Self-similar solutions of the p-Laplace heat equation: the fast diffusion case*

Marie Françoise Bidaut-Véron†

.

Abstract

We study the self-similar solutions of the equation

$$u_t - div(|\nabla u|^{p-2} \nabla u) = 0$$

in $\mathbb{R}^N, N \geq 1, p \in (1,2)$. We provide a complete description of the signed solutions of the form

$$u(x,t) = (\pm t)^{-\alpha/\beta} w((\pm t)^{-1/\beta} |x|)$$

with $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$, regular or singular at $x = 0$, and possibly not defined on whole $\mathbb{R}^N \times (0, \pm\infty)$.

Contents

*to appear in PACIFIC JOURNAL

†Laboratoire de Mathématiques et Physique Théorique, CNRS UMR 6083, Faculté des Sciences, Parc Grandmont, 37200 Tours, France. e-mail:veronmf@univ-tours.fr

.

1 Introduction and main results

In this article we study the existence of self-similar solutions of the degenerate parabolic equation involving the p-Laplace operator in $\mathbb{R}^N$, $N \geq 1$,

$$u_t - div(|\nabla u|^{p-2} \nabla u) = 0, \qquad (\mathbf{E}_u)$$

with $1 < p < 2$. In the sequel we set

$$\delta = \frac{p}{2-p},$$

thus $\delta > 1$. Two critical values P_1, P_2 are involved in the problem, see for example [10]:

$$P_1 = \frac{2N}{N+1}, \qquad P_2 = \frac{2N}{N+2};$$

they are connected to δ by the relations

$$p > P_1 \Longleftrightarrow \delta > N, \qquad p > P_2 \Longleftrightarrow \delta > \frac{N}{2}.$$

If $u(x,t)$ is a solution then for any $\alpha, \beta \in \mathbb{R}$, $u_\lambda(x,t) = \lambda^\alpha u(\lambda x, \lambda^\beta t)$ is a solution of $(\mathbf{E}_u)$ if and only if

$$\beta = p - (2-p)\alpha = (2-p)(\delta - \alpha); \qquad (1.1)$$

thus $\beta > 0 \Longleftrightarrow \alpha < \delta$. For given $\alpha \in \mathbb{R}$ such that $\alpha \neq \delta$, the natural way to construct particular solutions is to search for self-similar solutions, radially symmetric in x, of the form:

$$u = u(x,t) = (\varepsilon\beta t)^{-\alpha/\beta} w(r), \qquad r = (\varepsilon\beta t)^{-1/\beta} |x|, \qquad (1.2)$$

where $\varepsilon = \pm 1$. By translation, for any real T, we obtain solutions defined for any $t > T$ when $\varepsilon\beta > 0$, or $t < T$ when $\varepsilon\beta < 0$. The hypersurfaces $\{r = \text{constant}\}$ are of "focussing" type if $\beta > 0$ and "spreading" one if $\beta < 0$. We are lead to the equation

$$\left(|w'|^{p-2} w'\right)' + \frac{N-1}{r} |w'|^{p-2} w' + \varepsilon(rw' + \alpha w) = 0 \qquad \text{in } (0,\infty). \qquad (\mathbf{E}_w)$$

Furthermore, if we look for solutions of $(\mathbf{E}_u)$ under the form

$$u = Ae^{-\varepsilon\mu t} w(r), \qquad r = Me^{-\varepsilon\mu t/\delta} |x|, \qquad \mu > 0,$$

Then w solves $(\mathbf{E}_w)$, provided $M = \delta/\alpha$ and $A = (\delta^p/\alpha^{p-1}\mu)^{1/(2-p)}$, where $\alpha > 0$ is arbitrary. This is another motivation for studying equation $(\mathbf{E}_w)$ for any real α.

In the huge litterature on self-similar solutions of parabolic equations, many results deal on positive solutions u defined and smooth on $\mathbb{R}^N \times (0,\infty)$. Equation $(\mathbf{E}_w)$ was studied in [16] when

$\alpha > 0$, $\varepsilon = 1$ with that topic. In our work we provide an exhaustive description of the self-similar solutions of equation ($\mathbf{E}_u$), possibly not defined on whole $(0, \infty)$, with constant or changing sign. In particular for suitable values of α, we prove the existence of solutions w oscillating with respect to 0 as r tends to 0 or ∞, or constant sign solutions oscillating with respect to some nonzero constant. Our main tool is the reduction of the problem to an autonomous system with two variables and two parameters: p and α. We are lead to a problem of dynamical systems, which we study by phase-plane techniques. When $p = 3/2$, this system is nearly quadratic, and many devices from the theory of algebraic dynamical systems could be used. In the general case such structures do not exist, then we use energy functions associated to the system. The behaviour of the solutions presents a great diversity, according to the possible values of p and α.

In the sequel we set

$$\eta = \frac{N-p}{p-1},$$

thus $\eta > 0$ if $N \geq 2$, and $\eta = -1$ if $N = 1$. Observe the relation which connects η, δ and N :

$$\frac{\delta - N}{p-1} = \delta - \eta = \frac{N - \eta}{2-p}. \tag{1.3}$$

1.1 Explicit solutions

Obviously if w is a solution of ($\mathbf{E}_w$), then also $-w$. Many particular solutions are well-known.

(i) The infinite point source solution U_∞. The simplest positive solutions of equation ($\mathbf{E}_w$), which exist for any α such that $\varepsilon(\delta - N)(\delta - \alpha) > 0$, are given by

$$w(r) = \ell r^{-\delta}, \tag{1.4}$$

where

$$\ell = \left(\varepsilon \delta^{p-1} \frac{\delta - N}{\delta - \alpha}\right)^{1/(2-p)} > 0. \tag{1.5}$$

They correspond to a unique solution u of ($\mathbf{E}_u$) called U_∞ in [8], singular at $x = 0$, for any $|t| > 0$:

$$U_\infty(x,t) = (\frac{Ct}{|x|^p})^{1/(2-p)}, \qquad C = (2-p)\delta^{p-1}(\delta - N). \tag{1.6}$$

(i) Case $\alpha = N$. Then the equation ($\mathbf{E}_w$) has a first integral

$$w + \varepsilon r^{-1} \left|w'\right|^{p-2} w' = C r^{-N}. \tag{1.7}$$

All the solutions corresponding to $C = 0$ are given by

$$w = w_{K,\varepsilon}(r) = \pm \left(\varepsilon \delta^{-1} r^{p'} + K\right)^{-\delta/p'},$$

$$u = \pm u_{K,\varepsilon}(x,t) = (\varepsilon \beta_N t)^{-N/\beta_N} \left(\varepsilon \delta^{-1} (\varepsilon \beta_N t)^{-p'/\beta_N} |x|^{p'} + K\right)^{-(p-1)/(2-p)}, \qquad K \in \mathbb{R}, \tag{1.8}$$

with $\beta = \beta_N = (N+1)(p-P_1)$. For $p > P_1, \varepsilon = 1, K > 0$, the solutions are usually called *Barenblatt solutions,* see [3]. For given $c > 0$, the function $u_{K,1}$, defined on $\mathbb{R}^N \times (0,\infty)$, is the unique solution of equation $(\mathbf{E}_u)$ with initial data $u(0) = c\delta_0$, where δ_0 is the Dirac mass at 0, where K is determined by $\int_{\mathbb{R}^N} u_K(x,t)dt = c$, see for example [19]. Moreover the $u_{K,1}(K>0)$ are the only nonnegative solutions defined on $\mathbb{R}^N \times (0,\infty)$, such that $u(x,0) = 0$ for any $x \neq 0$, see [14]. In the case $K = 0$, we find again the function U_∞ given at (1.4), and U_∞ is the limit of the functions $u_{K,1}$ as $K \to 0$, that means $c \to \infty$.

(ii) Case $\alpha = \eta$. We exhibit a family of solutions of $(\mathbf{E}_w)$:

$$w(r) = Cr^{-\eta}, \qquad u(t,x) = C\,|x|^{-\eta} = C\,|x|^{(p-N)/(p-1)}, \qquad C \neq 0, \tag{1.9}$$

The solutions u, independent of t, are the fundamental p-harmonic solutions the equation when $p > P_1$.

(iii) Case $\alpha = -p'$. Equation $(\mathbf{E}_w)$ admits solutions of the form

$$w(r) = \pm K\left(N(Kp')^{p-2} + \varepsilon r^{p'}\right), \qquad u(x,t) = \pm K\left(N(Kp')^{p-2}t + \varepsilon\,|x|^{p'}\right), \qquad K > 0. \tag{1.10}$$

and the functions u are the only solutions of the form $\psi(t) + \Phi(|x|)$ with Φ nonconstant. They have a constant sign when $\varepsilon = 1$, and a changing sign when $\varepsilon = -1$.

(iv) Case $\alpha = 0$. Here equation $(\mathbf{E}_w)$ can be explicitely solved: either $w' \equiv 0$, thus $w \equiv a \in \mathbb{R}$, u is a constant solution of $(\mathbf{E}_u)$, or there exists $K \in \mathbb{R}$ such that

$$|w'| = r^{(1-N)/(p-1)} \times \begin{cases} \left(K + \frac{\varepsilon}{\delta - N} r^{N-\eta}\right)^{-1/(2-p)}, & \text{if } \delta \neq N, \\ \left(\frac{2-p}{p-1}\left(K + \varepsilon \ln r\right)\right)^{-1/(2-p)}, & \text{if } \delta = N, \end{cases} \tag{1.11}$$

which gives w by integration, up to a constant, and then $u(x,t) = w(|x|/(\varepsilon pt)^{1/p})$.

(v) Case $N = 1$ **and** $\alpha = (p-1)/(2-p) > 0$. Here again we obtain explicit solutions:

$$w(r) = \pm\left(\varepsilon K(r - (K\alpha)^{p-1}\right)^{-\alpha}, \qquad u(x,t) = \pm\left(\varepsilon K(|x| - \varepsilon(K\alpha)^{p-1}t\right)^{-\alpha}, \qquad K > 0.$$

Observe that all the functions w above are defined on intervals of the form $(R,0)$, $R \geq 0$ if $\varepsilon = 1$, $(0,S)$, $S \leq \infty$ if $\varepsilon = -1$.

Remark 1.1 *When* $\alpha = \delta$, *equation* $(\boldsymbol{E}_u)$ *is invariant under the transformation* $u_\lambda(x,t) = \lambda^\alpha u(\lambda x,t)$*; searching solutions of the form* $u(x,t) = |x|^{-\delta}\,\psi(t)$*, we find again the function* U_∞.

1.2 Different kinds of singularities

Consider equation $(\mathbf{E}_w)$. It is easy to get local existence and uniqueness near any point $r_1 > 0$, thus any solution w is defined on a maximal interval (R_w, S_w), with $0 \leq R_w < S_w \leq \infty$; and in fact $S_w = \infty$ when $\varepsilon = 1$, and $R_w = 0$ when $\varepsilon = -1$, see Theorem 2.5. Returning to solution u of equation $(\mathbf{E}_u)$ associated to w by (1.2), it is defined on a subset of $\mathbb{R}^N \backslash \{0\} \times (0, \pm\infty)$:

$$D_w = \left\{(x,t) : x \in \mathbb{R}^N, \varepsilon\beta t > 0, (\varepsilon\beta t)^{1/\beta} R_w < |x| < (\varepsilon\beta t)^{1/\beta} S_w \right\}.$$

When w is defined on $(0, \infty)$, then u is defined on $\mathbb{R}^N \backslash \{0\} \times (0, \pm\infty)$.

(i) Regular solutions Among the solutions of $(\mathbf{E}_w)$ defined near 0, we also show the existence and uniqueness of solutions $w = w(., a) \in C^2([0, S_w))$ such that for some $a \in \mathbb{R}$,

$$w(0) = a, \qquad w'(0) = 0, \tag{1.12}$$

called *regular solutions.* Obviously, they are defined on $[0, \infty)$ when $\varepsilon = 1$. If w is regular, then $D_w = \mathbb{R}^N \times (0, \pm\infty)$, and $u(., t) \in C^1(\mathbb{R}^N)$ for $t \neq 0$; we will say that u is *regular;* this does not imply the regularity up to $t = 0$*:* indeed u presents a singularity at time $t = 0$ if and only if $0 < \alpha < \delta$. In the sequel we shall not mention the trivial solution $w \equiv 0$, corresponding to $a = 0$.

(ii) Singular solutions If $R_w = 0$, and w is not regular, then u presents a singularity at $x = 0$ for $t \neq 0$, called *standing singularity.* Using the terminology of [17] and [8], for such a solution, we say that $x = 0$ is a *weak singularity* if $x \mapsto w(|x|) \in L^1_{loc}(\mathbb{R}^N)$, or equivalently $u(., t) \in L^1_{loc}(\mathbb{R}^N)$ for $t \neq 0$; and a *strong singularity* if not. If u has a strong (resp. weak) singularity, and $\lim_{t\to 0} u(t, x) = 0$ for any $x \neq 0$, u is called a *strong* (resp. *weak*) *razor blade.* If $u(., t) \in L^1(\mathbb{R}^N)$ for $t \neq 0$, u is called *integrable.*

(iii) Solutions with a reduced domain If $R_w > 0$ or $S_w < \infty$, we will say that u and w have a *reduced domain.* Then D_w has a lateral boundary of the form $\Sigma_w = \left\{|x| = C(\varepsilon\beta t)^{1/\beta}\right\}$, of parabolic type if $\beta > 0$, of hyperbolic type if $\beta < 0$, and u has an explosion near Σ_w. We precise the blow-up rate, of the order of $d(x,t)^{-(p-1)/(2-p)}$, where $d(x,t)$ is the distance to Σ_w, at Proposition 2.20.

1.3 Main results

Let us give a summary of our main results, expressed in terms of function u, and, *for simplicity*, we avoid the particular cases (for example $N = 1$, or $\alpha = \delta$, or $p = P_1$) and do not mention the existence of solutions with a reduced domain, although there exist many such solutions. All of them and the detailed results in terms of function w can be found inside each section. An important critical value of α is involved:

$$\alpha^* = \delta + \frac{\delta(N-\delta)}{(p-1)(2\delta - N)}; \tag{1.13}$$

it appears when $\varepsilon = 1$, $p > P_2$, and then $\alpha^* > 0$, or $\varepsilon = -1$, $p < P_2$, and then $\alpha^* < 0$.

Remark 1.2 *In order to return from w to u, consider any solution w of ($\boldsymbol{E}_w$) defined on $(0,\infty)$, such that for some $\lambda \geq 0$ and $\mu \in \mathbb{R}$, $\lim_{r\to 0} r^\lambda w = c \neq 0$ and $\lim_{r\to 0} r^\mu w = c' \neq 0$. Then*

(i) For fixed t, u has a singularity in $|x|^{-\lambda}$ near $x = 0$, and a behaviour in $|x|^{-\mu}$ for large $|x|$. Thus $x = 0$ is a weak singularity if and only if $\lambda < N$, and u is integrable if and only if $\lambda < N < \mu$.

(ii) For fixed $x \neq 0$, the behaviour of u near $t = 0$, depends on the sign of β:

$$\lim_{t\to 0} |x|^\mu |t|^{(\alpha-\mu)/\beta} u(x,t) = C \neq 0 \text{ if } \alpha < \delta, \qquad \lim_{t\to 0} |x|^\lambda |t|^{(\alpha-\lambda)/\beta} u(x,t) = C \neq 0 \text{ if } \delta < \alpha.$$

(i) Solutions defined for $t > 0$

Here we look for solutions u of ($\mathbf{E}_u$) on $\mathbb{R}^N \setminus \{0\} \times (0,\infty)$ of the form (1.2). That means $\varepsilon\beta > 0$, or equivalently $\varepsilon = 1, \alpha < \delta$ (see Section 3) or $\varepsilon = -1, \delta < \alpha$ (see Section 4). We begin by the case $\varepsilon = 1$, and discuss with respect of the sign of $p - P_1$. For the proofs, see Theorems 3.2, 3.5 and 3.7.

Theorem 1.3 *Assume $\varepsilon = 1, -\infty < \alpha < \delta$, and $p > P_1$ ($N \geq 2$). Then U_∞ is a solution on $\mathbb{R}^N \setminus \{0\} \times (0,\infty)$, it is a strong razor blade. There exist also positive solutions with a strong singularity in $|x|^{-\delta}$, and $\lim_{t\to 0} |x|^\alpha u = L > 0$ (for $x \neq 0$). For $\alpha \leq N$, any function $u(.,t)$ has at most one zero at time t.*

(1) For $\alpha < N$, the regular solutions on $\mathbb{R}^N \times (0,\infty)$ have a constant sign, are not integrable,and they are solutions of ($\boldsymbol{E}_u$) with initial data $L|x|^{-\alpha} \in L^1_{loc}(\mathbb{R}^N)$. There exist positive integrable razor blades, with a singularity in $|x|^{-\eta}$. There exist also positive solutions with a weak regularity in $|x|^{-\eta}$, with $\lim_{t\to 0} |x|^\alpha u = L$ (in particular if $\alpha = \eta$, then $u \equiv C|x|^{-\eta}$). There exist solutions with one zero and a weak or a strong singularity.

(2) For $\alpha = N$, the regular (Barenblatt) solutions have a constant sign and are integrable. There exist solutions with one zero and a weak singularity.

(3) For $N < \alpha$, the regular solutions have at least one zero. If $\alpha < \alpha^*$, then any solution has a finite number of zeros. If $N < \alpha^*$, there exists $\breve{\alpha} \in (\alpha^*, \delta)$ such that if $\breve{\alpha} < \alpha$, the regular solutions are oscillating around 0 for large $|x|$, and $r^\delta w$ is asymptotically periodic in $\ln r$; and there exists precisely a solution u such that $r^\delta w$ is periodic in $\ln r$.

Theorem 1.4 *Assume $\varepsilon = 1, -\infty < \alpha < \delta$, and $p < P_1$. Then the regular solutions on $\mathbb{R}^N \times (0,\infty)$ have a constant sign, are not integrable, and are solution of ($\boldsymbol{E}_u$) with initial data $L|x|^{-\alpha} \in L^1_{loc}(\mathbb{R}^N)$. There is no other solution on $\mathbb{R}^N \setminus \{0\} \times (0,\infty)$.*

Observe that if $\alpha > 0$, all the solutions w tend to 0 at ∞, whereas if $\alpha < 0$, some of the solutions are unbounded near ∞. Next we come to the case $\varepsilon = -1$, which is treated at Theorems 4.1 and 4.2.

Theorem 1.5 *Assume $\varepsilon = -1$, $\delta < \alpha$, $p > P_1$ $(N \geq 2)$. There is no regular solution on $\mathbb{R}^N \times (0, \infty)$. Besides function U_∞, which is a strong razor blade, there exist positive integrable razor blades, with a singularity in $|x|^{-\eta}$, and positive solutions with a strong singularity in $|x|^{-\alpha}$, and $\lim_{t \to 0} |x|^\alpha u = L$.*

Theorem 1.6 *Assume $\varepsilon = -1$, $\delta < \alpha$, $p < P_1$ $(N \geq 2)$. There is no regular solution on $\mathbb{R}^N \times (0, \infty)$. There exists a positive solution on $\mathbb{R}^N \setminus \{0\} \times (0, \infty)$ with a singularity in $|x|^{-\alpha}$ (a strong one if and only if $N \leq \alpha$), and $\lim_{t \to 0} |x|^\alpha u = L$.*

Remark 1.7 *Weak singularities can occur even if $p > P_1$. For example, the solutions $u(t,x) = C|x|^{-\eta} = C|x|^{(p-N)/(p-1)}$ $(N \geq 2)$ given at (1.9) have a weak singularity. There even exist positive solutions u with a standing singularity, and integrable, see Theorems 1.3, 1.5,. This is not contradictory with the regularizing effect $L^1_{loc}(\mathbb{R}^N) \to L^\infty_{loc}(\mathbb{R}^N)$, which concerns solutions in $(0, \infty) \times \mathbb{R}^N$. The functions constructed above are solutions in $(0, \infty) \times \mathbb{R}^N \setminus \{0\}$, and the singularity $x = 0$ is not removable.*

(ii) Solutions defined for $t < 0$.

Next we consider the solutions defined for $t < 0$, and more generally for $t < T$. They correspond to $\varepsilon = 1, \delta < \alpha$ (see Section 5), or $\varepsilon = -1, \alpha < \delta$ (see Section 6). A main question in that case is the *extinction problem*: does there exist regular solutions u vanishing identically on $\mathbb{R}$ at time T? Does there exist singular razor blades, vanishing on $\mathbb{R}^N \setminus \{0\}$ at time T? Are they integrable?

One of our most significative results is the existence of two critical values $\alpha_{crit} > 0$ (when $P_2 < p < P_1$) and $\alpha^{crit} < 0$ (when $1 < p < P_2$), for which the regular solutions $u_{\alpha_{crit}}$ are positive, integrable, and vanish identically at time 0. Another new phenomena is the existence of positive solutions such that $C_1 U_\infty \leq u \leq C_2 U_\infty$ for some $C_1, C_2 > 0$, with a periodicity property, see Theorems 1.9 and 1.11.

First assume $\varepsilon = 1$. From Theorems 5.1 when $p > P_1$ and 5.4, 5.8 and 5.10 when $p < P_1$, we deduce the following.

Theorem 1.8 *Assume $\varepsilon = 1$, $\delta < \alpha$, $p > P_1$ $(N \geq 2)$. Then any solution u on $\mathbb{R}^N \setminus \{0\} \times (0, -\infty)$, in particular the regular ones, is oscillating around 0 for fixed $t < 0$ and large $|x|$, and $r^\delta w$ is asymptotically periodic in $\ln r$. There exists a solution such that $r^\delta w$ is periodic in $\ln r$. There exist weak integrable razor blades, with a singularity in $|x|^{-\eta}$.*

Theorem 1.9 *Assume $\varepsilon = 1$, $\delta < \alpha$, $p < P_1$. Then U_∞ is a solution on $\mathbb{R}^N \setminus \{0\} \times (0, -\infty)$, it is a weak razor blade. Moreover*

(1) If $p < P_2$, the regular solutions on $\mathbb{R}^N \times (0, -\infty)$ have a constant sign, are not integrable, and vanish identically at $t = 0$, with $\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leq C|t|^{\alpha/|\beta|}$. All the solutions have a finite number of zeros.

(2) For $\alpha < \eta$, the regular solutions have a constant sign, with the same behaviour (given by (1.8) if $\alpha = N$). There exists a positive solution u, which is not integrable, with a singularity in $|x|^{-\alpha}$ (a strong one if and only if $\alpha \geq N$), and $\lim_{t\to 0} |x|^{\alpha} u = L$. If $\alpha = \eta$, then $u(t,x) = C|x|^{-\eta}$ is a solution with a strong singularity.

(3) If $p > P_2$, there exists a critical value α_{crit} such that $\eta < \alpha_{crit} < \alpha^*$ and the regular solutions $u_{\alpha_{crit}}$ have a constant sign, are integrable, and vanish identically at $t = 0$, with $\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leq C|t|^{\alpha/|\beta|}$.

(4) If $\alpha \in (\alpha_{crit}, \alpha^*)$, there exist positive solutions u such that $r^\delta w$ is periodic in $\ln r$, thus

$$C_1 U_\infty \leq u \leq C_2 U_\infty \quad \textit{for some } C_1, C_2 > 0.$$

There exist positive solutions u, with the same bounds, such that $r^\delta w$ is asymptotically periodic near 0. There exist positive integrable solutions u such that $r^\delta w$ is asymptotically periodic near 0.

(5) If $\alpha_{crit} < \alpha$, the regular solutions are oscillating around 0 for fixed $t < 0$ and large $|x|$, and $r^\delta w$ is asymptotically periodic in $\ln r$. There exist solutions oscillating around 0, such that $r^\delta w$ is periodic. If $\alpha^* < \alpha$, there exist positive integrable razor blades, with a singularity in $|x|^{-\delta}$.

Finally suppose $\varepsilon = -1$. From Theorems 6.1, 6.2 when $p > P_1$ and 6.4, 6.6, 6.8, 6.9 when $p < P_1$, we obtain the following:

Theorem 1.10 *Assume $\varepsilon = -1$, $\alpha < \delta$ and $p > P_1$ $(N \geq 2)$. If $\alpha > 0$, there exist positive solutions u with a weak singularity in $|x|^{-\eta}$, integrable if and only if $\alpha > N$, and $\lim_{t\to 0} |x|^{\alpha} u = L$. If $\alpha < 0$, any solution has at least a zero. If $-p' < \alpha$, there is no regular solution on $\mathbb{R}^N \times (0, -\infty)$. If $\alpha = -p'$, the regular solutions, given by (1.10), have one zero.*

Theorem 1.11 *Assume $\varepsilon = -1$, $\alpha < \delta$ and $p < P_1$. Then U_∞ is a solution on $\mathbb{R}^N \setminus \{0\} \times (0, -\infty)$, it is a weak razor blade. Moreover*

(1) If $p > P_2$, all the solutions have a finite number of zeros. There exist positive integrable razor blades, with a singularity in $|x|^{-\delta}$.

(2) If $-p' < \alpha$, there is no regular solution on $\mathbb{R}^N \times (0, -\infty)$. There exist positive integrable razor blades as above. If $\alpha > 0$, there exist positive solutions u with a weak singularity in $|x|^{-\delta}$, integrable if and only if $\alpha > N$, and $\lim_{t\to 0} |x|^{\alpha} u = L$. If $-p' < \alpha < 0$, there exist solutions with one zero and the same behaviour. If $\alpha = -p'$, the regular solutions, given by (1.10), have one zero.

(3) If $p < P_2$, there exists a critical value α^{crit} such that $\alpha^* < \alpha^{crit} < -p'$ for which the regular solutions $u_{\alpha^{crit}}$ have a constant sign, are integrable, vanishing identically at $t = 0$, with $\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leq C|t|^{\alpha/|\beta|}$.

(4) If $p < P_2$ and $\alpha \in \left(\alpha^*, \alpha^{crit}\right)$, there exist positive solutions u such that $r^\delta w$ is periodic in $\ln r$, and thus

$$C_1 U_\infty \leq u \leq C_2 U_\infty \quad \text{for some } C_1, C_2 > 0.$$

There exist positive solutions with a weak singularity in $|x|^{-\delta}$, with the same bounds, such that $r^\delta w$ is asymptotically periodic near ∞. The regular solutions have a constant sign, are not integrable, vanishing identically at $t = 0$, and $r^\delta w$ is asymptotically periodic near ∞.

(5) If $p < P_2$ and $\alpha < \alpha^{crit}$, there exist solutions oscillating around 0, such that $r^\delta w$ is periodic. There exists solutions oscillating around 0, integrable, such that $r^\delta w$ is asymptotically periodic. If $\alpha \leq \alpha^*$ the regular solutions have a constant sign, are not integrable, and vanish identically at $t = 0$.

Remark 1.12 *If $p < P_1$, recall that the Harnack inequality does not hold, as it can be shown by the regular positive solutions constructed at Theorems 1.9, in particular those given by (1.8) when $\alpha = N$. Notice that the two kinds of regular, integrable, solutions constructed for the critical values $\alpha_{crit} > 0$ and $\alpha^{crit} < 0$ are of different types: the first one, constructed for $p > P_2$, desappears in a spreading way, the second one, constructed for $p < P_2$ desappears in a focussing way.*

The case $p > 2$ will be treated in a second article, see [5], where we complete the results of [11].

2 General properties

2.1 Different formulations of the problem

In all the sequel we can assume

$$\alpha \neq 0,$$

since the solutions are given explicitely by (1.11) when $\alpha = 0$. Defining

$$J_N(r) = r^N \left(w + \varepsilon r^{-1} \left|w'\right|^{p-2} w'\right), \qquad J_\alpha(r) = r^{\alpha - N} J_N(r), \tag{2.1}$$

equation $(\mathbf{E}_w)$ can be written in an equivalent way under the forms

$$J_N'(r) = r^{N-1}(N - \alpha) w, \qquad J_\alpha'(r) = -\varepsilon(N - \alpha) r^{\alpha - 2} \left|w'\right|^{p-2} w'. \tag{2.2}$$

If $\alpha = N$, then J_N is constant, so we find again (1.7).

We shall often use the following logarithmic substitution; for given $d \in \mathbb{R}$, setting

$$w(r) = r^{-d} y_d(\tau), \qquad Y_d = -r^{(d+1)(p-1)} \left|w'\right|^{p-2} w', \qquad \tau = \ln r, \tag{2.3}$$

we obtain the equivalent system:

$$\left.\begin{array}{r}y_d' = dy_d - |Y_d|^{(2-p)/(p-1)}\, Y_d, \\ Y_d' = (p-1)(d-\eta)Y_d + \varepsilon e^{(p+(p-2)d)\tau}(\alpha y_d - |Y_d|^{(2-p)/(p-1)}\, Y_d).\end{array}\right\} \tag{2.4}$$

And y_d, Y_d satisfy the equations

$$y_d'' + (\eta - 2d)y_d' - d(\eta - d)y_d + \frac{\varepsilon}{p-1}e^{((p-2)d+p)\tau}\left|dy_d - y_d'\right|^{2-p}\left(y_d' + (\alpha - d)y_d\right) = 0, \tag{2.5}$$

$$\begin{array}{r}Y_d'' + (p-1)(\eta - 2d - p')Y_d' + \varepsilon e^{((p-2)d+p)\tau}\, |Y_d|^{(2-p)/(p-1)}\, (Y_d'/(p-1) + (\alpha - d)Y_d) \\ -(p-1)^2(\eta - d)(p' + d)Y_d = 0.\end{array} \tag{2.6}$$

2.2 Reduction to an autonomous system

In particular the substitution (2.3) with $d = \delta$ is the most performant: setting $y = y_d$,

$$w(r) = r^{-\delta}y(\tau), \qquad Y = -r^{(\delta+1)(p-1)}\left|w'\right|^{p-2} w', \qquad \tau = \ln r, \tag{2.7}$$

we are lead to the *autonomous* system

$$\left.\begin{array}{r}y' = \delta y - |Y|^{(2-p)/(p-1)}\, Y, \\ Y' = (\delta - N)Y + \varepsilon(\alpha y - |Y|^{(2-p)/(p-1)}\, Y).\end{array}\right\} \tag{\textbf{S}}$$

Since $N - \delta p = \eta - 2\delta$, and $N - \delta = (p-1)(\eta - \delta)$, equations (2.5), (2.6) take the form

$$(p-1)y'' + (N - \delta p)y' + \delta(\delta - N)y + \varepsilon\left|\delta y - y'\right|^{2-p}\left(y' + (\alpha - \delta)y\right) = 0, \tag{$\mathbf{E}_y$}$$

$$Y'' + (N-2\delta)Y' + \frac{\varepsilon}{p-1}\,|Y|^{(2-p)/(p-1)}\, Y' + \varepsilon(\alpha - \delta)\,|Y|^{(2-p)/(p-1)}\, Y + \delta(\delta - N)Y = 0, \tag{$\mathbf{E}_Y$}$$

When w has a constant sign, we define two functions associated to (y, Y) :

$$\zeta(\tau) = \frac{|Y|^{(2-p)/(p-1)}\, Y}{y}(\tau) = -\frac{rw'(r)}{w(r)}, \qquad \sigma(\tau) = \frac{Y}{y}(\tau) = -\frac{|w'(r)|^{p-2}\, w'(r)}{rw(r)}. \tag{2.8}$$

They play an essential role in the asymptotic behaviour: indeed ζ describes the behaviour of w'/w and σ is the slope in the phase plane (y, Y). They satisfy the equations

$$\zeta' = \zeta(\zeta - \eta) + \frac{\varepsilon}{p-1}\,|\zeta y|^{2-p}\,(\alpha - \zeta)) = \zeta(\zeta - \eta + \frac{\varepsilon(\alpha - \zeta)}{(p-1)\sigma}), \tag{2.9}$$

$$\sigma' = \varepsilon(\alpha - N) + \left(|\sigma y|^{(2-p)/(p-1)}\, \sigma - N\right)(\sigma - \varepsilon) = \varepsilon(\alpha - N) + (\zeta - N)\,(\sigma - \varepsilon). \tag{2.10}$$

Remark 2.1 *Since (**S**) is autonomous, for any solution w of ($\boldsymbol{E}_w$) of the problem, all the functions $w_\xi(r) = \xi^\delta w(\xi r), \xi > 0$, are also solutions. From uniqueness, all the regular solutions are completly described from one of them: $w(r,a) = aw(a^{1/\delta}r, 1)$, thus they present the same behaviour at infinity.*

System (**S**) will be studied by using phase plane techniques, which was not done in [16], and gives our main results. Notice that the set of trajectories of system (**S**) in the phase plane (y, Y) is symmetric with respect to $(0, 0)$. In the phase plane (y, Y) we define

$$\mathcal{M} = \left\{(y, Y) \in \mathbb{R}^2 : |Y|^{(2-p)/(p-1)}\, Y = \delta y\right\}, \tag{2.11}$$

which is the set of the extremal points of y. We denote the four quadrants by

$$\mathcal{Q}_1 = (0, \infty) \times (0, \infty), \quad \mathcal{Q}_2 = (-\infty, 0) \times (0, \infty), \quad \mathcal{Q}_3 = -\mathcal{Q}_1, \quad \mathcal{Q}_4 = -\mathcal{Q}_2.$$

Remark 2.2 *The vector field at any point $(\xi, 0), \xi > 0$ satisfies $y' = -\xi^{1/(p-1)} < 0$, thus points to $\mathcal{Q}_2$. The field at any point $(\varphi, 0), \varphi > 0$ satisfies $Y' = \varepsilon\alpha\varphi$, thus points to $\mathcal{Q}_1$ if $\varepsilon\alpha > 0$ and to $\mathcal{Q}_4$ if $\varepsilon\alpha < 0$.*

Remark 2.3 *The couple (y, Y) is related to J_N by the identity*

$$J_N(r) = r^{N-\delta}(y(\tau) - \varepsilon Y(\tau)), \qquad \tau = \ln r, \tag{2.12}$$

and the formulae (2.2) can be found again from the relations

$$(y-\varepsilon Y)' = (\delta-\alpha)y+\varepsilon(N-\delta)Y = (\delta-\alpha)(y-\varepsilon Y)+\varepsilon(N-\alpha)Y = (\delta-N)(y-\varepsilon Y)+(N-\alpha)y. \tag{2.13}$$

Remark 2.4 *In the sequel the sense of variations of the functions y_d, Y_d, in particular y, Y, and ζ and σ plays an important role. At any extremal point τ, they satisfy respectively*

$$y_d''(\tau) = y_d(\tau)\left(d(\eta - d) - \frac{\varepsilon(\alpha - d)}{p-1} e^{((p-2)d+p)\tau} \left|dy_{d(\tau)}\right|^{2-p}\right), \tag{2.14}$$

$$Y_d''(\tau) = Y_d(\tau)\left((p-1)^2(\eta - d)(p' + d) - \varepsilon(\alpha - d)e^{((p-2)d+p)\tau} \left|Y_d(\tau)\right|^{(2-p)/(p-1)}\right), \tag{2.15}$$

$$(p-1)y''(\tau) = \delta^{2-p}y(\tau)\left(\delta^{p-1}(N-\delta) - \varepsilon(\alpha-\delta)\left|y(\tau)\right|^{2-p}\right) = -\left|Y(\tau)\right|^{(2-p)/(p-1)} Y'(\tau), \tag{2.16}$$

$$Y''(\tau) = Y(\tau)\left(\delta(N-\delta) - \varepsilon(\alpha-\delta)\left|Y(\tau)\right|^{(2-p)/(p-1)}\right) = \varepsilon\alpha y'(\tau), \tag{2.17}$$

$$(p-1)\zeta''(\tau) = \varepsilon(2-p)((\alpha-\zeta)\left|\zeta\right|^{2-p}\left|y\right|^{-p} yy')(\tau) = \varepsilon(2-p)((\alpha-\zeta)(\delta-\zeta)\left|\zeta y\right|^{2-p})(\tau), \tag{2.18}$$

$$(p-1)\sigma''(\tau) = (2-p)((\sigma-\varepsilon)\left|\sigma\right|^{(2-p)/(p-1)} Y \left|y\right|^{(4-3p)/(p-1)} y')(\tau) = \zeta'(\tau)(\sigma(\tau)-\varepsilon). \tag{2.19}$$

2.3 Energy functions for system (S)

A classical energy function is associated to equation $(\mathbf{E}_w)$:

$$E(r) = \frac{1}{p'} |w'|^p + \varepsilon \frac{\alpha}{2} w^2, \tag{2.20}$$

which is is nonincreasing when $\varepsilon = 1$, since $E'(r) = -(N-1)r^{-1} |w'|^p - \varepsilon r w'^2$. It is not sufficient in the study: we need energy functions adapted to y and Y. Using the ideas of [4], we construct two of them by using the Anderson and Leighton formula, see [2].

