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On a semilinear parabolic system of reaction–diffusion with absorption [1]

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Abstract. We consider the semilinear parabolic system with absorption terms in a bounded domain Ω of $\mathbb{R}^N$

$$\begin{cases} u_t - \Delta u + |v|^p|u|^{k-1}u = 0, & \text{in } \Omega \times (0,\infty),\\ v_t - \Delta v + |u|^q|v|^{\ell-1}v = 0, & \text{in } \Omega \times (0,\infty),\\ u(0) = u_0, \quad v(0) = v_0, & \text{in } \Omega, \end{cases}$$

where $p, q > 0$ and $k, \ell \geqslant 0$, with Dirichlet or Neuman conditions on $\partial\Omega \times (0,\infty)$. We study the existence and uniqueness of the Cauchy problem when the initial data are L^1 functions or bounded measures. We find invariant regions when $u_0,\ v_0$ are nonnegative, and give sufficient conditions for positivity or extinction in finite time.

1. Introduction and main results

Let Ω be a bounded regular domain of $\mathbb{R}^N$ ($N \geqslant 1$). We consider the parabolic system with absorption terms in $Q_\infty = \Omega \times (0,\infty)$,

$$\begin{cases} u_t - \Delta u + |v|^p|u|^{k-1}u = 0,\\ v_t - \Delta v + |u|^q|v|^{\ell-1}v = 0, \end{cases} \tag{1.1}$$

where $p, q > 0$, and $k, \ell \geqslant 0$, with the convention in case $k = 0$ (resp. $\ell = 0$):

$$|u|^{-1}u = \mathrm{sign}_0\, u = \begin{cases} 1 & \text{if } u > 0,\\ 0 & \text{if } u = 0,\\ -1 & \text{if } u < 0. \end{cases} \tag{1.2}$$

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We are concerned with the existence and uniqueness of the Cauchy problem, with Dirichlet or Neuman conditions on $\partial\Omega \times (0,\infty)$, and initial data

$$u(0) = u_0, \qquad v(0) = v_0,$$

where $u_0,\ v_0$ lie in $L^1(\Omega)$ or in the set $\mathcal{M}_b(\Omega)$ of bounded Radon measures in Ω. We also study the existence of invariant regions, and the properties of strict positivity or extinction in finite time of the solutions, when $u_0, v_0 \geqslant 0$.

In the case of nonnegative solutions, this system serves as a simple model for the joint evolution of two interacting biological species with densities $u,\ v$, competing for a common resource, see [25,19,24]. Then it is interesting to prove the existence of a global solution and study the uniqueness. Also assuming that at initial time one species dominates the other one, will it continue to dominate it all the time long? This corresponds to finding the existence of invariant regions, that means subsets Σ of $\mathbb{R}^2$ such that if $(u_0, v_0) \in \Sigma$, a.e. in Ω, then $(u, v) \in \Sigma$ in Q_∞. Another question is to know also under what conditions one of the species disappear in finite time, and what happens with the other.

Independently of the biological applications, system (1.1) offers a special interest due to the multiple difficulties that it brings up. It is the direct extension to two functions $u,\ v$ of the scalar equation with an absorption term

$$w_t - \Delta w + |w|^{Q-1} w = 0, \tag{1.3}$$

where $Q > 0$. Nevertheless, the situation is much more complex: contrarily to the case of Eq. (1.3), no comparison principle holds for the system, as we will see below, and thus we cannot use any technique of supersolutions, or pass to the limit by monotone convergence arguments. One of the most striking results is the existence of a minimal–maximal solution and a maximal–minimal one and a principle of cross comparisons when $u_0,\ v_0$ are nonnegative, see Theorem 1.3 below, which show the specific character of the system.

Also the problem of uniqueness is quite involved, in particular because of the lack of monotonicity. It is easy to solve whenever $u_0, v_0 \in L^\infty(\Omega)$ and $p, q, k, \ell \geqslant 1$. In the general case the nonlinear parts of system (1.1) may be non-Lipschitz on the sets $\{u = 0\}$ or $\{v = 0\}$. Thus uniqueness was qualified as an open problem by Kalashnikov in [3], and still open up to now, despite on some announcements in [28,29], which deal with porous media operators but restricted to the Lipschitzian case. We solve the problem in a great number of cases, see Theorems 1.6 and 1.7.

