone C_S and vanishing on ∂C_S are associated to elements of $\mathcal{E}$, we can prove the existence counterpart of Theorem 1 in dimension 2.

Corollary 1. *Let $N = 2$ and S be any angular sector of S^1. Then there exists a positive solution of* (1.8) *vanishing at the two end points of S if and only if $\beta_q < \beta_S$. Furthermore this solution is unique. In particular, existence holds for any sector if $p < 2$ and $q \geqslant 2(p-1)/(2-p)$.*

The case $p = 1$ appears as a limiting case of the preceding one. In that case we observe that u is a positive solution of (1.13) if and only if $v = u^q$ is a solution of the same equation relative to $q = 1$,

$$\operatorname{div}(|Dv|^{-1}Dv) + v - \frac{c}{|x|} = 0. \tag{1.20}$$

The initial case $c = 0$ is easily treated, but the case $c \neq 0$, that we shall analyse in full generality, is much richer and delicate and shows a large variety of solutions depending on various parameters.

Theorem 3. *Assume $p = 1$ and $q > 0$. Then*:

(i) *If $c \neq 0$, or $c = 0$ and $q > 1$, $\mathcal{E}$ is empty. If $c = 0$ and $q \leqslant 1$, $\mathcal{E} = \{\omega_0(.+\psi)\colon \psi \in S^1\}$, where $\sigma \mapsto \omega_0(\sigma) := 2^{1/q}|\sin\sigma|^{(1-q)/q}\sin\sigma$ is a C^1 solution of* (1.14).

(ii) *If $c \leqslant -1$, $\mathcal{E}^+$ is empty. If $-1 < c < 0$, $\mathcal{E}^+$ is reduced to the constant function $(c+1)^{1/q}$. If $c > 0$,*

$$\mathcal{E}^+ = \left\{(c+1)^{1/q}\right\} \cup \bigcup_{\substack{k\in\mathbb{N}\\ k=k_1}}^{k_2} \left\{\omega_k^+(.+\psi):\ \psi \in S^1\right\},$$

in which expression ω_k^+ is a positive function with least period $2\pi/k$, k_2 is the largest integer strictly smaller than $(c+1)^{1/2}$ and k_1 is the smallest integer greater than $\frac{\pi}{2}\int_0^{\pi/2}\sqrt{\frac{\cos\theta}{\cos\theta+2c}}\,d\theta$. Finally, if $c = 0$,

$$\mathcal{E}^+ = \{1\} \cup \bigcup_{K\in(0,1)} \left\{\omega_K^+(.+\psi):\ \psi \in S^1\right\} \cup \begin{cases} \emptyset & \text{if } q \geqslant 1,\\ \{\omega_0^+(.+\psi):\ \psi \in S^1\} & \text{if } q < 1,\end{cases}$$

where the functions ω_K^+ and ω_0^+ are explicitly given by

$$\omega_K^+ = \left(\sqrt{1-K^2\sin^2\sigma} - K\cos\sigma\right)^{1/q} \quad \text{and} \quad \omega_0^+ = \left(2|\sin\sigma|\right)^{1/q} \quad \forall\sigma \in S^1.$$

A striking phenomenon is the existence of a 2-parameter family of solutions when $c = 0$.

Our paper is organized as follows: Section 1—Introduction. Section 2—The N-dimensional case. Section 3—The 2-dim dynamical system. Section 4—The case $p > 1$. Section 5—The case $p = 1$.

2. The N-dimensional case

2.1. *The spherical p-harmonic spectral problem*

If $p \geqslant 1$, $\beta > 0$ and $\lambda \in \mathbb{R}$ we denote by $\mathfrak{T}_{\beta,\lambda}$ the operator defined on $C^1(S^{N-1})$ by

$$\varphi \mapsto \mathfrak{T}_{\beta,\lambda}[\varphi] = -\nabla'.\left(\left(\beta^2\varphi^2 + |\nabla'\varphi|^2\right)^{(p-2)/2}\nabla'\varphi\right) - \lambda\left(\beta^2\varphi^2 + |\nabla'\varphi|^2\right)^{(p-2)/2}\varphi. \quad (2.1)$$

Let $q > p - 1 > 0$, S be a smooth connected domain on S^{N-1} and C_S the cone with vertex 0 generated by S. If u is a positive solutions of

$$-\operatorname{div}\left(|Du|^{p-2}Du\right) = u^q, \quad (2.2)$$

in $C_S \setminus \{(0)\}$ vanishing on $\partial C_S \setminus \{(0)\}$, under the form

$$u(r,\sigma) = r^{-\beta}\omega(\sigma), \quad (2.3)$$

then $\beta = p/(q+1-p) := \beta_q$ and ω solves

$$\begin{cases} \mathfrak{T}_{\beta_q,\lambda_{q,p}}[\omega] - \omega^q = 0 & \text{in } S,\\ \omega = 0 & \text{on } \partial S,\end{cases} \quad (2.4)$$

where

$$\lambda_{q,p} = \beta_q(q\beta_q - N).$$

We denote by β_S the exponent corresponding to the first spherical singular p-harmonic function and by ϕ_S the corresponding function. Thus $\beta_S > 0$ and $u(r,\sigma) = r^{-\beta_S}\phi_S(\sigma)$ is p-harmonic in $C_S \setminus \{(0)\}$ and vanishes on $\partial C_S \setminus \{(0)\}$. Furthermore $\phi = \phi_S > 0$ and satisfies

$$\begin{cases} \mathfrak{T}_{\beta_S,\lambda_S}[\phi] = 0 & \text{in } S, \\ \phi = 0 & \text{on } \partial S, \end{cases} \tag{2.5}$$

where

$$\lambda_S = \beta_S\big(\beta_S(p-1) + p - N\big).$$

We recall that (β_S, ϕ_S) is unique up to a homothety upon ϕ. Furthermore ϕ_S is positive in S, $\partial\phi_S/\partial\nu < 0$ on ∂S and

$$S' \subset S,\ S' \neq S \quad \Longrightarrow \quad \beta_{S'} > \beta_S.$$

2.2. Non-existence

Proof of Theorem 1. We put

$$\theta = \frac{\beta_q}{\beta_S} \quad \text{and} \quad \eta = \phi_S^\theta.$$

Then $\theta \geqslant 1$ and

$$\begin{aligned}
\nabla'\eta &= \theta\phi_S^{\theta-1}\nabla'\phi_S, \\
\beta_q^2\eta^2 + |\nabla'\eta|^2 &= \theta^2\phi_S^{2(\theta-1)}\big(\beta_S^2\phi_S^2 + |\nabla'\phi_S|^2\big), \\
\big(\beta_q^2\eta^2 + |\nabla'\eta|^2\big)^{(p-2)/2} &= \theta^{p-2}\phi_S^{(p-2)(\theta-1)}\big(\beta_S^2\phi_S^2 + |\nabla'\phi_S|^2\big)^{(p-2)/2}, \\
\nabla'.\big(\beta_q^2\eta^2 + |\nabla'\eta|^2\big)^{(p-2)/2}\nabla'\eta & \\
= \theta^{p-1}\phi_S^{(p-1)(\theta-1)}\nabla'.\big(\beta_S^2\phi_S^2 &+ |\nabla'\phi_S|^2\big)^{(p-2)/2}\nabla'\phi_S \\
+ (p-1)(\theta-1)\theta^{p-2}\phi_S^{(p-1)(\theta-1)-1}&\big(\beta_S^2\phi_S^2 + |\nabla'\phi_S|^2\big)^{(p-2)/2}|\nabla'\phi_S|^2.
\end{aligned}$$

Using (2.5) with $\phi = \phi_S$, we derive

$$\mathfrak{T}_{\beta_q,\lambda_{q,p}}[\eta] = -(p-1)\theta^{p-1}(\theta-1)\phi_S^{(p-1)(\theta-1)-1}\big(\beta_S^2\phi_S^2 + |\nabla'\phi_S|^2\big)^{p/2} \quad \text{in } S. \tag{2.6}$$

Because ω is a nonnegative nontrivial solution of (2.4), it is nonpositive in S. Furthermore $\partial\omega/\partial\nu < 0$ on ∂S. Therefore we can choose ϕ_S as the maximal positive solution of (2.5) such that $\eta \leqslant \omega$. If $\theta > 1$ there exists $\sigma^* \in S$ such that

$$\omega\big(\sigma^*\big) = \eta\big(\sigma^*\big) > 0 \quad \text{and} \quad \omega(\sigma) \geqslant \eta(\sigma) \quad \forall\sigma \in \bar{S}. \tag{2.7}$$

If $\theta = 1$, the graphs of ω and η could be tangent only on ∂S. This means that either (2.7) holds, or there exists $\bar{\sigma} \in \partial S$ such that

$$\partial\omega(\bar{\sigma})/\partial\nu = \partial\eta(\bar{\sigma})/\partial\nu < 0 \quad \text{and} \quad \omega(\sigma) < \eta(\sigma) \quad \forall\sigma \in S. \tag{2.8}$$

Let $\psi = \omega - \eta$ and we first consider the case where (2.7) holds. Let $g = (g_{ij})$ be the metric tensor on S^{N-1}. We recall the following expressions in local coordinates σ_j around σ^*,

$$|\nabla'\varphi|^2 = \sum_{j,k} g^{jk} \frac{\partial\varphi}{\partial\sigma_j}\frac{\partial\varphi}{\partial\sigma_k},$$

for any $\varphi \in C^1(S)$, and

$$\nabla'.X = \frac{1}{\sqrt{|g|}} \sum_{\ell} \frac{\partial}{\partial\sigma_\ell}\big(\sqrt{|g|}X^\ell\big) = \frac{1}{\sqrt{|g|}} \sum_{\ell,i} \frac{\partial}{\partial\sigma_\ell}\big(\sqrt{|g|}g^{\ell i}X_i\big),$$

for any vector field $X \in C^1(TS^{N-1})$, if we lower the indices by setting $X^\ell = \sum_i g^{\ell i}X_i$. We derive from the mean value theorem

$$\big(\beta_q^2\omega^2 + |\nabla'\omega|^2\big)^{(p-2)/2}\frac{\partial\omega}{\partial\sigma_i} - \big(\beta_q^2\eta^2 + |\nabla'\eta|^2\big)^{(p-2)/2}\frac{\partial\eta}{\partial\sigma_i} = \sum_j \alpha_j^i \frac{\partial(\omega-\eta)}{\partial\sigma_j} + b^i(\omega-\eta),$$

where

$$\begin{aligned} b^i = {} & (p-2)\big(\beta_q^2\big(\eta + t(\omega-\eta)\big)^2 + |\nabla'\big(\eta + t(\omega-\eta)\big)|^2\big)^{(p-4)/2} \\ & \times \big(\eta + t(\omega-\eta)\big)\frac{\partial(\eta + t(\omega-\eta))}{\partial\sigma_i}, \end{aligned}$$

and

$$\begin{aligned} \alpha_j^i = {} & (p-2)\big(\beta_q^2\big(\eta + t(\omega-\eta)\big)^2 + \big|\nabla'\big(\eta + t(\omega-\eta)\big)\big|^2\big)^{(p-4)/2} \\ & \times \frac{\partial(\eta + t(\omega-\eta))}{\partial\sigma_i}\sum_k g^{jk}\frac{\partial(\eta + t(\omega-\eta))}{\partial\sigma_k} \\ & + \delta_i^j\big(\beta_q^2\big(\eta + t(\omega-\eta)\big)^2 + \big|\nabla'\big(\eta + t(\omega-\eta)\big)\big|^2\big)^{(p-2)/2}. \end{aligned}$$

Since the graphs of η and ω are tangent at σ^*,

$$\eta\big(\sigma^*\big) = \omega\big(\sigma^*\big) = P_0 > 0 \quad \text{and} \quad \nabla'\eta\big(\sigma^*\big) = \nabla'\omega\big(\sigma^*\big) = Q.$$

Thus

$$b^i\big(\sigma^*\big) = (p-2)\big(\beta_q^2P_0^2 + |Q|^2\big)^{(p-4)/2}P_0Q_i,$$

and

$$\alpha^i_j(\sigma^*) = \left(\beta_q^2 P_0^2 + |Q|^2\right)^{(p-4)/2}\left(\delta_i^j\left(\beta_q^2 P_0^2 + |Q|^2\right) + (p-2)Q_i\sum_k g^{jk}Q_k\right).$$