• We find a first function W, given by

$$W(\tau) = \mathcal{W}(y(\tau), Y(\tau)), \text{ where } \mathcal{W}(y, Y) = \varepsilon \left(\frac{(2\delta - N)\delta^{p-1}}{p} |y|^p + \frac{|Y|^{p'}}{p'} - \delta y Y \right) + \frac{\alpha - \delta}{2} y^2. \tag{2.21}$$

It satisfies

$$\begin{aligned} W'(\tau) &= \varepsilon(2\delta - N) \left(\delta y - |Y|^{(2-p)/(p-1)} Y \right) \left(|\delta y|)^{p-2} \delta y - Y \right) - \left(\delta y - |Y|^{(2-p)/(p-1)} Y \right)^2 \\ &= \left(\delta y - |Y|^{(2-p)/(p-1)} Y \right) \left(|\delta y|)^{p-2} \delta y - Y \right) (\varepsilon(2\delta - N) - \frac{\delta y - |Y|^{(2-p)/(p-1)} Y}{|\delta y|^{p-2} \delta y - Y}). \end{aligned} \tag{2.22}$$

When $\varepsilon(2\delta - N) \leq 0$, then W is nonincreasing. When $\varepsilon(2\delta - N) > 0$, we consider the curve

$$\mathcal{L} = \left\{ (y, Y) \in \mathbb{R}^2 : H(y, Y) = \frac{\delta y - |Y|^{(2-p)/(p-1)} Y}{|\delta y|^{p-2} \delta y - Y} = \varepsilon(2\delta - N) \right\}, \tag{2.23}$$

where by convention the quotient takes the value $|\delta y|^{2-p} / (p-1)$ if $|\delta y|)^{p-2} \delta y = Y$. It is a closed curve surrounding $(0,0)$, symmetric with respect to $(0,0)$. Let $\mathcal{S}_{\mathcal{L}}$ be the domain with boundary $\mathcal{L}$ and containing $(0,0)$:

$$\mathcal{S}_{\mathcal{L}} = \left\{ (y, Y) \in \mathbb{R}^2 : H(y, Y) \leq \varepsilon(2\delta - N) \right\}. \tag{2.24}$$

Then $W'(\tau) \geq 0$ if $(y(\tau), Y(\tau)) \in \mathcal{S}_{\mathcal{L}}$ and $W'(\tau) \leq 0$ if $(y(\tau), Y(\tau)) \notin \mathcal{S}_{\mathcal{L}}$. Observe that $\mathcal{S}_{\mathcal{L}}$ is bounded: indeed for any $(y, Y) \in \mathbb{R}^2$,

$$H(y, Y) \geq \frac{1}{2}((\delta y)^{2-p} + |Y|^{(2-p)/(p-1)}). \tag{2.25}$$

Also $\mathcal{S}_{\mathcal{L}}$ is connected, more precisely for any $(y, Y) \in \mathcal{S}_{\mathcal{L}}$ and any $\theta \in [0,1]$, $(\theta y, \theta^{p-1} Y) \in \mathcal{S}_{\mathcal{L}}$.

• A second function, denoted by V, is also given by Anderson formula (or by multiplication by Y' in $(\mathbf{E}_Y)$): let

$$V(\tau) = \mathcal{V}(Y(\tau), Y'(\tau)), \text{ where } \mathcal{V}(Y, Z) = \varepsilon \left(\frac{\delta(\delta - N)}{2} Y^2 + \frac{1}{2} Y'^2 \right) + \frac{\alpha - \delta}{p'} |Y|^{p'}, \tag{2.26}$$

then V satisfies

$$V'(\tau) = \left(\varepsilon(2\delta - N) - \frac{1}{p-1}|Y|^{(2-p)/(p-1)}\right)Y'^2. \tag{2.27}$$

When $\varepsilon(2\delta - N) \leq 0$, then V is nonincreasing. When $\varepsilon(2\delta - N) > 0$, then $V'(\tau) \geq 0$ whenever $|Y(\tau)| \leq D$, where

$$D = (\varepsilon(2\delta - N)(p-1))^{(p-1)/(2-p)}. \tag{2.28}$$

The function W gives more informations on the system, because $\mathcal{S}_{\mathcal{L}}$ is bounded, whereas the set of zeros of V' is unbounded.

2.4 Stationary points of system (S)

If $\alpha = \delta = N$, then (**S**) has an infinity of stationary points, given by $\pm(k, (\delta k)^{p-1})$, $k \geq 0$. Apart from this case, if $\varepsilon(\delta - N)(\delta - \alpha) \leq 0$, it has a unique stationary point $(0,0)$. If $\varepsilon(\delta - N)(\delta - \alpha) > 0$, it admits three stationary points:

$$(0,0), \qquad M_\ell = (\ell, (\delta\ell)^{p-1}) \in \mathcal{Q}_1, \qquad M'_\ell = -M_\ell \in \mathcal{Q}_3, \tag{2.29}$$

where ℓ is defined at (1.5). In that case, we find again that $w \equiv \ell r^{-\delta}$ is a particular solution of equation ($\mathbf{E}_w$).

(i) Local behaviour at $(0,0)$: the linearized problem at $(0,0)$ is given by

$$y' = \delta y, \qquad Y' = (\delta - N)Y + \varepsilon\alpha y,$$

and has the eigenvalues $\mu_1 = \delta - N$ and $\mu_2 = \delta$. Thus $(0,0)$ is a saddle point when $\delta < N$, and a source when $N < \delta$. One can choose a basis of eigenvectors $v_1 = (0,-1)$ and $v_2 = (N, \varepsilon\alpha)$.

(ii) Local behaviour at M_ℓ. Setting

$$y = \ell + \overline{y}, \qquad Y = (\delta\ell)^{p-1} + \overline{Y}, \tag{2.30}$$

system (**S**) is equivalent in $\mathcal{Q}_1$ to

$$\overline{y}' = \delta\overline{y} - \varepsilon\nu(\alpha)\overline{Y} - \Psi(\overline{Y}), \qquad \overline{Y}' = \varepsilon\alpha\overline{y} + (\delta - N - \nu(\alpha))\overline{Y} - \varepsilon\Psi(\overline{Y}), \tag{2.31}$$

where

$$\nu(\alpha) = \frac{\delta(N-\delta)}{(p-1)(\alpha-\delta)}, \text{ and } \Psi(\vartheta) = ((\delta\ell)^{p-1} + \vartheta)^{1/(p-1)} - \delta\ell - \frac{(\delta\ell)^{2-p}}{p-1}\vartheta, \qquad \vartheta > -(\delta\ell)^{p-1}. \tag{2.32}$$

The linearized problem is given by

$$\overline{y}' = \delta\overline{y} - \varepsilon\nu(\alpha)\overline{Y}, \qquad \overline{Y}' = \varepsilon\alpha\overline{y} + (\delta - N - \nu(\alpha))\overline{Y}.$$

Its eigenvalues $\lambda_1 \leq \lambda_2$ are the solutions of equation

$$\lambda^2 - (2\delta - N - \nu(\alpha))\lambda + p'(N - \delta) = 0 \tag{2.33}$$

The discriminant Δ of the equation (2.33) is given by

$$\Delta = (2\delta - N - \nu(\alpha))^2 - 4p'(N - \delta) = (N + \nu(\alpha))^2 - 4\nu(\alpha)\alpha. \tag{2.34}$$

The critical value α^* of α given at (1.13) appears when $\varepsilon(\delta - N/2) > 0$:

$$\alpha = \alpha^* \Longleftrightarrow \lambda_1 + \lambda_2 = 0.$$

When $\delta < N$, and $\varepsilon = 1$, then $\delta < \alpha$ and M_ℓ is a sink when $\delta \leq N/2$ or $\delta > N/2$ and $\alpha < \alpha^*$, and a source when $\delta > N/2$ and $\alpha > \alpha^*$. When $\delta < N$, and $\varepsilon = -1$, then $\alpha < \delta$, M_ℓ is a source when $\delta \geq N/2$ or $\delta < N/2$ and $\alpha > \alpha^*$, and a sink when $\delta < N/2$ and $\alpha < \alpha^*$. When $N < \delta$, then M_ℓ is always a saddle point, but, as we will see after, the value α^* also plays a part.

In the sequel the sign of α^* and its position with respect to N or η plays a role. By computation,

$$\alpha^* = \frac{p'(\delta^2 - 3\delta + 2N)}{2(2\delta - N)} = \eta + \frac{(\delta - N)^2}{(p-1)(2\delta - N)} = N + \frac{(\delta - N)(\delta^2 - (N+3)\delta + N)}{(2\delta - N)(\delta - 1)}. \tag{2.35}$$

Thus if $\varepsilon = 1$, then $\alpha^* > \eta > 0$ if $N \geq 2$; if $N = 1$, $\alpha^* > 0$ if $p > 4/3$. If $\varepsilon = -1$, then $\alpha^* < -p' < 0$.

Otherwise, when $\Delta > 0$ one can choose a basis of eigenvectors $u_1 = (-\varepsilon\nu(\alpha), \lambda_1 - \delta)$ and $u_2 = (\varepsilon\nu(\alpha), \delta - \lambda_2)$. If $\Delta \geq 0$, then δ is exterior to the roots if $\varepsilon\alpha > 0$, and $\lambda_1 < \delta < \lambda_2$ if $\varepsilon\alpha < 0$.

2.5 Existence of solutions of equation ($\mathbf{E}_w$)

Theorem 2.5 *(i) Let $r_1 > 0$ ($r_1 \geq 0$ if $N = 1$) and $a, a' \in \mathbb{R}$. Then there exists a unique solution w of equation ($\boldsymbol{E}_w$) in a neighborhood $\mathcal{V}$ of r_1, such that $w \in C^2(\mathcal{V})$ and $w(r_1) = a$, $w'(r_1) = a'$. It has a unique extension to a maximal interval of the form*

$$(R_w, \infty), \quad 0 \leq R_w, \quad \text{if } \varepsilon = 1; \qquad (0, S_w), \quad S_w \leq \infty, \quad \text{if } \varepsilon = -1.$$

Moreover if $0 < R_w$ (resp. $S_w < \infty$), then w is monotone near this point with an infinite limit.

(ii) For any $a \in \mathbb{R}$, there exists a unique regular solution of equation ($\boldsymbol{E}_w$) satisfying (1.12); and

$$\lim_{r\to 0} |w'|^{p-2} w'/rw = -\varepsilon\alpha/N. \tag{2.36}$$

(iii) If $N \geq 2$, any solution defined near 0 and bounded is regular. If $N = 1$, it satisfies $\lim_{r\to 0} w' = b \in \mathbb{R}$, and $\lim_{r\to 0} w = a \in \mathbb{R}$.

Proof. (i) Local existence and uniqueness near $r_1 > 0$ follow directly from the Cauchy theorem applied to equation $(\mathbf{E}_w)$ or to system $(\mathbf{S})$, since the map $\xi \mapsto f_p(\xi) = |\xi|^{(2-p)/(p-1)}\,\xi$ is of class C^1. If $N = 1$, we can take $r_1 = 0$, obtain a local solution in a neighborhood of 0 in $\mathbb{R}$ and reduce it to $[0, \infty)$.

Any local solution around r_1 has a unique extension to a maximal interval (R_w, S_w). Suppose $0 < R_w$ (resp. $S_w < \infty$), and w is oscillating around 0 near this point. Making the substitution (2.3), whith $d \neq 0$, if τ is a maximal point of $|y_d|$, then (2.14) holds. Taking d such that $\varepsilon\,(d - \alpha) > 0$, then $(y_d(\tau))$ stays bounded since the exponential has a positive limit; for that reason y_d stays bounded, w is bounded near R_w (resp. S_w) and then also J_N', J_N and w', which is contradictory. Thus w keeps a constant sign, for example $w > 0$ near R_w(resp. S_w). At each extremal point r such that $w(r) > 0$, we find $(|w'|^{p-2}\,w')'(r) = -\varepsilon\alpha w(r)$, thus r is unique since $\alpha \neq 0$. Thus w is strictly monotone near R_w (resp. S_w), and w and $|w'|$ tend to ∞.

First suppose $\varepsilon = 1$. Let us show that $S_w = \infty$. It is easy when $\alpha > 0$: since E is nondecreasing, w and w' are bounded for $r > r_1$. Assume $\alpha < 0$ and $S_w < \infty$; then for example $w > 0$ near S_w, and w is nondecreasing, and $\lim_{r \to S_w} w = \infty$. Then J_α is nonincreasing and nonnegative near S_w, hence again w and w' are bounded, which is contradictory. Next suppose $\varepsilon = -1$. If $R_w > 0$, as above, for example $w > 0$ and w is nonincreasing and $\lim_{r \to R_w} w = \infty$. Then either $\alpha < N$ and J_N is nonnegative and nondecreasing near R_w, thus bounded, or $\alpha \geq N$ and J_α is nonnegative and nondecreasing near R_w, and still bounded. In any case we reach a contradiction, then $R_w = 0$.

(ii) By symmetry we can suppose $a \geq 0$. Let $\rho > 0$. From (2.1) and (2.2), any regular solution w on $[0, \rho]$ satisfies

$$w(r) = a - \varepsilon \int_0^r f_p(sT(w))ds, \qquad T(w)(r) = w(r) + (\alpha - N) \int_0^1 \theta^{N-1} w(r\theta) d\theta. \tag{2.37}$$

Reciprocally, any function $w \in C^0\left([0, \rho]\right)$ solution of (2.37) satisfies $w \in C^1\left((0, \rho]\right)$, and $|w'|^{p-2}\,w'(r) = rT(w)$, hence $|w'|^{p-2}\,w' \in C^1\left((0, \rho]\right)$ and w satisfies $(\mathbf{E}_w)$ in $(0, \rho]$. And $\lim_{r \to 0} rT(w) = 0$, thus $w \in C^1\left([0, \rho]\right)$ and $|w'|^{p-2}\,w' \in C^1\left([0, \rho]\right)$. Then w satisfies $(\mathbf{E}_w)$ in $[0, \rho]$ and $w'(0) = 0$. Moreover from $(\mathbf{E}_w)$, $\lim_{r \to 0} |w'|^{p-2}\,w'/rw = -\varepsilon\alpha/N$, therefore, $w - a = O(r^{p'})$ near 0. We search w under the form $w = a + r^{p'}\zeta(r)$, with

$$\zeta \in \mathcal{B}_{\rho,M} = \left\{ \zeta \in C^0\left([0, \rho]\right) : \|\zeta\|_{C^0([0,\rho])} = \max_{r \in [0,\rho]} |\zeta(r)| \leq M \right\}.$$

We are lead to the problem $\zeta = \Theta(\zeta)$, where

$$\Theta(\zeta)(r) = -\varepsilon \int_0^1 \theta^{1/(p-1)} f_p(T(a + (r\theta)^{p'}\zeta(r\theta)))d\theta = -\varepsilon \int_0^1 \theta^{1/(p-1)} f_p(\frac{\alpha a}{N} + T((r\theta)^{p'}\zeta(r\theta)))d\theta.$$

Taking for example $M = (|\alpha|\, a)^{1/(p-1)}$, it follows that Θ is a strict contraction from $\mathcal{B}_{\rho,M}$ into itself for ρ small enough, hence existence and uniqueness hold in $[0, \rho]$.

(iii) If w is defined in $(0,\rho)$ and bounded, then J_N' is integrable; let $l = \lim_{r\to 0} J_N(r)$; then $|w'|^{p-2} w' = \varepsilon l r^{1-N}(1+o(1)$; if $N \geq 2$ it implies $l = 0$, thus from above, w is regular. If $N = 1$, then $\lim_{r\to 0} w' = b \in \mathbb{R}$, and $\lim_{r\to 0} w = a \in \mathbb{R}$. ■

Remark 2.6 *Let w be any solution of ($\boldsymbol{E}_w$) such that $w(r) > 0$ on some interval I.*

(i) Then w has at most one extremal point on I, since it satisfies $(|w'|^{p-2} w')' = -\varepsilon\alpha w$, and it is a maximum if $\varepsilon\alpha > 0$, a minimum if $\varepsilon\alpha < 0$.

(ii) From (2.36), if w is regular and $w > 0$ in $(0,r_1)$, $r_1 \leq \infty$, then $w' < 0$ in $(0,r_1)$ when $\varepsilon\alpha > 0$, thus $\mathcal{T}_r$ is in $\mathcal{Q}_1$, and $w' > 0$ in $(0,r_1)$ when $\varepsilon\alpha < 0$, thus $\mathcal{T}_r$ is in $\mathcal{Q}_3$ in $(-\infty, \ln r_1)$.

Remark 2.7 *In the case $\delta \neq N$, we can give a shorter proof of (ii) using the dynamical system. Indeed $(0,0)$ is either a source, or a saddle point. Thus there exists precisely one trajectory starting from $(0,0)$ at $-\infty$, with $y > 0$, with the slope $\varepsilon\alpha/N$. The corresponding solutions are regular: they satisfy $\lim_{\tau\to-\infty} \sigma = \varepsilon\alpha/N$, then $\lim_{r\to 0} |w'|^{p-2} w'/rw = -\varepsilon\alpha/N$; thus $w^{(2-p)/(p-1)}$ has a limit $a > 0$. Moreover $\lim_{r\to 0} w' = 0$, thus w satisfies (1.12), and any a is obtained by scaling.*

Definition 2.8 *For any $p > 1$, The trajectory $\mathcal{T}_r$ in the plane (y,Y) starting from $(0,0)$ at $-\infty$, with $y > 0$, with the slope $\varepsilon\alpha/N$ and its opposite $-\mathcal{T}_r$ will be called regular trajectories. We shall say that y is regular. Observe that $\mathcal{T}_r$ starts in $\mathcal{Q}_1$ if $\varepsilon\alpha > 0$, and in $\mathcal{Q}_4$ if $\varepsilon\alpha < 0$.*

Notation 2.9 *For any point $P_0 = (y_0, Y_0) \in \mathbb{R}^2 \backslash \{(0,0)\}$, the unique trajectory in the phase plane going through P_0 is denoted by $\mathcal{T}_{[P_0]}$. Notice that $\mathcal{T}_{[-P_0]} = -\mathcal{T}_{[P_0]}$, from the symmetry of system ($\boldsymbol{S}$).*

2.6 First sign properties

Proposition 2.10 *Let $w \not\equiv 0$ be any solution of ($\boldsymbol{E}_w$).*

(i) If $\varepsilon = 1$ and $\alpha \leq \max(N,\eta)$, then w has at most one zero, and no zero if w is regular.

(ii) If $\varepsilon = 1$ and $N < \min(\delta,\alpha)$ and w is regular, then w has at least one zero.

(iii) If $\varepsilon = -1$ and $\alpha \geq \min(0,\eta)$, then w has at most one zero. If $\alpha > 0$ and w is regular, then it has no zero.

(iv) If $\varepsilon = -1$ and $-p' \leq \alpha < \min(0,\eta)$, then w' has at most one zero, consequently w has at most two zeros, and at most one if w is regular.

Proof. (i) Let $\varepsilon = 1$. Consider two consecutive zeros $\rho_0 < \rho_1$ of w, with $w > 0$ on (ρ_0, ρ_1) thus $w'(\rho_0) < 0 < w'(\rho_2)$. If $\alpha \leq N$, we find

$$J_N(\rho_1) - J_N(\rho_0) = -\rho_1^{N-1} \left|w'(\rho_1)\right|^{p-2} - \rho_0^{N-1} w'(\rho_0)^{p-1} = (N-\alpha)\int_{\rho_0}^{\rho_2} s^{N-1} w ds,$$

which is contradictory; thus w has at most one zero. If w is regular with $w(0) > 0$, and ρ_1 is a first zero, then

$$J_N(\rho_1) = -\rho_1^{N-1} \left|w'(\rho_1)\right|^{p-1} = (N-\alpha)\int_0^{\rho_1} s^{N-1} w ds \geq 0,$$

hence again a contradiction. Next suppose $0 < \alpha \leq \eta$ and use the substitution (2.3), with $d = \alpha$. Then y_α has at most one zero. Indeed if y_α has a maximal point τ where it is positive, and is not constant, then from (2.14),

$$y''_\alpha(\tau) = \alpha(\eta - \alpha) y_d(\tau); \tag{2.38}$$

hence $y''_\alpha(\tau) < 0$, which is impossible. In the same way the regular solution satisfies $\lim_{\tau\to-\infty} y_\alpha = 0$ since $\alpha > 0$, and y_α has no maximal point, thus y_α is positive and increasing.

(ii) Let $\varepsilon = 1$ and $w > 0$ on $[0, \infty)$. If $N < \alpha$, then $J_N(r) = (N-\alpha)\int_0^r s^{N-1} w ds < 0$. The function $r \longmapsto \delta r^{p'} - w^{(p-2)/(p-1)}$ is nonincreasing, thus $w = O(r^{-\delta})$ at ∞, then y is bounded at ∞. For any $r \geq 1$, one gets $J_N(r) \leq J_N(1) < 0$, hence $y(\tau) + |J_N(1)| e^{(\delta-N)\tau} \leq Y(\tau)$ for any $\tau \geq 0$, from (2.12). Then $\lim_{\tau\to\infty} Y = \infty$, thus $\lim_{\tau\to\infty} y' = -\infty$ from $(\mathbf{S})$, which is impossssible.

(iii) Let $\varepsilon = -1$ and $\alpha \geq \min(\eta, 0)$. Here we use again the substitution (2.3) from some $d \neq 0$. If y_d has a maximal point, where it is positive, and is not constant, then (2.14) holds. Taking $d \in (0, \min(\alpha, \eta))$, if $N \geq 2$ and $\alpha > 0$, and $d = -1$ if $N = 1$ and $\eta = -1 \leq \alpha$, we are lead to a contradiction. Suppose w regular and $\alpha > 0$. Then $w' > 0$ near 0, from Theorem 2.5, and as long as w stays positive, any extremal point r is a strict minimum; thus in fact $w' > 0$ on $[0, S_w)$.

(iv) Let $\varepsilon = -1$ and $-p' \leq \alpha < \min(0, \eta)$. Suppose that w' has two consecutive zeros $\rho_1 < \rho_2$, and use again (2.3) with $d = \alpha$. If the function Y_α has a maximal point τ, where it is positive and is not constant, then from (2.15),

$$Y''_\alpha(\tau) = (p-1)^2(\eta - \alpha)(p' + \alpha) Y_\alpha(\tau) \tag{2.39}$$

thus $Y''_\alpha(\tau) < 0$, and Y_α has at most one zero. Next consider the regular solutions: they satisfy $Y_\alpha = e^{(\alpha(p-1)+p)\tau}(|\alpha|\, a/N)(1 + o(1)$ near $-\infty$, from Theorem 2.5 and (2.3), thus $\lim_{\tau\to-\infty} Y_\alpha = 0$; as above Y_α cannot cannot have any extremal point, thus Y_α is positive and increasing; then $w' < 0$ from (2.3), thus w has at most one zero. ∎

Remark 2.11 *From (2.38) and (2.39), if $0 < \alpha \leq \eta$ (resp. $-p' \leq \alpha \leq \min(\eta, 0)$) then y_α (resp. Y_α) has only minimal points on any set where it is positive.*

Proposition 2.12 *Let y be any solution of ($\boldsymbol{E}_y$), of constant sign near $\ln R_w$ or $\ln S_w$.*

(i) Suppose that y is not strictly monotone near this point, then $R_w = 0$ or $S_w = \infty$; if y is not constant, then either $\varepsilon = 1$ and $\delta < N < \alpha$, or $\varepsilon = -1$ and $\alpha < \delta < N$; in any case, y oscillates around ℓ.

(ii) If y is strictly monotone near $\ln R_w$ (resp. $\ln S_w$), then also Y, ζ, σ are monotone near this point.

Proof. Let $s = R_w$ or S_w, and y of constant sign near s, then also Y, from Remark 2.6.

(i) At each point τ where $y'(\tau) = 0$, then $y''(\tau) \neq 0$, and (2.16) holds with $y > 0$. Suppose that y is not strictly monotone near s. Then there exists a sequence (τ_n) strictly monotone, converging to s, such that $y'(\tau_n) = 0$, $y''(\tau_{2n}) > 0$, $y''(\tau_{2n+1}) < 0$. Then either $\varepsilon = 1$ and $\delta < \min(\alpha, N)$, or $\varepsilon = -1$ and $\alpha < \delta < N$; and $y(\tau_{2n}) < \ell < y(\tau_{2n+1})$. It cannot happen if s is finite, because y tends to ∞. It is also impossible when $\varepsilon = 1$ and $\alpha \leq N$. Indeed there exist at least two points $\theta_1 < \theta_2$, such that $y(\theta_1) = y(\theta_2) = \ell$ and $y \geq \ell$ on (θ_1, θ_2), $y'(\theta_1) > 0 > y'(\theta_2)$. Then from (**S**), $Y(\theta_1) < (\delta\ell)^{p-1} < Y(\theta_2)$. And from (2.13), $(e^{(N-\delta)\tau}(y - Y))' = (N - \alpha)e^{(N-\delta)\tau}y$; and the constant $(\ell, (\delta\ell)^{p-1})$ is also solution of (**S**), hence

$$(e^{(N-\delta)\tau}(y - \ell - Y + (\delta\ell)^{p-1}))' = (N - \alpha)e^{(N-\delta)\tau}(y - \ell) \geq 0 \tag{2.40}$$

on (θ_1, θ_2). A contradiction follows by integration on this interval.

(ii) Suppose y strictly monotone near s. At any extremal point τ of Y, we find $Y''(\tau) = \varepsilon\alpha y'(\tau)$ from (2.17), then $y'(\tau) \neq 0$, $Y''(\tau)$ has a constant sign; thus τ is unique, and Y is strictly monotone near s. Next consider the function ζ, which satisfies (2.9). If there exists τ_0 such that $\zeta(\tau_0) = \alpha$, then $\zeta'(\tau_0) = \alpha(\alpha - \eta)$. If $\alpha \neq \eta$, then τ_0 is unique, thus $\alpha - \zeta$ has a constant sign near s. Then also $\zeta''(\tau)$ has a constant sign at any extremal point τ of ζ, from (2.18). Then ζ is strictly monotone near s. If $\alpha = \eta$, then $\zeta \equiv \alpha$. At last consider σ, which satisfies (2.10). At each point τ such that $\sigma'(\tau) = 0$, one finds (2.19) and Y has a constant sign. If there exists τ_0 such that $\sigma(\tau_0) = \varepsilon$, then $\sigma'(\tau_0) = \varepsilon(\alpha - N)$. If $\alpha \neq N$, then τ_0 is unique, and $\sigma - \varepsilon$ has a constant sign near s. Thus $\sigma''(\tau)$ has a constant sign at any extremal point τ of σ, from (2.19), since Y has a constant sign near s. If $\alpha = N$, then $\sigma \equiv \varepsilon$. ■

2.7 Behaviour of w near 0 or ∞

Here we suppose w defined near 0 or ∞, that means y is defined near $\pm\infty$. We study the behaviour of y and then return to w. First we suppose y monotone, thus we can assume $y > 0$ near $\pm\infty$. We do not look for a priori estimates, which could be obtained by successive approximations as in [6]. Our method is based on the monotonicity and the L'Hospital's rule, much more rapid and efficient.

Proposition 2.13 *Let (y, Y) be any solution of (**S**), such that y is strictly monotone and $y > 0$ near $s = \pm\infty$. Then ζ has a finite limit λ near s, equal to $0, \alpha, \eta, \delta$. More precisely, one of the eventualities holds:*

(i) (y, Y) converges to a stationary point different from $(0, 0)$; then $\lambda = \delta$, and $\varepsilon(\delta - N)(\delta - \alpha) > 0$ or $\alpha = \delta = N$.

(ii) (y, Y) converges to $(0, 0)$; then

- *either $\lambda = 0$, $s = -\infty$, and y is regular, or $N = 1$;*

• *or $\lambda = \eta$; then either $(s = \infty, \delta < N)$ or $(s = \infty, \delta = N, \varepsilon(\alpha - N) < 0))$ or $(s = -\infty, N < \delta)$ or $(s = -\infty, \delta = N, \varepsilon(\alpha - N) > 0))$.*

(iii) $\lim_{\tau\to s} y = \infty$, and $\lambda = \alpha$; then either $(s = \infty, \alpha < \delta)$ or $(s = \infty, \alpha = \delta, \varepsilon(\delta - N) < 0)$ or $(s = -\infty, \delta < \alpha)$ or $(s = -\infty, \alpha = \delta, \varepsilon(\delta - N) > 0)$.

Proof. From Proposition 2.12, the functions y, Y, σ, ζ are monotone, thus ζ has a limit $\lambda \in [-\infty, \infty]$ and σ has a limit $\mu \in [-\infty, \infty]$, and (y, Y) converges to a stationary point, or $\lim y = \infty$; then $\lim |Y| = \infty$, since $\alpha \neq 0$ from system (**S**). In order to apply the L'Hospital's rule, we consider the two quotients

$$\frac{Y'}{y'} = \frac{(\delta - N)\sigma + \varepsilon(\alpha - \zeta)}{\delta - \zeta} \tag{2.41}$$

and

$$\frac{(|Y|^{(2-p)/(p-1)} Y)'}{y'} = \frac{\zeta(\delta - N + \varepsilon(\alpha - \zeta)/\sigma)}{(p-1)(\delta - \zeta)} = \frac{\zeta(\delta - N) + \varepsilon(\alpha - \zeta) |\zeta y|^{2-p}}{(p-1)(\delta - \zeta)}. \tag{2.42}$$

(i) First case: $\varepsilon(\delta - N)(\delta - \alpha) > 0$ and (y, Y) converges to $(\ell, (\delta\ell)^{p-1})$. then obviously $\lambda = \delta$; or $\alpha = \delta = N$ and $\lim_{\tau\to s} y = k > 0$; then $\lim_{\tau\to s} Y = (\delta k)^{p-1}$, thus $\lambda = \delta$.

(ii) Second case: (y, Y) converges to $(0, 0)$. Then λ is finite; indeed if $\lambda = \pm\infty$, the quotient (2.42) converges to $(N - \delta)/(p - 1)$, because $|\zeta y| = |Y|^{1/(p-1)} = o(1)$; thus $\zeta = |Y|^{(2-p)/(p-1)} Y/y$ has the same limit, from the L'Hospital's rule, which is contradictory.

• If $N < \delta$, then $(0, 0)$ is a source, thus $s = -\infty$. Using the eigenvectors, either $\mu = \varepsilon\alpha/N$, then $|\zeta|^{p-1} = |\mu| y^{2-p}(1 + o(1))$, thus $\lambda = 0$ and w is regular, from Remark 2.7. Or $\mu = \pm\infty$; then $\lambda = \lambda(\delta - N)/(p - 1)(\delta - \zeta)$ from (2.42), thus $\lambda = 0$ or $\lambda = \eta$. If $\lambda = 0$, then ζ'/ζ converges to $-\eta$ from (2.9), and $s = -\infty$, thus necessarily $\eta < 0$, which means $N = 1$.

• If $\delta < N$ (thus $N \geq 2$) then $(0, 0)$ is a saddle point, thus either $s = -\infty$ and $\mu = \varepsilon\alpha/N$, $\lambda = 0$ and w is regular. Or $s = \infty$, $\mu = \pm\infty$, and as above, $\lambda = 0$ or $\lambda = \eta$. Now if $\lambda = 0$ the quotient (2.41) converges to $\mp\infty$, which is contradictory. Thus $\lambda = \eta$.

• If $\delta = N$ (thus $N \geq 2$), either $\lambda = 0$, thus $y' > 0$, then $s = -\infty$, and $\mu = \varepsilon\alpha/N$ from (2.42). Or $\lambda > 0$; then $\lambda = N = \eta$ from (2.42). Moreover if $s = \infty$, then $\varepsilon(\alpha - N) < 0$; if $s = -\infty$, then $\varepsilon(\alpha - N) > 0$. Indeed $(\varepsilon y - Y)' = \varepsilon(N - \alpha)y$ and $y - \varepsilon Y$ converges to 0; thus if $s = \infty$ and $\varepsilon(N - \alpha) \geq 0$, or $s = -\infty$ and $\varepsilon(N - \alpha) \leq 0$, then $\mu \leq \varepsilon$, but $\mu = \infty$, we reach again a contradiction.

(iii) Third case: y tends to ∞. If $s = \infty$, then $y' > 0$, thus $\zeta < \delta$; if $s = -\infty$, then $\zeta > \delta$. If $\lambda = \pm\infty$, then the quotient (2.42) converges to $\varepsilon\infty$; thus $\lambda = \varepsilon\infty$ and $s = -\varepsilon\infty$. In any case, $\zeta' < 0$, thus $|\mu| \leq 1/(p-1)$ from (2.9), then $\mu = \varepsilon$ from (2.41), thus $Y' = -\varepsilon |Y|^{(2-p)/(p-1)} Y(1 + o(1))$; we reach a contradiction by integration. Thus λ is finite, and $\lambda \neq 0$. Indeed if $\lambda = 0$, then $\mu = 0$, seeing that $\sigma = |\zeta y|^{p-2} \zeta$, but $\mu = \alpha/\delta$ from (2.41).

• If $\alpha \neq \delta$, then $\lambda = \alpha$ or δ, from (2.42). In turn $\sigma = |\lambda y|^{p-2} \lambda(1 + o(1))$, thus $\mu = 0$. From (2.41), necessarily $\lambda = \alpha$. And if $s = \infty$, then $y' > 0$, thus $\zeta < \delta$, thus $\alpha < \delta$. If $s = -\infty$, then similarly $\alpha > \delta$.

• If $\alpha = \delta$, then $\lambda = \alpha = \delta \neq N$, and $\varepsilon(\delta - N)(\delta - \zeta) < 0$ from (2.42); thus if $s = \infty$, then $\varepsilon(\delta - N) < 0$ since $\zeta < \delta$; if $s = -\infty$, then $\varepsilon(\delta - N) > 0$. ■

Next we improve Proposition 2.14 by giving a precise behaviour of w in any case:

Proposition 2.14 *Under the assumptions of Proposition 2.13,*

(i) If $\lambda = \alpha \neq \delta$, then $\lim r^\alpha w = L > 0$ (near 0, or ∞).

(ii) If $\lambda = \eta > 0$, $\eta \neq N$, then $\lim r^\eta w = c > 0$.

(iii) If $\lambda = \alpha = \delta \neq N$, then

$$\lim r^\delta (\ln r)^{-1/(2-p)} w = \kappa = ((2-p)\delta^{p-1} |N - \delta|)^{1/(2-p)}. \tag{2.43}$$

(iv) If $\lambda = \eta = N = \delta \neq \alpha$, then

$$\lim r^N (\ln r)^{(N+1)/2} w = \rho = \frac{1}{N} \left(\frac{N(N-1)}{2|\alpha - N|} \right)^{(N+1)/2}. \tag{2.44}$$

(v) If $N = 1$, $\lambda = \eta = -1$ or $\lambda = 0$ (near 0) then

$$\lim_{r \to 0} w = a \in \mathbb{R}, \qquad \lim_{r \to 0} w' = b; \tag{2.45}$$

and $b \neq 0$, and $a = 0$ (thus $b > 0$) if and only if $\lambda = -1$.