System (1.1) with nonnegative u, v has also to be compared to the problem with the other sign,

$$\begin{cases} u_t - \Delta u = v^p u^k, \\ v_t - \Delta v = u^q v^\ell, \end{cases} \tag{1.4}$$

where the nonlinear parts are source terms. It is a cooperative system, so that some comparison principles hold, see [16], even when the nonlinear part is non-Lipschitz, see [14,12]. As a consequence, system (1.4) has given rise to many works about blow up properties or global existence in Ω or in $\mathbb{R}^N$, among them [2,27,13]; see also [18] for systems with porous media operators. On the contrary, system (1.1) has been little studied. Existence results in $\mathbb{R}^N$ and partial compact support properties are given in [21]. Some properties of shrinking of the support in the space variable of the solutions are shown in [22,23].

Also the existence of travelling waves is analyzed in [17] in case $k = \ell = 0$ and $N = 1$. In the elliptic case the behaviour near a singularity is described in [8].

In Section 2 we describe the solutions of the system of ordinary differential equations associated to (1.1), namely

$$\begin{cases} u_t + |v|^p |u|^{k-1} u = 0, \\ v_t + |u|^q |v|^{\ell-1} v = 0. \end{cases} \tag{1.5}$$

The solutions of (1.5) are also solutions of (1.1) in the case of Neuman problem. Thus their properties of extinction in finite time or strict positivity can give information about one can expect for system (1.1). System (1.5) can be completely solved. Setting

$$a = p + 1 - \ell, \qquad b = q + 1 - k,$$

the main point is that the region

$$\{u > 0,\ v > 0,\ au^b = bv^a\}$$

is invariant whenever $a, b \neq 0$. The study of particular solutions leads also to define

$$\delta = pq - (1 - k)(1 - \ell),$$

called the discriminant of the system. The condition $\delta > 0$ (resp. $\delta < 0$) means that, in a certain sense, the system is superlinear (resp. sublinear). When $a, b \neq 0$, we also introduce the quantities

$$\mathbf{P} = 1 + \frac{\delta}{b}, \qquad \mathbf{Q} = 1 + \frac{\delta}{a}, \tag{1.6}$$

which play a role in the sequel.

In Section 3 we study the existence and the regularity of weak solutions of the Cauchy problem with Dirichlet (resp. Neuman) data on the lateral boundary:

$$\begin{cases} u_t - \Delta u + |v|^p |u|^{k-1} u = 0, & \text{in } Q_\infty, \\ v_t - \Delta v + |u|^q |v|^{\ell-1} v = 0, & \text{in } Q_\infty, \\ u = v = 0 \text{ (resp. } \partial u/\partial\nu = \partial v/\partial\nu = 0), & \text{on } \partial\Omega \times (0, \infty), \\ u(0) = u_0, \quad v(0) = v_0, & \text{in } \Omega. \end{cases} \tag{1.7}$$

We first give an existence result for initial data in $L^1(\Omega)$, which needs some care in the case $k = 0$ or $\ell = 0$, because of the lack of continuity of the function sign_0. We denote by

$$\big(S(t)\big)_{t\geqslant 0} = \big(S_d(t)\big)_{t\geqslant 0} \quad \text{(resp. } \big(S(t)\big)_{t\geqslant 0} = \big(S_n(t)\big)_{t\geqslant 0}\big),$$

the linear heat flow the heat equation in $L^1(\Omega)$ with Dirichlet (resp. Neuman) conditions on the lateral boundary $\partial\Omega \times (0, \infty)$.

Theorem 1.1. *Assume that $u_0, v_0 \in L^1(\Omega)$, and there exists $s, \sigma \geqslant 1$ such that*

$$\left(S(\cdot)|u_0|\right)^k \left(S(\cdot)|v_0|\right)^p \in L^s_{\mathrm{loc}}\left(\overline{Q_\infty}\right), \qquad \left(S(\cdot)|u_0|\right)^q \left(S(\cdot)|v_0|\right)^\ell \in L^\sigma_{\mathrm{loc}}\left(\overline{Q_\infty}\right), \tag{1.8}$$

with $s > 1$ if $k = 0$, $\sigma > 1$ if $\ell = 0$. Then there exists a weak solution (u, v) of problem (1.7)*. And for any $t \geqslant 0$,*

$$\left|u(t)\right| \leqslant S(t)|u_0|, \quad \left|v(t)\right| \leqslant S(t)|v_0|, \quad \text{a.e. in } \Omega. \tag{1.9}$$

If $u_0 \geqslant 0$, a.e. in Ω, then $u(t) \geqslant 0$, a.e. in Ω. If $v_0 \geqslant 0$, a.e. in Ω, then $v(t) \geqslant 0$, a.e. in Ω.