Now

$$\begin{aligned}
&\mathfrak{T}_{\beta_q,\lambda_{q,p}}[\omega] - \mathfrak{T}_{\beta_q,\lambda_{q,p}}[\eta] \\
&\quad = \omega^q + (p-1)\theta^{p-1}(\theta-1)\phi_S^{(p-1)(\theta-1)-1}\left(\beta_S^2\phi_S^2 + |\nabla'\phi_S|^2\right)^{p/2} \\
&\quad = \frac{-1}{\sqrt{|g|}}\sum_{\ell,i}\frac{\partial}{\partial\sigma_\ell}\left[\sqrt{|g|}g^{\ell i}\left(\left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p}{2}-1}\frac{\partial\omega}{\partial\sigma_i} - \left(\beta_q^2\eta^2 + |\nabla'\eta|^2\right)^{\frac{p}{2}-1}\frac{\partial\eta}{\partial\sigma_i}\right)\right] \\
&\qquad - \lambda_{q,p}\left(\left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p}{2}-1}\omega - \left(\beta_q^2\eta^2 + |\nabla'\eta|^2\right)^{\frac{p}{2}-1}\eta\right), \\
&\quad = -\frac{1}{\sqrt{|g|}}\sum_{\ell,i}\frac{\partial}{\partial\sigma_\ell}\left[\sqrt{|g|}g^{\ell i}\left(\sum_j\alpha^i_j\frac{\partial(\omega-\eta)}{\partial\sigma_j} + b^i(\omega-\eta)\right)\right] + \sum_i C_i\frac{\partial(\omega-\eta)}{\partial\sigma_i} \\
&\quad = -\frac{1}{\sqrt{|g|}}\sum_{\ell,j}\frac{\partial}{\partial\sigma_\ell}\left[a^\ell_j\frac{\partial(\omega-\eta)}{\partial\sigma_j}\right] + \sum_i C_i\frac{\partial(\omega-\eta)}{\partial\sigma_i},
\end{aligned}$$

where the C_i are continuous functions and

$$a^\ell_j = \sqrt{|g|}\sum_i g^{\ell i}\alpha^i_j.$$

The matrix $(\alpha^i_j(\sigma_0))$ is symmetric, definite and positive since it is the Hessian of the strictly convex function

$$X = (X_1,\ldots,X_{n-1}) \mapsto \frac{1}{p}\left(P_0^2 + |X|^2\right)^{p/2} = \frac{1}{p}\left(P_0^2 + \sum_{j,k}g^{jk}X_jX_k\right)^{p/2}.$$

Therefore (α^i_j) has the same property in some neighborhood of σ^*, and the same holds true with (a^ℓ_j). Finally the function $\psi = \omega - \eta$ is nonnegative, vanishes at σ^* and satisfies

$$-\frac{1}{\sqrt{|g|}}\sum_{\ell,j}\frac{\partial}{\partial\sigma_\ell}\left[a^\ell_j\frac{\partial\psi}{\partial\sigma_j}\right] + \sum_i C_i\frac{\partial\psi}{\partial\sigma_i} \geqslant 0. \tag{2.9}$$

Then $\psi = 0$ in a neighborhood of S. Since S is connected, ψ is identically 0 which is a contradiction.

If (2.8) holds, then $\theta = 1$ and the graphs of η and ω are tangent at $\bar\sigma$. Proceeding as above and using the fact that $\partial\eta/\partial\nu$ exists and never vanishes on the boundary, we see that $\psi = \eta - \omega$ satisfies (2.9) with a strongly elliptic operator in a neighborhood $\mathcal{N}$ of $\bar\sigma$. Moreover $\psi > 0$ in $\mathcal{N}$, $\psi(\bar\sigma) = 0$ and $\partial\psi/\partial\nu(\bar\sigma) = 0$. This is a contradiction, which ends the proof. □

Remark. If $p=2$, the proof of nonexistence is straightforward by multiplying the equation in ω by the first eigenfunction ϕ_S and get

$$\int_S \big((\lambda_S-\lambda_{q,2})\omega-\omega^q\big)\phi_S\,d\sigma=0,$$

a contradiction since $\lambda_S\leqslant\lambda_{q,2}$.

2.3. Existence results

Let us consider the case $q=q_c=(N(p-1)+p)/(N-p)$ $(N>p>1)$, and let S be any smooth subdomain of S^{N-1}. Since in that case $\lambda_{q,p}=-\beta_{q_c}^2$, the research of solutions of (1.5) under the form (1.6) vanishing on ∂C_S leads to

$$\begin{cases}\mathfrak{T}_{\beta_{q_c},-\beta_{q_c}^2}[\omega]-|\omega|^{q_c-1}\omega=0 & \text{in } S,\\ \omega=0 & \text{in } \partial S,\end{cases}\tag{2.10}$$

where $\beta_{q_c}=N/p-1$. This equation is the Euler–Lagrange variation of the functional J defined on $W_0^{1,p}(S)$ by

$$J(\psi)=\int_S\left(\frac{1}{p}\big(\beta_{q_c}^2\psi^2+|\nabla'\psi|^2\big)^{p/2}-\frac{1}{q_c+1}|\psi|^{q_c+1}\right)d\sigma.\tag{2.11}$$

Theorem 2.1. *Problem* (2.10) *admits a positive solution.*

Proof. Clearly the functional is well defined on $W_0^{1,p}(S)$ since q_c is smaller than the Sobolev exponent p_{N-1}^* for $W^{1,p}$ in dimension $N-1$. For any $\psi\in W_0^{1,p}(S)$, $\lim_{t\to\infty}J(t\psi)=-\infty$. Furthermore there exist $\delta>0$ and $\epsilon>0$ such that $J(\psi)\geqslant\epsilon$ for any $\psi\in W_0^{1,p}(S)$ such that $\|\psi\|_{W^{1,p}}=\delta$. Assume now that $\{\psi_n\}$ is a sequence of $W_0^{1,p}(S)$ such that $J(\psi_n)\to\alpha$ and $\|DJ(\psi_n)\|_{W^{-1,p'}}\to 0$ as $n\to\infty$. Then

$$\mathfrak{T}_{\beta_{q_c},-\beta_{q_c}^2}[\psi_n]-|\psi_n|^{q_c-1}\psi_n=\epsilon_n\to 0.$$

Then

$$\int_S\big(\big(\beta_{q_c}^2\psi_n^2+|\nabla'\psi_n|^2\big)^{p/2}-|\psi_n|^{q_c+1}\big)\,d\sigma=\langle\epsilon_n,\psi_n\rangle.$$

Since $J(\psi_n)\to\alpha$ it follows

$$\int_S\big(\beta_{q_c}^2\psi_n^2+|\nabla'\psi_n|^2\big)^{p/2}\,d\sigma\to p(q_c+1)\alpha/(q_c+1-p).$$

Therefore $\{\psi_n\}$ remains bounded in $L^{q_c+1}(S)$, and relatively compact in $L^r(S)$, for any $1 < r < q_c + 1$. Multiplying the equation $DJ(\psi_n) - \epsilon_n$ by $T_{k,\theta}(\psi_n)$ where $\theta \in (1, (p^*_{N-1} - 1)/q_c)$, $k > 0$ and $T_{k,\theta}(r) = \operatorname{sgn} \min\{|r|, k\}$ and using standard bootstrap arguments yields to the boundedness of $\{\psi_n\}$ in $L^\infty(S)$. Combining this fact with the compactness in $L^r(S)$, we derive the compactness in any L^s, for $s < \infty$. Therefore $\{\psi_n\}$ is relatively compact in $W_0^{1,p}(S)$. This means that J satisfies the Palais–Smale condition. □

3. The 2-dim dynamical system

3.1. Extension of the data

Due to possible applications and similarly to what is done in the semilinear case $p = 2$ (see [3,7,8]), we shall consider the existence problem for 2π-periodic solutions of a more general quasilinear equation than (1.14),

$$\frac{d}{d\sigma}\left[\left(\beta^2\omega^2 + \left(\frac{d\omega}{d\sigma}\right)^2\right)^{p/2-1}\frac{d\omega}{d\sigma}\right] + \lambda\left[\beta^2\omega^2 + \left(\frac{d\omega}{d\sigma}\right)^2\right]^{p/2-1}\omega + g(\omega) - c|\omega|^{p-2}\omega = 0, \tag{3.1}$$

where λ, β, c are real parameters, with $\beta > 0$, and $g \in C^0(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ is odd and satisfies

$$\lim_{s\to 0+} g(s)/s^q = 1, \qquad \lim_{s\to\infty} g(s)/s^{p-1} = \infty, \qquad \frac{d}{ds}\big(g(s)/|s|^{p-1}\big) > 0 \quad \text{on } (0,\infty), \tag{3.2}$$

with $q > p - 1 \geqslant 0$. In fact we can easily reduce the problem to a simpler form, and particularly in the case $p = 1$, where the equation has a remarkable homogeneity property. The next statement is a straightforward computation which transforms the equation satisfied by ω into two more canonic forms.

Lemma 3.1. *Let ω be a solution of* (3.1).

(i) *Assume $p > 1$. If we set*

$$\tau = \beta\sigma, \quad \omega(\sigma) = \beta^{p/(q+1-p)} w(\tau) \quad \textit{and} \quad w' = \frac{dw}{d\tau}, \tag{3.3}$$

then w satisfies

$$\frac{d}{d\tau}\big(\big(w^2 + w'^2\big)^{p/2-1}w'\big) - b\big(w^2 + w'^2\big)^{p/2-1}w + f(w) - d|w|^{p-2}w = 0, \tag{3.4}$$

where

$$b = \frac{-\lambda}{\beta^2}, \qquad d = \frac{c}{\beta^p}, \qquad f(s) = \beta^{-pq/(q+1-p)} g\big(\beta^{p/(q+1-p)} s\big). \tag{3.5}$$

In particular f satisfies the same assumptions (3.2) *as g.*

(ii) *Assume* $p > 1$. *If on any open interval* $I \subset (0, 2\pi)$ *where* $\omega(\sigma) \neq 0$, *we set*

$$\tau = \beta q \sigma \quad \text{and} \quad \omega(\sigma) = (\beta q)^{1/q} \left|w(\tau)\right|^{1/q-1} w(\tau), \tag{3.6}$$

then w *satisfies* (3.4) *on* I, *with*

$$b = -\lambda/\beta^2 q, \qquad d = c/\beta q, \qquad f_1(s) = \beta^{-q} g\big((\beta q s)^{1/q}\big). \tag{3.7}$$

Furthermore f_1 *satisfies the assumptions* (3.2) *with* $q = 1$, *i.e.*

$$\lim_{s\to 0+} f_1(s)/s = 1, \qquad \lim_{s\to\infty} f_1(s) = \infty, \qquad f_1'(s) > 0 \quad \text{on } (0,\infty). \tag{3.8}$$

Due to this result, the changes of variables (3.3) and (3.6) reduce the problem to the study both of existence of periodic solutions of Eq. (3.4), and to characterizing the period function of these solutions, in the range $q > p - 1$ if $p > 1$, and $q > 0$ if $p = 1$.

3.2. Reduction to dynamical systems

We rewrite (3.4) as the system,

$$\begin{cases} w' = F(w, y) = y, \\ y' = G(w, y) = \dfrac{bw^3 + (b + 2 - p)w y^2 - (f(w) - d|w|^{p-2}w)(w^2 + y^2)^{2-p/2}}{w^2 + (p-1)y^2}, \end{cases} \tag{3.9}$$

and we denote by h the odd function defined on $\mathbb{R}$ by

$$h(s) = \begin{cases} f(s)/|s|^{p-2}s & \text{if } s \neq 0, \\ 0 & \text{if } s = 0. \end{cases} \tag{3.10}$$

If $b + d \leqslant 0$, (3.9) has no nontrivial stationary point, while if $b + d > 0$, it admits the two stationary points $\pm P_0$, with $P_0 = (a, 0)$ and $a = h^{-1}(b + d)$. Furthermore P_0 is a center since the linearized system at P_0 is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ -ah'(a) & 0 \end{pmatrix}.$$

System (3.9) is clearly singular at $(0, 0)$. Furthermore it could singular be along the line $w = 0$ if $p = 1$, if $q < 1$, and if $p < 2$ and $d \neq 0$. Actually, for $p > 1$ it is not singular at any points $(0, \sigma)$ with $\sigma \neq 0$. This can be checked as follows: consider the Cauchy problem

$$\begin{cases} w'' = G(w, w'), \quad t \in (-\delta, \delta), \\ w(0) = 0, \quad w'(0) = \sigma, \end{cases} \tag{3.11}$$

and let w be any local solution; since near $(0,\sigma)$, G is continuous with respect to w and C^1 with respect to y, w is C^2; because $\sigma \neq 0$, t can be expressed locally in terms of w. Defining $w'(t) = p(w)$, then p is C^1 near 0, $p(0) = 1$ and satisfies

$$\frac{dp}{dw} = \frac{G(w,p)}{p},$$

with $J(w,p) = G(w,p)/p$. Clearly is C^1 with respect to p and continuous with respect to w, thus one gets local uniqueness of p. and then the local uniqueness of problem $w'(t) = p(w(t))$, $w(0) = 1$, since p is of class C^1.

The phase plane of the system (3.9) is equivariant under symmetries with respect to the two axes of coordinates, because F is even with respect to w and odd with respect to y, and G is odd with respect to w and even with respect to y. Thus from now we can restrict the study to the first quadrant

$$\overline{\mathcal{Q}} \setminus \{(0,0)\}, \quad \text{where } \mathcal{Q} = (0,\infty) \times (0,\infty),$$

where, in particular, $w \geqslant 0$. Due to the symmetries, in the case $p > 1$, any trajectory which meets the two axes in finite times $\tau, \tau + T$ is a closed orbit of period $4T$.