Proof. (i) Let $\lambda = \alpha \neq \delta$. From (2.8), $rw'(r) = -\alpha w(r)(1 + O(1)$. Next apply Proposition 2.13:

• Either $s = \infty$ and $\alpha < \delta$; thus for any $\gamma > 0$, $w = O(r^{-\alpha+\gamma})$ and $1/w = O(r^{\alpha+\gamma})$ near ∞ and $w' = O(r^{-\alpha-1+\gamma})$; then $J'_\alpha(r) = O(r^{\alpha(2-p)-p-1+\gamma})$ thus J'_α is integrable, hence J_α has a limit L, and $\lim r^\alpha w = L$, seeing that $J_\alpha(r) = r^\alpha w(1 + o(1))$. If $L = 0$, then $r^\alpha w = O(r^{\alpha(2-p)-p+\gamma})$, which contradicts the estimate of $1/w = O(r^{\alpha+\gamma})$ for γ small enough. Thus $L > 0$.

• 0r $s = -\infty$, and $\delta < \alpha$, and $\lim_{\tau \to s} y = \infty$, $w = O(r^{-\alpha-\gamma}), 1/w = O(r^{\alpha-\nu})$, $w' = O(r^{-\alpha-1-\gamma})$ near 0, and $J'_\alpha(r) = O(r^{\alpha(2-p)-p-1-\gamma})$, thus J'_α is still integrable; hence $\lim r^\alpha w = L \geq 0$. If $L = 0$, then $r^\alpha w = O(r^{\alpha(2-p)-p-\gamma})$, which contradicts the estimate of $1/w$. Then again $L > 0$.

(ii) Let $\lambda = \eta > 0$, $\eta \neq N$. From Proposition 2.13, either ($s = \infty$, $\delta < N$) or ($s = -\infty$ and $N < \delta$). As above we get $w = O(r^{-\eta\pm\gamma})$ and $1/w = O(r^{\eta\pm\gamma})$ near ∞ or 0. Here we make the substitution (2.3) with $d = \eta$. We find $y_\eta = O(e^{\pm\gamma\tau})$, $1/y_\eta = O(e^{\pm\gamma\tau})$, $y'_\eta = O(e^{\pm\gamma\tau})$, thus $Y_\eta = O(e^{\pm\gamma\tau})$, and from (2.4), $Y'_\eta = O(e^{\pm\gamma\tau})$. Reporting in (2.4), we deduce $Y'_\eta = O(e^{(2-p)((\delta-\eta)\pm\gamma)\tau})$. When

$s = \infty$, then $\delta < \eta$, when $s = -\infty$, then $\delta > \eta$ from (1.3). In any case, Y_η' is integrable, hence Y_η has a limit k, and $Y_\eta - k = O(e^{(2-p)((\delta-\eta)\pm\gamma)\tau})$. Now $(e^{-\eta\tau}y_\eta)' = -e^{-\eta\tau}Y_\eta^{1/(p-1)}$, thus y_η has a limit $c = k^{1/(p-1)}/\eta$; in other words, $\lim r^\eta w = c$. If $c = 0$, then $Y_\eta = O(e^{(2-p)((\delta-\eta)\pm\gamma)\tau})$, $y_\eta = O(e^{((2-p)((\delta-\eta)\pm\gamma)/(p-1))\tau})$, which contradicts $1/y_\eta = O(e^{\gamma\tau})$ for γ small enough.

(iii) Let $\lambda = \alpha = \delta \neq N$, thus ($s = \infty$ and $\varepsilon(\delta - N) < 0$ or ($s = -\infty$ and $\varepsilon(\delta - N) > 0$), and $\lim_{\tau\to s} y = \infty$. Then $Y = (\delta y)^{p-1}(1 + o(1))$, and $\mu = 0$, thus $y - \varepsilon Y = y(1 + o(1))$, and from (2.13),

$$(y - \varepsilon Y)' = \varepsilon(N - \delta)Y = \varepsilon(N - \delta)\delta^{p-1}\,(y - \varepsilon Y)^{p-1}\,(1 + o(1)).$$

Then $y = (|N - \delta|)\delta^{p-1}(2 - p)\,|\tau|)^{1/(2-p)}(1 + o(1))$, which is equivalent to (2.43).

(iv) Let $\lambda = \eta = N = \delta \neq \alpha$, thus ($s = \infty$ and $\varepsilon(\alpha - N) < 0$) or ($s = -\infty$ and $\varepsilon(\alpha - N) > 0$), and $\lim_{\tau\to s} y = 0$. Then $Y = (Ny)^{p-1}(1 + o(1))$ and $\mu = \infty$, thus $Y - \varepsilon y = Y(1 + o(1))$, and from (2.13))

$$(Y - \varepsilon y)' = \varepsilon(\alpha - N)y = \varepsilon(\alpha - N)N^{-1}(Y - \varepsilon y)^{1/(p-1)}(1 + o(1)).$$

As a consequence $y = c\,|\tau|^{-(N+1)/2}\,(1 + o(1))$, with $c = (1/N)\,(N(N-1)/2\,|\alpha - N|)^{(N+1)/2}$, and (2.44) follows.

(v) Let $\lambda = 0$, then also $rw' = o(w)$, thus by integration $w + |w'| = O(r^{-k})$ for any $k > 0$. Then J_1' is integrable, thus J_1 has a limit at 0, and $\lim_{r\to 0} rw = 0$, thus $\lim_{r\to 0} w' = b \in \mathbb{R}$, $\lim_{r\to 0} w = a \geq 0$. Then $b \neq 0$, since the regular solutions satisfy (2.36), and $a \neq 0$, since $a = 0$ implies $w = -br(1 + o(1)$, $\zeta = -1$. If $\lambda = \eta = -1$, then from (2.8), w is nondecreasing, thus it has a limit $a \geq 0$ at 0, thus $w' = -a\lambda r^{-1}(1 + o(1))$, and by integration $a = 0$. And $((w')^{p-1})' = \varepsilon(1 - \alpha)w(1 + o(1))$, thus w' has a limit $b \neq 0$. ■

Next we consider the cases where y is not monotone, and possibly changing sign.

Proposition 2.15 *Assume $\varepsilon = 1$. (i) Assume that $N \leq \delta < \alpha$, or $N < \delta \leq \alpha$. Then any solution y has a infinite number of zeros near ∞.*

(ii) Suppose that y has a infinite number of zeros near $\pm\infty$. Then

either ($N < \alpha < \delta$ and $|y| < \ell$, $|Y| < (\delta\ell)^{p-1}$ near $\pm\infty$), or $N < \delta = \alpha$, or $\max(\delta, N, \eta) < \alpha$.

If moreover $\delta < N < \alpha$, then $|y| > \ell$ at its extremal points, $|Y| > (\delta\ell)^{p-1}$ at its extremal points.

Proof. (i) Suppose that is is not the case. Then for example $y > 0$ for large τ; and y is monotone, from Proposition 2.12, (i). Applying Proposition 2.13 with $s = \infty$, we reach a contradiction.

(ii) Suppose that y is oscillating around 0 near $\pm\infty$. Then from (2.16), at the extremal points,

$$|y(\tau)|^{2-p}\,(\delta - \alpha) < (\delta - N)\delta^{p-1}, \tag{2.46}$$

and the inequality is strict: if one equality holds, then y is constant, from uniqueness. Similarly Y is oscillating around 0, and at the extremal points, from (2.17), one finds

$$|Y(\tau)|^{(2-p)/(p-1)}(\delta-\alpha) < (\delta-N)\delta. \tag{2.47}$$

Then $\max(N,\eta) < \alpha$, from Proposition 2.10; and the conclusions follow from (2.46) and (2.47). ■

We can complete these results according to the sign of $\delta - N/2$:

Proposition 2.16 *Suppose that $\varepsilon(\delta - N/2) \leq 0$. Then any solution y has a finite number of zeros near $\ln R_w$ or $\ln S_w$. If it is defined near $\pm\infty$, and non monotone, then it converges to $\pm M_\ell$. There is no cycle in $\mathbb{R}^2$, and no homoclinic orbit in $\mathbb{R}^2$.*

Proof. (i) Suppose that y has an infinity of zeros. Then $R_w = 0$ or $S_w = \infty$, and there exists a strictly monotone sequence (r_n) of consecutive zeros of w, converging to 0 or ∞. Since $\varepsilon(\delta - N/2) \leq 0$, the energy function V defined at (2.26) is nonincreasing. We claim that V is bounded, which is not easy to prove. For that purpose, we introduce the function U defined by

$$U(r) = r^N(\frac{1}{2}w^2 + \varepsilon r^{-1}\left|w'\right|^{p-2}w'w) = e^{(N-2\delta)\tau}y(\frac{1}{2}y - \varepsilon Y);$$

we find

$$U'(r) = r^{N-1}((\frac{N}{2}-\alpha)w^2 + \varepsilon\left|w'\right|^p) = e^{(N-1-2\delta)\tau}((\frac{N}{2}-\alpha)y^2 + \varepsilon|Y|^{p'})$$

If $\varepsilon = 1$, then $\delta \leq N/2 < N < \alpha$. If $\varepsilon = -1$, then $\alpha < 0$, from Proposition 2.15. Then $U(r_n) = 0$, and $\varepsilon U'(r_n) > 0$. Therefore there exists another sequence (s_n), such that $s_n \in (r_n, r_{n+1})$, and $U(s_n) = 0$, and $\varepsilon U'(s_n) \leq 0$. At point $\tau_n = e^{s_n}$, we find $2^{1-p'}y^{2p'} = 2|Y|^{p'} \leq \varepsilon(2\alpha - N)y^2$, then $(y(\tau_n), Y(\tau_n))$ is bounded, $(V(\tau_n))$ is bounded, thus V is bounded near $\pm\infty$. Therefore V has a finite limit χ, and Y and Y' are bounded because $\varepsilon(\alpha-\delta) > 0$, and in turn (y, Y) is bounded. Otherwise $(0,0)$, $\pm M_\ell$, are not in the limit set at $\pm\infty$, since $(0,0)$ is a saddle point, and $\pm M_\ell$ is a source or a sink. Then the trajectory has a limit cycle $\mathcal{O}$, thus there exists a periodic solution (y, Y). The corresponding function V is periodic, and monotone, then it is constant, $V' \equiv 0$, thus Y is constant, and y is constant from $(\mathbf{S})$, which is contradictory.

(ii) Suppose that y is positive near $\pm\infty$, and non monotone. If $\varepsilon = 1$, then $\delta \leq N/2 < N < \alpha$; if $\varepsilon = -1$, then $\alpha < \delta < N$, from Proposition 2.12, and y oscillates around ℓ. There exists a sequence of minimal points (τ_n), where $y(\tau_n) < \ell$, and $|Y(\tau_n)| = \delta y(\tau_n)$, thus again $(y(\tau_n), Y(\tau_n))$ is bounded, and as above (y, Y) is bounded. The trajectory has no limit cycle, thus converges to M_ℓ. Finally if there is an homoclinic orbit, then $\mathcal{T}_r$ is homoclinic. Then $\lim_{\tau\to-\infty} V = \lim_{\tau\to\infty} V = 0$, thus $V \equiv 0$, as above (y, Y) is constant, hence $(y, Y) \equiv (0,0)$, which is contradictory. ■

Proposition 2.17 *Suppose that y is not monotone near $\varepsilon\infty$ (positive or changing sign) then y and Y are bounded.*

Proof. From Proposition 2.16, it follows that $\varepsilon(\delta - N/2) > 0$. When $\varepsilon = 1$, and y is changing sign and $N < \alpha < \delta$, then $|y|$ is bounded by ℓ from above. Apart from this case, if y is changing sign, then $\varepsilon(\alpha - \delta) > 0$, from Proposition 2.16. If y stays positive, either $\varepsilon = 1, \delta < \min(\alpha, N)$, or $\varepsilon = -1$, $\alpha < \delta < N$, fom Proposition 2.12. In any case $\varepsilon(\alpha - \delta) > 0$. Here we use the energy function W defined by (2.21). We can write $\mathcal{W}(y, Y)$ under the form

$$\mathcal{W}(y,Y) = \varepsilon(F(y,Y) + G(y)),$$
$$\text{with} \quad F(y,Y) = \frac{|Y|^{p'}}{p'} - \delta y Y + \frac{|\delta y|^p}{p}, \quad G(y) = \frac{(\delta - N)\delta^{p-1}}{p}|y|^p + \frac{\varepsilon(\alpha - \delta)}{2}y^2. \tag{2.48}$$

Observe that $F(y, Y) \geq 0$, thus $\varepsilon\mathcal{W}(y, Y) \geq G(y) > 0$ for large $|y|$. Then $W'(\tau) \leq 0$ whenever $(y(\tau), Y(\tau)) \notin \mathcal{S}_{\mathcal{L}}$, where $\mathcal{S}_{\mathcal{L}}$ is given at (2.24). Let τ_0 be arbitrary in the interval of definition of y. Since $\mathcal{S}_{\mathcal{L}}$ is bounded, there exists $k > 0$ large enough such that $\varepsilon W(\tau) \leq k$ for any τ such that $\varepsilon(\tau - \tau_0) \geq 0$ and $(y(\tau), Y(\tau)) \in \mathcal{S}$, and we can choose $k > W(\tau_0)$. Then $\varepsilon W(\tau) \leq k$ for $\varepsilon(\tau - \tau_0) \geq 0$, hence y and Y are bounded near $\varepsilon\infty$. ■

2.8 Further sign properties

From Propositions 2.13 and 2.14 we can improve Proposition 2.10:

Proposition 2.18 *Assume $\varepsilon = 1, -\infty < \alpha \leq \delta$ and $\alpha < N$. Then the regular solutions have a constant sign, y is strictly monotone and $\lim_{\tau\to\infty}\zeta = \alpha$. Moreover any solution has at most one zero, and then $\lim_{\tau\to\infty}\zeta = \alpha$.*

Proof. (i) The regular solutions have a constant sign from Proposition 2.10. Moreover J_N is increasing from 0, thus it is positive for $r > 0$, which means $Y < y$. And y is monotone near ∞ from Proposition 2.12. From Proposition 2.13, we have three possibilities: either $\alpha < N < \delta$, and $\lim_{\tau\to\infty}\zeta = \delta$, then $\lim_{\tau\to\infty} Y/y = (\delta - \alpha)/(\delta - N) > 1$, which is impossible; or $\delta \leq N$, and $\lim_{\tau\to\infty}\zeta = \eta \geq N$, then $\lim_{\tau\to\infty} Y/y = \infty$, which is also contradictory, or (finally) $\lim_{\tau\to\infty}\zeta = \alpha$. Moreover y is increasing on $\mathbb{R}$ from 0 to ∞. Indeed if y has a local maximum for some τ, then from (2.16), $\alpha < N < \delta$ and $y(\tau) \leq \ell$; and $\ell < \delta^{(p-1)/(2-p)}$; but $\delta y(\tau) = Y(\tau)^{1/(p-1)} < y(\tau)^{1/(p-1)}$, which is contradictory.

(ii) From Proposition 2.10, any solution $w \not\equiv 0$ has at most one zero. If $w(r_1) = 0$ and for example $w > 0$ on (r_1, ∞), then $w'(r_1) > 0$, thus $J_N(r) \geq J_N(r_1) > 0$ for $r \geq r_1$; we conclude as above. ■

Proposition 2.19 *Assume $\varepsilon = -1$.*

(i) If $\alpha < 0$ and $N \leq \delta$, the regular solutions have at least one zero.

(ii) If $0 < \alpha$, the regular solutions have a constant sign and satisfy $S_w < \infty$.

(iii) If $-p' < \alpha < \min(0, \eta)$, the regular solutions have precisely one zero and $S_w < \infty$.

Proof. (i) Let $\alpha < 0$ and $N \leq \delta$. Since $\varepsilon\alpha > 0$, the trajectory $\mathcal{T}_r$ starts in $\mathcal{Q}_1$. Suppose that y stays positive. Then $\mathcal{T}_r$ stays in $\mathcal{Q}_1$, from Remark 2.6. If $N \leq \delta$, then y is monotone, since it can only have minimal points, from (2.16); and $(0,0)$ is the only stationary point. Then $\lim_{\tau\to\infty} y = \infty$, and $\lim_{\tau\to\infty} \zeta = \alpha < 0$ from Proposition 2.13, thus (y, Y) is in $\mathcal{Q}_4$ for large τ, which is impossible.

(ii) Let $0 < \alpha$. Then $\varepsilon\alpha < 0$, so that $\mathcal{T}_r$ starts in $\mathcal{Q}_4$. Moreover $y > 0$ on $\mathbb{R}$, from Proposition 2.10. And $\mathcal{T}_r$ stays in $\mathcal{Q}_4$, from Remark 2.2. Thus $y' = \delta y + |Y|^{1/(p-1)} > 0$. If $S_w = \infty$, from Proposition 2.13, then $\lim_{\tau\to\infty} \zeta = \alpha > 0$, hence (y, Y) ends up in $\mathcal{Q}_1$, which is false. Then $S_w < \infty$.

(iii) Let $-p' < \alpha < \min(0, \eta)$. Then $\mathcal{T}_r$ starts in $\mathcal{Q}_1$. From Proposition 2.10, Y_α stays positive, $\mathcal{T}_r$ stays in $\mathcal{Q}_1 \cup \mathcal{Q}_2$, and Y_α is increasing:

$$Y_\alpha' = -(p-1)(\eta - \alpha)Y_\alpha + e^{(p-(2-p)\alpha)\tau}(Y_\alpha^{1/(p-1)} - \alpha y_\alpha) > 0.$$

Suppose that $S_w = \infty$. Then $\lim_{\tau\to\infty} Y_\alpha(\tau) \geq C > 0$, then $r^{\alpha+1}w'(r) \leq -C^{1/(p-1)}$ for large r, and by integration, $r^\alpha w(r) \leq -C^{1/(p-1)}/2$, thus from (2.3), in particular $\lim_{\tau\to\infty} y = -\infty$. From Propositions 2.12, 2.13, and 2.14, it follows that $\lim_{r\to\infty} r^\alpha w = L < 0$, thus $\lim_{\tau\to\infty} Y_\alpha(\tau) = (\alpha L)^{p-1}$. And there exists a unique τ_0 such that $y_\alpha(\tau_0) = 0$, from Remark 2.2. But

$$\begin{aligned} Y_\alpha''(\tau) - (p-1)^2(\eta - \alpha)(\alpha + p')Y_\alpha &= \frac{Y_\alpha'}{Y_\alpha}\left(\frac{1}{p-1}e^{(p-(2-p)\alpha)\tau}Y_\alpha^{1/(p-1)} - (p-1)(\eta - 2\alpha - p')Y_\alpha\right) \\ &\geq \frac{Y_\alpha'}{Y_\alpha}\left(\frac{\alpha}{p-1}e^{(p-(2-p)\alpha)\tau}y_\alpha + (\eta - \alpha)(2-p) + (p-1)(\alpha + p')Y_\alpha\right). \end{aligned} \tag{2.49}$$

Thus $Y_\alpha''(\tau) > 0$, for any $\tau \geq \tau_0$, which is impossible. Then $S_w < \infty$, $\lim_{\tau\to\ln S_w} Y/y = -1$, and y has a zero. ∎

2.9 Behaviour of w near $R_w > 0$ or $S_w < \infty$

Proposition 2.20 *(i) Let w be any solution of $(\boldsymbol{E}_w)$ with a reduced domain ($\varepsilon = 1, R_w > 0$, or $\varepsilon = -1, S_w < \infty$). Let $s = R_w$ or S_w. Then*

$$\lim_{r\to s}(|r - s|^{(p-1)/(2-p)}\, s^{1/(2-p)} w = \pm((p-1)/(2-p))^{(p-1)/(2-p)}, \quad \textit{and} \quad \lim_{\tau\to\ln s} \sigma = \varepsilon. \tag{2.50}$$

Proof. From Proposition 2.10, we can suppose that εw is decreasing near s and $\lim_{r\to s} w = \infty$, thus $y > 0$, $\varepsilon Y > 0$ near $\ln s$, and $\lim_{\tau\to\ln s} y = \infty$. And σ is monotone near $\ln s$, from Proposition 2.12; thus it has a limit μ such that $\varepsilon\mu \in [0, \infty]$. Suppose that $\mu = 0$. Then $Y = o(y) = o(y - \varepsilon Y)$; from (2.13),

$$(y - \varepsilon Y)' = (\delta - \alpha)(y - \varepsilon Y) + \varepsilon(N - \alpha)Y = (\delta - \alpha + o(1))(y - \varepsilon Y);$$

then y cannot blow up in finite time. In the same way, if $\mu = \infty$, then $y = o(\varepsilon Y) = o(\varepsilon Y - y)$, and

$$(y - \varepsilon Y)' = (\delta - N)(y - \varepsilon Y) + (N - \alpha)y = (\delta - N + o(1))(y - \varepsilon Y),$$

hence again a contradiction; thus $\varepsilon\mu \in (0, \infty)$. Therefore $\lim_{\tau \to \ln R_w} \zeta = \varepsilon\infty$, $\mu = \varepsilon$ from (2.41), then $w'w^{-1/(p-1)} + (\varepsilon + o(1))r^{1/(p-1)} = 0$, and (2.50) holds. ∎

2.10 More informations on the stationary points

(i) The Höpf bifurcation point.

A Höpf bifurcation appears at the critical value $\alpha = \alpha^*$. Then some cycles do appear near α^*, from the Poincaré-Andronov-Hopf theorem, see [12, P.344]. We get more precise results by using the Lyapounov test for a week sink or source; it requires an expansion up to the order 3 near M_ℓ, in a suitable basis of eigenvectors, where the linearized problem has a rotation matrix.

Theorem 2.21 *Let $\varepsilon(\delta - N/2) > 0$. If $\varepsilon = -1$, and $\alpha = \alpha^*$, then M_ℓ is a week source; moreover if $\alpha < \alpha^*$ and $\alpha^* - \alpha$ is small enough, then there exists a unique limit cycle in $\mathcal{Q}_1$, attracting at $-\infty$. If $\varepsilon = -1$, and $\alpha = \alpha^*$, M_ℓ is a week sink; moreover if $\alpha > \alpha^*$ and $\alpha - \alpha^*$ is small enough, then there exists a unique limit cycle in $\mathcal{Q}_1$, attracting at ∞.*

Proof. The eigen values are given by $\lambda_1 = -ib$, $\lambda_2 = ib$, with $b = \sqrt{p'(N - \delta)}$, and from (2.32),

$$\nu(\alpha^*) = 2\delta - N = \frac{\delta(N - \delta)}{(p-1)(\alpha^* - \delta)} = \frac{\varepsilon(\delta\ell)^{2-p}}{(p-1)}.$$

First we make the substitution (2.30) as above, which leads to (2.31). The function Ψ defined at (2.32) has an expansion near $t = 0$ of the form $\Psi(\vartheta) = B_2\vartheta^2 + B_3\vartheta^3 + ..$, where

$$B_2 = \frac{(2-p)(\delta\ell)^{3-2p}}{2(p-1)^2}, \qquad B_3 = \frac{(2-p)(3-2p)(\delta\ell)^{4-3p}}{6(p-1)^6} = \frac{2(3-2p)B_2^2}{3(2-p)\nu(\alpha^*)}.$$

Next we make the substitution

$$\tau = -\theta/b, \qquad \overline{y}(\tau) = \varepsilon\nu(\alpha)x_1(\theta), \qquad \overline{Y}(\tau) = \delta x_1(\theta) + bx_2(\theta),$$

and obtain

$$x_1'(\theta) = x_2 + \frac{\varepsilon}{b\nu(\alpha)}\Psi(\delta x_1 + bx_2), \qquad x_2'(\theta) = -x_1 - \frac{\varepsilon(N-\delta)}{b^2\nu(\alpha)}\Psi(\delta x_1 + bx_2).$$

We write the expansion of order 3 under the form

$$\begin{aligned} x_1' &= x_2 + \varepsilon(a_{2,0}x_1^2 + a_{1,1}x_1x_2 + a_{0,2}x_2^2 + a_{3,0}x_1^3 + a_{2,1}x_1^2x_2 + a_{1,2}x_1x_2^2 + a_{0,3}x_2^3 + ...) \\ x_1' &= -x_1 + \varepsilon(b_{2,0}x_1^2 + b_{1,1}x_1x_2 + b_{0,2}x_2^2 + b_{3,0}x_1^3 + b_{2,1}x_1^2x_2 + b_{1,2}x_1x_2^2 + b_{0,3}x_2^3 + ...), \end{aligned}$$

and compute the (pretty awful) Lyapounov coefficient

$$L_C = \varepsilon(3a_{3,0} + a_{1,2} + b_{2,1} + 3b_{0,3}) - a_{2,0}a_{1,1} + b_{1,1}b_{0,2} - 2a_{0,2}b_{0,2} - a_{0,2}a_{1,1} + 2a_{2,0}b_{2,0} + b_{1,1}b_{2,0}.$$

After simplifications, we obtain

$$\frac{(2-p)b\nu(\alpha)^2}{2B_2^2(\delta^2+b^2)}L_C = (N-2\delta)(1-\varepsilon(3-2p) = \begin{cases} 2(N-2\delta)(p-1) < 0, & \text{if } \varepsilon = 1, \\ 2(N-2\delta)(2-p) > 0, & \text{if } \varepsilon = -1, \end{cases}$$

The nature of M_ℓ follow from [13, p.292], taking in account the fact that θ has the opposite sign of τ. Moreover there exists a small limit cycle attracting at $-\infty$ for all α near α^* such that M_ℓ is a sink, that means $\alpha < \alpha^*$. If $\varepsilon = -1$, M_ℓ is a week sink and there exists a small limit cycle attracting at ∞ for all α near α^* such that M_ℓ is a source, that means $\alpha^* < \alpha$. ■

(ii) Node points or spiral points.

When system (**S**) has three stationary points, and M_ℓ is a source or a sink, thus $\delta < N$, it is interesting to know if M_ℓ is a node point. When α^* exists, it is a spiral point, from (2.33).

If $\varepsilon = 1$, from (2.34), then M_ℓ is a node point when $\delta \leq N/2 - \sqrt{p'(N-\delta)}$ or $\delta > N/2 - \sqrt{p'(N-\delta)}$ and $\alpha \leq \alpha_1$, or $\delta > N/2 + \sqrt{p'(N-\delta)}$ and $\alpha_2 \leq \alpha$, where

$$\alpha_1 = \delta + \frac{\delta(N-\delta)}{(p-1)(2\delta - N + 2\sqrt{p'(N-\delta)})}, \qquad \alpha_2 = \delta + \frac{\delta(N-\delta)}{(p-1)(2\delta - N - 2\sqrt{p'(N-\delta)})}. \quad (2.51)$$

If $\varepsilon = -1$, then M_ℓ is a node point when $\delta \geq N/2 + \sqrt{p'(N-\delta)}$, or $\delta < N/2 + \sqrt{p'(N-\delta)}$ and $\alpha_2 \leq \alpha$, or $\delta < N/2 - \sqrt{p'(N-\delta)}$ and $\alpha \leq \alpha_1$. In any case $\alpha_1 < \alpha_2$.

Remark 2.22 *(i) Let $\varepsilon = 1$. One verifies that $N \leq \alpha_1$, and $N = \alpha_1 \Longleftrightarrow N = \delta/(p-1) = p'/(2-p)$. Also $\alpha_1 < \eta \Longleftrightarrow \delta^2 + (7-N)\delta + N > 0$, which is true for $N \leq 14$, but not always.*

(ii) Let $\varepsilon = -1$. It is easy to see that $\alpha_2 \leq 0$. And $\alpha_2 = 0 \Longleftrightarrow N(2-p) = \delta \Longleftrightarrow N = p/((2-p)^2$. Also $\alpha_2 > -p' \Longleftrightarrow \delta^2 + 7\delta - 8N < 0$, which is true for $\delta < N/2 < 9$, but not always.

(iii) Nonexistence of cycles.

If system (**S**) admits a cycle $\mathcal{O}$ in $\mathbb{R}^2$, then $\mathcal{O}$ surrounds at least one stationary point. If it surrounds $(0,0)$, the corresponding solutions y are changing sign. If it only surrounds M_ℓ, then it stays in $\mathcal{Q}_1$, thus y stays positive. Indeed $\alpha \neq 0$ from (1.11), and $\mathcal{O}$ cannot intersect $\{(\varphi,0),\varphi > 0\}$ at two points, and similarly $\{(0,\xi),\xi > 0\}$, from Remark 2.2.

For suitable values of α,δ,N, we can show that cycles cannot exist, by using Bendixon'criterium or Poincaré map. Writing system (**S**) under the form

$$y' = f_1(y,Y), \qquad Y' = f_2(y,Y), \quad (2.52)$$

we obtain

$$\frac{\partial f_1}{\partial y}(y,Y)+\frac{\partial f_2}{\partial Y}(y,Y)=2\delta-N-\varepsilon\,|Y|^{(2-p)/(p-1)}\,. \tag{2.53}$$

For example, as a direct consequence of Bendixon'criterium, if $\varepsilon\,(\delta-N/2)<0$, we find again the nonexistence of any cycle in $\mathbb{R}^2$, which was obtained at Proposition 2.16. Now we consider cycles in $\mathcal{Q}_1$.

First we extend to system (**S**) a general property of quadratic systems, proved in [9], stating that there cannot exist a closed orbit surrounding a node point. Notice that the restriction of our system to $\mathcal{Q}_1$ is quadratic whenever $p=3/2$.

Theorem 2.23 *Let $\delta<N$ and $\varepsilon(\delta-\alpha)<0$. When M_ℓ is a node point, there is no cycle, and no homoclinic orbit in $\mathcal{Q}_1$.*

Proof. Let us use the linearization (2.30), (2.31), (2.32). Consider the line L given by the equation $A\overline{y}+\overline{Y}=0$, where A is a real parameter. The points of L are in $\mathcal{Q}_1$ whenever $-(\delta\ell)^{p-1}<\overline{Y}$ and $-\ell<\overline{y}$. As in [9], we study the orientation of the vector field along L : we find

$$A\overline{y}'+\overline{Y}'=\left(\varepsilon\nu(\alpha)A^2+(N+\nu(\alpha))A+\varepsilon\alpha\right)\overline{y}-(A+\varepsilon)\Psi(\overline{Y}).$$

From (2.34), up to the case $\varepsilon=1,\alpha=N=\alpha_1$, we can find an A such that $\varepsilon\nu(\alpha)A^2+(N+\nu(\alpha))A+\varepsilon\alpha=0$, and $A+\varepsilon\neq 0$. Moreover $\Psi(\overline{Y})\geq 0$ on $L\cap\mathcal{Q}_1$. Indeed $(p-1)\Psi'(t)=((\delta\ell)^{p-1}+t)^{(2-p)/(p-1)}-t^{(2-p)/(p-1)}$, thus Ψ has a minimum on $\left(-(\delta\ell)^{p-1},\infty\right)$ at point 0, thus it is nonnegative on this interval. Then the orientation of the vector field does not change along $L\cap\mathcal{Q}_1$, in particular no cycle can exist in $\mathcal{Q}_1$; and similarly no homoclinic trajectory can exist. In the case $\varepsilon=1,\alpha=N=\alpha_1$, then $Y\equiv y\in[0,\ell)$ defines the trajectory $\mathcal{T}_r$, corresponding to the solutions given by (1.8) with $K>0$, and again no cycle can exist in $\mathcal{Q}_1$: it would intersect $\mathcal{T}_r$. ■

Next we prove the nonexistence on one side of the Höpf bifurcation point:

Theorem 2.24 *Assume $\delta<N$ and $\varepsilon(\delta-\alpha)<0<\varepsilon(\delta-N/2)$. If $\varepsilon(\alpha-\alpha^*)\geq 0$, there exists no cycle and no homoclinic orbit in $\mathcal{Q}_1$.*

Proof. Here M_ℓ is a source or a weak source for $\varepsilon=1$ (resp. a sink or a weak sink for $\varepsilon=-1$). Suppose that there exists a cycle in $\mathcal{Q}_1$. Then any trajectory starting from M_ℓ at $-\varepsilon\infty$ has a limit cycle in $\mathcal{Q}_1$, which is attracting at $\varepsilon\infty$. Such a cycle is not unstable (resp. not stable); in other words the Floquet integral on the period $[0,\mathcal{P}]$ is nonpositive (resp. nonnegative). Thus from (2.53),

$$\varepsilon\int_0^{\mathcal{P}}(\frac{\partial f_1}{\partial y}(y,Y)+\frac{\partial f_2}{\partial Y}(y,Y))d\tau=\int_0^{\mathcal{P}}(|2\delta-N|-\frac{1}{p-1}Y^{(2-p)/(p-1)})d\tau\leq 0. \tag{2.54}$$

Now from (2.31),

$$0=\delta\int_0^{\mathcal{P}}\overline{y}d\tau-\nu(\alpha)\int_0^{\mathcal{P}}\overline{Y}d\tau-\int_0^{\mathcal{P}}\Psi(\overline{Y})d\tau,\qquad 0=\alpha\int_0^{\mathcal{P}}\overline{y}d\tau+(\delta-N-\nu(\alpha))\int_0^{\mathcal{P}}\overline{Y}d\tau-\int_0^{\mathcal{P}}\Psi(\overline{Y})d\tau;$$

besides, since Ψ is nonnegative,

$$\int_0^{\mathcal{P}}\Psi(\overline{Y})d\tau=-p'\int_0^{\mathcal{P}}\overline{y}d\tau=-\frac{p'(N-\delta)}{\alpha-\delta}\int_0^{\mathcal{P}}\overline{Y}d\tau>0;$$

and $y'=\delta y-Y^{1/(p-1)}$, hence

$$\int_0^{\mathcal{P}}Y^{1/(p-1)}dt=\delta\int_0^{\mathcal{P}}ydt<\delta\ell\mathcal{P}. \tag{2.55}$$

From (2.54),(2.55) and the Jensen inequality, it follows that

$$(p-1)\left|2\delta-N\right|\leq\int_0^{\mathcal{P}}Y^{(2-p)/(p-1)})d\tau\leq\mathcal{P}^{p-1}\left(\int_0^{\mathcal{P}}Y^{1/(p-1)}d\tau\right)^{2-p}<(\delta\ell)^{2-p}=\frac{\varepsilon\delta(N-\delta)}{\alpha-\delta},$$

thus $\varepsilon(\alpha-\alpha^*)<0$, which is contradictory. Next suppose that there is an homoclinic orbit. Then from [13, p.303], Theorem 9.3, the saddle connection is repelling (resp. attracting), because the sum of the eigenvalues μ_1,μ_2 of the linearized problem at $(0,0)$ is $2\delta-N$. That means that the solutions just inside it spiral toward the loop near $-\varepsilon\infty$. Since M_ℓ is a source, or a week source (resp a sink, or a weak sink), such solutions have a limit cycle attracting at $\varepsilon\infty$. As before, we reach a contradiction. ■

Finally we get the nonexistence in nonobvious cases, where we have shown that any solution has at most one or two zeros.

Theorem 2.25 *Assume $\delta<N$ and $\varepsilon(\delta-\alpha)<0<\varepsilon(\delta-N/2)$. If $\varepsilon=1$ and $\alpha\leq\eta$, or $\varepsilon=-1$ and $-p'\leq\alpha<0$, there exists no cycle and no homoclinic orbit in $\mathcal{Q}_1$.*

Proof. (i) Suppose that there exists at least one cycle.

• Let $\varepsilon=1$ and $\alpha\leq\eta$. Since $\alpha<\alpha^*$, then M_ℓ is a sink, any trajectory converging to M_ℓ at ∞ has a limit cycle $\mathcal{O}$ in $\mathcal{Q}_1$, attracting at $-\infty$. Let (y,Y) be any solution of orbit $\mathcal{O}$, of period $\mathcal{P}$. Then $\mathcal{O}$ is not stable, thus the Floquet integral is nonnegative, and from (2.54),

$$\int_0^{\mathcal{P}}(2\delta-N-\frac{1}{p-1}Y^{(2-p)/(p-1)})d\tau\geq 0.$$

Otherwise y is bounded from above and below; thus y_α, defined by (2.3) with $d = \alpha$, satisfies $\lim_{\tau\to-\infty} y_\alpha = 0$, $\lim_{\tau\to\infty} y_\alpha = \infty$. It has only minimal points, from (2.38), since $\alpha \le \eta$; thus $y'_\alpha > 0$ on $\mathbb{R}$. From (2.5) and (2.4) with $d = \alpha$,

$$\frac{y''_\alpha}{y'_\alpha} + (\eta - 2\alpha) + \frac{1}{p-1} Y^{(2-p)/(p-1)} = \frac{\alpha(\eta-\alpha)y_\alpha}{y'_\alpha} = \frac{\alpha(\eta-\alpha)y_\alpha}{\alpha y_\alpha - Y_\alpha^{1/(p-1)}} > \eta - \alpha.$$

Integrating on $[0,\mathcal{P}]$ it implies $\eta - 2\alpha + 2\delta - N > \eta - \alpha$, which is impossible, since $\delta - N + \delta - \alpha < 0$.