As a consequence, we get existence results for initial data in some L^θ spaces ($1 \leqslant \theta \leqslant \infty$) and by extension in $\mathcal{M}_b(\Omega)$, related to the Fujita exponent $(N + 2)/N$:

Corollary 1.1. *Assume that $u_0 \in L^{\theta_1}(\Omega)$ and $v_0 \in L^{\theta_2}(\Omega)$ for some $1 \leqslant \theta_1, \theta_2 \leqslant \infty$, and*

$$\max\left(\frac{k}{\theta_1} + \frac{p}{\theta_2}, \frac{q}{\theta_1} + \frac{\ell}{\theta_2}\right) < \frac{N+2}{N}. \tag{1.10}$$

Then there exists a weak solution (u, v) to problem (1.7).

Theorem 1.2. *Assume that $u_0, v_0 \in \mathcal{M}_b(\Omega)$, and*

$$\max(k + p, q + \ell) < \frac{N+2}{N}. \tag{1.11}$$

Then there exists a weak solution (u, v) to problem (1.7).

Finally we prove our cross comparison principle when initial data are nonnegative:

Theorem 1.3. *Assume that $u_0, v_0 \in L^1(\Omega)$ and* (1.8) *holds* (*or $u_0, v_0 \in \mathcal{M}_b(\Omega)$*) *and* (1.11) *holds*)*, and $u_0, v_0 \geqslant 0$. Then there exists two nonnegative solutions (U, V) and $(\widetilde{U}, \widetilde{V})$ of problem* (1.7)*, such that any nonnegative solution (u, v) satisfies*

$$U \leqslant u \leqslant \widetilde{U} \quad \text{and} \quad \widetilde{V} \leqslant v \leqslant V. \tag{1.12}$$

Moreover if $0 \leqslant u_0 \leqslant u_0'$ and $0 \leqslant v_0' \leqslant v_0$ with the same assumptions on u_0', v_0', then the corresponding solutions (U', V') and $(\widetilde{U}', \widetilde{V}')$ are ordered in the same way:

$$U \leqslant U', \quad \widetilde{U} \leqslant \widetilde{U}', \quad V' \leqslant V, \quad \widetilde{V}' \leqslant \widetilde{V}. \tag{1.13}$$

Section 4 concerns the question of invariant regions. It is interesting to see that pointwise correspondence can remain in Q_∞ between the two functions u and v, despite the lack of comparison principle:

Theorem 1.4. *Assume that $u_0, v_0 \in L^\infty(\Omega)$, $u_0, v_0 \geqslant 0$, and $ab > 0$. Let (u, v) be any solution of problem* (1.7).

(1) *Assume that*

$$au_0^b \leqslant bv_0^a, \quad \text{a.e. in } \Omega.$$

Then

$$au^b \leqslant bv^a, \quad \text{in } Q_\infty,$$

in any of the following cases:

$$0 < a \leqslant b \quad \text{and} \quad a \leqslant 1, \tag{1.14}$$

$$0 < a \leqslant b \quad \text{and} \quad 1 < a \quad \text{and} \quad \delta \geqslant 0, \tag{1.15}$$

$$b \leqslant a < 0. \tag{1.16}$$

Moreover if one of the eventualities holds, with the restriction ($\delta \geqslant 0$ *or* $1 \leqslant b$) *in the case* $0 < a < 1$, *and if* $au_0^b \not\equiv bv_0^a$, *then* $au^b < bv^a$ *in* Q_∞.

(2) *Assume that the inequality is strict*:

$$au_0^b < bv_0^a \quad \text{a.e. in } \Omega,$$

and $1 < a \leqslant b$ *and* $u, v \in C(\overline{Q_\infty})$. *Then* $au^b < bv^a$ *in* Q_∞.