Remark. It is useful to introduce the slope $\xi = w'/w$ (or a function of the slope) as a new variable. This was first used for $p > 1$ in [16] for the homogeneous problem

$$\frac{d}{d\tau}\left(\left(w^2 + w'^2\right)^{p/2-1} w'\right) - b\left(w^2 + w'^2\right)^{p/2-1} w = 0.$$

In that case the function ξ satisfies

$$\frac{d}{d\tau}\left(\left(1+\xi^2\right)^{p/2-1}\xi\right) = -\left((p-1)\xi^2 - b\right)\left(1+\xi^2\right)^{p/2-1},$$

for $w > 0$, and this equation is completely integrable in terms of $u = (1+\xi^2)^{p/2-1}\xi$.

By using polar coordinates in $\mathcal{Q}$

$$(w,y) = (\rho\cos\theta, \rho\sin\theta), \quad \rho > 0,\ \theta \in (0,\pi/2),$$

we transform (3.9) into

$$\begin{cases} \theta' = \dfrac{b - (p-1)\tan^2\theta + (d - h(\rho\cos\theta))\cos^{p-2}\theta}{1 + (p-1)\tan^2\theta}, \\ \rho' = \rho(1+\theta')\tan\theta. \end{cases} \tag{3.12}$$

Equivalently, if we introduce the slope $\xi = \tan\theta \in (0,\infty)$, and set

$$u = \phi(\xi) = \cos^{1-p}\theta\sin\theta, \qquad \phi(\xi) = \left(1+\xi^2\right)^{(p-2)/2}\xi, \tag{3.13}$$

then $\phi'(\xi) = (1+\xi^2)^{(p-4)/2}(1+(p-1)\xi^2)$; thus ϕ is strictly increasing: from $(0,\infty)$ into $(0,\infty)$ when $p > 1$, and from $(0,\infty)$ into $(0,1)$ when $p = 1$. Defining

$$\varphi = \phi^{-1} \quad \text{and} \quad E(\xi) = \big((p-1)\xi^2 - b\big)\big(1+\xi^2\big)^{p/2-1}, \tag{3.14}$$

we obtain

$$\begin{cases} w' = w\varphi(u), \\ u' = -E\big(\varphi(u)\big) - h(w) + d. \end{cases} \tag{3.15}$$

This system is still singular on the line $w = 0$ if $h \notin C^1([0,\infty))$ near 0. In the sequel we set

$$\Psi(u) = \int_0^u \varphi(s)\,ds. \tag{3.16}$$

Noticing that

$$E'(\xi) = \big(p(p-1)\xi^2 + 2(p-1) - (p-2)b\big)\big(1+\xi^2\big)^{(p-2)/2}\xi, \tag{3.17}$$

we derive that E is increasing on $(0,\infty)$ when $(p-2)b \leqslant 2(p-1)$. When $(p-2)b > 2(p-1)$, E is decreasing on $(0,\eta)$ and then increasing, where η is defined by

$$p(p-1)\eta^2 = (p-2)b - 2(p-1), \tag{3.18}$$

and

$$\min E = E(\eta) = -\frac{2}{p-2}\left(\frac{(p-2)(b+p-1)}{p(p-1)}\right)^{p/2}. \tag{3.19}$$

In the case of initial problem (1.14), E is increasing.

Remark. If $p > 1$, system (3.9) is singular at $(0,0)$. If we replace the assumption $\lim_{s\to 0+} f(s)/s^q = 1$, by the stronger one

$$\lim_{s\to 0+} f'(s)/s^{q-1} = q, \tag{3.20}$$

we can transform system (3.15) in $(0,\infty)\times\mathbb{R}$ in a system of the same type, but without singularity: this is obtained by performing the substitution $v = w^{q+1-p}$. Then

$$\begin{cases} v' = (q+1-p)v\varphi(u), \\ u' = -E\big(\varphi(u)\big) - \tilde{h}(v) + d, \end{cases} \tag{3.21}$$

where $v \mapsto \tilde{h}(v) = h(v^{1/(q+1-p)}) \in C^1([0,1))$. In particular, if $f(w) = |w|^{q-1}w$, we find

$$\begin{cases} v' = (q+1-p)v\varphi(u), \\ u' = -E\big(\varphi(u)\big) - v + d. \end{cases} \tag{3.22}$$

Remark. In the case $f(w) = |w|^{q-1}w$, we can differentiate the equation relative to u' and obtain that u satisfies the following equation

$$u'' = B(\varphi(u))u' + (q+1-p)(E(\varphi(u)) - d)\varphi(u), \tag{3.23}$$

where E is given above, and

$$B(\xi) = \frac{(p-2)b + q - 3(p-1) + (q+1-2p)(p-1)\xi^2}{1+(p-1)\xi^2}\xi. \tag{3.24}$$

Notice that Eq. (3.23) has no singularity for $p > 1$.

4. The case $p > 1$

4.1. Existence of a first integral

A natural question is to see if Eq. (3.4) admits a variational structure. When $p = 2$, it is the case, for any b and d. Since (3.4) takes the form

$$w'' - (b+d)w + f(w) = 0,$$

it is the Euler equation of the functional

$$\mathcal{H}_2(w, w') = \frac{w'^2}{2} + (b+d)\frac{w^2}{2} - \mathcal{F}(w),$$

where $\mathcal{F}(w) = \int_0^w f(s)\,ds$. Thus the function $w'^2 = (b+d)w^2 - 2\mathcal{F}(w)$ is constant along the trajectory. When $p \neq 2$, $p > 1$, we find that a first integral exists only in the case $b = 1$. In such a case (3.4) is the Euler equation of the functional

$$\mathcal{H}(w, w') = \frac{(w^2 + w'^2)^{p/2}}{p} + d\frac{|w|^p}{p} - \mathcal{F}(w).$$

Therefore, the associated Painlevé integral

$$\mathcal{P}(w, w') = \frac{1}{p}(w^2 + w'^2)^{p/2-1}((p-1)w'^2 - w^2) - \frac{d|w|^p}{p} + \mathcal{F}(w) \tag{4.1}$$

is constant along the trajectories. Using the function E introduced at (3.14), then (4.1) is equivalent to

$$E\left(\frac{w'}{w}\right) = E(\varphi(u)) = d - p\frac{K + \mathcal{F}(w)}{w^p} \tag{4.2}$$

for $w > 0$. Hence E is increasing on $(0, \infty)$ from $-b = -1$ to $+\infty$.

In the general case, we cannot use a first integral for studying the periodicity properties of the solutions, while it was the main tool in [3] for $p = 2$. This is the reason for which we are lead to use phase plane techniques. Notice that, for the initial problem (1.14), the value $b = 1$ corresponds to the case $p < 2$ and $q = (3p-2)/(2-p)$.

4.2. Description of the solutions

In this section we describe in full details the trajectories of system (3.9) in the phase plane (w, y). Notice that the system can be singular on the axis $w = 0$.

Proposition 4.1. *Assume $p > 1$. Then all the orbits of system* (3.9) *are bounded. Any trajectory $\mathcal{T}_{[P]}$ issued from a point P in $\mathcal{Q}$ is*

(i) *either a closed orbit surrounding* $(0, 0)$*, or*
(ii) *if $b + d > 0$, a closed orbit surrounding P_0 but not* $(0, 0)$*, or*
(iii) *a homoclinic orbit defined on $\mathbb{R}$, starting from* $(0, 0)$ *with initial slope*

$$\lim_{t\to-\infty} \frac{w'(t)}{w(t)} = m,$$

where m is defined $E(m) = d$, and ending at $(0, 0)$ *with*

$$\lim_{t\to\infty} \frac{w'(t)}{w(t)} = -m.$$

Proof. We recall that E and u are defined by (3.13) and (3.14), by using polar coordinates (ρ, θ) in the (w, y)-plane.

First look at the vector field on the boundary of $\mathcal{Q}$. At any point $(0, \sigma)$ with $\sigma > 0$, it is given by $(\sigma, 0)$, thus it is transverse and inward. At any point $(\bar{w}, 0)$ with $\bar{w} > 0$, it is given by $(0, \bar{w}(b + d) - h(\bar{w})))$. Thus it is transverse and outward whenever $b + d \leqslant 0$ or $b + d > 0$ and $\bar{w} > a$, and inward whenever $b + d > 0$ and $\bar{w} < a$.

Consider any solution (w, y) of the system, such that $P = (w(0), y(0)) \in \mathcal{Q}$, and let (τ_1, τ_2) be its maximal interval existence in $\mathcal{Q}$. At any point τ where $u'(\tau) = 0$ and $u(\tau) > 0$, there holds $u''(\tau) = -h'(w)w\varphi(u) < 0$ from (3.15). Thus if τ exists, it is unique, and it is a maximum for u.

Since $w' = y > 0$, w has the limits $\ell_2 \in (0, \infty]$ as $\tau \uparrow \tau_2$ and $\ell_1 \in [0, \infty)$ as $\tau \downarrow \tau_1$. Therefore u is strictly monotone near τ_1 and τ_2, thus it has limits $u_1, u_2 \in [0, \infty]$, in other words θ has limits $\theta_1, \theta_2 \in [0, \pi/2]$.

(i) Let us go forward in time. On any interval where u is increasing, one has $E(\varphi(u)) \leqslant d$, thus u is bounded and, consequently, u_2 is finite. If $\ell_2 = \infty$, then $\theta'(\tau) \to -\infty$, as $\tau \uparrow \tau_2$; by (3.13), ρ is decreasing, thus it is bounded, which is contradictory; thus ℓ_2 is finite. If $u_2 > 0$ then $(\ell_2, \ell_2\varphi(u_2))$ is stationary, which is impossible. Thus u is decreasing to 0, and the trajectory converges to $(\ell_2, 0)$. If $b + d > 0$ and $\ell_2 = a$, u' tends to 0 from (3.15), and

$$u'' = -(E \circ \varphi)'(u)u' - h'(w)w\varphi(u) = -h'(a)a\varphi(u)\bigl(1 + o(1)\bigr);$$

therefore $u'' < 0$ near τ_2, which is impossible. Finally, either $b+d \leqslant 0$, or $b+d > 0$ and $\bar{w} > a$, and τ_2 is finite, the trajectory leaves $\mathcal{Q}$ transversally at τ_2.

(ii) Next let us go backward in time.

• Suppose $u_1 = 0$. Clearly the trajectory converges to $(\ell_1, 0)$; then necessarily $b + d > 0$ and $\ell_1 \leqslant a$, thus $\ell_1 < a$ as above. The trajectory enters $\mathcal{Q}$ transversally at τ_2, and from the symmetries it is a closed orbit surrounding only the stationary point P_0.

• Next, suppose $u_1 = \infty$. It means that θ tends to $\pi/2$. Then from (3.12), θ' tends to 1, thus τ_1 is finite,

$$\pi/2 - \theta = (\tau - \tau_1)\big(1 + o(1)\big), \qquad \tan\theta = (\tau - \tau_1)^{-1}\big(1 + o(1)\big),$$

and

$$(p-1)(\tau - \tau_1)^{-1}\frac{\rho'}{\rho} = \big(b + 1 + \big(d - h(\rho\cos\theta)\big)\cos^{p-2}\theta\big)\big(1 + o(1)\big).$$

If $p \geqslant 2$, then $\rho'/\rho = O((\tau - \tau_1))$; if $p < 2$ then $\rho'/\rho = O((\tau - \tau_1)^{p-1})$. In any case, $\ln\rho$ case is integrable, thus ρ has a finite limit $\bar{y} > 0$. Then the trajectory enters $\mathcal{Q}$ transversally at τ_1 and from the symmetries it is a closed orbit surrounding $(0,0)$. From the considerations in Section 3.2, for any $\bar{y} > 0$ there exists such an orbit, and it is unique. Moreover in $\mathcal{Q}$ the slope $w'/w = \xi = \varphi(u)$ is decreasing from ∞ to 0; indeed it decreases near τ_1 and τ_2 and can only have a maximal point.

• At end, suppose $0 < u_1 < \infty$. If $\ell_1 > 0$, then $(\ell_1, \ell_1\varphi(u_1))$ is stationary, which is impossible. Thus (y, w) converges to $(0,0)$. And w'/w tends to $\varphi(u_1)$, thus $\tau_1 = -\infty$. And u' converges to $d - E(\varphi(u_1))$, thus $\tan\theta = \varphi(u)$ has a limit $m \geqslant 0$ such that $E(m) = d$. From the symmetries the trajectory is homoclinic and the solution w is defined on $\mathbb{R}$. □

The next theorem studies the precise behaviour of solutions according to the sign of $b + d$.