• Let $\varepsilon = -1$ and $-p' \le \alpha < 0$. Since $\alpha^* < \alpha$, M_ℓ is a source, any trajectory converging to it at $-\infty$ has a limit cycle attracting $\mathcal{O}'$ at ∞. Let (y, Y) be any solution of orbit $\mathcal{O}'$, of period $\mathcal{P}$. Then $\mathcal{O}$ is not unstable, thus the Floquet integral is nonpositive, hence

$$\int_0^{\mathcal{P}} (2\delta - N + \frac{1}{p-1} Y^{(2-p)/(p-1)}) d\tau \le 0.$$

Moreover Y is bounded from above and below; thus Y_α satisfies $\lim_{\tau\to-\infty} Y_\alpha = \infty$, $\lim_{\tau\to\infty} Y_\alpha = 0$. It has only minimal points, from (2.39), since $-p' \le \alpha < 0$; thus $Y'_\alpha < 0$ on $\mathbb{R}$. From (2.6) and (2.4),

$$\frac{Y''_\alpha}{Y'_\alpha} + (p-1)(\eta - 2\alpha - p') - \frac{1}{p-1} Y^{(2-p)/(p-1)} = \frac{(p-1)^2(\eta-\alpha)(p'+\alpha)Y_\alpha}{Y'_\alpha} < -(p-1)(p'+\alpha).$$

Integrating on $[0,\mathcal{P}]$ it implies $(p-1)(\eta - 2\alpha - p') + 2\delta - N < -(p-1)(p'+\alpha)$, which means $p\delta + (p-1)\left|\alpha\right| < 0$; but this is false.

(ii) Suppose that there exists an homoclinic orbit. Since $\delta < N$, $(0,0)$ is a saddle point, thus $\mathcal{T}_r$ is the only trajectory starting from $(0,0)$ in $\mathcal{Q}_1$, and there exists a unique trajectory $\mathcal{T}_s$ converging to $(0,0)$, in $\mathcal{Q}_1$ for large τ, with an infinite slope at $(0,0)$, and $\lim_{r\to0} r^\eta w = c > 0$.

• If $\varepsilon = 1$, then $\mathcal{T}_r$ satisfies $\lim_{\tau\to-\infty} e^{-\alpha\tau} y_\alpha = a > 0$, thus $\lim_{\tau\to-\infty} y_\alpha = 0$; and y_α has only minimal points, thus it is increasing and positive; and $\mathcal{T}_s$ satisfies $\lim_{\tau\to\infty} e^{(\eta-\alpha)\tau} y_\alpha = c > 0$. If $\alpha < \eta$, then $\lim_{\tau\to\infty} y_\alpha = 0$, thus $\mathcal{T}_r \neq \mathcal{T}_s$. If $\alpha = \eta$, $\mathcal{T}_s$ is given explicitely by (1.9), that means y_α is constant, thus again $\mathcal{T}_r \neq \mathcal{T}_s$.

• If $\varepsilon = -1$, then $\mathcal{T}_s$ satisfies $\lim_{\tau\to-\infty} e^{(\eta-\alpha)(p-1)\tau} Y_\alpha > 0$, because $\lim_{\tau\to-\infty} \zeta = \eta$, thus $\lim_{\tau\to\infty} Y_\alpha = 0$; and Y_α has only minimal points, thus it is increasing and positive; otherwise $\mathcal{T}_r$ satisfies $\lim_{\tau\to-\infty} e^{-(\alpha(p-1)+p)\tau} Y_\alpha = -a\alpha/N > 0$, from (2.36). If $\alpha > -p'$, thus $\lim_{\tau\to\infty} Y_\alpha = 0$ which implies $\mathcal{T}_r \neq \mathcal{T}_s$. If $\alpha = -p'$, then $\mathcal{T}_r$ is given explicitely by (1.10), in other words Y_α is constant, thus again $\mathcal{T}_r \neq \mathcal{T}_s$. ■

(iv) Boundeness of cycles.

When there exist cycles, except for a few cases, we cannot prove their uniqueness, but we show the following:

Theorem 2.26 *When it is nonempty, the set $\mathcal{C}$ of all the cycles of system $(\boldsymbol{S})$ is bounded in $\mathbb{R}^2$.*

Proof. Suppose that there exists a cycle in $\mathbb{R}^2$. From Propositions 2.10, 2.12, 2.15, 2.16 and Theorem 2.25, it can happen only in four cases: $\varepsilon = 1$ and $N < \alpha < \delta$, $\varepsilon = 1$ and $N < \delta = \alpha$, $\varepsilon = 1$ and $\max(\delta, N, \eta) < \alpha$ and $N/2 < \delta$, $\varepsilon = -1$ and $\delta < N/2$ and $\alpha < -p'$. In the first case, then $\mathcal{C}$ is bounded, contained in $(-\ell, \ell) \times (-(\delta\ell)^{p-1}, (\delta\ell)^{p-1})$, from Proposition 2.15. In the other cases we use the energy function W. Let (y, Y) be a solution of trajectory $\mathcal{O}$. Then W is periodic, and its maximum and minimum points are precisely the points of the curve $\mathcal{L}$. Indeed if $W'(\tau_1) = 0$ and $(y(\tau_1), Y(\tau_1) \notin \mathcal{L}$, then it is on the curve $\mathcal{M}$ defined at (2.11); hence $y'(\tau_1) = 0$, and $y''(\tau_1) \neq 0$, since $\mathcal{O}$ is not reduced to a stationary point. As a consequence, $\left(\delta y - |Y|^{(2-p)/(p-1)} Y\right) \left(|\delta y|)^{p-2}\delta y - Y\right) > 0$ near τ_1; then W' has a constant sign, and τ_1 is not a maximum or a minimum. In this way we obtain estimates for W independent of the trajectory:

$$\max_{\tau \in \mathbb{R}} |W(\tau)| = M = \max_{(y,Y) \in \mathcal{L}} |\mathcal{W}(y, Y)|.$$

At the maximal points τ of y, one has $|Y(\tau)|^{(2-p)/(p-1)} Y(\tau) = \delta y(\tau)$, thus

$$W(\tau) = \frac{\varepsilon(\delta - N)\delta^{p-1}}{p} |y(\tau)|^p + \frac{\alpha - \delta}{2} y^2(\tau).$$

In any case, from Hölder inequality, y is bounded by a constant K independent of the trajectory, and

$$\frac{|Y|^{p'}}{p'} \leq \delta y Y + \frac{|2\delta - N| \, \delta^{p-1}}{p} |y|^p + \frac{|\alpha - \delta|}{2} y^2 + M,$$

thus Y is also uniformly bounded, and $\mathcal{C}$ is bounded. ■

3 The case $\varepsilon = 1, \alpha < \delta$ or $\alpha = \delta < N$

3.1 General properties

Lemma 3.1 *Assume $\varepsilon = 1$ and $-\infty < \max(\alpha, N) < \delta (\alpha \neq 0)$. Then in the phase plane (y, Y), there exist*

(i) a trajectory $\mathcal{T}_1$ converging to M_ℓ at ∞, such that y is increasing as long as it is positive;

(ii) a trajectory $\mathcal{T}_2$ in $\mathcal{Q}_1 \cup \mathcal{Q}_4$ converging to M_ℓ at $-\infty$, and unbounded at ∞, with $\lim_{\tau \to \infty} \zeta = \alpha$;

(iii) a trajectory $\mathcal{T}_3$ converging to M_ℓ at $-\infty$, such that y has at least one zero;

(iv) a trajectory $\mathcal{T}_4$ in $\mathcal{Q}_1$, converging to M_ℓ at ∞, with $\lim_{\tau \to \ln R_w} Y/y = 1$;

(v) trajectories $\mathcal{T}_5$ in $\mathcal{Q}_1 \cup \mathcal{Q}_4$ unbounded at $\pm\infty$, with $\lim_{\tau \to \infty} \zeta = \alpha$, $\lim_{\tau \to \ln R_w} Y/y = 1$.

Proof. Here system (**S**) has three stationary points. The point $(0,0)$ is a source, and the point M_ℓ is a saddle point. The eigenvalues satisfy $\lambda_1 < 0 < \lambda_2 < \delta$. The eigenvectors $u_1 = (-\nu(\alpha), \lambda_1 - \delta)$ and $u_2 = (\nu(\alpha), \delta - \lambda_2)$ form a direct basis, and u_1 points to $\mathcal{Q}_3$, u_2 points to $\mathcal{Q}_1$. There exist four particular trajectories $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4$ converging to M_ℓ at $\pm\infty$:

• $\mathcal{T}_1$ converging to M_ℓ at ∞, with tangent vector u_1; then $y < \ell$ and $Y < (\delta\ell)^{p-1}$ and $y' > 0$ near ∞; as above, y cannot have a local minimum, so that $y' > 0$ whenever $y > 0$.

• $\mathcal{T}_2$ converging to M_ℓ at $-\infty$, with tangent vector u_2; then $y' > 0$ near $-\infty$. If y has a local maximum at some τ, then $y''(\tau) \leq 0$, so that $y(\tau) \leq \ell$ from (2.16), which is impossible. Then y is increasing on $\mathbb{R}$ and $\lim_{\tau\to\infty} y = \infty$, and $\lim_{\tau\to\infty} \zeta = \alpha$ from Proposition 2.13. In particular $\mathcal{T}_2$ stays in $\mathcal{Q}_1$ if $\alpha > 0$, and enters $\mathcal{Q}_4$ if $\alpha < 0$.

• $\mathcal{T}_3$ converging to M_ℓ at $-\infty$, with tangent vector $-u_2$; then $y' < 0$ near $-\infty$. If y has a local minimum at some τ, then $y(\tau) \geq \ell$, which is still impossible. Thus y is decreasing at long as the trajectory stays in $\mathcal{Q}_1$. It cannot stay in it, because it cannot converge to $(0,0)$. It cannot enter $\mathcal{Q}_4$ from Remark 2.2. Then it enters $\mathcal{Q}_2$ and y has at least one zero.

• $\mathcal{T}_4$ converging to M_ℓ at ∞, with tangent vector $-u_1$; then $y' < 0$ near ∞. As above, y cannot have a local maximum, it is decreasing and $\lim_{\tau\to\ln R_w} y = \infty$. From Proposition 2.13, y cannot be defined near $-\infty$, hence $R_w > 0$ and $\lim_{\tau\to\ln R_w} Y/y = 1$.

For any trajectory $\mathcal{T}$ in the domain delimitated by $\mathcal{T}_2, \mathcal{T}_4$, the function y is positive, and $\mathcal{T}$ cannot converge to M_ℓ at ∞, and y is monotone for large τ from Proposition 2.12, because $\alpha < \delta$; thus $\lim_{\tau\to\infty} \zeta = \alpha$ from Proposition 2.13, and y is not defined near $-\infty$, and $\mathcal{T}$ is of type (5). ∎

Next we study the global behaviours, according to the values of α. The results are expressed in terms of w.

3.2 Subcase $\alpha \leq N < \delta$

Theorem 3.2 *Assume $\varepsilon = 1$ and $-\infty < \alpha \leq N < \delta (\alpha \neq 0)$. Then the regular solutions w have a constant sign, and $\lim_{r\to\infty} r^\alpha |w| = L > 0$ if $\alpha < N$, $\lim_{r\to\infty} r^\delta |w| = \ell$ if $\alpha = N$. And $w(r) = \ell r^{-\delta}$ is also a solution. There exist solutions such that*

(1) (only if $\alpha < N$) w is positive, $\lim_{r\to 0} r^\eta w = c > 0$, if $N \geq 2$ (and (2.45) holds with $a > 0 > b$ if $N = 1$), and $\lim_{r\to\infty} r^\delta w = \ell$;

(2) w is positive, $\lim_{r\to 0} r^\delta w = \ell$, $\lim_{r\to\infty} r^\alpha w = L > 0$;

(3) w has precisely one zero, $\lim_{r\to 0} r^\delta w = \ell$, $\lim_{r\to\infty} r^\alpha w(r) = L < 0$;

(4) w is positive, $R_w > 0$, $\lim_{r\to\infty} r^\delta w = \ell$;

(5) w is positive, $R_w > 0$, $\lim_{r\to\infty} r^\alpha w = L > 0$;

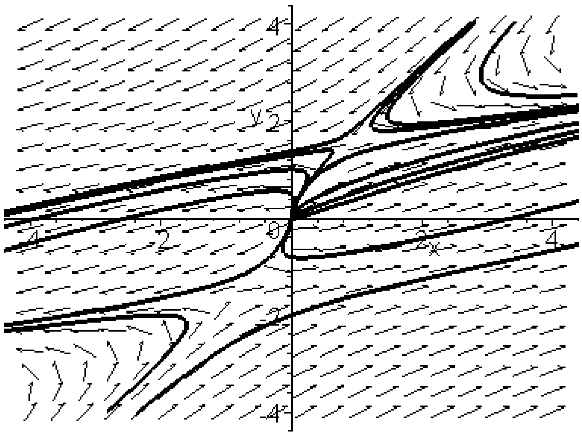
(6) w has one zero, $R_w > 0$, and $\lim_{r\to\infty} r^{\alpha} w = L \neq 0$;

(7) (only if $\alpha < N$) w is positive, $\lim_{r\to 0} r^{\eta} w = c > 0$ if $N \geq 2$ (and (2.45) holds with $a > 0 > b$ if $N = 1$) and $\lim_{r\to\infty} r^{\alpha} w = L > 0$;

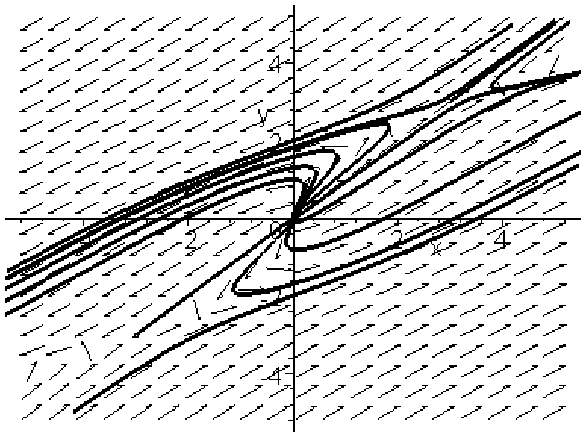
(8) w has one zero, with $\lim_{r\to 0} r^{\eta} w = c > 0$ if $N \geq 2$ ((and (2.45) holds with $a > 0 > b$ if $N = 1$), and $\lim_{r\to\infty} r^{\alpha} w = -L < 0$;

(9) $N = 1, w > 0$ and (2.45) holds with $a \geq 0$, $b > 0$ and $\lim_{r\to\infty} r^{\alpha} w = L$.

Up to a symmetry, all the solutions of ($\boldsymbol{E}_w$) are described.



th 3.2,fig I: $\alpha = 1 < N = 2 < \delta = 3$



th 3.2,figII: $\alpha = 2 = N = 2 < \delta = 3$

Proof. (i) Case $\alpha \neq N$ (see fig I). The trajectory $\mathcal{T}_r$ starts in $\mathcal{Q}_1$ for $\alpha > 0$, in $\mathcal{Q}_4$ for $\alpha < 0$, and y stays positive. Then $\lim_{\tau\to\infty} y = \infty$, and $\lim_{\tau\to\infty} \zeta = \alpha$, and $\lim_{r\to\infty} r^{\alpha} w = L > 0$, from Propositions 2.15 and 2.18, since $\alpha < N$. Moreover y is increasing: indeed if it has a local maximum, at this point $y \leq \ell$, and then y has no local minimum, since at such a point $y \geq \ell$, so that y cannot tend to ∞. Then $\mathcal{T}_r$ stays in $\mathcal{Q}_1$, and Y is increasing from 0 to ∞. Indeed each extremal point τ of Y is a local minimum, from (2.17). If $\alpha < 0$, in the same way, then Y is decreasing from 0 to $-\infty$, and $\mathcal{T}_r$ stays in $\mathcal{Q}_4$.

• Let us follow the trajectory $\mathcal{T}_1$: it does not intersect $\mathcal{T}_r$, and cannot enter $\mathcal{Q}_2$ from Remark 2.2. Thus y stays positive and increasing. It cannot enter $\mathcal{Q}_4$, seeing that it does not meet $\mathcal{T}_r$ if $\alpha > 0$, or from Remark 2.2 if $\alpha < 0$. Thus $\mathcal{T}_1$ stays in $\mathcal{Q}_1$, and (y, Y) converges necessarily to $(0, 0)$. If $N \geq 2$, then $\lim_{\tau\to-\infty} \zeta = \eta$, $\lim_{r\to 0} r^{\eta} w = c > 0$ from Proposition 2.13 and 2.14. If $N = 1$, since $\mathcal{T}_1$ stays in $\mathcal{Q}_1$, then necessarily $\lim_{\tau\to-\infty} \zeta = 0$, thus (2.45) holds with $a > 0 > b$.

• Next we follow $\mathcal{T}_3$: the function y has a zero, which is unique, since $\alpha < N$, from Proposition 2.10. Then $y < 0$, and $\lim_{\tau\to\infty} y = -\infty$, $\lim_{r\to\infty} r^{\alpha}w = -L < 0$ from Proposition 2.13 and 2.15. And $\mathcal{T}_3$ stays in $\mathcal{Q}_2$ if $\alpha < 0$, or goes from $\mathcal{Q}_2$ into $\mathcal{Q}_3$ if $\alpha > 0$.

• The trajectories $\mathcal{T}_2, \mathcal{T}_4, \mathcal{T}_5$ of Lemma 3.1 correspond to the solutions w of type (2),(4),(5).

• For any trajectories $\mathcal{T}_6$ in the domain delimitated by $\mathcal{T}_3, \mathcal{T}_4$, y has one zero, and $\lim_{r\to\infty} r^{\alpha}w = L \neq 0$; and w is of type (6).

• The solutions of type (7) correspond to the trajectories $\mathcal{T}$ in the domain delimitated by $\mathcal{T}_r, \mathcal{T}_1, \mathcal{T}_2$. Indeed $\lim_{\tau\to\infty} y = \infty$, and $\lim_{r\to\infty} r^{\alpha}w = L > 0$. And $\lim_{\tau\to-\infty} y = 0$. If $N \geq 2$, then $\lim_{\tau\to-\infty} \zeta = \eta$, $\lim_{r\to 0} r^{\eta}w = c > 0$, from Proposition 2.13 and 2.14. If $N = 1$, $\mathcal{T}$ cannot meet $\mathcal{T}_r$, thus necessarily $\lim_{\tau\to-\infty} \zeta = 0$, and (2.45) holds with $a > 0 > b$.

• Up to the change of w into $-w$, the solutions of type (8),(9) correspond to the trajectories in the domain delimitated by $-\mathcal{T}_r, \mathcal{T}_1, \mathcal{T}_3$. Indeed they satisfy $\lim_{\tau\to\infty} y = -\infty$, and $\lim_{r\to\infty} r^{\alpha}w = L < 0$; and $\lim_{\tau\to-\infty} y = 0$. If $N \geq 2$, then $\lim_{r\to 0} r^{\eta}w = c > 0$ and w has a zero. If $N = 1$, either (2.45) holds with $a = 0 > b$, and w stays negative, or $a < 0, b < 0$ and w has a zero. Such solutions exist from Theorem 2.5. By symmetry, all the solutions are described.

(ii) Case $\alpha = N$(see fig II). Then M_ℓ belongs to the line $y = Y$, and $u_1 = (-\delta/(p-1), -\delta/(p-1))$ has the same direction. Moreover J_N is constant, which means $y - Y = Ce^{(\delta-N)\tau}$, $C \in \mathbb{R}$. The solutions corresponding to $C = 0$ satisfy $y \equiv Y$, thus $\mathcal{T}_1 = \mathcal{T}_r = \{(\xi,\xi) : \xi \in [0,\ell))\}$, corresponding to the regular Barenblatt solutions. And $\mathcal{T}_4 = \{(\xi,\xi) : \xi > \ell)\}$ corresponds to the solutions defined by (1.8) for $K < 0$. All the other solutions exist as before, up to the solutions of type (7). ■

Remark 3.3 *The trajectory $\mathcal{T}_1$ is the unique one joining the stationary points $(0,0)$ and M_ℓ. As a consequence, for $\alpha < N$, the solutions w of type (1) are unique, up to the scaling given at Remark 2.1. The solutions of types (2),(4),(5) are also unique.*

3.3 Subcase $N < \alpha < \delta$.

Here we prove that some periodic trajectories can exist, according to the value of α with respect to α^*. Notice that $N < \alpha^*$ whenever $\delta^2 - (N+3)\delta + N > 0$, from (2.35), in particular $\alpha^* < N$ for any $p \leq 3/2$. Our main tool is the Poincaré-Bendixon theorem, using the level curves of the energy function $\mathcal{W}$:

Lemma 3.4 *Assume $\varepsilon = 1$ and $N < \alpha < \delta$. Consider the level curves $\mathcal{C}_k = \left\{(y,Y) \in \mathbb{R}^2 : \mathcal{W}(y,Y) = k\right\}$ $(k \in \mathbb{R})$ of function $\mathcal{W}$ defined at (2.21), which are symmetric with respect to $(0,0)$. Let*

$$k_\ell = \mathcal{W}(\ell, (\delta\ell)^{p-1}) = \frac{1}{2}(\delta - N)\delta^{p-2}\ell^p. \tag{3.1}$$

If $k > k_\ell$, then $\mathcal{C}_k$ has two unbounded connected components. If $0 < k < k_\ell$, $\mathcal{C}_k$ has three connected components and one of them is bounded. If $k = k_\ell$, $\mathcal{C}_{k_\ell}$ is connected with a double point at M_ℓ. If

$k=0$ and one of the three connected components of $\mathcal{C}_0$ is $\{(0,0)\}$. If $k<0$, $\mathcal{C}_k$ has two unbounded connected components.

Proof. The energy k_ℓ of M_ℓ, given by (3.1), is positive. We observe that $(y,Y)\in C_k$ if and only if $F(y)=k-G(y)$, where F,G are defined at (2.48). By symmetry we can reduce the study of C_k to the set $y>0$. Let $\varphi(s)=|s|^{p'}/p'-s+1/p$, for any $s\in\mathbb{R}$, and $\theta=Y/(\delta y)^{p-1}$. Then (2.48) reduces to the equation

$$\varphi(\theta)=\left(k-G(y)\right)/(\delta y)^p.$$

The function φ is decreasing on $(-\infty,1)$ from ∞ to 0, and increasing on $(1,\infty)$ from 0 to ∞. Let ψ_1 be the inverse of the restriction of φ to $(-\infty,1]$, and ψ_2 be inverse of the restriction of φ to $[1,\infty)$, both defined on $[0,\infty)$. For any $y>0$,

$$y\in\mathcal{C}_k\Leftrightarrow(Y<(\delta y)^{p-1}\text{ and }Y=\Phi_1(y)\quad\text{or }(Y\geq(\delta y)^{p-1}\text{ and }Y=\Phi_2(y)),$$

where

$$\Phi_i(y)=(\delta y)^{p-1}\psi_i(\frac{k-G(y)}{(\delta y)^p}),\quad i=1,2, \tag{3.2}$$

Φ_1 is under $\mathcal{M}$ and Φ_2 is above, and $\Phi_1,\Phi_2\in C^1\left((0,\infty)\right)$. The function G has a maximal point at $y=\ell$, and $G(\ell)=k_\ell$. Using the symmetry, either $k>k_\ell$ and y describes $\mathbb{R}$, and $\mathcal{C}_k$ has two unbounded connected components. Or $0<k<k_\ell$ and $\mathcal{C}_k$ has three connected components and one of them, $\mathcal{C}_k^b$, is bounded. Or $k=k_\ell$ and $\mathcal{C}_{k_\ell}$ is connected with a double point at M_ℓ. Or $k=0$ and one of the three connected components of $\mathcal{C}_0$ is $\{(0,0)\}$. Or $k<0$ and $\mathcal{C}_k$ has two unbounded connected components. The unbounded components satisfy $\lim_{|y|\to\infty}Y/y^{2/p'}=\pm(p'(\delta\alpha)/2)^{1/p'}$ from (3.2). The zeros of Φ_i' are contained in

$$\mathcal{N}=\left\{(y,Y)\in\mathbb{R}^2:y>0,\delta Y=-(\delta-\alpha)y+(2\delta-N)(\delta y)^{p-1}\right\},$$

and $\mathcal{N}$ is above $\mathcal{M}$ as long as $y<\ell$.

Let us describe $\mathcal{C}_k^b$ when $0<k\leq k_\ell$: the function Φ_1 is increasing on a segment $[0,\overline{y}]$, such that $\overline{y}<\ell$, and $\Phi_1(0)=-(kp')^{1/p'}$ and $(\overline{y},\Phi_1(\overline{y}))\in\mathcal{M}$, with an infinite slope at this point; the function Φ_2 is increasing on some interval $[0,\tilde{y})$ such that $(\tilde{y},\Phi_2(\tilde{y}))\in\mathcal{N}$ and then decreasing on $(\tilde{y},\overline{y}]$, and $\Phi_2(0)=(kp')^{1/p'}$and $\Phi_2(\overline{y})=\Phi_1(\overline{y})$. By symmetry with respect to $(0,0)$, the curve $\mathcal{C}_k^b$ is completly described.

Next consider $\mathcal{C}_{k_\ell}$ for $y>0$: the function Φ_2 is increasing on $[0,\infty)$ from $(p'k_\ell)^{1/p'}$ to ∞, and $\Phi_2(\ell)=(\delta\ell)^{p-1}$; the function Φ_1 is increasing on some interval $[0,\hat{y})$ such that $(\hat{y},\Phi_1(\hat{y}))\in\mathcal{N}$ thus $\hat{y}>\ell$; and $(\hat{y},\Phi_1(\hat{y}))$ is under $\mathcal{M}$, and $\Phi_1(\ell)=(\delta\ell)^{p-1}$, and Φ_1 is decreasing on $(\hat{y},\infty)$, and $\lim_{y\to\infty}\Phi_1=-\infty$. Setting $\mathcal{C}_{k_\ell,1}=\{(y,\Phi_1(y)):y>\ell\}$ and $\mathcal{C}_{k_\ell,2}=\{(y,\Phi_2(y)):y>\ell\}$, one has $\mathcal{C}_{k_\ell}=\mathcal{C}_{k_\ell}^b\cup\pm\mathcal{C}_{k_\ell,1}\cup\mathcal{C}_{k_\ell,2}$. ∎

Theorem 3.5 *Assume $\varepsilon = 1$ and $N < \alpha < \delta$. Then $w(r) = \ell r^{-\delta}$ is a solution. Moreover*

(i) If $\alpha \leq \alpha^*$, then any solution of ($\boldsymbol{E}_w$) has at most a finite number of zeros.

(ii) There exist $\breve{\alpha}$ such that $\max(N, \alpha^*) < \breve{\alpha} < \delta$, such that if $\alpha > \breve{\alpha}$, in the phase plane (y, Y), there exists a cycle surrounding $(0, 0)$.

(iii) Let any α such that there exists no such cycle. Then the regular solutions have a finite positive number of zeros and $\lim_{r\to\infty} r^\alpha w = L_r \neq 0$ or $\lim_{r\to\infty} r^\delta w = \pm\ell$. There exist solutions of types (2)(3)(4),(5),(6) of Theorem 3.2, and solutions such that

(1') (only if $L_r \neq 0$) $\lim_{r\to 0} r^\delta w = \ell$, and $\lim_{r\to 0} r^\eta w = c \neq 0$ (or (2.45) holds if $N = 1$);

(7') $\lim_{r\to 0} r^\eta w = c \neq 0$ (or (2.45) holds if $N = 1$) and $\lim_{r\to\infty} r^\alpha w = L \neq 0$.

(iv) Consider any α such that there exists such a cycle, thus there exist solutions w which oscillate near 0 and ∞, and $r^\delta w$ is periodic in $\ln r$. The regular solutions w oscillate near ∞, and $r^\delta w$ is asymptotically periodic in $\ln r$. There exist solutions of types (2),(4),(5), and solutions

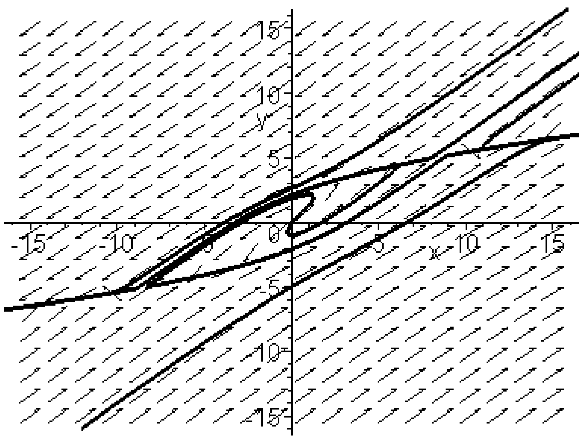
(1") with precisely one zero, $R_w > 0$, and $\lim_{r\to\infty} r^\delta w = \ell$;

(3") such that $\lim_{r\to 0} r^\delta w = \ell$, and oscillating near ∞;

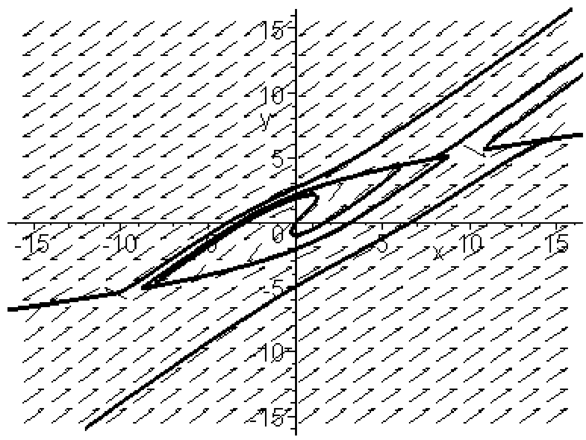
(9) such that $\lim_{r\to 0} r^\eta w = c \neq 0$ (or (2.45) holds if $N = 1$) and oscillating near ∞;

(10) with precisely one zero, $R_w > 0$, and $\lim_{r\to\infty} r^\alpha w = L \neq 0$;

(11) with $R_w > 0$ and oscillating near ∞.



th 3.5,figIII: $\varepsilon = 1, N = 2 < \alpha = 2.41 < \delta = 3$



th 3.5,figIV: $\varepsilon = 1, N = 2 < \alpha = 2.42 < \delta = 3$

Proof. First observe that there always exist solutions of type (2),(4),(5), from Lemma 3.1.

(i) Assume $\alpha \leq \alpha^*$(see fig III). Consider any trajectory $\mathcal{T}$. Suppose that y has an infinity of zeros near $\pm\infty$. From Proposition 2.15, $\mathcal{T}$ is contained in the set $\mathcal{D} = \left\{(y,Y) \in \mathbb{R}^2 : |y| < \ell, |Y| < (\delta\ell)^{p-1}\right\}$ near $\pm\infty$. Then it is is bounded near $\pm\infty$, hence the limit set at $\pm\infty$ is contained in $\mathcal{D}$. But $M_\ell \notin \mathcal{D}$; and $(0,0)$ is a source, and a node point, it cannot be in the limit set Γ at ∞. From the Poincaré-Bendixon theorem, Γ is a closed orbit, so that there exists a cycle. Moreover, from (2.52) and (2.53),

$$\frac{\partial f_1}{\partial y}(y,Y) + \frac{\partial f_2}{\partial Y}(y,Y) = \frac{1}{p-1}(D^{(2-p)/(p-1)} - |Y|^{(2-p)/(p-1)}),$$

thus from Bendixon's criterion, there is no cycle in the set $\{|Y| < D\}$. Now observe that

$$\alpha \leq \alpha^* \Longleftrightarrow (\delta\ell)^{p-1} \leq D. \tag{3.3}$$

Then there is no cycle in $\mathcal{D}$, and we reach a contradiction.

(ii) Assume $\alpha > \max(N, \alpha^*)$. The curve $\mathcal{L}$ intersects $\mathcal{M}$ at point $(\delta^{-1}D^{1/(p-1)}, D)$. Then

$$\mathcal{S}_{\mathcal{L}} \cap \mathcal{M} = \left\{(\delta^{-1}(\theta D)^{1/(p-1)}, \theta D) : \theta \in [0,1]\right\};$$

and $D < (\delta\ell)^{p-1}$ from (3.3), thus $\mathcal{S}_{\mathcal{L}}$ does not contain M_ℓ. We can find $k_1 > 0$ small enough such that $\mathcal{C}^b_{k_1}$ is interior to $\mathcal{S}_{\mathcal{L}}$. Next we search if there exists some $k \in (0, k_\ell)$ such that $\mathcal{L}$ is in the domain delimitated by C^b_k. By symmetry we only consider the points of $\mathcal{L}$ such that $y \geq 0$. In any case for any point of $\mathcal{L}$, from (2.25) and by convexity, $|\delta y|^p + |Y|^{p'} \leq M = (2(2\delta - N))^\delta$. By a straightforward computation it implies $\mathcal{W}(y,Y) \leq KM$, where $K = \max(2/p', (3\delta - N)/\delta p)$. Let $\breve{\alpha} = \breve{\alpha}(\delta, N)$ be given by $KM = k_\ell$, that means

$$\delta - \breve{\alpha} = (\frac{\delta - N}{2K\delta^{2-p}})^{1/\delta} \frac{\delta^{p-1}(\delta - N)}{2(2\delta - N)}.$$

If $\alpha > \breve{\alpha}$, there exists $k_2 < k_\ell$ such that $\mathcal{L}$ is contained in the set $\left\{(y,Y) \in \mathbb{R}^2 : \mathcal{W}(y,Y) < k_2\right\}$, which has three connected components; inasmuch $\mathcal{S}_{\mathcal{L}}$ is connected, it is is contained in the interior to $\mathcal{C}_{k^b_2}$. Then the domain delimitated by $\mathcal{C}^b_{k_1}$ and $\mathcal{C}^b_{k_2}$ is bounded and positively invariant. It does not contain any stationary point, thus contains a cycle, from the Poincaré-Bendixon theorem (see fig IV).

(iii) Let α such that there exists no cycle. Since $N < \alpha$, the regular solutions have at least one zero. They a finite number of zeros. Indeed in the other case, (y,Y) is bounded near ∞, thus it has a limit cycle. Then either $\lim_{\tau\to\infty} y = \pm\infty$, and $\lim_{\tau\to\infty}\zeta = \alpha > 0$, so that the trajectory $\mathcal{T}_r$ ends up in $\mathcal{Q}_1$ or $\mathcal{Q}_3$, and $\lim r^\alpha w = L_r \neq 0$, or $\lim_{\tau\to\infty} y = \pm\ell$ and $\lim_{r\to\infty} r^\delta w = \pm\ell$.

• The trajectory $\mathcal{T}_3$ cannot meet $\mathcal{T}_r$ or $-\mathcal{T}_r$, thus y has a unique zero, and $\lim_{\tau\to\infty} y = -\infty$, and $\lim_{\tau\to\infty}\zeta = \alpha$. The same happens for the trajectories $\mathcal{T}_6$ in the domain delimitated by $\mathcal{T}_3, \mathcal{T}_4$. Thus there exist solutions of types (3),(6).

• Suppose $L_r \neq 0$ and consider $\mathcal{T}_1$: the trajectories $\mathcal{T}_r, -\mathcal{T}_r, \mathcal{T}_1$ have a last intersection point at time τ_0 with the half axis $\{y = 0, Y < 0\}$ at some points P_r, P'_r, P_1, and $P_1 \in [P_r, P'_r]$. The domain delimitated by $\mathcal{T}_r, -\mathcal{T}_r$ and $[P_r, P'_r]$ is bounded and negatively invariant, from Remark 2.2. Then $\mathcal{T}_1$ stays in it for $\tau < \tau_0$, it has a finite number of zeros, and converges to $(0,0)$ near $-\infty$; thus w is of type $(1')$. If $N \geq 2$, then $\lim_{\tau\to\infty}\zeta = \eta$, so that y has at least one zero.