Hence, under the assumptions of Theorem 1.4, the region $\{au^b \leqslant bv^a\}$ of $\mathbb{R}^2$ is invariant. Notice that under the assumptions of Theorem 1.4, v is a supersolution of the scalar equation

$$w_t - \Delta w + \left(\frac{a}{b}\right)^{-q/b} w^{\mathbf{P}} = 0,$$

where coefficient **P** is defined in (1.6). As a consequence we give new existence results in Section 5: we can have existence beyond the critical case (1.11) when only one of the initial data is a measure, when **P** is less than the Fujita exponent:

Theorem 1.5. *Suppose* $0 < a \leqslant b$, *and that* (1.14) *or* (1.15) *holds. Assume that*

$$u_0 \in L^1(\Omega), \quad v_0 = V_0 + \mu_0, \quad V_0 \in L^1(\Omega),\ \mu_0 \in \mathcal{M}_b(\Omega),\ u_0, V_0, \mu_0 \geqslant 0,$$

and

$$au_0^b \leqslant bV_0^a, \quad \text{a.e. in } \Omega. \tag{1.17}$$

If

$$\mathbf{P} = \min\left(1 + \frac{\delta}{a}, 1 + \frac{\delta}{b}\right) < \frac{N+2}{N}, \tag{1.18}$$

there exists a weak nonnegative solution (u, v) *of problem* (1.7).

This is a typical result for such kind of systems. Conditions of a similar type appear also in problem (1.4), see [2,16], and in elliptic systems with multipowered absorption or source terms, see for example [8,6,9].

In Section 6 we give sufficient conditions for the strict positivity of at least one of the components u or v. This implies properties of extinction in finite time for the other solution, either directly, or by combining with the comparison results of Section 4. Some of them will depend on the nature of the lateral boundary conditions. Indeed the diffusion of the Laplacian plays its role, notably in the case of Dirichlet conditions.

Section 7 is devoted to the difficult question of uniqueness. First we give a general result, available without any assumption on the sign of the initial data. The proof shows that the terms $|u|^{k-1}u, |v|^{\ell-1}v$ play a real role of absorption terms, and the terms $|v|^p,\ |u|^q$ appear as trouble-makers to the absorption.

Theorem 1.6. *Let $p, q \geqslant 1$, and $k, \ell \geqslant 0$ be arbitrary. Assume that $u_0, v_0 \in L^\infty(\Omega)$, or more generally $u_0 \in L^{\theta_1}(\Omega)$ and $v_0 \in L^{\theta_2}(\Omega)$ for some $\theta_1, \theta_2 \in [1, \infty]$, and*

$$\max\left(\frac{k-1}{\theta_1} + \frac{p}{\theta_2}, \frac{\ell-1}{\theta_2} + \frac{q}{\theta_1}\right) < \frac{2}{N}. \tag{1.19}$$

Then problem (1.7) *admits a unique solution* (u, v). *And*

$$u \in C\big([0,\infty), L^{\theta_1}(\Omega)\big) \quad \textit{if } \theta_1 < \infty, \quad \textit{and} \quad v \in C\big([0,\infty), L^{\theta_2}(\Omega)\big) \quad \textit{if } \theta_2 < \infty. \tag{1.20}$$

In particular uniqueness holds for any $u_0, v_0 \in L^1(\Omega)$, under condition (1.11).

When $u_0, v_0 \in L^\infty(\Omega)$, and $u_0, v_0 \geqslant 0$, we obtain new results, where the four parameters $p,\ q,\ k,\ \ell$ are involved, by using Theorem 1.3, and also the positivity properties of Section 6. Our main result is the following.