Theorem 4.2. *Assume $p > 1$ and consider system* (3.9) *in the (w, y)-plane.*

(i) *Assume $b + d > 0$. Then there exists a unique homoclinic trajectory $\mathcal{H}$ starting from* $(0,0)$ *in $\mathcal{Q}$ with initial slope $m_d = E^{-1}(d)$ ($m_0 = \sqrt{b/(p-1)}$ if $d = 0$), ending at* $(0,0)$ *with the slope $-m_d$, and surrounding P_0. Up to the stationary points, the other orbits are closed, and either they surround only one of the points P_0 or $-P_0$, in the domain delimited by $\mathcal{H}$, corresponding to solutions w of constant sign, or they are exterior to $\pm\mathcal{H}$ and surround* $(0,0)$ *and $\pm P_0$, corresponding to sign changing solutions w.*

(ii) *Assume $b + d \leqslant 0$. Then*

- *if $(p-2)b \leqslant 2(p-1)$, or $[(p-2)b > 2(p-1)$ and $d < E(\eta)]$, there is no homoclinic trajectory*;
- *if $[(p-2)b > 2(p-1)$ and $E(\eta) < d \leqslant -b]$, then denoting by $m_{1,d} < m_{2,d}$ the two positive roots of equation $E(m) = d$, there exist infinitely many homoclinic trajectories $\mathcal{H}_1$ starting from* $(0,0)$ *in $\mathcal{Q}$ with the initial slope $m_{1,d}$ and ending at* $(0,0)$ *with the final slope $-m_{1,d}$, and a unique homoclinic trajectory $\mathcal{H}_2$ starting from* $(0,0)$ *in $\mathcal{Q}$ with initial slope $m_{2,d}$ and ending at* $(0,0)$ *with final slope $-m_{2,d}$.*

Proof. (i) *Case $b + d > 0$.* Then the equation $E(m) = d$ has a unique positive solution $m = E^{-1}(d)$; and w'/w tends to m; thus the trajectory starts from $(0,0)$ with a slope m. Then for any $P \in \mathcal{Q}$, the trajectory $\mathcal{T}_{[P]}$ passing through P meets the axis $y = 0$ after P at some point $(\mu, 0)$ with $\mu > a$. Denote

$$\mathcal{U} = \big\{P \in \mathcal{Q}\colon\ \mathcal{T}_{[P]} \cap \big\{(0,\sigma)\colon\ \sigma > 0\big\} \neq \emptyset\big\},$$
$$\mathcal{V} = \big\{P \in \mathcal{Q}\colon\ \mathcal{T}_{[P]} \cap \big\{(\mu,0)\colon\ 0 < \mu < a\big\} \neq \emptyset\big\}. \tag{4.3}$$

Then either $P \in \mathcal{U}$ and the trajectory is a closed orbit surrounding $(0,0)$ and $\pm P_0$, and in $\mathcal{Q}$. Or $P \in \mathcal{V}$ and the trajectory is a closed orbit surrounding only P_0. Or $\mathcal{T}_{[P]}$ is a homoclinic orbit $\mathcal{H}$ starting from $(0,0)$ with the slope m, where m is the unique solution of equation $E(m) = d$ (such that $m > \eta$ if E is not monotone, see (3.17)). Next $\mathcal{U}$ and $\mathcal{V}$ are open, since the vector field is transverse on the axes, thus $\mathcal{U} \cup \mathcal{V} \neq \mathcal{Q}$. This shows the existence of such an orbit $\mathcal{H}$.

(ii) *Case* $b + d \leqslant 0$.

• Either $b + d < 0$ and E is increasing, or E has a minimum at η and $d < E(\eta)$. In such a case equation $E(m) = d$ has no solution, and there is no homoclinic orbit. Or E is increasing and $b + d = 0$; then $E(\varphi(u)) > -b = d$, thus $u' < 0$, thus u cannot tend to 0, and the same conclusion holds.

• Or E has a minimum at η and $E(\eta) < d \leqslant -b$. In that case the equation $E(m) = d$ has two roots m_1, m_2 such that $0 \leqslant m_1 < \eta < m_2 \leqslant m_b$, where $m(b)$ is defined by $E(m_b) = -b$. Any trajectory $\mathcal{T}_{[P]}$ such that $P \in \mathcal{U}$ (see (4.3) for the definition) satisfies $u' < 0$, it means $h(w) > d - E(\varphi(u))$ and the range of u is $(0, \infty)$, therefore there exists τ such that $\varphi(u)(\tau) = \eta$, hence $h(w(\tau)) > d - E(\eta)$ and $y(\tau) = \eta w(\tau)$. Next consider any trajectory $\mathcal{T}_{[\tilde{P}]}$ starting from $\tilde{P} = (\tilde{w}, \eta\tilde{w})$ such that $h(\tilde{w}) \leqslant d - E(\eta)$. It cannot be a trajectory of the preceding type, thus $(y, w) \to (0,0)$ as $\tau \to \tau_1$, and θ tends to θ_1, with $\tan\theta_1 = m_1$ or m_2; moreover $u'(0) \geqslant 0$, and $u' < 0$ near τ_2, thus there exists a unique $\tau \geqslant 0$ such that $u'(\tau) = 0$; then $u' > 0$ in (τ_1, τ), therefore $\tan\theta_1 < \eta$, and finally $\tan\theta_1 = m_1$. Consequently there exist infinitely many such trajectories $\mathcal{H}_1$, with initial slope m_1. Next fix one trajectory $\mathcal{T}_{[\tilde{P}_0]}$ such that $h(\tilde{w}_0) \leqslant d - E(\eta)$. Let $\mathcal{R}$ be the subdomain of $\mathcal{Q}$ delimited by $\mathcal{T}_{[\tilde{P}_0]}$ and $\mathcal{T}_{[(0,1)]}$ and

$$\mathcal{V}=\bigl\{P \in \mathcal{R}\colon\ \mathcal{T}_{[P]} \cap \bigl\{(w, \eta w)\colon\ 0 < w < \tilde{w}_0\bigr\} \neq \emptyset\bigr\}.$$

The set $\mathcal{V}$ is open because the intersection with the line $y = \eta w$ for $w < \tilde{w}_0$ is transverse since at the intersection point, $h(\tilde{w}) < d - E(\eta)$, thus $u' > 0$, and $y/w = \varphi(u) = \eta$, and

$$\frac{y'}{y} = \varphi(u) + \frac{\varphi'(u)}{\varphi(u)}u' > \eta = \frac{w'}{w}.$$

Then $(\mathcal{U} \cap \mathcal{R}) \cup \mathcal{V} \neq \mathcal{R}$. Then there exists at least a trajectory $\mathcal{H}_{1,*}$ starting from $(0,0)$ with initial slope m_2.

(iii) *Uniqueness of* $\mathcal{H}$ *and* $\mathcal{H}_2$. Let $m = m_0$ or $m_{2,d}$. Suppose that system (3.9) has two solutions (w_1, y_1), (w_2, y_2) defined near $-\infty$, such that $w_i > 0$ and $w_i(\tau)$ tends to 0 and $y_i(\tau)/w_i(\tau)$ tend to m as $\tau \downarrow -\infty$. Then the system (3.15) has two local solutions (w_1, u_1), (w_2, u_2) such that $\varphi(u_i)$ tends to m at $-\infty$. Then $w_i' > 0$ locally and one can express u_i as a function of w_i. Then at the same point w,

$$w\frac{d\Psi(u_i)}{dw} = w\varphi(u_i)\frac{du_i}{dw} = -E\bigl(\varphi(u_i)\bigr) - h(w) + d,$$

$$w\frac{d(\Psi(u_2) - \Psi(u_1))}{dw} = -E\bigl(\varphi(u_2)\bigr) - E\bigl(\varphi(u_1)\bigr) = E'\bigl(\varphi\bigl(u^*\bigr)\bigr)\varphi'\bigl(u^*\bigr)(u_2 - u_1)$$

for some u^* between u_1 and u_2, and $E'(\varphi(u^*)) = E'(m)(1 + o(1))$; and $E'(m) > 0$. Then for small w

$$\frac{d(\Psi(u_2) - \Psi(u_1))}{dw}\bigl(\Psi(u_2) - \Psi(u_1)\bigr) < 0,$$

which implies that $(\Psi(u_2)-\Psi(u_1))^2$ is decreasing, with limit 0 at 0. Therefore $\Psi(u_2)=\Psi(u_1)$, thus $u_2\equiv u_1$ near $-\infty$; but from (3.15), $h(w_1)=h(w_2)$, and since h is one to one, it follows $w_1\equiv w_2$ near $-\infty$. The global uniqueness follows, since the system is regular except at $(0,0)$. All the trajectories are described. □

Remark. Under the assumption (3.20), existence and uniqueness of $\mathcal{H}$ and $\mathcal{H}_2$ can be obtained in a more direct way whenever $d\neq E(\eta)$. Indeed the system (3.21) relative to (v,u) is regular, with stationary points $(0,0)$, $(0,\pm\varphi^{-1}(m))$, where $m=m_0,m_1$ or m_2 and also $(\pm a,0)$ if $b+d>0$. The linearized system at $(0,\varphi^{-1}(m))$ is given by the matrix $\begin{pmatrix} m(q+1-p) & 0 \\ 0 & K(m)\end{pmatrix}$, with $K(m)=p(p-1)(\eta^2-m^2)/(1+(p-1)m^2)$. If $m=m_{1,d}$, then it is a source, and we find again the existence of an infinity of solutions. If $m=m_d$ or $m=m_{2,d}$, then $K(m)<0$, thus this point is a saddle point. Then in the phase plane (v,u), there exists precisely one trajectory defined near $-\infty$, such that $v>0$ and converging to $(0,m)$ at $-\infty$, and u/v converges to 0.

Remark. Suppose $f(w)=|w|^{q-1}w$, then we can study the critical case $(p-2)b>2(p-1)$ and $E(\eta)=d$: there exist infinitely many homoclinic trajectories $\mathcal{H}_1$ starting from $(0,0)$ in $\mathcal{Q}$ with an infinite initial slope and ending at $(0,0)$ with an infinite slope, and a unique homoclinic trajectory $\mathcal{H}_2$ starting from $(0,0)$ in $\mathcal{Q}$ with the initial slope η and ending at $(0,0)$ with the slope $-\eta$. Indeed using system (3.22) and setting $u=\varphi^{-1}(\eta)+z$, and $\zeta=(q+1-p)\eta z+v$, it can be written under the form

$$\zeta'=P(\zeta,v),\qquad v'=(q+1-p)\eta v+Q(\zeta,v),$$

where P and Q both start with quadratic terms. Moreover the quadratic part of $P(\zeta,v)$ is given by $p_{2,0}\zeta^2+p_{1,1}\zeta v+p_{0,2}v^2$, where by computation,

$$p_{2,0}=-\frac{p(p-1)}{q+1-p}\eta\varphi'^2\big(\varphi(\eta)\big)\big(1+\eta^2\big)^{(p-2)/2}<0.$$

The results follow from the description of saddle-node behaviour given in [13, Theorem 9.1.7].

Remark. In the case $b=1>-d$, we have a representation of the homoclinic trajectory: it corresponds to $K=0$ in (4.2). In the case $f(w)=|w|^{q-1}w$, in terms of u we obtain

$$u'=\frac{q+1-p}{p}\big(E\big(\varphi(u)\big)-d\big),$$

which allows to compute u by a quadrature.

4.3. Period of the solutions

First we consider the sign changing solutions.

Theorem 4.3. *Assume $p>1$. For any $\nu>0$ let $\mathcal{T}_{[(0,\nu)]}$ be the trajectory which starts from $(0,\nu)$, and let $T(\nu)$ be its least period. Then $\nu\mapsto T(\nu)$ is decreasing on $(0,\infty)$. Furthermore the range of $T(.)$ can be computed in the following way.*

(i) *If $b + d \leqslant 0$ and $m \mapsto E(m)$ is increasing, or if $d < \min E$, then $T(.)$ decreases from T_d to 0, where*

$$T_d = 4\int_0^\infty \frac{du}{E(\varphi(u)) - d} = 4\int_0^{\pi/2} \frac{1 + (p-1)\tan^2\theta}{(p-1)\tan^2\theta - b - d\cos^{p-2}\theta}\, d\theta, \tag{4.4}$$

and T_d is finite if and only if $b + d < 0$. If $b < 0 = d$, then

$$T_0 = 2\pi \frac{(p-1)\gamma + 1}{(p-1)\gamma(\gamma+1)} \quad \textit{with } \gamma = \sqrt{|b|/(p-1)}. \tag{4.5}$$

(ii) *If $b + d > 0$ or $b + d \leqslant 0$ and $d \geqslant \min E$, then $T(.)$ decreases from ∞ to 0.*

Proof. *Step 1. Monotonicity of T.* Consider the part of the trajectories $\mathcal{T}_{[(0,\nu)]}$ located in $\mathcal{Q}$, given by (w_ν, y_ν). We have already shown that u is decreasing with respect to τ from ∞ to 0, then $E(\varphi(u)) + h(w_\nu(u)) - d > 0$ and w_ν can be expressed in terms of u, and

$$T(\nu) = 4\int_0^\infty \frac{du}{E(\varphi(u)) + h(w_\nu(u)) - d}. \tag{4.6}$$

Let $\lambda > 1$. Since the trajectories $\mathcal{T}_{[(0,\nu)]}$ and $\mathcal{T}_{[(0,\lambda\nu)]}$ have no intersection point, $w_{\lambda\nu}(u) > w_\nu(u)$ for any $u > 0$, and h is nondecreasing, thus $T(\lambda\nu) < T(\nu)$, and T is decreasing.