• Since $(0,0)$ is a source, there exist other solutions converging to $(0,0)$ near $-\infty$, they have a finite number of zeros, and $\lim_{\tau\to\infty}\zeta = \alpha$, and w is of type $(7')$.

(iv) Let α such that there exists a cycle, thus $\mathcal{T}_r$ has a limit cycle $\mathcal{O}$.

• Consider again $\mathcal{T}_1$. Since $M_\ell \notin \mathcal{S}_\mathcal{L}$, the function W is decreasing near ∞, so that $W(\tau) > k_\ell$; thus $\mathcal{T}_1$ is exterior to $\mathcal{C}^b_{k_\ell}$ for large τ, in the domain exterior to $\mathcal{C}^b_{k_\ell}$ delimitated by $\mathcal{C}_{k_\ell,1}$ and $-\mathcal{C}_{k_\ell,2}$; and it cannot cut $\mathcal{C}_{k_\ell}$. Moreover y is decreasing at long as $y > 0$, then $\mathcal{T}_1$ enters $\mathcal{Q}_4$ as τ decreases. It cannot stay in it, because it would converge to $(0,0)$, which is impossible. Then y has at least one zero, and $\mathcal{T}_1$ enters $\mathcal{Q}_3$. It stays in it, since it cannot cross $-\mathcal{C}_{k_\ell,2}$. Thus y has a unique zero, and $\lim_{\tau\to-\infty} y = -\infty$, and $R_w > 0$ from Proposition 2.13, because $\mathcal{T}_1$ cannot converge to $(0,0)$ at $-\infty$, and w is of type $(1'')$.

• Next consider $\mathcal{T}_3$. Then W is decreasing near $-\infty$, hence $W(\tau) < k_\ell$; thus $\mathcal{T}_3$ is in the interior of $\mathcal{C}^b_{k_\ell}$ near $-\infty$. Now the domain delimitated by $\mathcal{C}^b_{k_1}$ and $\mathcal{C}^b_{k_\ell}$ is positively invariant, thus $\mathcal{T}_3$ stays in it; then it is bounded, and has a limit cycle at ∞, and w is of type (3").

• The solutions of type (9) correspond to trajectories $\mathcal{T}$ in the domain delimitated by $\mathcal{O}$, and distinct from $\mathcal{T}_r$. Indeed $\mathcal{T}$ is bounded, in particular the limit-set at $-\infty$ is $(0,0)$, or a closed orbit. But $\mathcal{T}$ cannot intersect $\mathcal{T}_r$. Then $\mathcal{T}$ converges to $(0,0)$ near $-\infty$.

• The solutions of type (10) correspond to a trajectory $\mathcal{T}$ in the domain delimitated by $\mathcal{T}_1 \cup \mathcal{T}_2$ (or its opposite): indeed y has a constant sign near ∞, and near $\ln R_w$ and $\lim_{r\to\infty} r^\alpha w = L \neq 0$, and $R_w > 0$, from Proposition 2.13. Then $\mathcal{T}$ starts in $\mathcal{Q}_3$, and ends up in $\mathcal{Q}_1$; and y has at most one zero, because at such a point $y' = -|Y|^{1/(p-1)} Y > 0$, thus precisely one zero.

• The solutions of type (11) correspond to a trajectory $\mathcal{T}$ in the domain delimitated by $\mathcal{T}_1, \mathcal{T}_4, -\mathcal{T}_1, -\mathcal{T}_4$. Then y cannot have a constant sign near ∞ : indeed this implies $\lim \zeta = \alpha > 0$; this is impossible since the line $Y = y$ is an asymptotic direction for $\mathcal{T}_1, \mathcal{T}_4$. Thus $\mathcal{T}$ is bounded near ∞, and it has a limit cycle at ∞. Near $-\infty$, y a constant sign, because $\mathcal{T}$ cannot meet $\mathcal{T}_3$; and $R_w > 0$ from Proposition 2.13, and $\mathcal{T}$ has the same asymptotic direction $Y = y$ as $\mathcal{T}_1, \mathcal{T}_4$. ■

Remark 3.6 *From the numerical studies, we conjecture that $\breve{\alpha}$ is unique, and the number of zeros of w is increasing with α in the range $(N, \breve{\alpha})$; and moreover there exists $\alpha_1 = N < \alpha_2 < .. < \alpha_n < \alpha_{n+1} < ..$, such that the regular solutions have n zeros for any $\alpha \in (\alpha_n, \alpha_{n+1})$, with* $\lim_{r\to\infty} r^\alpha w = L_r \neq 0$, *and $n+1$ zeros for $\alpha = \alpha_{n+1}$, with* $\lim_{r\to\infty} r^\delta w = \pm\ell$.

3.4 Subcase $\alpha \leq \delta \leq N, \alpha \neq N$

Here $(0,0)$ is the only stationary point, and $N \geq 2$.

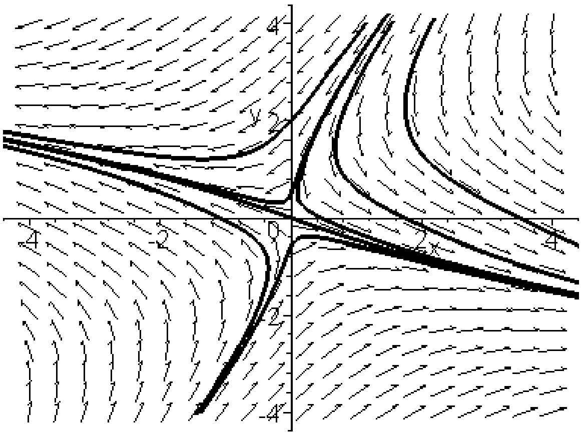
Theorem 3.7 *Assume* $\varepsilon = 1$ *and* $-\infty < \alpha \leq \delta \leq N, \alpha \neq 0, N$. *Then the regular solutions have a constant sign, and the positive ones satisfy* $\lim_{r\to\infty} r^\alpha w(r) = L > 0$ *if* $\alpha \neq \delta$, *or (2.43) holds if* $\alpha = \delta$. *All the other solutions have a reduced domain* ($R_w > 0$). *Among them, there exist solutions such that*

(1) w *is positive,* $\lim_{r\to\infty} r^\eta w = c \neq 0$ *if* $\delta < N$, *or* $\lim_{r\to\infty} r^N (\ln r)^{(N+1)/2} w = \varrho$ *defined at (2.44) if* $\delta = N$;

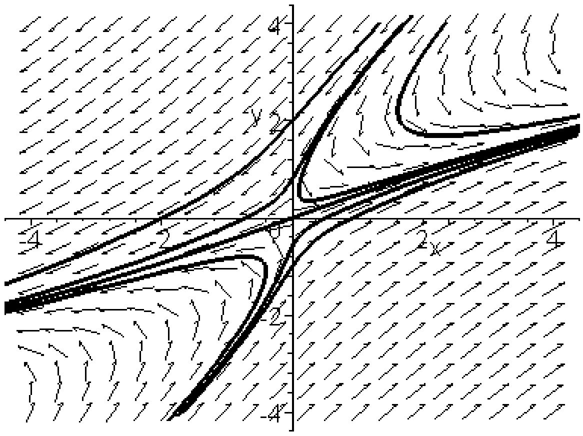
(2) w *is positive,* $\lim_{r\to\infty} r^\alpha w = L > 0$ *if* $\alpha \neq \delta$, *or (2.43) holds if* $\alpha = \delta$;

(3) w *has one zero, such that* $\lim_{r\to\infty} r^\alpha w = L \neq 0$ *if* $\alpha \neq \delta$, *or (2.43) holds if* $\alpha = \delta$.

Up to a symmetry, all the solutions are described.



th 3.7,figV: $\varepsilon = 1, \alpha = -2 < \delta = 3 < N = 4$



th 3.7,figVI: $\varepsilon = 1, \alpha = 2 < \delta = 3 < N = 4$

Proof. Any solution has at most one zero, from Proposition 2.10. The trajectory $\mathcal{T}_r$ starts in $\mathcal{Q}_4$ for $\alpha < 0$ (see fig V), in $\mathcal{Q}_1$ for $\alpha > 0$ (see fig VI) and y stays positive, and $\lim_{\tau\to\infty} y = \infty$, and $\lim_{\tau\to\infty} \zeta = \alpha$, from Proposition 2.18. Then $\lim_{r\to\infty} r^\alpha w(r) = L > 0$ if $\alpha < \delta$, or (2.43) holds if $\alpha = \delta$, from Proposition 2.14. Moreover y is increasing: indeed it has no local maximum from (2.16). As a consequence $\mathcal{T}_r$ does not meet $\mathcal{M}$, thus stays under $\mathcal{M}$. If $\alpha > 0$, then$\mathcal{T}_r$ stays in $\mathcal{Q}_1$, and Y is increasing from 0 to ∞. Indeed each extremal point τ of Y is a local minimum, from (2.17). If $\alpha < 0$, in the same way, then Y is decreasing from 0 to $-\infty$, and $\mathcal{T}_r$ stays in $\mathcal{Q}_4$. The only solutions y defined on $(0, \infty)$ are the regular ones, from Proposition 2.13.

• For any point $P = (\varphi, (\delta\varphi)^{p-1}) \in \mathbb{R}^2$ with $\varphi > 0$, in other words on the curve $\mathcal{M}$, the trajectory $\mathcal{T}_{[P]}$ intersects $\mathcal{M}$ transversally: the vector field is $(0, -(N-\alpha)\ \varphi)$. Moreover the solution going through this point at time τ_0 satisfies $y''(\tau_0) > 0$ from $(\mathbf{E}_y)$, then τ_0 is a point of local minimum. From (2.16), τ_0 is unique, so that it is a minimum. Then $y > 0$, $\lim_{\tau\to\infty} \zeta = \alpha$, $\lim_{\tau\to\ln R_w} Y/y = 1$, and $\mathcal{T}_{[P]}$ stays in $\mathcal{Q}_1$ if $\alpha > 0$, or goes from $\mathcal{Q}_1$ into $\mathcal{Q}_3$ if $\alpha < 0$. The corresponding w is of type (2).

• For any point $P = (0, \xi), \xi > 0$, the trajectory $\mathcal{T}_{[P]}$ goes through P from $\mathcal{Q}_1$ into $\mathcal{Q}_2$, from Remark 2.2. Then y has only one zero, and as above, it is decreasing on $\mathbb{R}$ and $\lim_{\tau\to\infty} y = -\infty$, and $\lim_{\tau\to\infty} \zeta = \alpha$, $\lim_{\tau\to\ln R_w} Y/y = 1$. Thus $\mathcal{T}_{[P]}$ starts in $\mathcal{Q}_1$, then stays in $\mathcal{Q}_2$ if $\alpha < 0$, and enters $\mathcal{Q}_3$ and stays in it if $\alpha > 0$. The corresponding w is of type (3).

• It remains to prove the existence of a solution of type (1). If $\delta < N$, then $(0,0)$ is a saddle point. There exists a trajectory $\mathcal{T}_1$ converging to $(0,0)$ at ∞, with $y > 0$, and $\lim_{\tau\to\infty} \zeta = \eta > 0$, thus in $\mathcal{Q}_1$ near ∞, with $y' < 0$. As above, y has no local maximum, it is increasing, so that $y > 0$. If $\delta = N$, we consider the sets

$$\mathcal{A} = \left\{P \in (0,\infty) \times \mathbb{R} : \mathcal{T}_{[P]} \cap \mathcal{M} \neq \emptyset\right\}, \qquad \mathcal{B} = \left\{P \in (0,\infty) \times \mathbb{R} : \mathcal{T}_{[P]} \cap \{(0,\xi) : \xi > 0\} \neq \emptyset\right\}.$$

They are nonempty, and open, because the intersections are transverse. Since $\mathcal{T}_r$ is under $\mathcal{M}$, the sets $\mathcal{A}$ and $\mathcal{B}$ are contained in the domain $\mathcal{R}$ of $\mathcal{Q}_1 \cup \mathcal{Q}_2$ above $\mathcal{T}_r$, and $\mathcal{A} \cup \mathcal{B} \neq \mathcal{R}$. As a result there exists at least a trajectory $\mathcal{T}_1$ above $\mathcal{T}_r$, which does not intersects $\mathcal{M}$ and the set $\{(0,\xi) : \xi > 0\}$. The corresponding y is monotone. Suppose that y is increasing, then $\lim_{\tau\to-\infty} y = 0$; it is impossible since $\mathcal{T}_1 \neq \mathcal{T}_r$. Then y is decreasing, and $\lim_{\tau\to\infty} y = 0$. In any case w is of type (1), from Propositions 2.13 and 2.44. All the solutions are described, because any solution has at most one zero, and at most one extremum point. And $\mathcal{T}_1$ is unique when $\delta < N$. ■

4 The case $\varepsilon = -1, \delta < \alpha$

4.1 Subcase $N < \delta < \alpha$

Theorem 4.1 *Assume $\varepsilon = -1$ and $N < \delta < \alpha$. Then the regular solutions have a constant sign and satisfy $S_w < \infty$. And $w \equiv \ell r^{-\delta}$ is a solution. There exist solutions such that*

(1) w is positive, $\lim_{r\to 0} r^\eta w = c \neq 0$ if $N \geq 2$ ($\lim_{r\to 0} w = a > 0$, and $\lim_{r\to 0} w' = b$ for any $a > 0$ and some $b = b(a) < 0$ if $N = 1$) and $\lim_{r\to\infty} r^\delta w = \ell$;

(2) w is positive, $\lim_{r\to 0} r^\delta w = \ell$ and $S_w < \infty$;

(3) w has one zero, $\lim_{r\to 0} r^\delta w = \ell$ and $S_w < \infty$;

(4) w is positive, $\lim_{r\to 0} r^\alpha w = L \neq 0$ and $\lim_{r\to\infty} r^\delta w = \ell$;

(5) w is positive, $\lim_{r\to 0} r^\alpha w = L \neq 0$ and $S_w < \infty$;

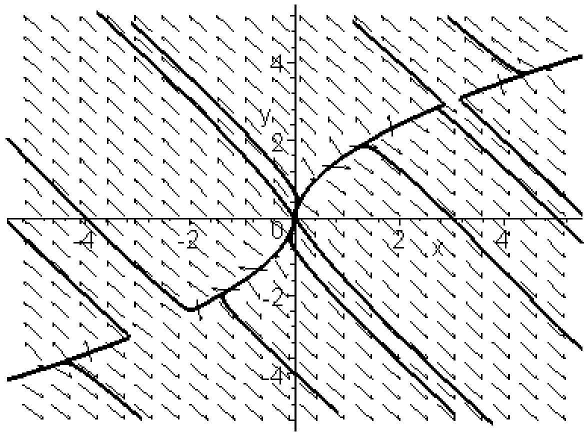
(6) w has one zero, $\lim_{r\to 0} r^{\alpha} w = L \neq 0$ *and* $S_w < \infty$;

(7) w is positive, $\lim_{r\to 0} r^{\eta} w = c \neq 0$ *if* $N \geq 2$ *(*$\lim_{r\to 0} w = a > 0$, *and* $\lim_{r\to 0} w' = b$, *for any* $a > 0$ *and some* $b < 0$ *if* $N = 1$*) and* $S_w < \infty$;

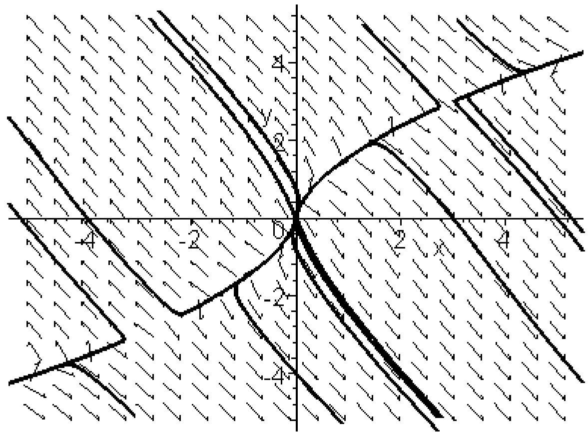
(8) w has one zero and the same behaviour;

(9) (only if $N = 1$*) w is positive,* $\lim_{r\to 0} w = a > 0$, *and* $\lim_{r\to 0} w' = b$, *for any* $a \geq 0$ *and any* $b > 0$, *and* $S_w < \infty$.

Up to a symmetry, all the solutions are described.



th 4.1,figVII: $\varepsilon = -1, N = 2 < \delta = 3 < \alpha = 4$



th 4.1,figVIII: $\varepsilon = -1, N = 1 < \delta = 3 < \alpha = 5$

Proof. Here we still have three stationary points, and $(0,0)$ is a source and M_ℓ is a saddle point (see fig V and VI). From Propositions 2.10 and 2.19, the regular solutions have a constant sign and satisfy $S_w < \infty$. And $\mathcal{T}_r$ stays in $\mathcal{Q}_4$, from Remark 2.6, and $\lim_{\tau\to\ln S_w} Y/y = -\infty$ from Proposition 2.20. Since $\alpha > 0$, any solution y has at most one zero from Proposition 2.10, and y is monotone near $\ln S_w$ (finite or not) and near $-\infty$, from Proposition 2.12. In the linearization near M_ℓ the eigenvectors $u_1 = (\nu(\alpha), \lambda_1 - \delta)$ and $u_2 = (-\nu(\alpha), \delta - \lambda_2)$ form a direct basis, where now $\nu(\alpha) < 0$, and $\lambda_1 < \delta < \lambda_2$; thus u_1 points to $\mathcal{Q}_3$ and u_2 points to $\mathcal{Q}_4$. There exist four particular trajectories $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4$ converging to M_ℓ near $\pm\infty$:

• $\mathcal{T}_1$ converging to M_ℓ at ∞, with tangent vector u_1. Here y is increasing near ∞, and as long as $y > 0$; indeed if there exists a minimal point τ, then from $(\mathbf{E}_y)$, $y(\tau) > \ell$. And $\mathcal{T}_1$ stays in $\mathcal{Q}_1$ on $\mathbb{R}$, from Remark 2.2. Then $\mathcal{T}_1$ converges to $(0,0)$ at $-\infty$, and w is of type (1).

• $\mathcal{T}_2$ converging to M_ℓ at $-\infty$, with tangent vector u_2. Here again $y' > 0$ as long as $y > 0$. And $Y' < 0$ near $-\infty$, and Y is decreasing as long as $Y > 0$: if there exists a minimal point of Y in $\mathcal{Q}_1$, then from $(\mathbf{E}_Y)$, $Y(\tau) > (\delta\ell)^{p-1}$. But (y,Y) cannot stay in $\mathcal{Q}_1$: it would imply $\lim_{\tau\to\infty} y = \infty$, which is impossible, from Proposition 2.13. Thus $\mathcal{T}_2$ enters $\mathcal{Q}_4$ at some point $(\xi_2, 0)$, $\xi_2 > 0$ and stays in it, since $y' > 0$. Then $S_w < \infty$ and $\lim_{\tau\to\infty} Y/y = -1$, and w is of type (2).

• $\mathcal{T}_3$ converging to M_ℓ at $-\infty$, with tangent vector $-u_2$. Here again $y' < 0$ as long as $y > 0$. And $Y' > 0$ as long as $Y > 0$; thus $Y' > 0$ on $\mathbb{R}$. Then again (y,Y) cannot stay in $\mathcal{Q}_1$, thus y has a unique zero, and $\mathcal{T}_3$ enters $\mathcal{Q}_2$ at some point $(0, \xi_3)$, $\xi_3 > 0$, and stays in it. Hence $S_w < \infty$ and $\lim_{\tau\to\infty} Y/y = -1$, and w is of type (3).

• $\mathcal{T}_4$ converging to M_ℓ at ∞, with tangent vector $-u_1$. In the same way, y is decreasing near ∞, and y is everywhere decreasing: if there exists a maximal point τ, then $y(\tau) < \ell$ from $(\mathbf{E}_y)$. Then Y stays positive, thus $\mathcal{T}_4$ stays in $\mathcal{Q}_1$. From Proposition 2.13, $\lim_{\tau\to-\infty} y = \infty$ and $\lim_{\tau\to-\infty} \zeta = \alpha$, so that w is of type (4).

Next we describe all the other trajectories $\mathcal{T}_{[P]}$ with one point P in the domain $\mathcal{R}$ above $\mathcal{T}_r \cup (-\mathcal{T}_r)$.

• If $P = (\varphi, 0)$, $\varphi > \xi_2$, then $\mathcal{T}_{[P]}$ stays in $\mathcal{Q}_4$ after P, because it cannot meet $\mathcal{T}_2$; before P it stays in $\mathcal{Q}_1$, from Remark 2.2. Thus again $S_w = \infty$, and $\lim_{\tau\to-\infty} \zeta = \alpha > 0$, and y has a unique minimal point, and w is of type (5). For any P is in the domain delimitated by $\mathcal{T}_2, \mathcal{T}_4$, the trajectory $\mathcal{T}_{[P]}$ is of the same type.

• If $P = (0, \xi)$, $\xi > \xi_3$, then $\mathcal{T}_{[P]}$ stays in $\mathcal{Q}_2$ after P, in $\mathcal{Q}_1$ before P, since it cannot meet $\mathcal{T}_2, \mathcal{T}_4$. Then $\lim_{\tau\to-\infty} \zeta = \alpha > 0$, and $S_w = \infty$, and w is of type (6). If P is in the domain delimitated by $\mathcal{T}_3, \mathcal{T}_4$, then $\mathcal{T}_{[P]}$ is of the same type.

• If $P = (\varphi, 0)$,$\varphi \in (0, \xi_2)$, then $\mathcal{T}_{[P]}$ stays in $\mathcal{Q}_4$ after P, in $\mathcal{Q}_1$ before P; it cannot meet $\mathcal{T}_r$, thus $S_w < \infty$; and $\mathcal{T}_{[P]}$ converges to $(0,0)$ in $\mathcal{Q}_1$ at $-\infty$, thus w is of type (7). If P is in the domain delimitated by $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_r$, then $\mathcal{T}_{[P]}$ is of the same type.

• If $P = (0, \xi)$ for some $\xi \in (0, \xi_3)$, then $\mathcal{T}_{[P]}$ stays in $\mathcal{Q}_2$ after P, in $\mathcal{Q}_1$ before P; and $\mathcal{T}$ cannot meet $-\mathcal{T}_r$, so that $S_w < \infty$. Then $\mathcal{T}_{[P]}$ converges to $(0,0)$ in $\mathcal{Q}_1$ at $-\infty$, and w is of type (8).

• If P is is in the domain delimitated by $\mathcal{T}_1, \mathcal{T}_3, -\mathcal{T}_r$, either y has one zero, and $\mathcal{T}_{[P]}$ is of the same type; or $y < 0$ on $\mathbb{R}$, and $y' = \delta y - Y^{1/(p-1)} < 0$. Hence $S_w < \infty$ and is $\mathcal{T}_{[P]}$ converges to $(0,0)$ in $\mathcal{Q}_2$ at $-\infty$. It implies $N = 1$ (see fig VI), and $-w$ is of type (9), from Propositions 2.13 and 2.14; and such a solution does exist from Theorem 2.5. Up to a symmetry, all the solutions are obtained. Here again, up to a scaling, the solutions w of types (1),(2),(3),(4) are unique. ■

4.2 Subcase $\delta \le \min(\alpha, N)$ (apart from $\alpha = \delta = N$)

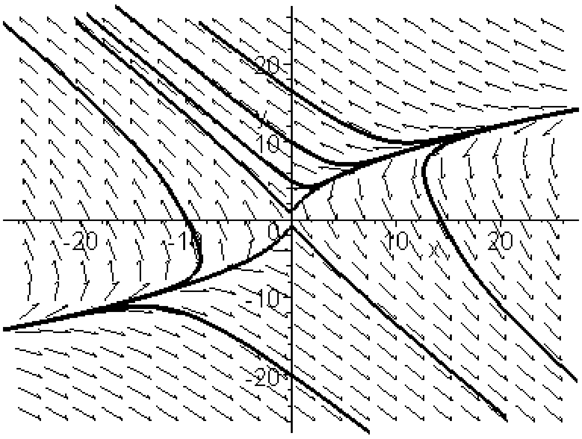
Theorem 4.2 *Suppose $\varepsilon = -1$ and $\delta \le \min(\alpha, N)$ (apart from $\alpha = \delta = N$). Then the regular solutions have a constant sign and a reduced domain ($S_w < \infty$). There exist solutions such that*

(1) w is positive, $\lim_{r\to 0} r^\alpha w = L \neq 0$ and $\lim_{r\to\infty} r^\eta w = c \neq 0$ if $\delta < N$, or (2.44) holds if $\delta = N < \alpha$;

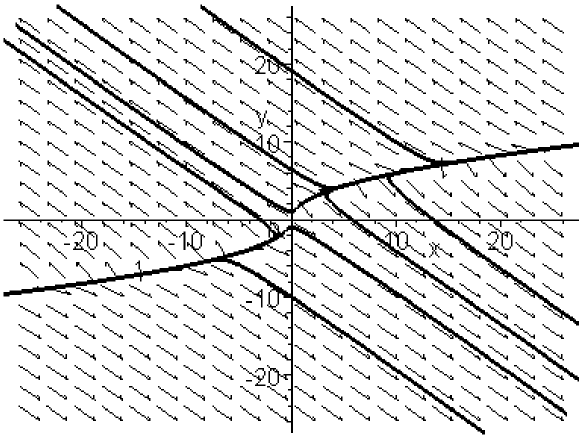
(2) w is positive, $\lim_{r\to 0} r^{\alpha} w = L \neq 0$ if $\delta < \alpha$, or (2.43) holds if $\alpha = \delta < N$, and $S_w < \infty$;

(3) w has one zero and the same behaviour.

Up to a symmetry, all the solutions are described.



th 4.2,figIX: $\varepsilon = -1, \delta = 3 < N = 4 < \alpha = 8$



th 4.2,figX: $\varepsilon = -1, \delta = 3 < \alpha = 3.1 < N = 4$

Proof. Here $(0,0)$ is the only one stationary point, and $N \geq 2$, (see fig IX and X) From Propositions 2.10 and 2.19, the regular solutions have a constant sign, and $S_w < \infty$. Moreover $w' > 0$ near 0 from Theorem 2.5; and w can only have minimal points, see Remark 2.6, thus $w' > 0$ on $(0, S_w)$; in other words $\mathcal{T}_r$ stays in $\mathcal{Q}_4$, and $\lim_{\tau\to\ln S_w} Y/y = -1$. From Propositions 2.10 and 2.12, any solution y has at most one zero and is monotone at the extremities. From Proposition 2.13, apart from $\mathcal{T}_r$, any trajectory $\mathcal{T}$ satisfies $\lim_{\tau\to-\infty} |y| = \infty$, then $\lim_{\tau\to-\infty} \zeta = \alpha > 0$, thus $\mathcal{T}$ starts from $\mathcal{Q}_1$ or $\mathcal{Q}_3$ at $-\infty$.

• For any $P = (\varphi, 0)$, $\varphi > 0$, $\mathcal{T}_{[P]}$ goes from $\mathcal{Q}_1$ into $\mathcal{Q}_4$ at P, from Remark 2.2, stays in $\mathcal{Q}_4$ after P, since it cannot meet $\mathcal{T}_r$, and in $\mathcal{Q}_1$ before P. Indeed it cannot start from $\mathcal{Q}_3$, because it does not meet $-\mathcal{T}_r$. Then y stays positive and w is of type (2).

• For any $P = (0, \xi)$, $\xi > 0$, $\mathcal{T}_{[P]}$ goes from $\mathcal{Q}_1$ into $\mathcal{Q}_2$ from Remark 2.2, thus $\mathcal{T}_{[P]}$ stays in $\mathcal{Q}_2$ after P, since it cannot meet $-\mathcal{T}_r$, and in $\mathcal{Q}_1$ before P, and w is of type (3).

• It remains to prove the existence of solutions of type (1). If $\delta < N$, $(0,0)$ is a saddle point, thus there exists a trajectory $\mathcal{T}_1$ converging to $(0,0)$ at ∞; and $\lim_{\tau\to\infty} \zeta = \eta > 0$ from Proposition 2.13. Then $\mathcal{T}_1$ is in $\mathcal{Q}_1$ for large τ, and stays in it, because $\mathcal{Q}_1$ is negatively invariant, and the conclusion follows. If $\delta = N$, we consider the sets

$$\mathcal{A} = \left\{P \in \mathcal{Q}_1 : \mathcal{T}_{[P]} \cap \{(\varphi, 0) : \varphi > 0\} \neq \emptyset\right\}, \qquad \mathcal{B} = \left\{P \in \mathcal{Q}_1 : \mathcal{T}_{[P]} \cap \{(0, \xi) : \xi > 0\} \neq \emptyset\right\}.$$

They are nonempty, and open, since the vector field is transverse at $(\varphi, 0)$ and $(0, \xi)$; thus $\mathcal{A} \cup \mathcal{B} \neq \mathcal{Q}_1$. Then there exists a trajectory $\mathcal{T}_1$ staying in $\mathcal{Q}_1$; therefore $S_w = \infty$ and $\mathcal{T}_1$ converges to $(0,0)$ at ∞, and w is of type (1), from Proposition 2.14. All the solutions are described, up to a symmetry. ■

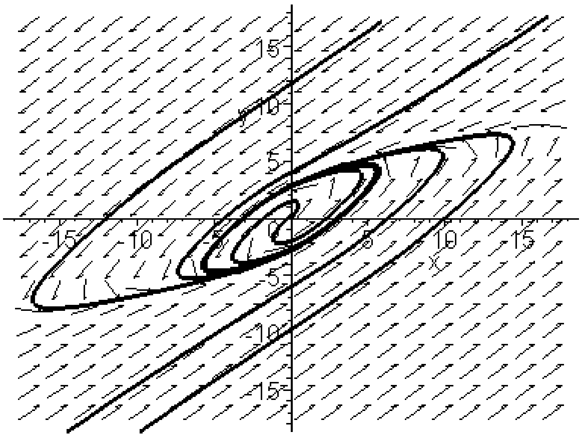
5 The case $\varepsilon = 1, \delta \leq \alpha$

5.1 Subcase $N \leq \delta \leq \alpha$.

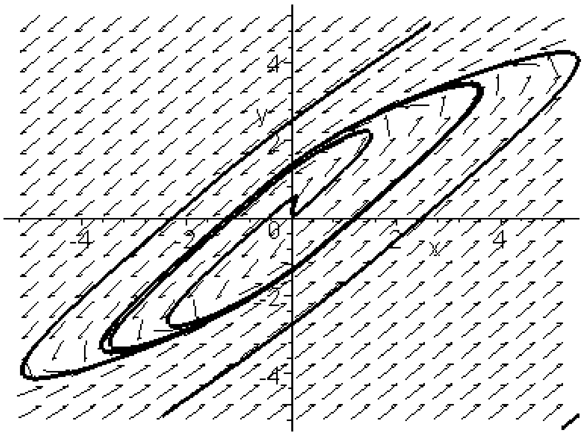
Theorem 5.1 *Assume $\varepsilon = 1$, $N \leq \delta \leq \alpha$ and $\alpha \neq N$. Then*

(i) There exists a cycle surrounding $(0,0)$, thus changing sign solutions such that $r^\delta w$ is periodic in $\ln r$. All the other solutions w, in particular the regular ones, are oscillating near ∞, and $r^\delta w$ is asymptotically periodic in $\ln r$. There exist solutions w such that $\lim_{r\to 0} r^\eta w = c \neq 0$ if $2 \leq N < \delta$ and (2.44) holds if $N = \delta$, or (2.45) holds if $N = 1$.

(ii) There exist solutions such that $R_w > 0$, or $\lim_{r\to 0} r^\alpha w = L \neq 0$ if $\alpha \neq \delta$, or (2.43) holds if $\alpha = \delta$.



th 5.1,figXI: $\varepsilon = 1, N = 2 < \delta = 3 < \alpha = 3.5$



th 5.1,figXII: $\varepsilon = 1, N = \delta = 3 < \alpha = 3.5$

Proof. (i) Here $(0,0)$ is the only stationary point. From Proposition 2.13, any trajectory is bounded and y is oscillating around 0 near ∞.

First assume $N < \delta < \alpha$ (see fig XI). Then $(0,0)$ is a source, all the trajectories have a limit cycle at ∞, or are periodic. In particular there exists at least a cycle, of orbit $\mathcal{O}_p$. In particular $\mathcal{T}_r$ presents a limit cycle $\mathcal{O} \subseteq \mathcal{O}_p$. There exists also trajectories $\mathcal{T}_s$ starting from $(0,0)$ with an infinite slope, satisfying $\lim_{r\to 0} r^\eta w = c \neq 0$ if $N \geq 2$, or (2.45) if $N = 1$, and they have the same limit cycle $\mathcal{O}$.

Next assume $N = \delta < \alpha$(see fig XII). Then $\mathcal{T}_r$ cannot converge to $(0,0)$, since it would intersect itself. Thus again the limit set at ∞ is a closed orbit $\mathcal{O}$. And no trajectory can converge to $(0,0)$ at ∞: it would be spiraling around this point, and then intersect $\mathcal{T}_r$. Consider any trajectory $\mathcal{T} \neq \mathcal{T}_r$ in the connected component of $\mathcal{O}$ containing $(0,0)$. It is bounded, in particular the limit set at $-\infty$ is $(0,0)$, or a closed orbit. The second case is impossible, since $\mathcal{T}$ does not meet $\mathcal{T}_r$. Then $\mathcal{T}$ is one of the $\mathcal{T}_s$, and the corresponding w satisfies (2.44).

(ii) From Theorem 2.26, all the cycles are contained in a ball B of $\mathbb{R}^2$. Take any point P_0 exterior to B. Then $\mathcal{T}_{[P_0]}$ has a limit cycle at ∞ contained in B. If it has a limit cycle at $-\infty$, then it is contained in B, so that $\mathcal{T}_{[P_0]}$ is contained in B, which is impossible. As a result y has constant sign near $\ln R_w$. From Proposition 2.13, either $R_w > 0$ or y is defined near $-\infty$. ■

Theorem 5.2 *Assume $\varepsilon = 1$ and $\alpha = \delta = N$. Then the regular solutions have a constant sign, and are given by (1.8). For any $k \in \mathbb{R}$, $w(r) = kr^{-N}$ is a solution. There exist solutions such that*

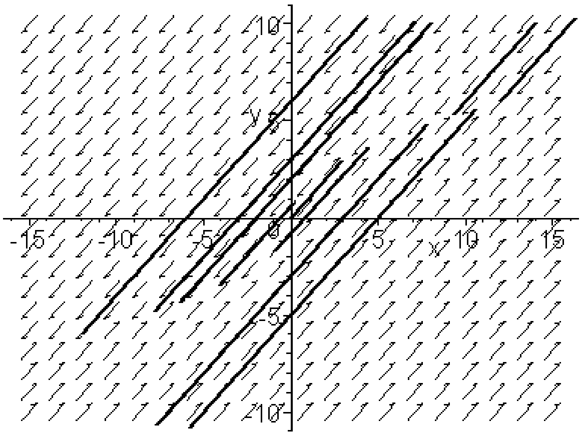
(1) w is positive, $\lim_{r\to 0} r^N w = c_1 > 0$, $\lim_{r\to 0} r^N w = c_2 > 0$ $(c_2 \neq c_1)$;

(2) w has one zero, $\lim_{r\to 0} r^N w = c_1 > 0$ and $\lim_{r\to\infty} r^N w = c_2 < 0$;

(3) w is positive, $R_w > 0$, and $\lim_{r\to 0} r^N w = c \neq 0$;

(4) w has one zero and the same behaviour.