Theorem 1.7. *Assume that $u_0, v_0 \in L^\infty(\Omega)$ and $u_0, v_0 \geqslant 0$. Then uniqueness holds for problem* (1.7), *in any of the following cases*:

(i) $p, q \geqslant 1$ (*see Theorem* 1.6);

(ii) $0 < k \leqslant q < 1$, $0 < \ell \leqslant p < 1$, *and*

$$p(1-q) \leqslant \ell(1-k) \quad \textit{and} \quad q(1-p) \leqslant k(1-\ell); \tag{1.21}$$

(iii) $p, k \geqslant 1,\ \ell > 0$, *and*

$$1 - \ell \leqslant q \quad \textit{and} \quad v_0^\ell(x) \leqslant c u_0^{1-q}(x), \quad \textit{a.e. in } \Omega,\ \textit{for some } c > 0; \tag{1.22}$$

(iv) $q, \ell \geqslant 1,\ k > 0$, *and*

$$1 - k \leqslant p \quad \textit{and} \quad u_0^k(x) \leqslant c v_0^{1-p}(x), \quad \textit{a.e. in } \Omega,\ \textit{for some } c > 0; \tag{1.23}$$

(v) $k, \ell \geqslant 1$ *and* (1.22), (1.23) *hold.*

In particular if $\inf_{x\in\Omega} u_0(x) > 0$ and $\inf_{x\in\Omega} v_0(x) > 0$, then uniqueness holds when $k,\ell \geqslant 1$. We give more complete results in the case of Neuman problem, see Theorem 7.1, or when the comparison properties of Section 4 hold, see Theorem 7.2.

Notice that uniqueness can hold with all the parameters $p,q,k,\ell < 1$, for example when $p=\ell$ and $q=k$, or when $p=1-k$ and $q=1-\ell$ and $p+q\geqslant 1$. The problem remains open for some ranges of the parameters, for example in the case of the Hamiltonian system

$$\begin{cases} u_t - \Delta u + v^p \operatorname{sign}_0 u = 0,\\ v_t - \Delta v + u^q \operatorname{sign}_0 v = 0,\end{cases}$$

where $k=\ell=0$, when $p<1$ or $q<1$.

2. The o.d.e. problem

Here we study the ordinary differential system (1.5). We will discuss according to the different values of $a,\ b,\ \delta$. Notice that

$$a,b>0 \text{ and } \delta<0 \quad\Longrightarrow\quad k,\ell<1, \tag{2.1}$$

$$a,b<0 \quad\Longrightarrow\quad k,\ell>1 \text{ and } \delta<0. \tag{2.2}$$

The role of δ is enlighted by the existence of particular solutions:

$$u^*=A^*t^{-a/\delta},\quad v^*=B^*t^{-b/\delta},\quad \text{on } (0,\infty),\quad \text{if } a,b>0 \text{ and } \delta>0,\ \text{or if } a,b<0, \tag{2.3}$$

$$u^*=A^*(t_-)^{a/|\delta|},\quad v^*=B^*(t_-)^{b/|\delta|},\quad \text{on } \mathbb{R},\quad \text{if } a,b>0 \text{ and } \delta<0, \tag{2.4}$$

are solutions of (1.5), where $A^*=|b/\delta|^{p/\delta}|a/\delta|^{(1-\ell)/\delta}$ and $B^*=|a/\delta|^{q/\delta}|b/\delta|^{(1-k)/\delta}$. When $a,b>0$ and $\delta=0$, one gets a family of solutions, where $c>0$ is arbitrary:

$$u_c^*=c\,\mathrm{e}^{-(a/b)^{p/a}t},\qquad v_c^*=\left(c\frac{a}{b}\right)^{b/a}\mathrm{e}^{-(b/a)^{q/b}t}. \tag{2.5}$$

The solutions given by (2.3) or (2.5), remain positive, but those given by (2.4) have a compact support.

Now consider the Cauchy problem with data $u(0)=u_0\in\mathbb{R},\ v(0)=v_0\in\mathbb{R}$, and unknown $(u,v)\in C^1([0,\infty))$.

Proposition 2.1. *The problem* (1.5) *with initial data* $u_0,v_0\in\mathbb{R}$, *has a unique global solution* (u,v) *on* $[0,\infty)$. *If* $u_0,v_0\geqslant 0$, *then* $u,v\geqslant 0$. *Whenever* $u_0,v_0>0$, *the following properties hold*:

(i) *If* $k\geqslant 1$ (*resp.* $\ell\geqslant 1$), *then* u (*resp.* v) *remains positive on* $[0,\infty)$. *In particular if* $a,b\leqslant 0$, *then* u,v *are positive.*