Step 2. Behaviour near ∞. Let $\nu_n \geqslant 1$, such that $\lim \nu_n = \infty$. Observe that for fixed u, for any integer $n \geqslant 1$, there exists a unique $\tilde{\nu}_n > 0$ (depending on u), such that $w_{\tilde{\nu}_n}(u) = n$; let $\hat{\nu}_n = \max(\tilde{\nu}_n, n)$. Then $h(w_{\hat{\nu}_n}(u)) \geqslant h(n)$, thus $h(w_{\hat{\nu}_n}(u))$ converges to ∞; since $\nu \mapsto h(w_\nu(u))$ is nondecreasing then $h(w_{\nu_n}(u))$ converges to ∞, and $T(\nu_n)$ converges to 0, using the Beppo–Levi theorem.

Step 3. Behaviour near 0.

• First assume $b + d \leqslant 0$, and E is increasing, or $d < E(\eta)$. Then all the orbits are of the type $\mathcal{T}_{[(0,\nu)]}$. Let $\nu_n \in (0, 1)$, such that $\lim \nu_n = 0$. For fixed u and any integer $n \geqslant 1$, there exists a unique $\bar{\nu}_n > 0$ (depending on u), such that $w_{\bar{\nu}_n}(u) = 1/n$; let $\check{\nu}_n = \min(\bar{\nu}_n, 1/n)$. Then $h(w_{\check{\nu}_n}(u)) \leqslant h(1/n)$, thus $h(w_{\check{\nu}_n}(u))$ converges to 0, and again $h(w_{\nu_n}(u))$ converges to 0. Then $T(\nu_n)$ converges to T_d given by (4.4), using the Beppo–Levi theorem. If $b + d < 0$, then T_d is finite: indeed near ∞, $E(\varphi(u)) = (p-1)u^{p/(p-1)}(1 + o(1))$; if E is increasing, then $E(\varphi(u)) - d > -(b + d) > 0$; if $d < E(\eta)$, then $E(\varphi(u)) - d \geqslant E(\eta) - d > 0$.

If $b + d = 0$ and E is increasing, then $T_d = \infty$: indeed near 0,

$$E\big(\varphi(u)\big) - d = u^2\big(E''(0)/2 + o(1)\big)$$

and

$$E''(0) = 2(p-1) - (p-2)b \quad \text{if } (p-2)b = 2(p-1).$$

Therefore

$$E\big(\varphi(u)\big) - d = \big(p(p-1)/4\big)u^4\big(1 + o(1)\big).$$

In all the cases the integral (4.6) giving T is divergent.

When $b<0=d$, one can compute T_0:

$$\frac{T_0}{4}=\int_0^\infty \frac{du}{E(\varphi(u))}=\int_0^\infty \frac{\phi'(\xi)\,d\xi}{E(\xi)}=\int_0^\infty \frac{1+(p-1)\xi^2}{(|b|+(p-1)\xi^2)(1+\xi^2)}\,d\xi$$

$$=\frac{\pi}{2}+\left(\frac{1}{p-1}-\gamma^2\right)\int_0^\infty \frac{ds}{(\gamma^2+s^2)(1+s^2)}=\frac{\pi}{2}\left(1+\frac{1-(p-1)\gamma^2}{(p-1)\gamma(\gamma+1)}\right).$$

Hence (4.5) holds.

• Next assume $d>E(\eta)$. Considering ν_n as above, for any fixed u such that $\varphi(u)>m_2$, there exists a unique $\bar\nu_n>0$ (depending on u), such that $w_{\bar\nu_n}(u)=1/n$. As above,

$$\int_{\varphi^{-1}(m_2)}^\infty \frac{du}{E(\varphi(u))+h(w_{\bar\nu_n})-d}\to\int_{\varphi^{-1}(m_2)}^\infty \frac{du}{E(\varphi(u))-d}=\infty,$$

since $E'(m_2)$ is finite. As a consequence, $T(\nu_n)$ tends to ∞. If $d=E(\eta)$, the same proof still works with m_2 replaced by η: the integral is still divergent because the denominator is of order 2 in $u-\varphi^{-1}(\eta)$, as, near 0, there holds

$$E\big(\varphi(u)\big)-d=\frac{1}{2}E''(\eta)\big(\varphi(u)-\eta\big)^2\big(1+o(1)\big)=\frac{1}{2}E''(\eta)\big(\varphi(u)-\eta\big)^2\big(1+o(1)\big),$$

and

$$E''(\eta)=2p(p-1)\eta^2\big(1+\eta^2\big)^{(p-2)/2}>0.$$

At last suppose $b+d>0$; the same proof with m_2 replaced by m shows that $T(\nu)$ converges to ∞ as ν tends to 0, since $E'(m)$ at $m=E^{-1}(d)$ is finite. □

The monotonicity of the period function is a more general property, since we have the following result.

Proposition 4.4. *Let $F,G\in C^1(\mathbb{R}^2\setminus(0,0))$ are such that F (respectively G) is odd with respect to y (respectively x) and even with respect to x (respectively y), with $F(w,y)>0$ in $\mathcal{Q}$. Assume that for any $(w,y)\in\mathcal{Q}$ and any $\lambda>0$,*

$$\frac{\partial}{\partial\lambda}\left(\frac{F(\lambda w,\lambda y)}{\lambda}\right)\geqslant 0\quad(\textit{respectively}\leqslant 0)\quad\textit{and}$$
$$\frac{\partial}{\partial\lambda}\left(\frac{G(\lambda w,\lambda y)}{\lambda}\right)<0\quad(\textit{respectively}>0).\tag{4.7}$$

Assume also that for any σ in some interval (σ_1,σ_2) (where $0<\sigma_1<\sigma_2$), the trajectory $\mathcal{T}_{[(0,\sigma)]}$ of solution of the system

$$\begin{cases} w' = F(w, y), \\ y' = G(w, y) \end{cases} \tag{4.8}$$

passing through $(0, \sigma)$ (*necessarily entering* $\mathcal{Q}$ *since* $F(0, \sigma) > 0$) *leaves* $\mathcal{Q}$ *transversally in a finite time* $T(\sigma)/4$ *at some point* $(c(\sigma), 0)$ (*thus* $G(c(\sigma), 0) < 0$). *Then* (*from the symmetries*), $\mathcal{T}_{[(0,\sigma)]}$ *is a closed orbit surrounding* $(0, 0)$, *with period* $T(\sigma)$, *and* $\sigma \mapsto T(\sigma)$ *is decreasing* (*respectively increasing*) *on* (σ_1, σ_2).

Remark. We can notice the condition on F is equivalent to $F(\lambda w, \lambda y) \geqslant \lambda F(w, y)$ for any $\lambda > 1$. The second condition implies that for any $\lambda > 1$,

$$G(\lambda w, \lambda y) < \lambda G(w, y) \quad \big(\text{respectively } G(\lambda w, \lambda y) > \lambda G(w, y)\big).$$

Proof of Proposition 4.4. In polar coordinates (ρ, θ) in $\mathcal{Q}$, we get

$$\rho' = F \cos\theta + G \sin\theta, \qquad \theta' = \frac{1}{\rho}(G \cos\theta - F \sin\theta).$$

At each point τ where $\theta'(\tau) = 0$, there holds

$$\rho\theta''(\tau) = \left(\frac{\partial G}{\partial \rho} \cos\theta - \frac{\partial F}{\partial \rho} \sin\theta\right), \qquad \rho'(\tau) = \frac{F}{\cos\theta}\left(\frac{\partial G}{\partial \rho} \cos\theta - \frac{\partial F}{\partial \rho} \sin\theta\right).$$

But (4.7) is equivalent to $\partial F/\partial\rho \geqslant F/\rho$ and $\partial G/\partial\rho < G/\rho$ (respectively $>$), thus

$$\rho\theta''(\tau) < \frac{F}{\rho\cos\theta}(G \cos\theta - F \sin\theta) = 0 \quad (\text{respectively } >).$$

In both cases θ'' has a constant sign. But $\theta'(0) = -F(0, \sigma) < 0$ and $\theta'(\sigma) = G(c(\sigma), 0) < 0$ thus we get a contradiction by considering the first (respectively the last) point where $\theta'(\tau) = 0$, which satisfies $\theta''(\tau) \geqslant 0$ (respectively $\leqslant 0$). Thus θ is decreasing from $\pi/2$ to 0. Then the curves can be represented in function of θ by $(\rho(\sigma, \theta), \theta(\sigma))$, and

$$T(\sigma) = 4 \int_0^{\pi/2} \frac{d\theta}{H(\rho(\sigma, \theta), \theta)}$$

with

$$H(\rho, \theta) = \frac{1}{\rho}\big(F(\rho\cos\theta, \rho\sin\theta)\sin\theta - G(\rho\cos\theta, \rho\sin\theta)\cos\theta\big).$$

Let $\lambda > 1$. Since the trajectories $\mathcal{T}_{[(0,\sigma)]}$ and $\mathcal{T}_{[(0,\lambda\sigma)]}$. have no intersection point, then $\rho(\lambda\sigma, \theta) > \rho(\sigma, \theta)$ for any $\theta \in (0, \pi/2)$; by assumption, for fixed θ, the function $\rho \mapsto F(\rho\cos\theta, \rho\sin\theta)/\rho$ is nondecreasing (respectively nonincreasing) and $\rho \mapsto G(\rho\cos\theta, \rho\sin\theta)/\rho$ is decreasing (respectively increasing), thus $H(\rho(\lambda\sigma, \theta), \theta) > H(\rho(\sigma, \theta), \theta)$, which yields to $T(\lambda\sigma) < T(\sigma)$ (respectively $>$). This implies that T is decreasing (respectively increasing). $\square$

Next we consider the positive solutions w of Eq. (3.4).

Proposition 4.5. *Assume $p > 1$ and $b + d > 0$. Consider the trajectories $\mathcal{T}_{[(\mu,0)]}$ in the phase plane (w, y) which goes through $(\mu, 0)$, for some $\mu \in (0, a)$. Let $T^+(\mu)$ be their least period. Then*

$$\lim_{\mu\to 0} T^+(\mu) = \infty, \qquad \lim_{\mu\to a} T^+(\mu) = \frac{2\pi}{\sqrt{ah'(a)}}.$$

In particular if $f(w) = |w|^{q-1}w$, then $\lim_{\mu\to a} T^+(\mu) = 2\pi/(q + 1 - p)(b + d)$.

Proof. We notice that the trajectory $\mathcal{T}_{[(\mu,0)]}$ intersects the line $y = 0$ at $(\mu, 0)$ and another point $(g(\mu), 0)$, with $\mu < a < g(\mu)$, and g is decreasing.

Step 1. Behaviour near a. When μ tends to a, then also $g(\mu)$ tends to a. Indeed for any small $\varepsilon > 0$, then $g(\mu) - a < \varepsilon$ as soon as $\mu - a < \min(\varepsilon, a - g^{-1}(a + \varepsilon))$. Since, along such a trajectory in $\mathcal{Q}$, $\xi = \varphi(u)$ varies from 0 to 0, it has a maximal ξ^*, where $u' = 0$, thus $E(\xi^*) = h(w^*)$. When μ tends to a, then $h(w^*)$ tends to b, thus ξ^* tends to $E^{-1}(b) = 0$, thus also $\max_{y\in\mathcal{T}_{[(\mu,0)]}} |y|$ tends to 0. Using the linearized form of the system at P_0, and polar coordinates with center $(a, 0)$, $w = a + r\cos\eta$, $y = \sqrt{ah'(a)}r\sin\eta$, then r tends to 0 as μ tends to a, and one finds $\eta' = -\sqrt{ah'(a)} + R/r$, where R involves the derivatives of G of order 2, which are bounded near the point $(a, 0)$, thus R/r^2 is bounded. Therefore η' tends to $-\sqrt{ah'(a)}$, and finally $T^+(\mu)$ tends to $2\pi/\sqrt{ah'(a)}$.