Up to a symmetry, all the solutions are described.



th 5.2,figXIII: $\varepsilon = 1, \alpha = \delta = N = 3$

Proof. Since $\alpha = N$, equation $(\mathbf{E}_w)$ admits the first integral (1.7), which means $J_N \equiv C, C \in \mathbb{R}$, and we have given at (1.8) the regular (Barenblatt) solutions relative to the case $C = 0$. Since $\delta = N$, (1.7) is equivalent to the equation $Y \equiv y - C$, from (2.12) (see fig XIII). For any $k \in \mathbb{R}$,

$(y,Y) \equiv (k, |Nk|^{p-2} Nk)$ is a solution of system (**S**), located on the curve $\mathcal{M}$, so that $w(r) = kr^{-N}$ is a solution. Any solution has at most one zero, from Proposition 2.10. From Propositions 2.13, and 2.15, any trajectory converges to a point $(k, |Nk|^{p-2} Nk)$ of $\mathcal{M}$ at ∞. Let $\overline{C} < 0$ such that the line $Y = y - \overline{C}$ is tangent to $\mathcal{M}$. Then for any $C \in \left(\overline{C}, 0\right)$, the line $Y = y - C$ cuts $\mathcal{M}$ at three points $k_1 < 0 < k_2 < k_3$. And $y' > 0$ if the trajectory is below $\mathcal{M}$ and $y' < 0$ if it is above $\mathcal{M}$. We find two solutions defined on $\mathbb{R}$: one is positive such that $\lim_{\tau \to -\infty} y = k_2$, $\lim_{\tau \to -\infty} y = k_3$, and the other has one zero. All the other solutions satisfy $R_w > 0$, $\lim_{\tau \to \ln R_w} Y/y = 1$, some of them are positive, the other have one zero. ∎

5.2 Subcase $\delta < \min(\alpha, N)$

Here the system has three stationary points, $(0,0)$, is a saddle point, and M_ℓ, M'_ℓ are sinks when $\delta \leq N/2$, or $N/2 < \delta$ and $\alpha < \alpha^*$, and sources when $N/2 < \delta$ and $\alpha > \alpha^*$, and node points whenever $\alpha \leq \alpha_1$, or $\alpha_2 \leq \alpha$, where α_1, α_2 are defined at (2.51). recall that α_1 can be greater or less than η. This case is one of the most delicate, since two types of periodic trajectories can appear, either surrounding $(0,0)$, corresponding to changing sign solutions, or located in $\mathcal{Q}_1$ or $\mathcal{Q}_3$, corresponding to constant sign solutions. Notice that $\delta < N$ implies $\delta < N < \eta$ from (1.3). And $N/2 < \delta$ implies $\eta < \alpha^*$ from (2.35). We begin by some general properties of the phase plane.

Remark 5.3 *(i) The trajectory $\mathcal{T}_r$ starts in $\mathcal{Q}_1$. Since $(0,0)$ is a saddle point, there exists a unique trajectory $\mathcal{T}_s$ converging to $(0,0)$, in $\mathcal{Q}_1$ for large τ, with an infinite slope at $(0,0)$, and $\lim_{r \to 0} r^\eta w = c > 0$, from Propositions 2.13 and 2.14. Moreover if $\mathcal{T}_r$ does not stay in $\mathcal{Q}_1$, then $\mathcal{T}_s$ stay in it, and it is bounded, contained in the domain delimitated by $\mathcal{Q}_1 \cap \mathcal{T}_r$, from Remark 2.2. Thus if $\mathcal{T}_r$ is homoclinic, it stays in $\mathcal{Q}_1$.*

(ii) Any trajectory $\mathcal{T}$ is bounded near ∞ from Propositions 2.13 and 2.17. From the strong form of the Poincaré-Bendixon theorem, see [13, p.239], any trajectory $\mathcal{T}$ bounded at $\pm\infty$ either converges to $(0,0)$ or $\pm M_\ell$, or its limit set $\Gamma_\pm$ at $\pm\infty$ is a cycle, or it is homoclinic hence $\mathcal{T} = \mathcal{T}_r$ and $\Gamma_\pm = \overline{\mathcal{T}_r}$ (indeed for any $P \in \Gamma_\pm$, $\mathcal{T}_{[P]}$ converges at ∞ and $-\infty$ to $(0,0)$ or $\pm M_\ell$; if one of them is $\pm M_\ell$, then $\pm M_\ell \in \overline{\mathcal{T}_{[P]}} \subset \Gamma_\pm$, and M_ℓ is a source or a sink, thus $\mathcal{T}$ converges to $\pm M_\ell$; otherwise $\mathcal{T}$ is homoclinic and $\mathcal{T}_{[P]} = \mathcal{T}_r$).

(iii) If there exists a limit cycle surrounding $(0,0)$, then from (2.46), it also surrounds the points $\pm M_\ell$.

We begin by the case $\alpha \leq \eta$, where there exists no cycle in $\mathcal{Q}_1$, and no homoclinic orbit, from Theorem 2.25.

Theorem 5.4 *Assume that $\varepsilon = 1$ and $\delta < \min(\alpha, N)$, and $\alpha \leq \eta$. Then the regular solutions have a constant sign, and $\lim_{r \to \infty} r^\delta |w(r)| = \ell$. And $w(r) = \ell r^{-\delta}$ is a solution. Moreover*

(i) If $\alpha < \eta$, There exist solutions such that

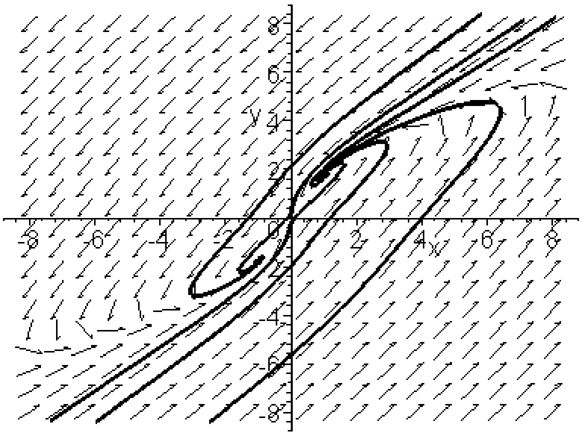
(1) w is positive, $\lim_{r\to 0} r^{\alpha} w = L$ *and* $\lim_{r\to\infty} r^{\delta} w = \ell$;

(2) w is positive, $R_w > 0$ *and* $\lim_{r\to\infty} r^{\eta} w = c > 0$;

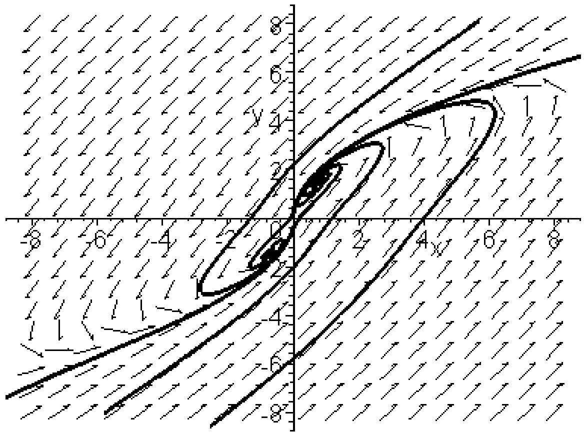
(3) w is positive, $R_w > 0$ *and* $\lim_{r\to\infty} r^{\delta} w = \ell$;

(4) w has one zero, $R_w > 0$ *and* $\lim_{r\to\infty} r^{\delta} w = \ell$;

(ii) If $\alpha = \eta$, then $w = Cr^{-\eta}$ is a solution and there exist solutions of type (4), but not of type (2) or (3).



th 5.4,figXIV:
$\varepsilon = 1, \delta = 3 < N = 4 < \alpha = 4.7 < \eta = 5$



th 5.4,figXV:
$\varepsilon = 1, \delta = 3 < N = 4 < \alpha = \eta = 5$

Proof. From Proposition 2.10 and Remark 2.6, $\mathcal{T}_r$ stays in $\mathcal{Q}_1$, and converges to M_ℓ at ∞; indeed there is no cycle in$\mathcal{Q}_1$, from Propositions 2.13, 2.17 and 2.25.

(i) Assume $\alpha < \eta$ (see fig XIV) Consider any trajectory in $\mathcal{Q}_1$, thus in particular $Y_\alpha > 0$. If there exists τ such that $Y_\alpha'(\tau) = 0$, then at this point $Y_\alpha''(\tau) \geq 0$ from (2.39), and τ is a local minimum. The trajectory $\mathcal{T}_r$ satisfies $\lim_{\tau\to-\infty} Y_\alpha = 0$, and consequently $Y_\alpha' > 0$ on $\mathbb{R}$. This is equivalent to

$$\alpha y > Y^{1/(p-1)} + (p-1)(\eta - \alpha)Y.$$

Therefore $\mathcal{T}_r$ stays strictly under the curve

$$\mathcal{M}_\alpha = \left\{ (y, Y) \in \mathcal{Q}_1 : \alpha y = Y^{1/(p-1)} + (p-1)(\eta - \alpha)Y \right\}.$$

• First consider $\mathcal{T}_s$. Since $\alpha < \eta$, it satisfies $\lim_{\tau\to\infty} Y_\alpha = 0$. Then $Y_\alpha' < 0$ on $(\ln R_w, \infty)$, so that $\mathcal{T}_s$ stays strictly above $\mathcal{M}_\alpha$. Then it stays above $\mathcal{M}$. Indeed if it meets $\mathcal{M}$ at a first point $(y_1, (\delta y_1)^{p-1})$, then y has a maximum at this point, thus from (2.16), $\ell < y_1$, and

$$(\alpha - \delta) y_1^{2-p} = \delta^{p-1}(p-1)(\eta - \alpha) < \delta^{p-1}(p-1)(\eta - \delta),$$

and we reach a contradiction from (1.3) and (1.5). Thus $y' < 0$. Suppose that y is defined on $\mathbb{R}$, then $\lim_{\tau\to-\infty} y = \infty, \lim_{\tau\to-\infty} \zeta = \alpha$. If $\zeta' > 0$ on $\mathbb{R}$, then $\zeta(\mathbb{R}) = (\alpha, \eta)$, which contradicts (2.9). Then ζ has at least an extremal point τ, and $\zeta(\tau)$ is exterior to (α, η) from (2.9); if it is a minimum, then $\zeta(\tau) > \alpha$ from (2.18), since $y' < 0$; if it is a maximum, then $\zeta(\tau) < \alpha$. Thus we reach again a contradiction. Then $R_w > 0$ and $\lim_{\tau\to\ln R_w} Y/y = 1$, and corresponding w are of type (2).

• For any $P = (\varphi, 0), \varphi > 0$, the trajectory $\mathcal{T}_{[P]}$ stays in $\mathcal{Q}_1$ after P. The solution (y, Y) issued from P at time 0 satisfies $Y_\alpha(0) = 0$, thus $Y'_\alpha(\tau) > 0$ for any $\tau \geq 0$. Thus $\mathcal{T}_{[P]}$ stays under $\mathcal{M}_\alpha$. Moreover it enters $\mathcal{Q}_4$ as τ decreases, and $y' > 0$ in $\mathcal{Q}_4$, from (**S**), thus it does not stays in it from Proposition 2.13, enters in $\mathcal{Q}_3$ and cannot meat $-\mathcal{T}_s$; it stays in $\mathcal{Q}_3$ and $R_w > 0$, and y has precisely one zero, and w is of type (4).

• Consider any trajectory $\mathcal{T}_{[P_1]}$ going through some point $P_1 = (y_1, Y_1)$ in $\mathcal{Q}_1$, under $\mathcal{T}_s$ and such that $\alpha y_1 < Y_1^{1/(p-1)}$. There exist such one, because the line $y = Y$ is an asymptotic direction of $\mathcal{T}_s$. Let (y, Y) be the solution issued from P_1 at time 0. Suppose that y is defined on $\mathbb{R}$, then $\lim_{\tau\to-\infty} y = \infty, \lim_{\tau\to-\infty} \zeta = \alpha$. And $\zeta(0) > \alpha$. Then $\zeta > \delta$ on $(-\infty, 0)$: otherwise there exists $\tau < 0$ such that $\zeta(\tau) = \alpha$ and $\zeta'(\tau) \geq 0$, which contradicts (2.9). Thus $y' < 0$ on $(-\infty, \tau_1)$. Either $\zeta' > 0$ on $(-\infty, 0)$, then $\zeta > \eta > 0$, from (2.9), which is impossible. Or ζ has at least an extremal point τ, and if it is a minimum, then $\zeta(\tau) > \alpha$ from (2.18); if it is a maximum, then $\zeta(\tau) < \alpha$; and we are lead to a contradiction. Therefore $R_w > 0$, and the trajectory stays in $\mathcal{Q}_1$, and converges to M_ℓ ; indeed there is no cycle in $\mathcal{Q}_1$, from Theorem 2.25; then w is of type (3).

• Let $\mathcal{O}$ be the domain of $\mathcal{Q}_1$ located under $\mathcal{T}_s$. It is positively invariant. Any trajectory going through any point of $\mathcal{O}$ converges to M_ℓ at ∞. Either it meets the axis $Y = 0$ at some point $(\xi, 0), \xi > 0$, or it stays in $\mathcal{O}$ and satisfies $R_w > 0$, $\lim_{\tau\to\ln R_w} T/y = 1$, and it meets $\mathcal{M}_\alpha$, since M_ℓ is strictly under $\mathcal{M}_\alpha$. Let

$$\mathcal{A} = \left\{P \in \mathcal{O} : \mathcal{T}_{[P]} \cap \{(\varphi, 0) : \varphi > 0\} \neq \emptyset\right\}, \qquad \mathcal{B} = \left\{P \in \mathcal{O} : \mathcal{T}_{[P]} \cap \mathcal{M}_\alpha \neq \emptyset\right\}.$$

Then $\mathcal{A}, \mathcal{B}$ nonempty, and open: indeed one verifies that the intersection with $\mathcal{M}_\alpha$ is transverse, because $\alpha \neq \eta$. Thus $\mathcal{A} \cup \mathcal{B} \neq \mathcal{O}$. Then there exists a trajectory $\mathcal{T}_1$ such that w is of type (1).

(ii) Assume $\alpha = \eta$ (see fig XV). Then there is no positive solution with $R_w > 0$, thus no solution of type (2) or (3). Indeed all the trajectories stay under $\mathcal{T}_s$, and $\mathcal{T}_s$ is defined by the equation $\zeta \equiv \eta$, that means $w \equiv Cr^{-\eta}$, or equivalently $Y_\eta \equiv C$; thus $Y'_\eta \equiv 0, \mathcal{T}_s = \mathcal{M}_\eta$. Consider any trajectory $\mathcal{T}_{[P]}$ going through some point $P = (\varphi, 0), \varphi > 0$, and the solution (y, Y) issued from P at time 0. Then $Y_\eta(0) = 0$, and $Y_\eta < 0$ thus $Y'_\eta = \eta y - |Y|^{(2-p)/(p-1)} Y > 0$ on $(-\infty, 0)$, seeing that $\mathcal{T}_{[P]}$ does not meet $-\mathcal{T}_s$. Suppose that it satisfies $R_w = 0$. Then $\mathcal{T}_{[P]}$ starts from $\mathcal{Q}_3$, with $\lim_{\tau\to-\infty} \zeta = \alpha = \eta$. Then $\lim_{\tau\to-\infty} y_\eta = L < 0$, thus $\lim_{\tau\to-\infty} Y_\eta = -(\alpha |L|)^{(2-p)/(p-1)}$. And by a straightforward computation,

$$Y''_\eta = Y'_\eta \left(N - \frac{1}{p-1} |Y|^{(2-p)/(p-1)}\right).$$

Hence $Y''_\eta < 0$ near $-\infty$, which is impossible; then $R_w < \infty$ and w is of type (4). ■

Remark 5.5 *Observe that for $\alpha \leq \eta$, both trajectories $\mathcal{T}_r$ and $\mathcal{T}_s$ stay in $\mathcal{Q}_1$.*

Remark 5.6 *(i) When $\alpha \leq N$, one can verify that the regular positive solution y is increasing and $y \leq \ell$ on $\mathbb{R}$, so that $r^{\delta}w(r) \leq \ell$ for any $r \geq 0$.*
(ii) When $\alpha = N$, then $\mathcal{T}_r = \{(\xi, \xi) : \xi \in [0, \ell))\}$, and the corresponding solutions w are given by (1.8) with $K > 0$. And $\mathcal{T}_3 = \{(\xi, \xi) : \xi > \ell)\}$ is a trajectory corresponding to particular solutions w of type (3), given by (1.8) with $K < 0$.

Next we come to the most interesting case, where $\eta < \alpha$.

Lemma 5.7 *Assume $\varepsilon = 1$ and $\delta < \min(\alpha, N)$ and $\eta < \alpha$. If $N/2 < \delta$ and $\alpha < \alpha^*$ and $\mathcal{T}_s$ stays in $\mathcal{Q}_1$, then it has a limit cycle at $-\infty$ in $\mathcal{Q}_1$, or it is homoclinic. If $\delta \leq N/2$, then $\mathcal{T}_s$ does not stay in $\mathcal{Q}_1$.*

Proof. In any case M_ℓ is a sink, thus $\mathcal{T}_s$ cannot converge to M_ℓ at $-\infty$. Suppose that $\mathcal{T}_s$ has no limit cycle in $\mathcal{Q}_1$, and is not homoclinic and stays in $\mathcal{Q}_1$. In particular it happens when $\delta \leq N/2$, from Proposition 2.16. Then either $\lim_{\tau\to-\infty} y = \infty$, $\lim_{r\to 0} r^{\alpha}w = \Lambda \neq 0$, or $R_w > 0$. In any case, for any $d \in (\eta, \alpha)$, the function $y_d(\tau) = r^d w = r^{d-\delta}y$ satisfies $\lim_{\tau\to\ln R_w} y_d = \infty = \lim_{\tau\to\infty} y_d$. Then it has a minimum point, which contradicts (2.5). ■

Theorem 5.8 *Assume $\varepsilon = 1$ and $N/2 < \delta < \min(\alpha, N)$. Then $w(r) = \ell r^{-\delta}$ is still a solution. Moreover*

(i) There exists a (maximal) critical value α_{crit} of α, such that

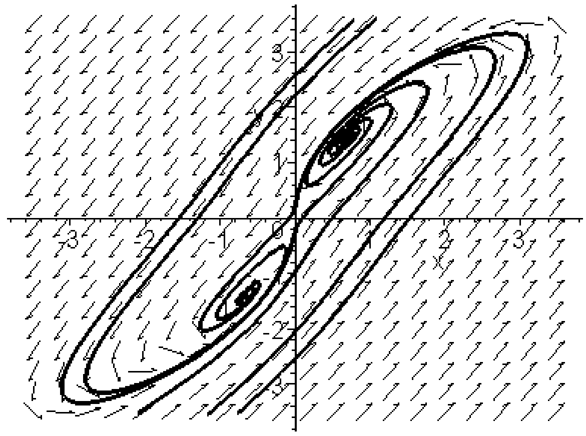
$$\max(\eta, \alpha_1) < \alpha_{crit} < \alpha^*,$$

and the regular trajectory is homoclinic: the regular solutions have a constant sign and satisfy $\lim_{r\to\infty} r^{\eta}w = c \neq 0$.

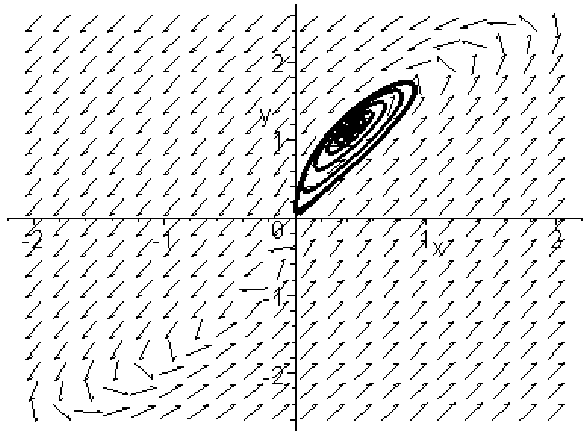
(ii) For any $\alpha \in (\alpha_{crit}, \alpha^*)$, there does exist a cycle in $\mathcal{Q}_1$, in other words there exist positive solutions w such that $r^{\delta}w$ is periodic in $\ln r$. There exist positive solutions such that $r^{\delta}w$ is asymptotically periodic in $\ln r$ near 0 and $\lim_{r\to\infty} r^{\delta}w = \delta$. There exist positive solutions such that $r^{\delta}w$ is asymptotically periodic in $\ln r$ near 0 and $\lim_{r\to\infty} r^{\eta}w = c \neq 0$.

(iii) For any $\alpha \geq \alpha^*$ there does not exist such a cycle, but there exist positive solutions such that $\lim_{r\to 0} r^{\delta}w = \ell$ and $\lim_{r\to\infty} r^{\eta}w = c > 0$.

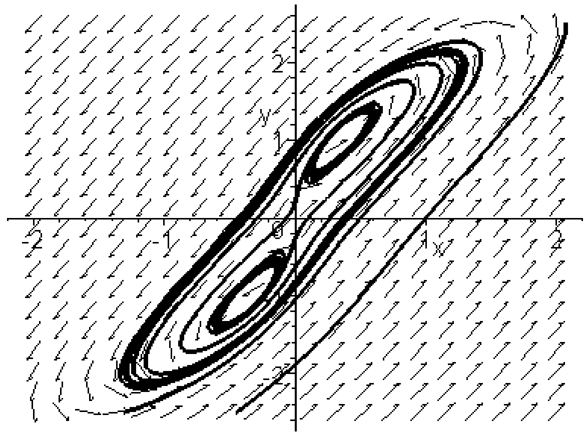
(iv) For any $\alpha > \alpha_{crit}$, there exists also a cycle, surrounding $(0,0)$ and $\pm M_\ell$, thus $r^{\delta}w$ is changing sign and periodic in $\ln r$. The regular solutions, are changing sign, and oscillating at ∞, and $r^{\delta}w$ is asymptotically periodic in $\ln r$. There exist solutions such that $R_w > 0$, or $\lim_{r\to 0} r^{\alpha}w = L \neq 0$, and oscillating at ∞, and $r^{\delta}w$ is asymptotically periodic in $\ln r$.



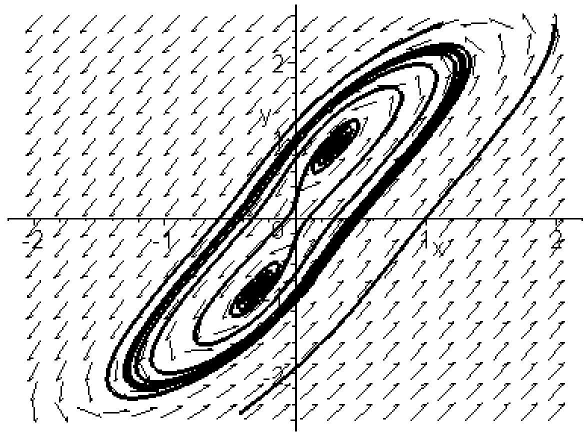
th 5.8,figXVI: $\varepsilon = 1$,
$N/2 < \delta = 3 < N = 4 < \eta = 5 < \alpha = 5.1$



th 5.8,figXVII: $\varepsilon = 1$,
$N/2 < \delta = 3 < N = 4 < \eta = 5 < \alpha = 5.62$



th 5.8,figXVIII: $\varepsilon = 1$,
$N/2 < \delta = 3 < N = 4 < \alpha = 5.9 < \alpha^* = 6$



th 5.8, figXIX: $\varepsilon = 1$,
$N/2 < \delta = 3 < N = 4 < \alpha^* = 6 < \alpha = 6.2$

Proof. (i). For any $\alpha \in (\alpha_1, \alpha_2)$ such that $\eta \leq \alpha$, from Remark 5.3, we have three possibilities for the regular trajectory $\mathcal{T}_r$:

• $\mathcal{T}_r$ is converging to M_ℓ and turns around this point, since α is a spiral point, or it has a limit cycle in $\mathcal{Q}_1$ around M_ℓ. Then $\mathcal{T}_r$ meats the set $\mathcal{E} = \left\{(\ell, Y) : Y > (\delta\ell)^{p-1}\right\}$ at a first point $(\ell, Y_r(\alpha))$. Notice that ℓ and $\mathcal{E}$ depend continuously of α. Then $\mathcal{T}_s$ meats $\mathcal{E}$ at a last point $(\ell l, Y_s(\alpha))$, such that $Y_s(\alpha) - Y_r(\alpha) > 0$ (see fig XVI)

• $\mathcal{T}_r$ does not stay in $\mathcal{Q}_1$, and then $\mathcal{T}_s$ is bounded at $-\infty$, thus converges to M_ℓ at $-\infty$ and turns around this point, or it has a limit cycle around M_ℓ. Then $\mathcal{T}_s$ meats $\mathcal{E}$ at a last point $(\ell l, Y_s(\alpha))$, $\mathcal{T}_r$ meats $\mathcal{E}$ at a first point $(\ell l, Y_r(\alpha))$, such that $Y_s(\alpha) - Y_r(\alpha) < 0$ (see fig XVIII and XIX)

• $\mathcal{T}_r$ is homoclinic, which is equivalent to $Y_s(\alpha) - Y_r(\alpha) = 0$ (see fig XVII).

Now the function $\alpha \mapsto g(\alpha) = Y_s(\alpha) - Y_r(\alpha)$ is continuous. If $\alpha_1 < \eta$, then $g(\eta)$ is defined and $g(\eta) > 0$, from Theorem 5.4. If $\eta \leq \alpha_1$, we observe that for $\alpha = \alpha_1$, the trajectory $\mathcal{T}_s$ leaves $\mathcal{Q}_1$, from Theorem 2.23, because α_1 is a sink, and transversally from Remark 2.2; thus also for $\alpha = \alpha_1 + \gamma$ for γ small enough, by continuity, thus $\mathcal{T}_r$ stays in $\mathcal{Q}_1$, and $g(\alpha_1 + \gamma) > 0$. If $\alpha \geq \alpha^*$(see fig XIX), then M_ℓ is a source, or a weak source, from Theorem 2.21, thus $\mathcal{T}_r$ cannot converge to M_ℓ. From Theorem 2.24, there exist no cycle in $\mathcal{Q}_1$, and no homoclinic orbit. From Remark 5.3, $\mathcal{T}_r$ cannot stay in $\mathcal{Q}_1$, thus $g(\alpha) < 0$ for $\alpha^* \leq \alpha < \alpha_2$. As a consequence, there exists at least an $\alpha_{crit} \in (\max(\eta, \alpha_1), \alpha^*)$ such that $g(\alpha_{crit}) = 0$. If it is not unique, we can choose the greatest one.

(ii) Let $\alpha < \alpha^*$. The existence and uniqueness of such a cycle $\mathcal{O}$ in $\mathcal{Q}_1$ follows from Theorem 2.21 when α is close to α^* (see fig XVIII). In fact the existence holds for any $\alpha \in (\alpha_{crit}, \alpha^*)$. Indeed $g(\alpha) < 0$ on this interval, and $\mathcal{T}_s$ cannot converge to M_ℓ at $-\infty$, thus it has a limit cycle around M_ℓ at $-\infty$. Since M_ℓ is a sink, there exist also trajectories converging to M_ℓ at ∞, with a limit cycle at $-\infty$ contained in $\mathcal{O}$. Now $\mathcal{T}_r$ does not stay in $\mathcal{Q}_1$, is bounded at ∞, thus it has a limit cycle at ∞, containing the three stationary points .

(iii) Let $\alpha \geq \alpha^*$. Then $\mathcal{T}_s$ stays in $\mathcal{Q}_1$, is bounded on $\mathbb{R}$, and converges at $-\infty$ to M_ℓ, and $\mathcal{T}_r$ does not stay in $\mathcal{Q}_1$ as above, thus it has a limit cycle at ∞, containing the three stationary points (see figXIX).

(iv) For any $\alpha > \alpha_{crit}$, apart from $\mathcal{T}_s$ and the cycles, all the trajectories have a limit cycle at ∞ containing the three stationary points. Moreover from Theorem 2.26, all the cycles are contained in a ball B of $\mathbb{R}^2$. Take any point P exterior to B. From Remark 6.5, $\mathcal{T}_{[P]}$ has a limit cycle at ∞ contained in B and cannot have a limit cycle at $-\infty$. Thus y has constant sign near $\ln R_w$. From Proposition 2.13, either $R_w > 0$ or y is defined near $-\infty$ and $\lim_{\tau \to -\infty} \zeta = L$, $\lim_{r \to 0} r^\alpha w = L$. ■

Remark 5.9 *An open question is the uniqueness of* α_{crit}. *It can be shown that if there exist two critical values* $\alpha^1_{crit} > \alpha^2_{crit}$, *then the first orbit is contained in the second one.*

In the case $\delta \leq N/2$, which means $p \leq P_2$, there exist no cycle in $\mathbb{R}^2$, and we obtain the following:

Theorem 5.10 *Assume* $\varepsilon = 1$ *and* $\delta \leq N/2$, $\delta < \alpha$. *Then the regular solutions have a constant sign, and* $\lim_{r\to\infty} r^\delta |w| = \ell$. *All the solutions have a finite number of zeros. And* $w(r) = \ell r^{-\delta}$ *is a solution. Moreover, if* $\alpha \leq \eta$, *theorem 5.4 applies. If* $\eta < \alpha$, *all the other solutions have at least one zero. There exist solutions, such that* $\lim_{r\to\infty} r^\eta w = c \neq 0$, *with a number* m *of zeros. All the other solutions satisfy* $\lim_{r\to\infty} r^\delta w = \pm\ell$, *and have* m *or* $m+1$ *zeros. There exist solutions with* $m+1$ *zeros.*

Proof. (i) From Proposition 2.16, all the solutions have a finite number of zeros. Since $\delta \leq N/2$, the function W defined at (2.21) is nonincreasing. The regular solutions (y,Y) satisfy $\lim_{\tau\to-\infty} W(\tau) = 0$, thus $W(\tau) \leq 0$ on $\mathbb{R}$. If $y(\tau_0) = 0$ for some real τ_0, then $W(\tau_0) = |Y(\tau_0)|^{p'} > 0$, and we reach a contradiction. From Propositions 2.13 and 2.16, then $\lim_{\tau\to\infty} y = \pm\ell$, thus $\lim_{r\to\infty} r^\delta w = \pm\ell$.

(ii) Assume $\eta < \alpha$. From Lemma 5.7, $\mathcal{T}_s$ does not stay in $\mathcal{Q}_1$. From Proposition 2.13 and 2.20, $\mathcal{T}_s$ cannot stay in $\mathcal{Q}_4$, thus y has at least one zero. Let m be the number of its zeros. Then $\mathcal{T}_s$ cuts the axis $y = 0$ at points $\xi_1, .., \xi_m$. From Remark 5.3, apart from $\mathcal{T}_s$, any trajectory converges to $\pm M_\ell$. For any $P = (0,\xi), \xi > |\xi_m|$, the trajectory $\mathcal{T}_{[P]}$ cannot intersect $\mathcal{T}_s$ and $-\mathcal{T}_s$, thus y has $m+1$ zeros. Any other solution has m or $m+1$ zeros, because the trajectory does not meet $\mathcal{T}_r$ and $-\mathcal{T}_r$ and $\mathcal{T}_{[P]}$. And $R_w > 0$ or $\lim_{r\to 0} r^\alpha w = L \neq 0$. ■

Remark 5.11 *Theorems 5.4, 5.8 and 5.10 cover in particular the results of [16, Theorem 2].*

6 Case $\varepsilon = -1$, $\alpha \leq \delta$

6.1 Subcase $\max(\alpha, N) \leq \delta$

Here $(0,0)$ is the only stationary point, and it is a source when $\delta \neq N$. We first suppose $0 < \alpha$.

Theorem 6.1 *Suppose $\varepsilon = -1, \max(\alpha, N) \leq \delta$ and $0 < \alpha$.*

(i) Suppose $\alpha \neq N$ or $\alpha \neq \delta$. Then the regular solutions have a constant sign and a reduced domain ($S_w < \infty$). Moreover there exist solutions such that

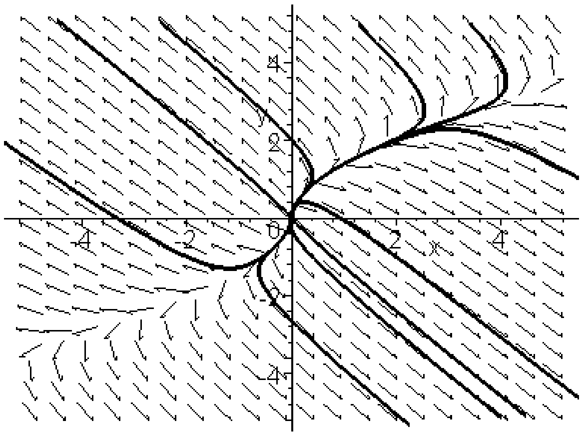
(1) w is positive, $\lim_{r\to 0} r^\eta w = c \neq 0$ if $N \geq 2$ ($\lim_{r\to 0} w = a > 0$, $\lim_{r\to 0} w' = b < 0$ if $N = 1$) and $\lim_{r\to\infty} r^\alpha w = L \neq 0$ if $\alpha \neq \delta$, or (2.43) holds if $\alpha = \delta$;

(2) w is positive, $\lim_{r\to 0} r^\eta w = c \neq 0$ if $N \geq 2$ ($\lim_{r\to 0} w = a > 0$, $\lim_{r\to 0} w' = b \neq 0$, or $a = 0 < b$ if $N = 1$) and $S_w < \infty$;

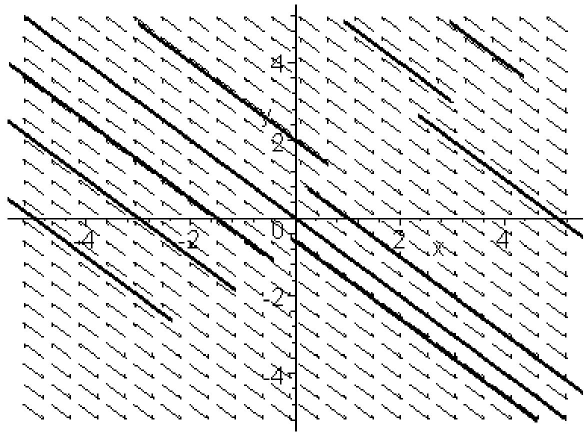
(3) w has one zero, $\lim_{r\to 0} r^\eta w = c \neq 0$ if $N \geq 2$ ($\lim_{r\to 0} w = a > 0$, $\lim_{r\to 0} w' = b < 0$ if $N = 1$) and $S_w < \infty$.

(ii) Suppose $\alpha = \delta = N$. Then the regular solutions, given by (1.8), have a constant sign, with $S_w < \infty$. For any $k \in \mathbb{R}$, $w(r) = kr^{-N}$ is a solution. Moreover there exist positive solutions such that $\lim_{r\to 0} r^N w = c > 0$ and $S_w < \infty$, and solutions with one zero, such that $\lim_{r\to 0} r^N w = c > 0$ and $S_w < \infty$.

Up to a symmetry, all the solutions are described.



th 6.1,figXX:
$\varepsilon = -1, N = 2 < \alpha = 2.5 < \delta = 3$



th 6.1,figXXI:
$\varepsilon = -1, \alpha = \delta = N = 3$

Proof. (i) Here $\alpha \neq N$ or $\alpha \neq \delta$ (see fig XX). Since $\alpha > 0$, from Propositions 2.10, 2.12 and 2.19, $\mathcal{T}_r$ satifies $y > 0$ and $S_w < \infty$; and any solution y has at most one zero, and y, Y are monotone near $-\infty$ and near $\ln S_w$. From Proposition 2.13, any trajectory $\mathcal{T}$ converges to $(0,0)$ at $-\infty$; and apart from $\mathcal{T}_r$, it is tangent to the axis $y = 0$. If $y > 0$ near $-\infty$, and $N \geq 2$, then $\mathcal{T}$ starts in $\mathcal{Q}_1$, since $\lim_{\tau \to -\infty} \zeta = \eta > 0$; if $N = 1$, then $\lim_{r \to 0} w = a \geq 0$, and $\lim_{r \to 0} w' = b$, and $\mathcal{T}$ starts in $\mathcal{Q}_1$ if $b < 0$, in $\mathcal{Q}_4$ if $b > 0$ (in particular when $a = 0$).

• For any $P = (\varphi, 0), \varphi > 0$, then $\mathcal{T}_{[P]}$ satisfies $y > 0$ on $\mathbb{R}$, and from Remark 2.2, $\mathcal{T}_{[P]}$ stays in $\mathcal{Q}_4$ after P, because it cannot meet $\mathcal{T}_r$, thus $S_w < \infty$, and it stays in $\mathcal{Q}_1$ before P, and w is of type (2). In the same way for any $P = (0, \xi), \xi > 0$, then $\mathcal{T}_{[P]}$ stays in $\mathcal{Q}_2$ after P, since it cannot meet $-\mathcal{T}_r$, thus $S_w < \infty$, and it stays in $\mathcal{Q}_1$ before P, and w is of type (3).

• Next consider the sets

$$\mathcal{A} = \left\{ P \in \mathcal{Q}_1 : \mathcal{T}_{[P]} \cap \{(\varphi, 0) : \varphi > 0\} \neq \emptyset \right\}, \qquad \mathcal{B} = \left\{ P \in \mathcal{Q}_1 : \mathcal{T}_{[P]} \cap \{(0, \xi) : \xi > 0\} \neq \emptyset \right\}.$$

From above, they are nonempty, and open, thus $\mathcal{A} \cup \mathcal{B} \neq \mathcal{Q}_1$. Then there exists a trajectory $\mathcal{T}_1$ starting from $(0,0)$ and staying in $\mathcal{Q}_1$. From Proposition 2.13, necessarily $\lim_{\tau \to \infty} y = \infty$ and $\lim_{\tau \to \infty} \zeta = \alpha > 0$, thus w is of type (1) from Proposition 2.14.

• Finally we describe all the other trajectories $\mathcal{T}_{[P]}$ with one point P in the domain $\mathcal{R}$ above $\mathcal{T}_r \cup (-\mathcal{T}_r)$. If P is in the domain delimitated by $\mathcal{T}_r, \mathcal{T}_1$, then w is still of the type (2). If P is in the domain delimitated by $-\mathcal{T}_r, \mathcal{T}_1$, then either y has a zero, and w is of type (3), or $N = 1$, $y < 0$ and $-w$ is of type (2). Up to a symmetry, all the solutions are obtained.

(ii) Here $\alpha = \delta = N$ ((see fig XXI). Since $\alpha = N$ (1.7) holds, and the regular solutions, relative to $C = 0$ are given by (1.8). Since $\delta = N$, (1.7) is equivalent to $y + Y \equiv C$, from (2.12). For any $k \in \mathbb{R}$, $(y, Y) \equiv P_k = (k, |Nk|^{p-2}\,Nk)$ is a solution of system (**S**), located on the curve $\mathcal{M}$, thus $w(r) = kr^{-N}$ is a solution of ($\mathbf{E}_w$). Any solution has at most one zero, from Proposition 2.10. From Propositions 2.13, and 2.15, any other trajectory converges to a point $P_k \in \mathcal{M}$ at ∞, and $S_w < \infty$. There exists trajectories such that y has a constant sign, and other ones such that y has one zero. All the solutions are obtained. ■

Next we suppose $\alpha < 0$, and distinguish the cases $N \geq 2$ and $N = 1$.

Theorem 6.2 *Suppose $\varepsilon = -1$ and $\alpha < 0 < 2 \leq N \leq \delta$. Then any solution has a finite number of zeros. The regular solutions have at least one zero, and precisely one if $-p' \leq \alpha$. Any solution has at least one zero, and any nonregular one satisfies $\lim_{r\to 0} r^{\eta} w = c \neq 0$. Moreover*

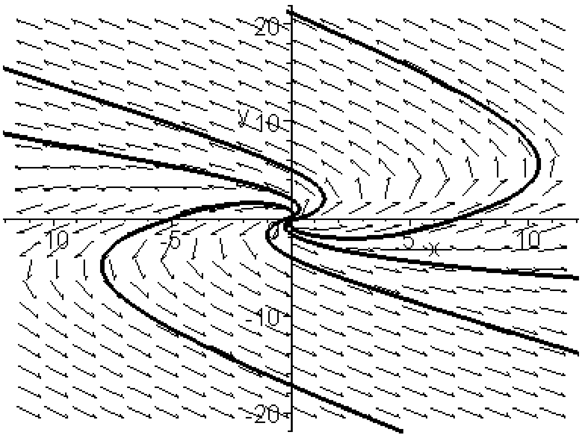
(i) If $-p' < \alpha$, then the regular solutions have a reduced domain ($S_w < \infty$); and there exist (exhaustively)

(1) solutions with two zeros and $S_w < \infty$;

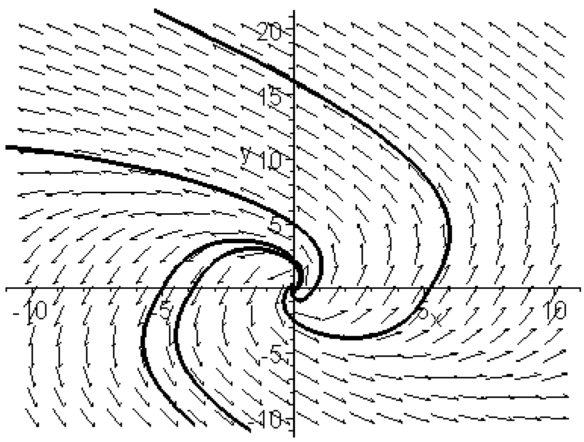
(2) solutions with one zero and $\lim_{r\to\infty} r^{\alpha} w = L \neq 0$;

(3) solutions with one zero and $S_w < \infty$.

(ii) If $\alpha = -p'$, the regular solutions satisfy $\lim_{r\to\infty} r^{\alpha} w = L \neq 0$. The other solutions are of type (1).



th 6.2,figXXII: $\varepsilon = -1$,
$-p' = -3 < \alpha = -2.5 < 0 < N = 2 < \delta = 3$



th 6.2figXXIII: $\varepsilon = -1$,
$-p' = -3 = \alpha < 0 < N = 2 < \delta = 3$

Proof. From Proposition 2.13, any trajectory converges necessarily to $(0,0)$ at $-\infty$, and apart from $\mathcal{T}_r$, it is tangent to the axis $y=0$. Any solution y has a finite number of zeros, and y is monotone near $-\infty$, and near S_w (finite or not), from Propositions 2.12 and 2.16, since $\delta > N/2$. And either $S_w < \infty$, thus $\lim_{\tau\to\ln S_w} Y/y = -1$, or $S_w = \infty$ and $\lim_{\tau\to\infty}\zeta = \alpha < 0$. In any case (y,Y) is in $\mathcal{Q}_2$ or $\mathcal{Q}_4$ for large τ. From Proposition 2.19, $\mathcal{T}_r$ has at least one zero, and starts in $\mathcal{Q}_1$. Since $N \geq 2$, any trajectory $\mathcal{T} \neq \pm\mathcal{T}_r$ satisfies $\lim_{\tau\to-\infty}\zeta = \eta > 0$. Thus it starts in $\mathcal{Q}_1$ (or $\mathcal{Q}_3$), and has at least one zero. Any trajectory $\mathcal{T}$ starting in $\mathcal{Q}_1$ enters $\mathcal{Q}_2$, from Remark 2.2. And $y' = \delta y - Y^{1/(p-1)}$, thus y decreases as long as $\mathcal{T}$ stays in $\mathcal{Q}_2$. Then either it enters $\mathcal{Q}_3$, thus necessarily $\mathcal{Q}_4$, and y has at least two zeros; or it stays in $\mathcal{Q}_2$, and either $S_w < \infty$, thus $\lim_{\tau\to\ln S_w} Y/y = -1$, or $S_w = \infty$ and $\lim_{\tau\to\infty}\zeta = \alpha$.

(i) Suppose $-p' < \alpha$ (see fig XXII). Then $\mathcal{T}_r$ has precisely one zero, from Proposition 2.19, thus it stays in $\mathcal{Q}_2$, and $S_w < \infty$, $\lim_{\tau\to\ln S_w} Y/y = -1$. Any other solution has at most two zeros, because the trajectory does not meet $\pm\mathcal{T}_r$. Recall that the function Y_α defined by (2.3) with $d = \alpha$ has only minimal points on the sets where it is positive, from Remark 2.11. From Proposition 2.19, $\mathcal{T}_r$ satisfies

$$Y_\alpha' = -(p-1)(\eta-\alpha)Y_\alpha + e^{(p-(2-p)\alpha)\tau}(Y_\alpha^{1/(p-1)} - \alpha y_\alpha) > 0,$$

which is equivalent to

$$Y^{1/(p-1)} - (p-1)(\eta-\alpha)Y > \alpha y. \tag{6.1}$$

And $\mathcal{T}_r$ stays strictly at the right of the curve

$$\mathcal{N}_\alpha = \left\{(y,Y) \in \mathbb{R}\times(0,\infty) : \alpha y = Y^{1/(p-1)} - (p-1)(\eta-\alpha)Y\right\}, \tag{6.2}$$

which intersects the axis $y=0$ at points $(0,0)$ and $(0,(p-1)(\eta-\alpha))$.

• For any $\bar{P} = (\varphi,0), \varphi < 0$, the trajectory $\mathcal{T}_{[\bar{P}]}$ enters $\mathcal{Q}_3$ after $\bar{P}$, from Remark 2.2; the solution passing through $\bar{P}$ at $\tau = 0$ satisfies and $Y_\alpha(0) = 0$, thus Y_α stays positive for $\tau < 0$, and $Y_\alpha'(\tau) < 0$, since it has no maximal point. Thus $\mathcal{T}_{[\bar{P}]}$ stays in $\mathcal{Q}_1 \cup \mathcal{Q}_2$ before P, at the left of $\mathcal{N}_\alpha$, and starts from $(0,0)$ in $\mathcal{Q}_1$, and ends up in $\mathcal{Q}_4$, thus y has two zeros. If $S_w = \infty$ then $\lim_{\tau\to\infty}|y| = \infty$, $\lim_{\tau\to\infty}\zeta = \alpha < 0$; it is impossible, becasue $\mathcal{T}_{[\bar{P}]}$ does not meet $-\mathcal{T}_r$; thus $S_w < \infty$, and w is of type (1).

• Next consider the trajectory $\mathcal{T}_{[P]}$, for any $P = (\varphi,\xi) \in \mathcal{N}_\alpha, \varphi \leq 0$. The solution going through P at $\tau = 0$ satisfies and $Y_\alpha'(0) = 0$, $Y_\alpha(0) > 0$, and 0 is a minimal point, thus $Y_\alpha''(0) > 0$. Indeed if $Y_\alpha''(0) = 0$, then from uniqueness, Y_α is constant on $\mathbb{R}$; in turn $Y_\alpha \equiv 0$, from (2.6), since $\alpha \neq -p'$, which is false. Then $Y_\alpha'(\tau) > 0$ for $\tau > 0$, $Y_\alpha'(\tau) < 0$ for $\tau < 0$. Thus $\mathcal{T}_{[P]}$ stays in $\mathcal{Q}_1 \cup \mathcal{Q}_2$, at the right of $\mathcal{N}_\alpha$ after P, with $y < 0$ from Remark 2.2, at its left before, and converges to $(0,0)$ at $-\infty$ in $\mathcal{Q}_1$. Suppose that $S_w = \infty$. Then $\lim_{\tau\to\infty}|y| = \infty$, $\lim_{\tau\to\infty}\zeta = \alpha$, and $\lim_{\tau\to\infty} y_\alpha = L < 0$ from Proposition 2.14, thus $\lim_{\tau\to\infty} Y_\alpha = (\alpha L)^{p-1}$. As in Proposition 2.19, one finds $Y_\alpha''(\tau) > 0$ for any $\tau > 0$, which is impossible. Then $\mathcal{T}_{[P]}$ satisfies $S_w < \infty$, thus $\lim_{\tau \to \ln S_w} Y/y = -1$. And the corresponding w is of type (3).

- Finally let $\mathcal{R}$ be the domain of $\mathcal{Q}_1 \cup \mathcal{Q}_2$ delimitated by $\mathcal{T}_r$ and containing $\mathcal{N}_\alpha$, and the sets

$$\mathcal{A}= \left\{P \in \mathcal{R} : \mathcal{T}_{[P]} \cap \{(\varphi, 0) : \varphi < 0\} \neq \emptyset\right\}, \qquad \mathcal{B}= \left\{P \in \mathcal{R} : \mathcal{T}_{[P]} \cap \mathcal{N}_\alpha \neq \emptyset\right\}, \tag{6.3}$$

corresponding to the trajectories of type (1) or (3). Then $\mathcal{A}, \mathcal{B}$ are nonempty, and open: here again the intersection with $\mathcal{N}_\alpha$ is transverse, because $\alpha \neq -p'$. Thus $\mathcal{A} \cup \mathcal{B} \neq \mathcal{R}$. There exists a trajectory in $\mathcal{R}$ which does not meet $\mathcal{N}_\alpha$, starting from $(0,0)$ in $\mathcal{Q}_1$ and ending up in $\mathcal{Q}_2$. It cannot satisfy $\lim_{\tau \to \ln S_w} Y/y = -1$, thus $S_w = \infty$ and $\lim_{\tau \to \infty} \zeta = \alpha$, thus w is of type (2).

(ii) Suppose $\alpha = -p'$ (see fig XXIII). The regular solutions are given by (1.10), they have one zero, but $S_w = \infty$ and $\lim_{\tau \to \infty} \zeta = \alpha$. They satisfy $Y_{-p'} \equiv C$, thus $Y'_{-p'} \equiv 0$, thus $\mathcal{T}_r = \mathcal{M}_{-p'}$. Consider $\mathcal{T}_{[\bar{P}]}$; the solution passing through $\bar{P}$ at $\tau = 0$ satisfies and $Y_{-p'}(0) = 0$, thus $Y_{-p'}$ stays negative for $\tau > 0$ and $Y'_{-p'} < 0$. Suppose that $S_w = \infty$, then $\lim_{\tau \to \infty} y_\alpha = L > 0$, $\lim_{\tau \to \infty} Y_\alpha = -(|\alpha| L)^{p-1}$. But as at (2.49), $Y''_\alpha(\tau) < 0$ for any $\tau > 0$, which leads to a contradiction. Thus $S_w < \infty$, and w is of type (1). Finally suppose that there exists a trajectory $\mathcal{T} \neq \mathcal{T}_r$ staying in $\mathcal{Q}_1 \cup \mathcal{Q}_2$. Then $Y_\alpha > 0$, $\lim_{\tau \to \infty} Y_\alpha = 0$, and it cannot meet $\mathcal{T}_r$, thus $S_w = \infty$, and $\lim_{\tau \to -\infty} Y_\alpha = \infty$, $\lim_{\tau \to \infty} Y_\alpha = C > 0$. As in Proposition 2.19, it is impossible. Thus there does not exist solution of type (2) or (3). ■

Theorem 6.3 *Suppose $\varepsilon = -1$ and $\alpha < 0 < N = 1 < \delta$. Then any solution has still a finite number of zeros. The regular solutions have at least one zero, and precisely one if $-p' \leq \alpha$. Moreover*

(i) If $-1 < \alpha < 0$, then the regular solutions a reduced domain ($S_w < \infty$), and

(1) the solutions with $\lim_{r \to 0} w = a > 0, \lim_{r \to 0} w' = b < 0$ have one zero and $S_w < \infty$;

(2) the solutions with $\lim_{r \to 0} w = 0, \lim_{r \to 0} w' = b > 0$ are positive and $S_w < \infty$;

(3) there exist solutions with one zero and $\lim_{r \to 0} w = a > 0, \lim_{r \to 0} w' = b > 0$ and $S_w < \infty$;

(4) there exist positive solutions with $\lim_{r \to 0} w = a > 0, \lim_{r \to 0} w' = b > 0$ and $S_w < \infty$;

(5) for any $a > 0$ there exists $b = b(a) > 0$ such that w is positive and $\lim_{r \to \infty} r^\alpha w = L \neq 0$.

(ii) If $\alpha = -1$, for any $b > 0$, $w \equiv br$ is a solution. The other solutions such that $\lim_{r \to 0} w \neq 0$ have one zero, and $S_w < \infty$.

(iii) If $-p' < \alpha < -1$, then

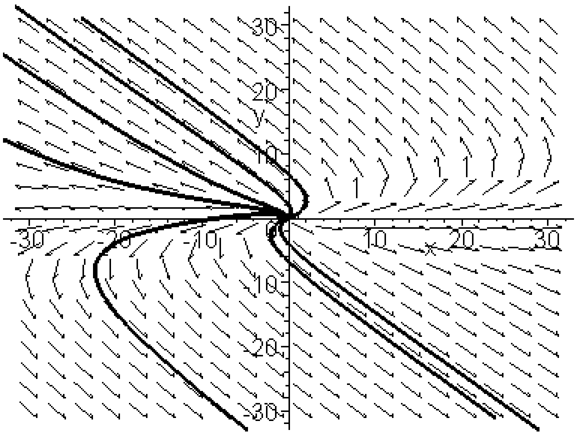
(6) there exist solutions with one zero, and $\lim_{r \to 0} w = a > 0, \lim_{r \to 0} w' = b < 0$, and $S_w < \infty$;

(7) the solutions with $\lim_{r \to 0} w = 0, \lim_{r \to 0} w' = b > 0$ have one zero and $S_w < \infty$;

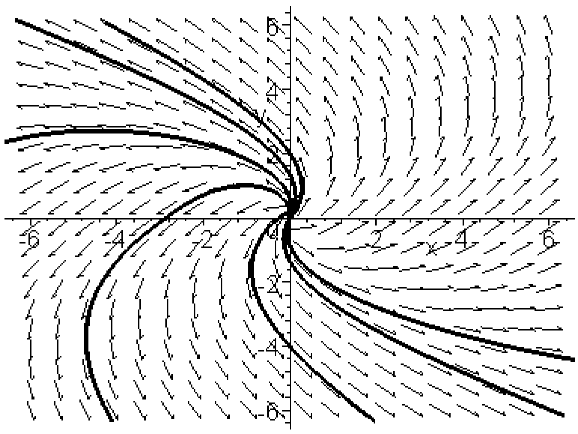
(8) there exist solutions with one zero, and $\lim_{r \to 0} w = a > 0, \lim_{r \to 0} w' = b > 0$ and $S_w < \infty$;

(9) there exist solutions with $\lim_{r \to 0} w = a > 0, \lim_{r \to 0} w' = b < 0$, two zeros and $S_w < \infty$;

(10) for any $a > 0$ there exists $b = b(a) > 0$ such that w has one zero and $\lim_{r \to \infty} r^\alpha w = L \neq 0$.



th 6.3,figXXIV: $\varepsilon = -1$,
$-1 < \alpha = -0.5 < N = 1 < \delta = 3$



th 6.3,figXXV: $\varepsilon = -1$,
$-p' = -3 < \alpha = -2.9 < -1 < N = 1 < \delta = 3$

Proof. The case $N = 1$ is still more complex, since some trajectories start in $\mathcal{Q}_2$ (or $\mathcal{Q}_4$), corresponding to the solutions such that $\lim_{r\to 0} w = a$ and $\lim_{r\to 0} w' = b$, with $b \neq 0$, $ab \geq 0$. Any solution has still a finite number of zeros, from Proposition 2.16.

(i) Suppose $-1 < \alpha < 0$ (see fig XXIV) From Proposition 2.10, any solution has at most one zero, thus the regular ones have precisely one zero. Thus $\mathcal{T}_r$ meets the axix $y = 0$ at some point $(0, \xi_r)$.

• Consider the trajectory $\mathcal{T}_s$ such that $\lim_{r\to 0} w = 0$ and $\lim_{r\to 0} w' = b < 0$, that means $\lim_{\tau\to-\infty} \zeta = \eta = -1$, starting from $(0,0)$ in $\mathcal{Q}_2$, thus $w < 0$ near 0. For any $d \in (-1, \alpha)$, the function y_d satisfies $y_d(\tau) = be^{(d+1)\tau}(1 + o(1))$ near $-\infty$, thus $\lim_{\tau\to-\infty} y_d = 0$. Then y_d has no zero, because $|y_d|$ has no maximal point from (2.14); thus $\mathcal{T}_s$ stays in $\mathcal{Q}_2$. Moreover if $\mathcal{T}_s$ satisfies $S_w = \infty$, then $\lim_{\tau\to\infty} y_\alpha = L < 0$, thus $\lim_{\tau\to\infty} y_d = 0$, which is impossible; thus w is of type (2). And $\mathcal{T}_r$ satisfies $S_w < \infty$, since it cannot meet $\mathcal{T}_s$.

• For any $\bar{P} = (\varphi, 0), \varphi < 0$, the trajectory $\mathcal{T}_{[\bar{P}]}$ does not meet $\mathcal{T}_s$, thus converges to $(0,0)$ at $-\infty$ in $\mathcal{Q}_2$; then $\lim_{r\to 0}(-w) = a > 0$ and $\lim_{r\to 0}(-w)' = b > 0$, and $\mathcal{T}_{[\bar{P}]}$ ends up in $\mathcal{Q}_4$; thus y has one zero and $-w$ is of type (3).

• For any $P = (0, \xi)$, $\xi \in (0, \xi_r)$, $\mathcal{T}_{[P]}$ has one zero, and converges to $(0,0)$ at $-\infty$ in $\mathcal{Q}_1$, hence $\lim_{r\to 0} w = a > 0$ and $\lim_{r\to 0} w' = b < 0$; and $\mathcal{T}_{[P]}$ cannot meet $\mathcal{T}_s$, thus $S_w < \infty$, and w is of type (1). Reciprocally any solution such that $\lim_{r\to 0} w = a > 0$ and $\lim_{r\to 0} w' = b < 0$ has one zero and $S_w < \infty$.

• Next consider any trajectory $\mathcal{T}$ such that $\lim_{r\to 0}(-w) = a > 0$ and $\lim_{r\to 0}(-w') = b > 0$, thus starting in $\mathcal{Q}_2$ under $\mathcal{T}_s$. Then $\zeta(\tau) = -(b/a)e^\tau(1 + o(1)$ near $-\infty$, thus $\lim_{\tau\to-\infty} \zeta = 0$. If ζ

has an extremal point θ, then from (2.18),

$$p-1)\zeta''(\theta) = (2-p)(\zeta-\alpha)(\delta-\zeta)\,|\zeta y|^{2-p};$$

thus it is a minimal point if $\zeta(\theta) > \alpha$, or a maximal one if $\zeta(\theta) < \alpha$; in case of equality, then $\zeta \equiv \alpha$, which is impossible. As a result, either ζ has a first zero τ_1 and $\alpha < \zeta(\tau) < 0$ for $\tau < \tau_1$, and $\mathcal{T}$ is one of the $\mathcal{T}_{[\bar{P}]}$. Or ζ stays negative; if $S_w = \infty$, then $\lim_{\tau\to\infty}\zeta = \alpha$; in that case ζ is necessarily decreasing, thus $\alpha < \zeta(\tau) < 0$ for any τ. In both cases, $\mathcal{T}$ stays under the curve

$$\mathcal{M}' = \left\{(y,Y) \in \mathbb{R}\times(0,\infty) : \alpha y = Y^{1/(p-1)}\right\},$$

as long as it is in $\mathcal{Q}_2$. As a consequence, for any $P \in \mathcal{Q}_2$ such that P is on or above $\mathcal{M}'$, the trajectory $\mathcal{T}_{[P]}$ satisfies $S_w < \infty$; in particular on finds again $\mathcal{T}_s$. For any P between $\mathcal{M}'$ and $\mathcal{T}_s$, the solution has a constant sign, $\mathcal{T}_{[P]}$ converges to $(0,0)$ at $-\infty$ and $\lim_{r\to0}(-w) = a > 0$ and $\lim_{r\to0}(-w') = b > 0$, and $\lim_{\tau\to\ln S_w} Y/y = -1$, thus $\mathcal{T}_{[P]}$ meets $\mathcal{M}_\alpha$; and $-w$ is of type (4).

• Finally let $\mathcal{R}_1$ be the domain of $\mathcal{Q}_2$ delimitated by $\mathcal{T}_s$ and the axis $Y=0$, and let

$$\mathcal{A}_1 = \left\{P \in \mathcal{R}_1 : \mathcal{T}_{[P]} \cap \{(\varphi,0) : \varphi < 0\} \neq \emptyset\right\}, \qquad \mathcal{B}_1 = \left\{P \in \mathcal{R}_1 : \mathcal{T}_{[P]} \cap \mathcal{N}_\alpha \neq \emptyset\right\}.$$

They are open, since again the intersection is transverse, because $\alpha \neq -1$. They are nonempty, thus $\mathcal{A}_1 \cup \mathcal{B}_1 \neq \mathcal{R}_1$, and there exists a trajectory such that y is defined on $\mathbb{R}$ and $\lim_{\tau\to\infty}\zeta = \alpha$. By scaling, for any $a > 0$ there is at least a $b = b(a)$ such that the corresponding w has a constant sign and $\lim_{r\to\infty} r^\alpha w = L \neq 0$, and $|w|$ is of type (5).

(ii) Suppose $\alpha = -1$, then $\mathcal{T}_s$ is given explicitely by $w \equiv br$, thus $Y \equiv -y^{p-1}$, or equivalently $Y_{-1} \equiv b$, thus $\mathcal{T}_s = \mathcal{N}_{-1}$. For any other solution, one finds $Y_{-1}'' = Y_{-1}'\left(1 + e^{2\tau}\,|Y_{-1}|^{(2-p)/(p-1)}\right)$, thus Y_{-1} is strictly monotone, from uniqueness, and Y_{-1}'' has the sign of Y_{-1}'. Any trajectory such that $\lim_{r\to0} w = a > 0$ and $\lim_{r\to0} w' = b < 0$, starting in $\mathcal{Q}_1$ satisfies $Y_{-1}' > 0$ and Y_{-1} is convex. Then Y_{-1} cannot have a finite limit, thus $S_w < \infty$, and the trajectory ends up in $\mathcal{Q}_2$, thus y has a zero. Any trajectory such that $\lim_{r\to0}(-w) = a > 0$ and $\lim_{r\to0}(-w)' = b > 0$, starting in $\mathcal{Q}_2$ satisfies $Y_{-1}' < 0$ and thus Y_{-1} has a zero, and the trajectory ends up in $\mathcal{Q}_4$. Then apart from $\mathcal{T}_s$, all the trajectories satisfy $S_w < \infty$, and y has one zero.

(iii) Suppose $-p' < \alpha < -1$ (see fig XXV). Then $\mathcal{T}_r$ starts in $\mathcal{Q}_1$, y has one zero from Proposition 2.19, and $\mathcal{T}_r$ ends up in $\mathcal{Q}_2$ and $S_w < \infty$. Any solution has at most two zeros.

• Consider $\mathcal{T}_s$: we claim that it cannot stay in $\mathcal{Q}_2$. Suppose that it stays in it, thus $y < 0 < Y$. Then $\zeta < 0$, and $\lim_{\tau\to-\infty}\zeta = \eta = -1$, and ζ is monotone near $-\infty$; if $\zeta' \leq 0$, then $\zeta \leq -1$ near $-\infty$, and we reach a contradiction from (2.9). Then $\zeta' \geq 0$ near $-\infty$; but any extremal point of ζ is a minimal point, from (2.18), thus ζ stays increasing, is defined on $\mathbb{R}$ and has a limit $\lambda \in [-1,0]$; but $\lambda = \alpha$, from Proposition 2.13, hence again a contradiction holds. Then $\mathcal{T}_s$ enters $\mathcal{Q}_3$ at some point $(\varphi_s, 0), \varphi_s < 0$, enters $\mathcal{Q}_4$, and y has precisely one zero; and w is of type (7).

• Any solution such that $\lim_{r\to 0}(-w) = a > 0$ and $\lim_{r\to 0}(-w)' = b > 0$, has also one zero, since its trajectory stays under $\mathcal{T}_s$ in $\mathcal{Q}_2$, and w is of type (8).

• As in the case $N \geq 2$, for any $P = (\varphi, \xi) \in \mathcal{N}_\alpha, \varphi \leq 0$, $\mathcal{T}_{[P]}$ stays in $\mathcal{Q}_1 \cup \mathcal{Q}_2$ and $S_w < \infty$. In particular for $P_0 = (0, \xi_0)$, $\xi_0 = ((p-1)(-1-\alpha))^{(p-1)/(2-p)}$, the trajectory $\mathcal{T}_{[P_0]}$ starts from $\mathcal{Q}_1$, thus $\lim_{r\to 0} w = a > 0, \lim_{r\to 0} w' = b_0(a) > 0$, and w has one zero, and $S_w < \infty$, and w is of type (6).

• Considering the sets $\mathcal{A}, \mathcal{B}$ defined at (6.3), they are still open, and $\mathcal{B}$ contains $\mathcal{T}_{[P_0]}$. And $\mathcal{A}$ contains $\mathcal{T}_s$, thus also any $\mathcal{T}_{[P]}$ such that $P = (\varphi, 0)$ with $\varphi < \varphi_s$. Such a trajectory satisfies $\lim_{r\to 0} w = a > 0$ and $\lim_{r\to 0} w' = b < 0$, and w is of type (9). Moreover $\mathcal{A} \cup \mathcal{B} \neq \mathcal{R}$, thus for any $a > 0$ there is a $b = b(a) < 0$ such that the corresponding w has one zero and $\lim_{r\to\infty} r^\alpha w = L \neq 0$, and w is of type (10). ■

6.2 Subcase $\alpha < \delta < N$

As in the case $\varepsilon = 1$, $\delta < \min(\alpha, N)$ of Section 5.2, here two kinds of periodic trajectories can appear, and the study is delicate. Here also $N \geq 2$, and we still have three stationary points, and $(0,0)$ is a saddle point. And M_ℓ is a source when $N/2 \leq \delta$ or $\delta < N/2$ and $\alpha^* < \alpha$, and a sink when $\delta < N/2$ and $\alpha < \alpha^*$; notice that $\alpha^* < -p' < 0$ from (2.35). Also M_ℓ is a node point whenever $\alpha \leq \alpha_1$, or $\alpha_2 \leq \alpha$, where α_1, α_2 are defined at (2.51), and α_2 can be greater or less than $-p'$. We begin by the simplest case where $\alpha > 0$.

Theorem 6.4 *Assume $\varepsilon = -1$ and $0 < \alpha < \delta < N$. Then the regular solutions have a constant sign and a reduced domain ($S_w < \infty$). And $w \equiv \ell r^{-\delta}$ is a solution. Moreover there exist solutions such that*

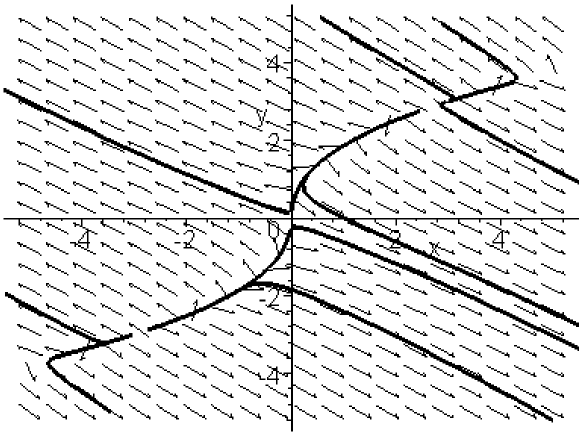
(1) w is positive, $\lim_{r\to 0} r^\delta w = \ell$ and $S_w < \infty$;

(2) w has one zero, $\lim_{r\to 0} r^\delta w = \ell$ and $S_w < \infty$;

(3) w is positive, $\lim_{r\to 0} r^\delta w = \ell$ and $\lim_{r\to\infty} r^\eta w = c > 0$;

(4) w is positive, $\lim_{r\to 0} r^\delta w = \ell$ and $\lim_{r\to\infty} r^\alpha w = L > 0$.

Up to a symmetry, all the solutions are described.



th 6.4,figXXVI: $\varepsilon = -1, 0 < \alpha = 2 < \delta = 3 < N = 4$

Proof. Since $\alpha > 0$, the regular solutions have a constant sign and satisfy $S_w < \infty$, from Propositions 2.10 and 2.19. Here $\mathcal{T}_r$ starts in $\mathcal{Q}_4$ and stays in it, from Remark 2.6 ((see fig XXVI). Any solution has at most one zero from Proposition 2.10. The point M_ℓ is a source, and a node point, from Remark 2.22, and $0 < \lambda_1 < \delta < \lambda_2$. The eigenvectors $u_1 = (\nu(\alpha), \lambda_1 - \delta)$ and $u_2 = (-\nu(\alpha), \delta - \lambda_2)$ form a direct basis, where now $\nu(\alpha) < 0$; thus u_1 points to $\mathcal{Q}_3$, u_2 points to $\mathcal{Q}_4$. There exist two particular trajectories $\mathcal{T}_1, \mathcal{T}_2$ starting from M_ℓ at $-\infty$, with respective tangent vectors u_2, and $-u_2$. All the other trajectories $\mathcal{T}$ which converge to M_ℓ at $-\infty$ have the direction of u_1; and y is monotone at the extremities, from Proposition 2.12, since $\mathcal{T}$ cannot meet $\mathcal{T}_1, \mathcal{T}_2$.

• First consider $\mathcal{T}_1$. The function y is nondecreasing near $-\infty$, and stays nondecreasing as long as $\mathcal{T}_1$ stays in $\mathcal{Q}_1$. Indeed Y is nonincreasing near $-\infty$, thus $Y(\tau) < (\delta\ell)^{p-1}$. If y has a maximal point τ, then $y(\tau) > \ell$ from (2.16), and $Y^{1/(p-1)} = \delta y$, thus $Y(\tau) > (\delta\ell)^{p-1}$, thus Y has a minimal point τ_1 in $\mathcal{Q}_1$; then $Y(\tau_1) < (\delta\ell)^{p-1}$ from $(\mathbf{E}_Y)$; and $Y'(\tau_1) = 0$, thus $\alpha\ell < \alpha y(\tau_1) < (N-\delta)\, \alpha Y(\tau_1)/(\delta - \alpha)$, which is contradictory. If $\mathcal{T}_1$ stays in $\mathcal{Q}_1$, then $\lim_{\tau \to -\infty} \zeta = \alpha > 0$ from Proposition 2.13, which is also contradictory. Thus $\mathcal{T}_1$ enters $\mathcal{Q}_4$ at some point $(\varphi_1, 0)$ and stays in it, does not meet $\mathcal{T}_r$, and thus $S_w < \infty$, and w is of type (1).

• Next consider $\mathcal{T}_2$. Near $-\infty$, the function Y is nondecreasing, and y nonincreasing, and y is monotone as long as $y > 0$: if there exists a minimal point τ, then $y(\tau) > \ell$ from (2.16). And Y is nondecreasing as long as $Y > 0$: if Y has a maximal point τ, $Y(\tau) > (\delta\ell)^{p-1}$ from $(\mathbf{E}_Y)$; and $\alpha\ell > \alpha y(\tau) > (N-\delta)\, \alpha Y(\tau)\delta - \alpha)$, which is still impossible. Thus $\mathcal{T}_2$ cannot stay in $\mathcal{Q}_1$, it enters $\mathcal{Q}_2$ at some point $(0, \xi_2)$ and does not meet $-\mathcal{T}_r$, then it stays in $\mathcal{Q}_2$, hence $S_w < \infty$, and w is of type (2).

• There also exists a unique trajectory $\mathcal{T}_3$ converging to $(0,0)$ at ∞, ending up in $\mathcal{Q}_1$, since $(0,0)$ is a saddle point. It stays in the domain of $\mathcal{Q}_1$ delimitated by $\mathcal{T}_1, \mathcal{T}_2$, because $\mathcal{Q}_1$ is negatively invariant. Thus $\mathcal{T}_3$ converges to M_ℓ at $-\infty$, tangentially to u_1. And y is increasing on $\mathbb{R}$: indeed $y' < 0$ near $\pm\infty$, and y cannot have two extremal points. Then w is of type (3).