Step 2. Behaviour near 0. On the trajectory $\mathcal{T}_{[(\mu,0)]}$, the function u is increasing up to a maximal value $u^*(\mu)$, and then decreasing; moreover u^* is a nonincreasing function of μ, because two different trajectories have no intersection. Let $\mu_n \in (0, a)$, such that $\lim \mu_n = 0$. For any n there exists $\tilde{\mu}_n \in (0, a)$ such that the orbit $\mathcal{T}_{[(\tilde{\mu}_n,0)]}$ contains a point above the line $y = \varphi^{-1}(m)(1 - 1/n)w$, let $\hat{\mu}_n = \min(\mu_n, 1/n)$. Then $u^*(\hat{\mu}_n) \geqslant \varphi^{-1}(m)(1 - 1/n)$, thus $u^*(\mu_n)$ tends to m; then from the Beppo–Levi theorem

$$\liminf T^+(\mu) \geqslant \lim \int_{u^*(\mu)}^{\infty} \frac{du}{E(\varphi(u)) - d + h(w(\mu, u))} = \int_{m}^{\infty} \frac{du}{E(\varphi(u)) - d + h(w(u))},$$

where w is the solution defining $\mathcal{H}$, and this integral is infinite. □

Remark. Here the question of the monotonicity of the period is difficult to answer, even for $p = 2$, where it is solved by using the first integral, see [3]. It is open in the general case. More generally, if a dynamical system a center, the description of the period function is still a challenging problem. For example, one can construct a quadratic dynamical system with a center, the associated period function of which is not monotone, and even with at least two critical points, see [7] and [8].

Remark. In the case $b = 1$, we can compute theoretically the period T^+ by using the first integral (4.1). The stationary point $P_0 = (h^{-1}(1), 0)$ is obtained for $K_a = a^p/p - \mathcal{F}(a) > 0$ (in case of a power, $K_a = (q + 1 - p)/p(q + 1)$). The positive solutions correspond to trajectories $\mathcal{T}_K$

with $K \in (0, K_a)$, intersecting the axis $y = 0$ at points $(w_1, 0)$, $(w_2, 0)$ with $w_1 < a < w_2$ defined by $w_i^p/p - \mathcal{F}(w_i) = K$, and the period is given by

$$T^+ = 2\int_{w_1}^{w_2} \frac{dw}{wE^{-1}(-p\frac{K+\mathcal{F}(w)}{w^p})}.$$

Unfortunately, this formula does not allow us to prove the monotonicity of the period function for $p \neq 2$.

It is remarkable that, in the case $f(w) = |w|^{q-1}w$, one can solve completely the problem in the particular case where $b = 1$ and $q = 2p - 1$, using Eq. (3.23) satisfied by u.

Proposition 4.6. *Suppose that $f(w) = |w|^{q-1}w$, and $b = 1$ and $q = 2p - 1$, $p > 1$ and $d + 1 > 0$. If $p > 2$ or $d + 1 < 1/(2 - p)$, then T^+ is decreasing on $(0, a)$.*

Proof. Since $B(\xi) = 0$ by (3.24), Eq. (3.23) turns to

$$u'' = (q + 1 - p)E\big(\varphi(u) - d\big)\varphi(u) = (q + 1 - p)\big(-(1 + d) + p\Psi(u)\big)\varphi(u).$$

Henceforth

$$\frac{1}{q + 1 - p}u''u' = -(1 + d)\Psi'(u)u' + p\Psi(u)\Psi'(u)u',$$

from which expression we derive the first integral,

$$\frac{1}{q + 1 - p}u'^2 = C - \mathcal{U}(u), \qquad \mathcal{U} = \mathcal{M} \circ \Psi, \qquad \mathcal{M}(t) = 2(1 + d)t - pt^2. \tag{4.9}$$

From (4.9) the integral curves $\mathcal{S}$ in the (u, u')-plane are symmetric with respect to the axis $u' = 0$. The times for going from $u = 0$ to $u = u^*$ and from u^* to 0 are equal, and u^* is given by $C = \mathcal{M}(\Psi(u^*))$. The computation of the period is reduced to the part relative to the first quadrant. Here we follow the method of [3]: we get

$$T^+\big(u^*\big) = 4\int_0^{u^*} \frac{d\eta}{\sqrt{\mathcal{U}(u^*) - \mathcal{U}(\eta)}} = 4\int_0^1 \frac{u^*\,ds}{\sqrt{\mathcal{U}(u^*) - \mathcal{U}(su^*)}}.$$

Then

$$\frac{dT^+(u^*)}{du^*} = 4\int_0^1 \frac{(\Theta(u^*) - \Theta(su^*))\,ds}{(\mathcal{U}(u^*) - \mathcal{U}(su^*))^{3/2}}, \quad \text{with } \Theta\big(u^*\big) = \mathcal{U}\big(u^*\big) - u^*\mathcal{U}'\big(u^*\big)/2,$$

and

$$2\frac{d\Theta(u^*)}{du^*} = 2\Theta'\big(u^*\big) = \mathcal{U}'\big(u^*\big) - u^*\mathcal{U}''\big(u^*\big).$$

In the interval of study, $\varphi(u^*) < E^{-1}(d)$, $(E \circ \varphi)(u) < d$ from (3.14), thus $\Psi(u) < (1+d)/p$, and $\mathcal{M}$ is increasing for $0 < t < (1+d)/p$, thus $\mathcal{U}' > 0$. Then at any point u, $\Theta'(u) > 0 \Leftrightarrow (\mathcal{U}'/u)' < 0$. Now

$$\frac{\mathcal{U}'(u)}{2pu} = \frac{(-E(\varphi(u)) + d)\varphi(u)}{u} = 1 - (p-1)\varphi^2(u) + d\big(1 + \varphi^2(u)\big)^{(2-p)/2},$$

hence $(\mathcal{U}'/u)' = 2X(u)\varphi(u)\varphi'(u)$, with

$$X(u) = -(p-1) + (2-p)\,d\big(1 + \varphi^2(u)\big)^{-p/2},$$

and $d > E(\varphi(u))$; it implies $X(u) < 0$ if $p > 2$ or $p < 2$ and $d < (p-1)/(2-p)$. Henceforth Θ is increasing, and the same holds for P as a function of u^*. Finally u^* is decreasing with respect to μ, and consequently P is decreasing with respect to μ. □

Remark. When $p = 2$, and $q = 2p - 1 = 3$, Eq. (3.23) reduces to $u'' = -2u + 2u^3$, which, surprisingly, is an equation corresponding to the problem with absorption, and (3.4) reduces to $w'' - w + w^3 = 0$. In this case, all the solutions can be expressed in terms of elliptic integrals, see [3].

4.4. Returning to the initial problem

Proof of Theorem 2. Here $\beta = \beta_q = p/(q+1-p)$, $\lambda = \lambda_q$ is given by (1.15) and $c_q = \beta_q^{p-2}\lambda_q$ by (1.16). Moreover $\omega(\sigma) = \beta_q^{\beta_q} w(\beta_q\sigma)$ from (3.3), $b = -\lambda_q/\beta_q^2 = -c_q/\beta_q^p$ and $d = c/\beta_q^p$ from (3.5). At end $f(s) = g(s) = |s|^{q-1}s$ and $h(s) = |s|^{q-p}s$. Thus $c > c_q$ is equivalent to $b + d > 0$, and then the constant solutions $w \equiv \pm(b+d)^{1/(q-p+1)}$ of (3.4) correspond to the constant solutions $\omega \equiv \pm(c - c_q)^{1/(q+1-p)}$ of Eq. (1.14). For any integer $k \geqslant 1$, we look for periodic solutions ω of smallest period $2\pi/k$, or equivalently solutions w of period $T_k = 2\pi\beta_q/k$. From (3.17), the function E is increasing. First consider the sign changing solutions: if $c \geqslant c_q$, then from Theorem 4.3, the period function T of w is decreasing from ∞ to 0, hence for any $k \geqslant 1$ it takes precisely once the value T_k. If $c < c_q$, then T decreases from T_d given by (4.4) to 0, thus it takes once the value T_k for any $k > M_q = T_d/2\pi\beta_q$ given at (1.18). Next consider the positive solutions: from Proposition 4.5, the period function of w takes any value between ∞ and $2\pi/\sqrt{(q+1-p)(b+d)}$, thus it takes the value T_k for any $k < (p\beta_q^{1-p}(c - c_q))^{1/2}$, which ends the proof. □

In the case of Eq. (1.14) (i.e. $c = 0$), we obtain the following description of the sets $\mathcal{E}$ and $\mathcal{E}^+$:

Corollary 4.7. *Assume $p > 1$, $q > p - 1$, and $c = 0$.*

(i) *Then the set $\mathcal{E}$ of changing sign solutions of* (1.14) *is given by* (1.17), *where $k_q = 1$ if $p < 2$ and $q \geqslant 2(p-1)/(2-p)$, and $k_q > M_q$ if $p \geqslant 2$ or ($p < 2$ and $q < 2(p-1)/(2-p)$), where*

$$M_q = 2/(q-1), \tag{4.10}$$

if $p = 2$, and

$$M_q = \frac{(p-2)m_q}{((p-1)m_q+1)(m_q-1)}, \quad \textit{with } m_q = \sqrt{\frac{2(p-1)+(p-2)q}{p(p-1)}}, \tag{4.11}$$

if $p \neq 2$.

(ii) *If* $p \geqslant 2$ *or* ($p < 2$ *and* $q < 2(p-1)/(2-p)$), *then* $\mathcal{E}^+ = \emptyset$. *If* $p < 2$ *and* $q \geqslant 2(p-1)/(2-p)$, *then* $\mathcal{E}^+ = \{(-c_q)^{1/(q+1-p)}\}$.

Proof. Here $c_q < 0$ is equivalent to $p < 2$ and $q > 2(p-1)/(2-p)$. Furthermore $M_q = T_0/2\pi\beta_q$ can be computed from (4.5), which gives (4.10), (4.11). Moreover in any case $c_q + \beta_q^{p-1}/p = \beta_q^p(p-1)(q+1)/p^2 > 0$ thus there exist no positive nonconstant periodic solutions. □

Proof of Corollary 1. Let S be a sector on S^1 with opening angle $\theta \in (0, 2\pi)$. From [14, Theorem 3.3], β_S is the positive solution of equation

$$\phi(\beta_S) = \left(1 + \frac{1}{k}\right)^2 \left(\beta_S^2 + \frac{p-2}{p-1}\beta_S - (\beta_S+1)^2\right) = 0,$$

where $k = \pi/\theta \geqslant 1$. Using Corollary 4.7 (applied without assuming that k is an integer) we distinguish two cases:

(i) $p < 2$ and $q \geqslant 2(p-1)/(2-p)$. Then there always exists a solution to the Dirichlet problem in S. Notice that $0 < \beta_q \leqslant (2-p)/(p-1)$, thus $\phi(\beta_S) < 0$ and consequently $\beta_q < \beta_S$.

(ii) $p > 2$ or $p < 2$ and $q < 2(p-1)/(2-p)$. The existence is equivalent to $k > M_q$ (see (4.11)). It means

$$\left(1 + \frac{1}{k}\right)^2 < \left(\frac{(p-1)m_q^2 - 1}{m_q(p-2)}\right)^2 = \frac{(\beta_q+1)^2}{\beta_q^2 m_q^2} = \frac{(\beta_q+1)^2}{\beta_q(\beta_q + (p-2)\beta_q/(p-1))}.$$

Thus $\phi(\beta_q) < 0$. Equivalently, $\beta_q < \beta_S$. □

5. The case $p = 1$

5.1. Existence of a first integral

As shown in Lemma 3.1, we can reduce the study to

$$\frac{d}{d\tau}\left(\frac{w'}{\sqrt{w^2 + w'^2}}\right) - b\frac{w}{\sqrt{w^2 + w'^2}} + f_1(w) - d|w|^{-1}w = 0, \tag{5.1}$$

where f_1 satisfies (3.8); in particular we are interested by the case $f_1(s) = s$.

Here the problem is variational: if $S_1(w)$ is any primitive of $w \mapsto |w|^{b-1} f_1(w)$ and $R(w) = |w|^b/b$ if $b \neq 0$, $R(w) = \ln|w|$ if $b = 0$, then (5.1) is the Euler equation of the functional

$$\mathcal{H}(w, w') = |w|^{b-1}\sqrt{w^2 + w'^2} - S_1(w) + dR(w).$$

Thus the following Painlevé first integral is constant along the trajectories

$$\mathcal{P}(w, w') = \frac{|w|^{b+1}}{\sqrt{w^2 + w'^2}} - S_1(w) + dR(w). \tag{5.2}$$

The system (3.9) reads as

$$\begin{cases} w' = y, \\ y' = G(w, y) = \dfrac{bw^3 + (b+1)wy^2 - (f(w) - d|w|^{-1}w)(w^2 + y^2)^{3/2}}{w^2}, \end{cases}$$

and it is singular on the line $w = 0$. For $w > 0$ system (3.15) reduces to

$$\begin{cases} w' = w\varphi(u) = w\dfrac{u}{\sqrt{1 - u^2}}, \\ u' = b\sqrt{1 - u^2} - f_1(w) + d. \end{cases} \tag{5.3}$$

In the case $f(w) = w$, the equation satisfied by u is

$$u'' = (1 - b)\frac{u}{\sqrt{1 - u^2}}u' - bu - d\frac{u}{\sqrt{1 - u^2}}. \tag{5.4}$$

5.2. Existence of periodic solutions

From the Painlevé integral (5.2), we can describe the solutions, in the phase plane (w, y). Since a complete description is rather long, we reduce it to the research of periodic solutions.

Proposition 5.1. *Let $p = 1$, and consider Eq.* (5.1).

(i) *If $d \neq 0$, there is no periodic sign changing solution. If $d = 0$ there exists such a solution if and only if $b > -1$, and then it is unique* (*up to a translation*).
(ii) *There exist periodic positive solutions if and only if $b + d > 0$.*
(iii) *Suppose moreover that $f_1(w) = w$. Then the sign changing solution is given by*

$$w(\tau) = (b + 1)\cos(\tau - \tau_1);$$

it has period 2π. The orbits $\mathcal{T}_{[(\mu,0)]}$ of the periodic solutions intersect the axis $y = 0$ at a first point $(\mu, 0)$ such that $\mu < a = b + d$, and μ describes $\mu \in (\bar{\mu}, a)$ with $\bar{\mu} = 0$ if $d \leqslant 0$, and $\bar{\mu} > 0$ if $d > 0$; it is given by (5.9), (5.7), (5.8).

Proof. By symmetry we reduce the study to the case $w \geqslant 0$ and the Painlevé integral (5.2) takes the form

$$w^b\sqrt{1 - u^2} - S_1(w) + dR(w) = C, \tag{5.5}$$

where we denote

$$S_1(w) = \int_0^w s^{b-1} f_1(s)\,ds \quad \text{if } b > -1,$$

$$S_1(w) = \int_1^w s^{b-1} f_1(s)\,ds + \frac{1}{b+1} \quad \text{if } b < -1,$$

and

$$S_1(w) = \int_1^w s^{-2} f_1(s)\,ds \quad \text{if } b = -1.$$

Step 1. Periodic sign changing solutions. The curves in the phase plane (w, y) are given, for $w > 0$, by

$$\begin{aligned} y^2 &= \left(\frac{w^{b+1}}{C - dR(w) + S_1(w)}\right)^2 - w^2 \\ &= \frac{w^2(w^b + dR(w) - S_1(w) - C)(w^b - dR(w) + S_1(w) + C)}{(S_1(w) + C - dR(w))^2}, \end{aligned}$$

which defines $\pm y$ in function of w. If there exists a sign changing periodic solution, the trajectory intersects the axis $w = 0$ at some point $(0, \ell)$ with $\ell \geqslant 0$, thus y needs to ℓ as w tends to 0. From (5.5), it is impossible if $b \leqslant -1$. Assume $d \neq 0$; if $-1 < b$, then near $w = 0$, in any case $y^2 \leqslant (b^2/d^2 + 1)w^2$, thus $\ell = 0$ and w'/w is bounded, thus the maximal interval of existence is infinite, and we reach a contradiction. If $d = 0$, and $C \neq 0$, then $y^2 = -w^2(1 + o(1))$, which is impossible. If $d = C = 0$, then $y^2 = w^2(w^{2b}/S_1^2(w) - 1)$; observing that the function $w \mapsto \chi(w) = w^{-b} S_1(w)$ is increasing from 0 to ∞, the curve intersects the two axes at $(0, b+1)$ and $(\chi^{-1}(1), 0)$ and this corresponds to a closed orbit.

Step 2. Existence of periodic positive solutions. If we look at the intersection points of any trajectory in the phase plane with the axis $y = 0$, we find that they are given by $H(w) = C$, where

$$H(w) = w^b + dR(w) - S_1(w).$$

Then $H'(w) = w^{b-1}(b + d - f_1(w))$. If $b + d \leqslant 0$, then H is decreasing, thus there exist no positive periodic solutions. If $b + d > 0$, the function H is increasing on $(0, a)$ where it reaches a maximum M, and decreasing on (a, ∞). The stationary point $(a, 0)$ with $a = f_1^{-1}(b + d)$ corresponds to $C = M$. If $b > 0$, then $\lim_{w\to 0} H = 0$, while, if $b \leqslant 0$, then $\lim_{w\to 0} H = -\infty$. Equation $H(w) = C$ has two roots $0 < w_1 < w_2$, if and only if $C \in (\max\{\lim_{w\to 0} H, \lim_{w\to\infty} H\}, M)$. Moreover, if there exists a trajectory going through $(w_1, 0)$ and $(w_2, 0)$ and if one denotes

$$K(w) = dR(w) - S_1(w) = H(w) - w^b,$$

one has $K(w) < C$ on (w_1, w_2), thus $C > M' = \max K = K(f_1^{-1}(d))$. Conversely, if

$$\max\Big\{\lim_{w\to 0} H, \lim_{w\to\infty} H, M'\Big\} < C < M, \quad M = H\big(f^{-1}(b+d)\big),\ M' = K\big(f^{-1}(d)\big), \tag{5.6}$$

then there exists a closed orbit going through $(w_1, 0)$ and $(w_2, 0)$.

Step 3. End of the proof. The sign changing solution is given by $w^2 + w'^2 = (b+1)^2$ and its trajectory is a circle with center 0 and radius $b+1$; for $w > 0$, $w = (b+1)\sqrt{1-u^2} = b\sqrt{1-u^2} - u'$, thus $u' = -\sqrt{1-u^2}$, and $\theta' = -1$, then $w(\tau) = (b+1)\cos(\theta - \tau_1)$, periodic solution with period 2π. Now consider the positive periodic solutions. Here $a = b+d$, and

$$H(w) = \begin{cases} (1+d/b)w^b - w^{b+1}/(b+1), & \text{if } b \neq 0, -1, \\ 1 + d\ln w - w, & \text{if } b = 0, \\ (1-d)w^{-1} - \ln w, & \text{if } b = -1, \end{cases} \tag{5.7}$$

$$\begin{cases} M = a^{b+1}/b(b+1), \quad M' = \big(d^+\big)^{b+1}/b(b+1), & \text{if } b \neq 0, -1, \\ M = 1 + d\ln d - d, \quad M' = d\ln d - d, & \text{if } b = 0, \\ M = -1 - \ln(d-1), \quad M' = -1 - \ln d, & \text{if } b = -1. \end{cases} \tag{5.8}$$

If $d \leqslant 0$, thus $b > 0$, then any $C \in (0, M)$ corresponds to a closed orbit, thus for any $\mu \in (0, a_1)$, one has a closed orbit passing through $(\mu, 0)$, of period still denoted by $T^+(\mu)$. If $d > 0$, in any case, any $C \in (M', M)$ corresponds to a closed orbit. If $-1 < b < 0$, then H is increasing on $(0, a)$ from $-\infty$ to $M < 0$, and then decreasing on (a, ∞) from M to $-\infty$. If $b < -1$, then $M > M' > 0$. If $b < -1$, then $d > 1$, and $\lim_{w\to 0} H = -\infty$, $\lim_{w\to\infty} H = 0$, $0 < M' < M$. If $b = 0$, thus $d > 0$, then $\lim_{w\to 0} H = -\infty$, then any $C \in (M-1, M)$ corresponds to a closed orbit. Then H is increasing on $(0, d)$ from $-\infty$ to $M = 1 + d\ln d - d \geqslant 0$ (notice that $M = 0 \Leftrightarrow d = 1$) and then decreasing on (a_1, ∞) from M to $-\infty$; let $\bar{\mu} \in (0, b+d)$ be defined by

$$H(\bar{\mu}) = M', \tag{5.9}$$

thus for any $\mu \in (\bar{\mu}, a)$, one has a closed orbit passing through $(\mu, 0)$, with a period still denoted by $T^+(\mu)$. If $-1 \leqslant b < 0$, thus $d > -b > 0$, then $\lim_{w\to 0} H = -\infty = \lim_{w\to\infty} H$, $M < 0$, and any $C \in (M', M)$ corresponds to a closed orbit (if $b = -1$, then $H(w) = (1-d)w^{-1} - \ln w$, $M = -1 - \ln(d-1)$, $M' = -1 - \ln d$). Returning to Eq. (1.14), the conclusion follows with $\bar{\mu}_q = \bar{\mu}^q$. $\square$

5.3. Period of the solutions

Let $p = 1, b+d > 0$. Consider Eq. (5.1). Let $T^+(\mu)$ be the least period of the periodic positive solutions corresponding to the orbit $\mathcal{T}_{[(\mu,0)]}$. As in the case $p > 1$, we have a general result:

$$\lim_{\mu\to a} T^+(\mu) = \frac{2\pi}{\sqrt{af_1'(a)}}. \tag{5.10}$$

Next we study the variations of the period in the case of a power $f_1(w) = w$.

Theorem 5.2. *Assume $p = 1$, $b+d > 0$ and $f_1(w) = w$. Then $\lim_{\mu\to a} T^+(\mu) = 2\pi/\sqrt{b+d}$. If $d < 0$, then $\lim_{\mu\to 0} T^+(\mu) = \infty$. If $d \geqslant 0$, then $\lim_{\mu\to\bar{\mu}} T^+(\mu) = \bar{T}^+$ is finite, and given by (5.12) if $b \notin \{0, -1\}$, by (5.13) if $b = 0$, and by (5.14) if $b = -1$. If $d = 0$, then $\bar{T}^+ = \pi(1 + 1/b)$.*

Proof. *Step 1. Assume* $b \notin \{0, -1\}$. From (5.5), the solutions of (5.1) satisfy

$$w^b\sqrt{1-u^2} - \frac{w^{b+1}}{b+1} + \frac{d}{b}w^b = C,$$

thus

$$u' = b\sqrt{1-u^2} - w + d = -\sqrt{1-u^2} - \frac{d}{b} + \frac{C(1+b)}{w^b}.$$

Eliminating w between the two relations, we find that $Cb(b+1) > 0$ and

$$\left(d + b\sqrt{1-u^2} - u'\right)^{b/(b+1)}\left(d + b\sqrt{1-u^2} + bu'\right)^{1/(b+1)} = \left(Cb(1+b)\right)^{1/(b+1)} := A.$$

When a solution goes through the half-part of its trajectory $\mathcal{T}$ located in $\mathcal{Q}$, the associated function u increases from 0 to some $u^* \in (0,1)$ where the derivative u' vanishes and $d + b\sqrt{1-u^{*2}} > 0$; next $d + b\sqrt{1-u^2}$ is monotone and positive at 0 and u^*, thus $d + b\sqrt{1-u^2} > 0$ everywhere. And $A = d + b\sqrt{1-u^{*2}} = w^*$ (the value of w when $u = u^*$). Let

$$z = \frac{u'}{d + b\sqrt{1-u^2}} \quad \text{and} \quad G(s) = (1-s)^{b/(b+1)}(1+bs)^{1/(b+1)}.$$

If $b > 0$, then $z \in (-1/b, 1)$; if $b < -1$ then $z \in (-\infty, 1)$; if $-1 < b < 0$ then $z \in (-\infty, 1/|b|)$, and

$$G(z) = \frac{A}{d + b\sqrt{1-u^2}}.$$

Since

$$G'(s) = -bs(1-s)^{-1/(b+1)}(1+bs)^{-b/(b+1)}$$

and

$$G''(s) = -b(1-s)^{-(b+2)/(b+1)}(1+bs)^{-(2b+1)/(b+1)},$$

it follows $G(0) = 1$, and 0 is a maximum if $b > 0$ and a minimum if $b < 0$: if $b > 0$, G increases on $(-1/b, 0)$ from 0 to 1 and decreases on $(0,1)$ from 1 to 0; if $b < 0$, G decreases on $(-\infty, 0)$ from ∞ to 1 and increases on $(0, \min(1, 1/|b|))$ from 1 to ∞. Thus it has two inverse functions $-L_1$ and L_2: for $b > 0$, L_1 maps $(0,1)$ into $(0, 1/b)$ and L_2 maps $(0,1)$ into $(0,1)$; for $b < 0$, L_1 maps $(1,\infty)$ into $(0,\infty)$ and L_2 maps $(1,\infty)$ into $(0, \min(1, 1/|b|))$. Then

$$T^+ = T_1^+ + T_2^+, \quad T_i^+ = \int_0^1 \psi_{i,u^*}(\lambda)\, d\lambda, \tag{5.11}$$

where

$$\psi_{i,u^*}(\lambda)=\frac{2u^*}{(d+b\sqrt{1-\lambda^2u^{*2}})L_i((d+b\sqrt{1-u^{*2}})/(d+b\sqrt{1-\lambda^2u^{*2}}))}.$$

• First suppose $d<0$ (thus $b>0$); then one looks at the case where $C\to 0$, thus $\sqrt{1-u^{*2}}\to -d/b$, thus $u^*\to\bar{u}=\sqrt{1-d^2/b^2}$. Near $\bar{u}$,

$$\psi_{i,u^*}(\lambda)\geqslant\frac{2u^*}{b(\sqrt{1-\lambda^2u^{*2}}-\sqrt{1-u^{*2}})L_i(0)}\geqslant\frac{-d}{bL_i(0)(1-\lambda^2)},$$

therefore T_i^+ tends to ∞.