• For any point $P = (\varphi, 0)\,, \varphi > \varphi_1$, the trajectory $\mathcal{T}_{[P]}$ goes from $\mathcal{Q}_1$ into $\mathcal{Q}_4$, from Remark 2.2. It does not meet $\mathcal{T}_r, \mathcal{T}_1$, thus it stays in $\mathcal{Q}_4$ after P, and $S_w < \infty$. Before P, it it stays in $\mathcal{Q}_1$ because it does not meet $\mathcal{T}_1$ or $\mathcal{T}_2$, and from Remark 2.2. From Proposition 2.13, either $\lim_{\tau\to-\infty} \zeta = \alpha < \delta$, thus $y' = y(\delta - \zeta) > 0$ near $-\infty$, and $\lim_{\tau\to-\infty} y = \infty$, which is impossible; or (necessarily) $\mathcal{T}_{[P]}$ converges to M_ℓ, tangentially to u_1, and $\mathcal{T}_{[P]}$ is of type (2). Similarly, for any $P' = (0, \xi), \xi > \xi_2$, the trajectory $\mathcal{T}_{[P']}$ goes from $\mathcal{Q}_1$ into $\mathcal{Q}_2$ and stays in $\mathcal{Q}_2$ after P, thus $S_w < \infty$, and in $\mathcal{Q}_1$ before P and converges to M_ℓ, tangentially to $-u_1$, at $-\infty$; and w is still of type (2).

• The sets

$$\mathcal{A}=\left\{P \in \mathcal{Q}_1 : \mathcal{T}_{[P]} \cap \{(\varphi, 0) : \varphi > 0\} \neq \emptyset\right\}, \qquad \mathcal{B}=\left\{P \in \mathcal{Q}_1 : \mathcal{T}_{[P]} \cap \{(0, \xi) : \xi > 0\} \neq \emptyset\right\},$$

are open, nonempty, thus $\mathcal{A} \cup \mathcal{B} \neq \mathcal{Q}_1$. There exists at least a trajectory $\mathcal{T}_4$ in $\mathcal{Q}_1$, as above converging to M_ℓ at $-\infty$, and such that $\lim_{\tau\to\infty} \zeta = \alpha$, and w is of type (4).

• For any point P in the bounded domain $\mathcal{R}'$ of $\mathcal{Q}_1$ delimitated by $\mathcal{T}_2, \mathcal{T}_3$, the trajectory $\mathcal{T}_{[P]}$ is confined in $\mathcal{R}'$ before P, and y has no maximal point, thus y is monotone, and $\mathcal{T}$ converge to M_ℓ at $-\infty$. It cannot stay in $\mathcal{Q}_1$ since it cannot converge to $(0,0)$. Then it goes from $\mathcal{Q}_1$ into $\mathcal{Q}_2$ and stays in it, because it does not meet $-\mathcal{T}_r$. Thus $S_w < \infty$, and w is still of type (2).

• For any P in the domain of $\mathcal{Q}_1$ delimitated by $\mathcal{T}_1, \mathcal{T}_3$, the trajectory $\mathcal{T}_{[P]}$ converge to M_ℓ at $-\infty$, tangentially to u_1, enters $\mathcal{Q}_4$ and stays in it. Thus $S_w < \infty$ and w is of type (1). No trajectory can stay in $\mathcal{Q}_4(\mathcal{Q}_2)$ except $\mathcal{T}_r(-\mathcal{T}_r)$; thus all the solutions are described, up to a symmetry. ∎

Now we come to the case $\alpha < 0$, and discuss according to the sign of $\alpha - p'$. Here also we begin by some remarks on the phase plane. Observe that *the situation is different from the one of Section* 5.2, from Remarks 6.5,(i) and 5.5.

Remark 6.5 *Assume $\varepsilon = -1$ and $\alpha < 0$. Then*

(i) The regular trajectory $\mathcal{T}_r$ starts in $\mathcal{Q}_1$. There exists a unique trajectory $\mathcal{T}_s$ converging to $(0,0)$, in $\mathcal{Q}_1$ for large τ, with an infinite slope at $(0,0)$, and $\lim_{r\to 0} r^\eta w = c > 0$. Moreover if $\mathcal{T}_s$ does not stay in $\mathcal{Q}_1$, then $\mathcal{T}_r$ stay in it, and it is bounded and contained in the domain delimitated by $\mathcal{Q}_1 \cap \mathcal{T}_s$, from Remark 2.2. If $\mathcal{T}_r$ is homoclinic, it stays in $\mathcal{Q}_1$.

Reciprocally if $\mathcal{T}_s$ stays in $\mathcal{Q}_1$, and is not homoclinic, then $\mathcal{T}_r$ does not stay in $\mathcal{Q}_1$ (indeed $\mathcal{T}_s$ converges to M_ℓ at $-\infty$ or has a limit cycle around it; if $\mathcal{T}_r$ stays in $\mathcal{Q}_1$, either the corresponding y is increasing, thus $\lim_{\tau\to\ln S_w} Y/y = -1$, or $\lim_{\tau\to\infty} \zeta = \alpha < 0$, from Propositions 2.20 and 2.13, thus $\mathcal{T}_r$ enters $\mathcal{Q}_4$ and we are led to a contradiction; or y oscillates around ℓ near ∞, from Proposition 2.12, thus it meets $\mathcal{T}_s$, which is impossible).

(ii) Any trajectory $\mathcal{T}$ is bounded near $-\infty$ from Propositions 2.13 and 2.15. Any trajectory $\mathcal{T}$ bounded at $\pm\infty$ converges to $(0,0)$ or $\pm M_\ell$, or its limit set $\Gamma_\pm$ at $\pm\infty$ is a limit cycle; or $\mathcal{T}_r$ is homoclinic and $\Gamma_\pm = \overline{\mathcal{T}_r}$.

(iii) If there exists a limit cycle surrounding $(0,0)$*, then from (2.46) and (2.47), it also surrounds the points* $\pm M_\ell$*.*

Next we first study the case $-p' \leq \alpha$, where there is no cycle and no homoclinic orbit in $\mathcal{Q}_1$, from Theorem 2.25.

Theorem 6.6 *(i) Assume* $\varepsilon = -1$ *and* $-p' < \alpha < 0 < \delta < N$*. Then the regular solutions have precisely one zero, and* $S_w < \infty$*. And* $w \equiv \ell r^{-\delta}$ *is a solution. There exist solutions such that*

(1) w *is positive,* $\lim_{r\to 0} r^\delta w = \ell$ *and* $\lim_{r\to\infty} r^\eta w = c > 0$;

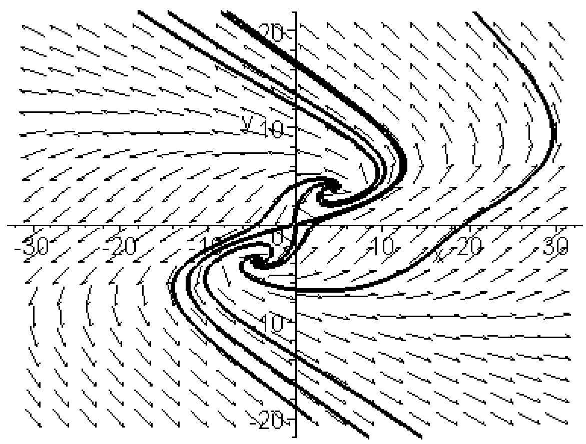
(2) w *has one zero,* $\lim_{r\to 0} r^\delta w = \ell$*, and* $\lim_{r\to\alpha} r^\alpha w = L < 0$;

(3) w *has one zero,* $\lim_{r\to 0} r^\delta w = \ell$*, and* $S_w < \infty$;

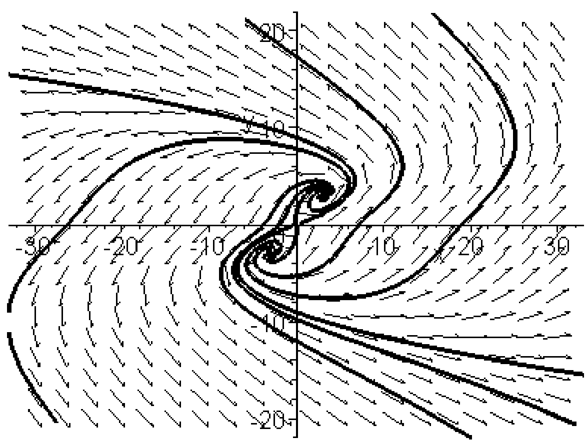
(4) w *has two zeros,* $\lim_{r\to 0} r^\delta w = \ell$*, and* $S_w < \infty$.

(ii) Assume $\alpha = -p'$*. Then the regular solutions, given by (1.10), have also one zero, and* $\lim_{r\to\alpha} r^\alpha w = L < 0$*, and there exist solutions of type (1) and (4).*

Up to a symmetry, all the solutions are described.



th 6.6,figXXVII: $\varepsilon = -1$,
$-p' = -3 < \alpha = -2 < \delta = 3 < N/2 < N = 9$



th 6.6,figXXVIII: $\varepsilon = -1$,
$-p' = -3 = \alpha < \delta = 3 < N/2 < N = 9$

Proof. (i) Assume $-p' < \alpha < 0$ (see fig XXVII). From Proposition 2.10, any solution y has at most two zeros, and Y has at most one zero.

• First consider $\mathcal{T}_s$. The function Y_α defined by (2.3) with $d = \alpha$ satisfies $Y_\alpha = O(e^{(\alpha-\eta)\tau})$ near ∞, thus $\lim_{\tau\to\infty} Y_\alpha = 0$. Then from Remark 2.11, Y_α is decreasing, thus $Y_\alpha > 0$, and $\mathcal{T}_s$ stays in

$\mathcal{Q}_1 \cup \mathcal{Q}_2$. In fact it stays in $\mathcal{Q}_1$, from Remark 2.2. From Propositions 2.13, 2.12, 2.16, and Theorem 2.25,$\mathcal{T}_s$ converges to M_ℓ at $-\infty$. Indeed if $\lim y = \infty$, then $\lim_{\tau\to\infty} \zeta = \alpha < 0$; if $S_w < \infty$, then $\lim Y/y = -1$; which contradicts $Y > 0$. Then w is of type (1).

• The trajectory $\mathcal{T}_r$ stays in $\mathcal{Q}_1 \cup \mathcal{Q}_2$, and y has precisely one zero, and $S_w < \infty$, thus $\lim_{\tau\to\ln S_w} Y/y = -1$. Now $\mathcal{T}_r$ cannot stay in $\mathcal{Q}_1$. Indeed it cannot converge to M_ℓ which is a source, or oscillate around $\mathcal{Q}_1$, becausee it does not meet $\mathcal{T}_s$, or tend to ∞, or satisfy $S_w < \infty$ with $Y > 0$. Then y has precisely one zero, $\mathcal{T}_r$ enters $\mathcal{Q}_2$ and stays in it. Moreover the corresponding Y_α satisfies $Y'_\alpha > 0$, or equivalently (6.1). Consider again the curve $\mathcal{N}_\alpha$ defined at (6.2). Here $\mathcal{T}_r$ stays strictly at the right of $\mathcal{N}_\alpha$, and $\mathcal{T}_s$ at the left of $\mathcal{N}_\alpha$.

• For any $\bar{P} = (\varphi, 0), \varphi < 0$, the trajectory $\mathcal{T}_{[\bar{P}]}$ enters $\mathcal{Q}_3$ after $\bar{P}$, from Remark 2.2. The solution going through $\bar{P}$ at $\tau = 0$ satisfies $Y_\alpha(0) = 0$; thus Y_α stays positive as before, and $Y'_\alpha < 0$, since Y_α has no maximal point, from Remark 2.11. Thus $\mathcal{T}_{[\bar{P}]}$ stays in $\mathcal{Q}_1 \cup \mathcal{Q}_2$ before $\bar{P}$, at the left of $\mathcal{N}_\alpha$. It cannot stay in $\mathcal{Q}_2$, from Propositions 2.12 and 2.13. As τ decreases, it enters $\mathcal{Q}_1$, and converges to M_ℓ, from Theorem 2.25. If $S_w = \infty$, then $\lim |y| = \infty$, $\lim_{\tau\to\infty} \zeta = \alpha < 0$; it is impossible, since $\mathcal{T}_{[\bar{P}]}$ does not meet $-\mathcal{T}_r$. Thus $S_w < \infty$, $\lim Y/y = -1$, $\mathcal{T}_{[\bar{P}]}$ goes from $\mathcal{Q}_3$ into $\mathcal{Q}_4$ and stays in it, and w is of type (4). The solution y has precisely two zeros.

• Next consider $\mathcal{T}_{[P]}$ for any $P = (\varphi, \xi) \in \mathcal{N}_\alpha$ with $\varphi < 0$. The solution passing through P at $\tau = 0$ satisfies and $Y'_\alpha(0) = 0$, $Y_\alpha(0) > 0$, and 0 is a minimal point, thus $Y''_\alpha(0) > 0$. Indeed if $Y''_\alpha(\tau) = 0$, then from uniqueness, Y_α is constant on $\mathbb{R}$; then from (2.6), $Y_\alpha \equiv 0$, since $\alpha \neq -p'$, which is false. Therefore $Y'_\alpha(\tau) > 0$ for $\tau > 0$, $Y'_\alpha(\tau) < 0$ for $\tau < 0$, thus $\mathcal{T}_{[P]}$ stays in $\mathcal{Q}_1 \cup \mathcal{Q}_2$, at the right of $\mathcal{N}_\alpha$ after P, with $y < 0$ from Remark 2.2, at its left before. As above it cannot stay in $\mathcal{Q}_2$ near $-\infty$, and converges to M_ℓ. Suppose that it satisfies $S_w = \infty$. Then $\lim |y| = \infty$, $\lim_{\tau\to\infty} \zeta = \alpha$, and $\lim_{\tau\to\infty} y_\alpha = L < 0$ from Proposition 2.14, thus $\lim_{\tau\to\infty} Y_\alpha = (\alpha L)^{p-1}$. As in Proposition 2.10,(iii) one finds $Y''_\alpha(\tau) > 0$ for any $\tau > 0$, which is impossible. Then $S_w < \infty$, thus $\lim_{\tau\to\ln S_w} Y/y = -1$, and w is of type (3).

• Finally consider the domain $\mathcal{R}$ of $\mathcal{Q}_1 \cup \mathcal{Q}_2$ delimitated by $\mathcal{T}_r, \mathcal{T}_s$ and containing $\mathcal{N}_\alpha$, and the sets

$$\mathcal{A}=\left\{P \in \mathcal{R} : \mathcal{T}_{[P]} \cap \mathcal{N}_\alpha \neq \emptyset\right\}, \qquad \mathcal{B}=\left\{P \in \mathcal{R} : \mathcal{T}_{[P]} \cap \{(\xi,0) : \xi > 0\} \neq \emptyset\right\},$$

corresponding to the trajectories of type (3) or (4). Then $\mathcal{A}, \mathcal{B}$ are nonempty, and open: here again the intersection with $\mathcal{N}_\alpha$ is transverse, because $\alpha \neq -p'$. Thus $\mathcal{A} \cup \mathcal{B} \neq \mathcal{R}$: there exists a trajectory in $\mathcal{R}$ which does not meet $\mathcal{M}_\alpha$; it converges to M_ℓ at $-\infty$ or oscillates around it, and it is located under $\mathcal{M}_\alpha$ in $\mathcal{Q}_2$. It cannot satisfy $\lim_{\tau\to\ln S_w} Y/y = -1$, thus $S_w = \infty$ and satisfies $\lim_{\tau\to\infty} \zeta = \alpha$, thus w is of type (2).

(ii) Assume $\alpha = -p'$ (see fig XXVIII). Then the regular solutions have a different behaviour: they are given explicitely at (1.10). They satisfy $Y_{-p'} \equiv C$, thus $Y'_{-p'} \equiv 0$, thus $\mathcal{T}_r = \mathcal{M}_{-p'}$. Here y has a zero, and $S_w = \infty$, and $\lim_{\tau\to\infty} \zeta = -p'$. As above $\mathcal{T}_s$ stays in $\mathcal{Q}_1$ and w is of type (1).

• Next consider again $\mathcal{T}_{[\bar{P}]}$. The solution going through $\bar{P}$ at $\tau = 0$ satisfies $Y_{-p'}(0) = 0$, thus $Y_{-p'}$ stays negative for $\tau > 0$ and $Y'_{-p'} < 0$. Suppose that $S_w = \infty$, and $\lim_{\tau\to\infty} \zeta = -p'$, then $\lim_{\tau\to\infty} y_\alpha = L > 0$, $\lim_{\tau\to\infty} Y_\alpha = -(|\alpha|\, L)^{p-1}$. But as at (2.49), $Y''_\alpha(\tau) < 0$ for any $\tau > 0$, which leads to a contradiction. Then $S_w < \infty$ and w is of type (4).

• Finally suppose that there exists a trajectory $\mathcal{T} \neq \mathcal{T}_r$ staying in $\mathcal{Q}_1 \cup \mathcal{Q}_2$. Then it converges to M_ℓ, thus $Y_\alpha > 0$, $S_w = \infty$, and $\lim_{\tau\to-\infty} Y_\alpha = \infty$, $\lim_{\tau\to\infty} Y_\alpha = C > 0$. If it has a minimal point, then it has an inflexion point where $Y'_\alpha > 0$, which as above is impossible. Then $Y'_\alpha < 0$, and from (2.6),

$$(p-1)Y''_{-p'} = Y'_{-p'}(e^{p'\tau}Y_{-p'}^{(2-p)/(p-1)} - N(p-1)) = Y'_{-p'}(Y - N(p-1)),$$

and $\lim_{\tau\to\infty} Y = \infty$, thus $Y''_{-p'} < 0$ for large τ, which is impossible. Thus there does not exist solution of type (2) or (3). ■

Let us come to the most difficult case: $\alpha < -p'$.

Lemma 6.7 *Assume $\varepsilon = -1$ and $\alpha < -p'$. If $\delta < N/2$ and $\alpha^* < \alpha$, either $\mathcal{T}_r$ has a limit cycle in $\mathcal{Q}_1$, or is homoclinic, or the regular solutions have at least two zeros. If $N/2 \leq \delta < N$, then they have at least two zeros.*

Proof. In any case M_ℓ is a source. Suppose that $\mathcal{T}_r$ has no limit cycle in $\mathcal{Q}_1$, or is not homoclinic (in particular it happens when $N/2 \leq \delta < N$, from Proposition 2.16), and stays in $\mathcal{Q}_1 \cup \mathcal{Q}_2$, thus Y stays positive. Then from Propositions 2.13, 2.14 and 2.20, either $\lim_{\tau\to-\infty} y = \infty$, $\lim_{\tau\to\infty} y_\alpha = L \neq 0$, $\lim_{\tau\to\infty} Y_\alpha = (\alpha L)^{p-1}$, or $S_w < \infty$. In any case, for any $d \in (\alpha, -p')$, the function $Y_d = e^{(d-\alpha)\tau}Y_\alpha$ satisfies $\lim_{\tau\to\ln S_w} Y_d = \infty = \lim_{\tau\to\infty} Y_d$. Then it has a minimum point, and this contradicts (2.15). Thus $\mathcal{T}_r$ enters $\mathcal{Q}_3$. If it stays in it, it has a limit cycle; then $-\mathcal{T}_r$ has a limit cycle in $\mathcal{Q}_1$. But $-\mathcal{T}_r$ does not meet $\mathcal{T}_r$, and M_ℓ is in the domain of $\mathcal{Q}_1$ delimitated by $\mathcal{T}_r$, since $\mathcal{T}_r$ meets $\mathcal{M}$ at the right of M_ℓ, from (2.16); this is impossible. Then $\mathcal{T}_r$ enters $\mathcal{Q}_4$, and y has at least two zeros. ■

Theorem 6.8 *Assume $\varepsilon = -1$ and $\delta < N/2$, $\alpha < -p'$. Then $w(r) = \ell r^{-\delta}$ is still a solution. Moreover*
(i) There exists a (minimal) critical value α^{crit} of α, such that

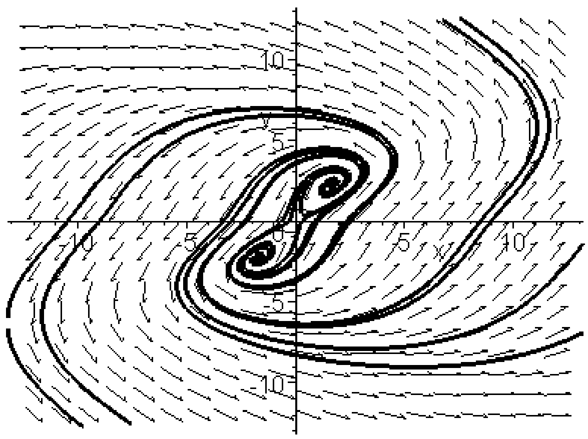
$$\alpha^* < \alpha^{crit} < \min(-p', \alpha_2) < 0,$$

and $\mathcal{T}_r$ is homoclinic: the regular solutions have a constant sign and $\lim_{r\to\infty} r^\eta w = c \neq 0$.

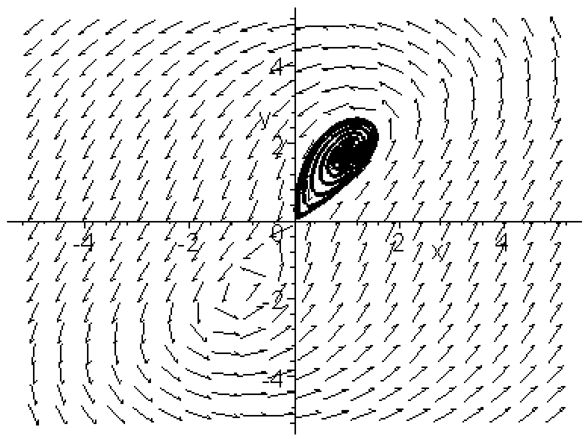
(ii) For any $\alpha \in (\alpha^*, \alpha^{crit})$ there does exist a cycle in $\mathcal{Q}_1$, equivalently there exist solutions such that $r^\delta w$ is periodic in $\ln r$. The regular solutions have a constant sign and $r^\delta w$ is asymptotically periodic in $\ln r$. There exist positive solutions such that $\lim_{r\to 0} r^\delta w = \ell$ and $r^\delta w$ is asymptotically periodic in $\ln r$.

(iii) For any $\alpha \leq \alpha^*$, there does not exist such a cycle, the regular solutions have a constant sign, and $\lim_{r\to\infty} r^{\delta} |w| = \ell$.

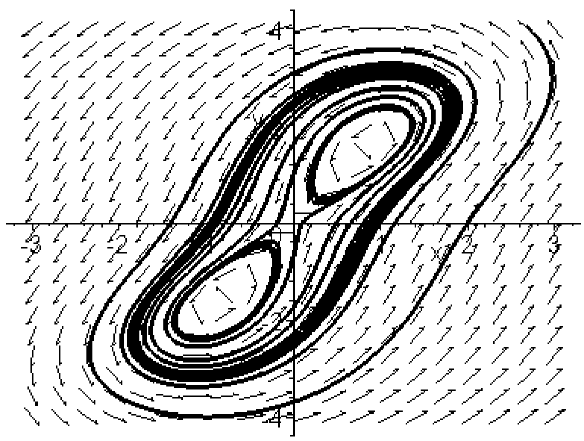
(iv) For any $\alpha < \alpha^{crit}$, there exists also a cycle surrounding $(0,0)$ and $\pm M_\ell$, thus w is changing sign and $r^{\delta}w$ is periodic in $\ln r$. There exist solutions oscillating near 0, and $r^{\delta}w$ is asymptotically periodic in $\ln r$, and $\lim_{r\to\infty} r^{\eta}w = c \neq 0$. There exist solutions oscillating near 0, and $r^{\delta}w$ is asymptotically periodic in $\ln r$, and $S_w < \infty$ or $\lim_{r\to\infty} r^{\alpha}w = L \neq 0$.



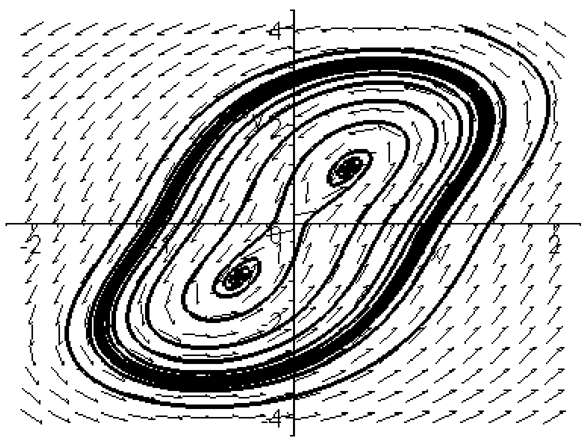
th 6.8,figXXIX: $\varepsilon = -1$,
$\alpha = -5 < -p' = -3 < 0 < \delta = 3 < N/2 < N = 9$



th 6.8,figXXX: $\varepsilon = -1$,
$\alpha = -7.4 < 0 < \delta = 3 < N/2 < N = 9$



th 6.8,figXXXI: $\varepsilon = -1$,
$\alpha^* = -9 < \alpha = -8 < \delta = 3 < N/2 < N = 9$



th 6.8,figXXXII: $\varepsilon = -1$,
$\alpha = -13 < \alpha^* = -9 < \delta = 3 < N/2 < N = 9$

Proof. (i) For any $\alpha \in (\alpha_1, \alpha_2)$, such that $\alpha \leq -p'$, from Remark (6.5), we still we have three possibilities:

• $\mathcal{T}_s$ converges to M_ℓ at $-\infty$, and turns around this point, since α is a spiral point, or it has a limit cycle around M_ℓ. Then $\mathcal{T}_r$ meats the set $\mathcal{E} = \left\{(\ell, Y) : Y > (\delta\ell)^{p-1}\right\}$ at a first point $(\ell, Y_r(\alpha))$. And $\mathcal{T}_r$ meats $\mathcal{E}$ at a last point $(\ell l, Y_r(\alpha))$, such that $Y_r(\alpha) - Y_s(\alpha) > 0$. Moreover $\mathcal{T}_r$ enters $\mathcal{Q}_2$, from Proposition 2.13 (see fig XXIX).

• $\mathcal{T}_s$ enters $\mathcal{Q}_4$, and then $\mathcal{T}_r$ is converging to M_ℓ at ∞ and turns around this point, or it has a limit cycle around M_ℓ. Then $\mathcal{T}_s$ meats $\mathcal{E}$ at a last point$(\ell l, Y_s(\alpha))$, $\mathcal{T}_r$ meats $\mathcal{E}$ at a first point $(\ell l, Y_r(\alpha))$, such that $Y_r(\alpha) - Y_s(\alpha) < 0$ (see fig XXXI and XXXII).

• Or $\mathcal{T}_r$ is homoclinic, which is equivalent to $Y_r(\alpha) - Y_s(\alpha) = 0$ (see fig XXX).

Now the function $\alpha \mapsto h(\alpha) = Y_r(\alpha) - Y_s(\alpha)$ is continuous. If $-p' < \alpha_2$, then $h(-p')$ is defined and $h(-p') > 0$, from Theorem 6.6. If $\alpha_2 \leq -p'$, we observe that for $\alpha = \alpha_2$, from Theorem 2.23, necessarily $\mathcal{T}_r$ leaves $\mathcal{Q}_1$, because α_2 is a source, and transversally; thus also for $\alpha = \alpha_2 - \gamma$ for $\gamma > 0$ small enough, thus $\mathcal{T}_s$ stays in it from Remark 6.5, hence $h(\alpha_2 - \gamma) > 0$. If $\alpha \leq \alpha^*$, then M_ℓ is a sink, or a weak sink, from Theorem 2.21, therefore $\mathcal{T}_s$ cannot converge to M_ℓ at $-\infty$. From Theorem 2.24, there exist no cycle in $\mathcal{Q}_1$, and no homoclinic orbit. From Remark 5.3, $\mathcal{T}_s$ cannot stay in $\mathcal{Q}_1$; then $\mathcal{T}_r$ stays in $\mathcal{Q}_1$ and is bounded and converges at ∞ to M_ℓ. Then $h(\alpha) < 0$ for $\alpha_1 < \alpha \leq \alpha^*$, thus there exists at least an $\alpha^{crit} \in (\alpha^*, \min(-p', \alpha_2)$ such that $h(\alpha^{crit}) = 0$. If it is not unique, we chose the smallest one.

(ii) Let $\alpha > \alpha^*$. The existence and uniqueness of such a cycle in $\mathcal{Q}_1$ follows from Theorem 2.21 if $\alpha - \alpha^*$ is small enough (see fig XXXI). For any $\alpha \in \left(\alpha^*, \alpha^{crit}\right)$, we still have existence. Indeed $h(\alpha) < 0$ on this interval, then $\mathcal{T}_r$ stays in $\mathcal{Q}_1$, and $\mathcal{T}_r$ cannot converge to M_ℓ at ∞, for that reason it has a limit cycle around M_ℓ at ∞. Since M_ℓ is a source, there also exist trajectories converging to M_ℓ at $-\infty$, with a limit cycle at ∞. And $\mathcal{T}_s$ does not stay in $\mathcal{Q}_1$, and it is bounded at $-\infty$. Then it has a limit cycle at $-\infty$ surrounding $(0,0)$ and $\pm M_\ell$.

(iii) Let $\alpha \leq \alpha^*$ (see fig XXXII). Then $\mathcal{T}_r$ stays in $\mathcal{Q}_1$, is bounded on $\mathbb{R}$, and converges to M_ℓ at ∞, and $\mathcal{T}_s$ does not stay in $\mathcal{Q}_1$ as above, thus it has a limit cycle at $-\infty$, containing the three stationary points.

(iv) For any $\alpha < \alpha^{crit}$, apart from $\mathcal{T}_r$ and the cycles, all the trajectories have a limit cycle at $-\infty$ containing the three stationary points. Moreover from Theorem 2.26, all the cycles are contained in a ball B of $\mathbb{R}^2$. Take any point P exterior to B. From Remark 5.3, $\mathcal{T}_{[P]}$ has a limit cycle at $-\infty$ contained in B and cannot have a limit cycle at ∞. Then y has constant sign near $\ln S_w$. From Proposition 2.13, either $S_w < \infty$ or y is defined near ∞ and $\lim_{\tau\to\infty} \zeta = L$, $\lim_{r\to\infty} r^\alpha w = L$. ■

Finally consider the case $N/2 \leq \delta$, where no cycle can exist.

Theorem 6.9 *Assume $\varepsilon = -1$ and $\alpha < 0 < N/2 \leq \delta < N$. Then all the solutions have a finite number of zeros. And $w(r) = \ell r^{-\delta}$ is a solution. Moreover if $-p' \leq \alpha$, theorem 6.6 applies. If $\alpha < -p'$, there exist positive solutions such that $\lim_{r\to 0} r^{\delta} w = \ell$, $\lim_{r\to\infty} r^{\eta} w = c > 0$. The regular solutions have a number $m \geq 2$ of zeros. All the other solutions satisfy $\lim_{r\to -\infty} r^{\delta} w = \pm\ell$, and have m ot $m+1$ zeros; there exist solutions with $m+1$ zeros.*

Proof. From Proposition 2.16, all the solutions have a finite number of zeros, and any solution is monotone near 0 and $\ln S_w$, or converges to $\pm M_\ell$. From Remark 6.5, apart from $\mathcal{T}_r$, any trajectory converges to $\pm M_\ell$ at $-\infty$. The functions V and W are nonincreasing. The trajectory $\mathcal{T}_s$ satisfies $\lim_{\tau\to\infty} V = \lim_{\tau\to\infty} W = 0$, thus $V \geq 0$, $W \geq 0$. If y has a zero at some point τ, then $W(\tau) = -\left|Y(\tau)\right|^{p'}/p'$, which is impossible. If Y has a zero at some point θ, then $V(\theta) = -Y'(\theta)^2/2$, hence also a contradiction. Thus $\mathcal{T}_s$ stays in $\mathcal{Q}_1$. From Remark 6.5 and Proposition 2.16, $\mathcal{T}_r$ does not stay in $\mathcal{Q}_1$, but enters $\mathcal{Q}_2$. From Lemma 6.7, $\mathcal{T}_r$ enters $\mathcal{Q}_4$, and y has at least two zeros. Let m be the number of its zeros. Then $\mathcal{T}_r$ cuts the axis $y = 0$ at points $\xi_1, .., \xi_m$. Consider any trajectory $\mathcal{T}_{[P]}$ with $P = (0, \xi), \xi > |\xi_m|$. It cannot intersect $\mathcal{T}_r$ and $-\mathcal{T}_r$, thus y has $m+1$ zeros. has $m+1$ zeros. And any trajectory has m or $m+1$ zeros, because it does not meet $\mathcal{T}_r$ and $-\mathcal{T}_r$ and $\mathcal{T}_{[P]}$. And $S_w < \infty$ or $\lim_{r\to\infty} r^{\alpha} w = L \neq 0$. ■

Acknowledgement *We want to thank Hector Giacomini (University of Tours) for interesting discussions on existence of cycles in dynamical systems.*

References

[1] D.G. Aronson and J. Graveleau, *A self-similar solution to the focusing problem for the porous medium equation,* Euro J. Math. Appl., 4 (92), 65-81.

[2] L.R. Anderson and W. Leighton, *Liapounov functions for autonomous systems of second order*, J. Math. Appl. Appl. 23 (1968), 645-664.

[3] G.I. Barenblatt, *On self-similar motions of incompressible fluids in porous media*, Prikl. Mat. Mach. 16 (1952), 679-698 (in Russian).

[4] M.F. Bidaut-Véron, *Local and global behavior of solutions of quasilinear equations of Emden-Fowler type,* Arc. Rat. Mech. Anal. 107 (1989), 293-324.

[5] M.F. Bidaut-Véron, *Self-similar solutions of the p-Laplace heat equation: the case $p > 2$,* in preparation.

[6] M.F. Bidaut-Véron, *The p-Laplace heat equation with a source term: self-similar solutions revisited,* Advances Nonlinear Studies, to appear.

[7] H. Brezis and A. Friedman, *Nonlinear parabolic equations involving measures as initial conditions*, J. Math. Pures Appl., 62 (1983), 73-97.

[8] E. Chasseigne and J.L. Vazquez, *Theory of extended solutions for fast diffusion equations in optimal classes of data. Radiation from singularities,* Arc. Rat. Mech. Anal. (2002), 133-187.

[9] C. Chicone and T. Jinghuang, *On general properties of quadratic systems,* Amer. Math. Monthly, 89 (1982), 167-178.

[10] E. Di Benedetto and M.A. Herrero, *Nonnegative solutions of the evolution p-Laplacian equation, initial traces and Cauchy problem when* $1 < p < 2$, Arc. Rat. Mech. Anal. 11 (1990), 225-290.

[11] O. Gil and J.L. Vazquez, *Focussing solutions for the p-Laplacian evolution equation,* Advances Diff. Equ. 2 (1997), 183-202.

[12] J. Hale and H. Koçak, Dynamics and bifurcations, Springer-Verlag, New-York, Berlin, Heidelberg (1991).

[13] J.H. Hubbard and B.H. West, Differential equations: A dynamical systems approach, Springer-Verlag (1995).

[14] S. Kamin and J.L. Vazquez, *Singular solutions of some nonlinear parabolic equations*, J. Anal. Math. 59 (1992), 51-74.

[15] S. Kamin and J.L. Vazquez, *Fundamental solutions and asymptotic behaviour for the p-Laplacian equation*, Rev. Mat. Iberoamericana, 4 (1988), 339-352.

[16] Y.W. Qi and M.Wang, *The global existence and finite time extinction of a quasilinear parabolic equation,* Advances Diff. Equ., 4 (1999), 731-753.

[17] J.L. Vazquez and L. Véron, *Different kinds of singular solutions of nonlinear parabolic equations*, in Nonliear problems of applied Mathematics Angell, Cook, Olmstead, SIAM (1996), 240-249

[18] L. Véron, *Effets régularisants de semi-groupes non-linéaires dans des espaces de Banach*, Ann. Fac. Sci. Toulouse, 1 (1979), 171-200.

[19] J. Zhao, *The asymptotic behaviour of solutions of a quasilinear degenerate equation,* J. Diff. Equ. 102 (1993), 33-52.

[20] J. Zhao, *The Cauchy problem for* $u_t = div(|\bigtriangledown u|^{p-2} \bigtriangledown u)$ *when* $2N/(N+1) < p < 2$, Nonlinear Anal., 24 (1995), 615-630.