• Suppose $d\geqslant 0$, $b>0$. One looks at the case where $C\to M'$, thus $u^*\to 1$. There exists a constant $m>0$ such that $0\leqslant 1-G(s)=G(0)-G(s)\leqslant m^2s^2$ on $[-1/b,1]$. Indeed $G'(0)=0$ and G'' is bounded on $[-1/2b,1/2]$, and on $[-1/b,-1/2b]\cup[1/2,1]$ the quotient $(G(0)-G(s))/s^2$ is bounded. Thus $1/L_i(\eta)\leqslant m/\sqrt{1-\eta}$ on $[0,1)$, hence taking $\eta=(d+b\sqrt{1-u^{*2}})/(d+b\sqrt{1-\lambda^2u^{*2}})$, and computing

$$1-\eta=\frac{b(1-\lambda^2)u^{*2}}{(d+b\sqrt{1-\lambda^2u^{*2}})(\sqrt{1-u^{*2}}+\sqrt{1-\lambda^2u^{*2}})},$$

one finds $\psi_{i,u^*}(\lambda)\leqslant 4m/\sqrt{b(1-\lambda^2)}$. From the Lebesgue theorem, as $u^*\to 1$, T^+ tends to the finite limit

$$\bar{T}^+=\bar{T}_1^++\bar{T}_2^+,\quad \bar{T}_i^+=2\int_0^1\frac{d\lambda}{(d+b\sqrt{1-\lambda^2})L_i(d/(d+b\sqrt{1-\lambda^2}))}\tag{5.12}$$

in particular if $d=0$, then $L_1(0)=1/b$, $L_1(0)=1$, thus $\bar{T}_{1,1}^+=\pi$ and $\bar{T}_{1,2}^+=\pi/b$.

• Suppose $b<0$, thus $d>-b>0$. Then again $C\to M'$, consequently $u^*\to 1$. The function

$$u^*\to Q(u^*,\lambda)=\eta=\frac{d+b\sqrt{1-u^{*2}}}{d+b\sqrt{1-\lambda^2u^{*2}}}=1-\frac{b(1-\lambda^2)u^{*2}}{(d+b\sqrt{1-\lambda^2u^{*2}})(\sqrt{1-u^{*2}}+\sqrt{1-\lambda^2u^{*2}})}$$

is increasing on $(0,1)$ from 1 to $d/(d+b\sqrt{1-\lambda^2})$ and $d/(d+b\sqrt{1-\lambda^2})\leqslant d/(d+b)=\alpha$. There exists $m>0$ such that $0\leqslant G(s)-1\leqslant m^2s^2$ on $[-L_1(\alpha),L_2(\alpha)]$, thus $1/L_i(\eta)\leqslant m/\sqrt{\eta-1}$ on $(1/d/(d+b)]$. Thus as above, $\psi_{i,u^*}(\lambda)\leqslant 4m/\sqrt{|b|(1-\lambda^2)}$, and T^+ tends to $\bar{T}^+$ defined at (5.12).

Step 2. Assume $b=0$. There exist periodic solutions for any $C\in(M-1,M)$. The solutions are given by

$$\sqrt{1-u^2}+H(w)=\sqrt{1-u^2}+d\ln w-w=C$$

and $u'=-w+d$, thus u is maximal $(=u^*)$ for $w=d$: therefore $\sqrt{1-u^{*2}}+H(d)=C$, then

$$H(d-u')=H(d)+\sqrt{1-u^{*2}}-\sqrt{1-u^2}$$

and H has two inverse functions H_i from $(-\infty, H(d))$ into $(0,d)$ and (d,∞), thus (5.11) holds with

$$\psi_{i,u^*}(\lambda) = \frac{2u^*\,d\lambda}{d - H_i(H(d) + \sqrt{1-u^{*2}} - \sqrt{1-\lambda^2 u^{*2}})}$$

and $\xi = H(d) + \sqrt{1-u^{*2}} - \sqrt{1-\lambda^2u^{*2}} = H(d) - k = H(d+h)$ stays in $(M-1, M) = (H(d)-1, H(d))$, and $H(d+h) - H(d) \geqslant -m^2h^2$ for $H(d+h) \in (M-1, M)$, thus $H(d) - \xi = k \leqslant m^2(d - H_i(\xi))^2$, thus

$$\psi_{i,u^*}(\lambda) \leqslant \frac{2m}{\sqrt{k}} = \frac{2m(\sqrt{1-u^{*2}} + \sqrt{1-\lambda^2u^{*2}})}{\sqrt{1-\lambda^2}} \leqslant \frac{4m}{\sqrt{1-\lambda^2}}.$$

Therefore, as $u^* \to 1$, T^+ tends to the finite limit

$$\bar{T}^+ = \bar{T}_1^+ + \bar{T}_2^+, \quad \bar{T}_i^+ = 2\int_0^1 \frac{d\lambda}{d - H_i(H(d) - \sqrt{1-\lambda^2})}. \tag{5.13}$$

Step 3. Assume $b=-1$. In that case $d > 1$; let $B = -(C+1) \in (\ln(d-1), \ln d)$ then $B \to \ln d$ and

$$u' + w = d - \sqrt{1-u^2} = (B+1)w - w\ln w = H_B(w),$$

where H_B is increasing on $(0, e^B)$ from 0 to e^B and decreasing on (e^B, ∞) from e^B to $-\infty$; it has two inverse functions $L_{B,i}$ from $(-\infty, e^B)$ into $(0, e^B)$ and (e^{B-1}, ∞); and $w^* = d - \sqrt{1-u^{*2}} = e^B$; then (5.11) holds with

$$\psi_{i,u^*}(\lambda) = \frac{2u^*}{d - \sqrt{1-\lambda^2u^{*2}} - L_{B,i}(d - \sqrt{1-\lambda^2u^{*2}})} = \frac{2u^*}{|H_{B-1}(L_{B,i}(d - \sqrt{1-\lambda^2u^{*2}}))|}.$$

Because $H_{B-1}(e^B) = 0$, $H_{B-1}(x) - H_{B-1}(e^B) = H'_{B-1}(\xi)(x - e^B)$ and x ranges onto $(H_{B,1}(d-1), H_{B,2}(d-1)) := (x_{1,B}, x_{2,B})$, when $B \to \ln d$, $(x_{1,B}, x_{2,B}) \to (x_{1,\ln d}, x_{2,\ln d})$, it follows $|H'_{B-1}(\xi)| \geqslant 1/\mu > 0$ independent on B. Moreover $H_B(x) - H_B(e^B) = (1/2)H''_B(\xi)(x-e^B)^2 = -(1/2\xi)(x-e^B)^2$. Thus there exists $m > 0$ such that

$$H_B(x) - H_B\big(e^B\big) \leqslant m^2\big(x - e^B\big)^2 \leqslant m^2\mu^2 H_{B-1}^2(x).$$

Therefore, near $\ln d$, taking $x = L_{B,i}(d - \sqrt{1-\lambda^2u^{*2}})$, one derive

$$\psi_{i,u^*}(\lambda) \leqslant \frac{2}{m\mu\sqrt{d - \sqrt{1-\lambda^2u^{*2}} - e^B}} = \frac{2}{m\mu\sqrt{\sqrt{1-u^{*2}} - \sqrt{1-\lambda^2u^{*2}}}} \leqslant \frac{4}{m\mu\sqrt{1-\lambda^2}}.$$

Consequently, as $u^* \to 1$, $T_{1,i}^+$ tends to the finite limit

$$\bar{T}^+ = \bar{T}_1^+ + \bar{T}_2^+, \qquad \bar{T}_{1,i}^+ = 2\int_0^1 \frac{d\lambda}{|H_{\ln d-1}(L_{\ln d,i}(d-\sqrt{1-\lambda^2}))|}. \qquad \square \tag{5.14}$$

Remark. In the case $d=0$, $b \neq 1$, notice that T_1^+ and T_2^+ converges to $\pi/\sqrt{b}$ as μ tends to b (one can verify it by linearizing the equation in u) and respectively to π and π/b as μ tends to 0. Thus if those functions are monotone, they vary in opposite senses and it is not easy to get the sense of variations of their sum T_1^+. Moreover in the phase plane (w, y_1), as μ tends to 0, one can observe that the trajectory tends to a limit curve constituted of a segment $[(0,0),(0,b)]$ and half of the unique closed orbit surrounding $(0,0)$, circle of center 0 and radius $b+1$, which is covered in a time π.

The case $b=1$ is the most interesting for (5.1), since it corresponds to the initial problem (1.14). In that case we improve the results by showing the monotonicity of the period function:

Theorem 5.3. *Assume $b=1$, $d>-1$. When $d=0$ the period function $T^+(\mu)$ is constant, with value 2π, thus there exists an infinity of positive solutions w of* (5.1)*, which are all 2π-periodic*; *they are explicitly given by*

$$w = \sqrt{1-K^{*2}\sin^2\tau} - K^*\cos\tau, \quad \tau \in [-\pi,\pi],\ K^* \in (0,1). \tag{5.15}$$

When $d \neq 0$, then $T^+(\mu)$ is strictly monotone; *if $d<0$ it decreases from ∞ to $2\pi/\sqrt{1+d}$*; *if $d>0$ it increases from*

$$\bar{T}^+ = 4\int_0^1 \frac{du}{\sqrt{(d+\sqrt{1-u^2})^2-d^2}} = 4\int_0^{\pi/2}\sqrt{\frac{\cos\theta}{\cos\theta+2d}}\,d\theta \tag{5.16}$$

to $2\pi/\sqrt{1+d}$.

Proof. • If $d=0$, then $u''=-u$, from (5.4), and $u=\sin\theta \in [0,1)$, thus the positive solutions w are given in $\mathcal{Q}$ by

$$u = K^*\sin\tau, \quad K^* \in [0,1),\ \tau \in [0,\pi],$$

and the period T^+ is constant, equal to 2π. We obtain an infinity of positive solutions w, given explicitly by

$$w = \sqrt{1-u^2} - u' = \sqrt{1-K^{*2}\sin^2\tau} - K^*\cos\tau, \quad K^* \in (0,1),$$

which intersect the axis $y=0$ at points $w_i = (1 \mp K^*)$.

• In the general case $d>-1$, we find

$$\bigl(d+\sqrt{1-u^2}-u'\bigr)\bigl(\sqrt{1-u^2}+u'+d\bigr) = A^2$$

that means G is symmetric: $G(s)=\sqrt{1-s^2}$, thus

$$u'^2=\big(d+\sqrt{1-u^2}\big)^2-A^2,$$

$\sqrt{1-u^{*2}}=A-d=\sqrt{2C}-d$; thus here $T_1^+=T_2^+$, and

$$T^+=4\int_0^1\frac{d\lambda}{\sqrt{\Psi(u^{*2},\lambda)}},\quad \text{where } \Psi(s,\lambda)=\frac{(d+\sqrt{1-\lambda^2 s}\,)^2-(d+\sqrt{1-s}\,)^2}{s}.$$

We show that the period function is strictly monotone with respect to u^*. Because

$$s^2\frac{\partial\Psi}{\partial s}(s,\lambda)=d(d+1)\big(1/\sqrt{1-s}-1/\sqrt{1-\lambda^2 s}\,\big)>0,$$

we see that T^+ is increasing if $d<0$ and decreasing if $d>0$ (and we find again that it is constant if $d=0$). Also μ can be expressed explicitly in terms of u^* by

$$\mu=d+1-\sqrt{(d+1)^2-\big(d+\sqrt{1-u^{*2}}\big)}.$$

Therefore μ is decreasing with respect to u^*, hence T^+ is decreasing with respect to μ if $d<0$ and increasing if $d>0$. □

5.4. Returning to the initial problem

Proof of Theorem 3. Here $\alpha_q=\beta_q=1/q$, the substitution (3.6) takes the form $\omega(\sigma)=|w(\sigma)|^{1/q-1}w(\sigma)$, and thus $b=1$, and $d=c$ from (3.7). Then the existence of sign changing solutions of (1.14) is given by Proposition 5.1. The constant solutions exist whenever $c+1>0$. Next we look for positive solutions of smallest period $2\pi/k$ applying Theorems 5.2 and 5.3. If $c<0$ the period function T^+ decreases from ∞ to $2\pi/\sqrt{1+c}>2\pi$, thus there exists no solution. If $c>0$, T^+ increases from $\bar{T}^+$ given by (5.16) to $2\pi/\sqrt{1+c}$, thus it takes once any intermediate value, which gives one solution (up to a translation) for any $k\in(k_1,k_2)$. If $c=0$, the solutions ω_K are given explicitly by (5.15), and ω_0^+ is obtained from ω_0; this means that system (3.9) does not satisfy the uniqueness property at $(0,0)$. □

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