(ii) *When* $a,b\neq 0$, *the region* $\{au^b=bv^a\}$ *is invariant, since*

$$bv^a-au^b=bv_0^a-au_0^b=C. \tag{2.6}$$

(iii) *If $a, b > 0$ and $\delta \geqslant 0$, then u, v are positive when $C = 0$; when $C > 0$ (resp. $C < 0$), v (resp. u) is positive, and u (resp. v) has a compact support if $k < 1$ (resp. $\ell < 1$).*

(iv) *If $a, b > 0$ and $\delta < 0$, then u and v have a compact support when $C = 0$; when $C > 0$ (resp. $C < 0$), v (resp. u) is positive, and u (resp. v) has a compact support.*

(v) *If $a \leqslant 0$ (resp. $b \leqslant 0$), then v (resp. u) is positive, and u (resp. v) has a compact support if $k < 1$ (resp. $\ell < 1$).*

Proof. First notice that any solution satisfies $(u^2)_t \leqslant 0$, a.e. in $(0, \infty)$. Therefore, if $u_0 = 0$, we find a unique solution on $(0, \infty)$: $u \equiv 0$, $v \equiv v_0$; if $v_0 = 0$, then $u \equiv u_0$, $v \equiv 0$. Thus we can suppose $u_0 \neq 0$ and $v_0 \neq 0$. Since u^2 and v^2 are nonincreasing, they stay bounded, then also u_t and v_t, and global existence and uniqueness follow from the Cauchy theorem.

Now let us assume $u_0 > 0$, $v_0 > 0$. This is not restrictive, since if (u, v) is a solution, then $(\pm u, \pm v)$ is also a solution.

(i) If $k \geqslant 1$, then $u_t + v_0^p u^k \geqslant 0$, thus

$$u(t) \geqslant \begin{cases} u_0\big(1 + (k-1)u_0^{k-1} v_0^p t\big)^{1/(k-1)}, & \text{if } k > 1, \\ u_0\, \mathrm{e}^{-v_0^p t}, & \text{if } k = 1, \end{cases}$$

hence $u(t) > 0$ for any $t \geqslant 0$. It happens in particular when $a \leqslant 0$.

(ii) As long as the solutions do not vanish, they satisfy

$$u^{q-k} u_t = -u^q v^p = v^{p-\ell} v_t.$$

If $a, b \neq 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(bv^a - au^b\big) = 0,$$

hence (2.6) holds, and we call $(u_{[u_0,C]}, v_{[u_0,C]})$ the corresponding solutions.

(iii) Let $a, b > 0$ and $\delta \geqslant 0$. In case $C = 0$ and $\delta > 0$, we find

$$u_{[u_0,0]} = u_0\left(1 + \frac{\delta}{a}\left(\frac{a}{b}\right)^{p/a} u_0^{\delta/a} t\right)^{-a/\delta}, \qquad v_{[u_0,0]} = \left(\frac{a}{b}\right)^{1/a} u_{[u_0,0]}^{b/a}; \tag{2.7}$$

notice that $(u_{[u_0,0]}, v_{[u_0,0]})$ is nothing but a translated of (u^*, v^*) given by (2.3). In case $C = 0$ and $\delta = 0$, then $(u_{[u_0,0]}, v_{[u_0,0]}) = (u^*_{u_0}, v^*_{v_0})$; in any case the two solutions remain positive on $(0, \infty)$. Now assume $C \neq 0$. By symmetry we can assume that $C > 0$. Then v stays positive. As long as u remains positive, it is given by

$$\int_u^{u_0} \frac{\mathrm{d}u}{u^k(au^b + C)^{p/a}} = b^{-p/a} t. \tag{2.8}$$

If T is the maximal value such that $u > 0$, then necessarily $u \to 0$ as $t \to T$. Then T is finite if and only if $\int_0^{u_0} \mathrm{d}u/u^k < \infty$, that means $k < 1$. In that case, $v \equiv (C/b)^{1/a} > 0$ on $[T, \infty)$. When $k \geqslant 1$, u remains positive, $u(t) \to 0$ and $v(t) \to (C/b)^{1/a}$ at infinity.

(iv) Let $a, b > 0$ and $\delta < 0$. In case $C = 0$, we find

$$u_{[u_0,0]} = u_0\left(\left(1 - \frac{|\delta|}{a}\left(\frac{a}{b}\right)^{p/a} u_0^{\delta/a} t\right)^+\right)^{a/|\delta|}, \qquad v_{[u_0,0]} = \left(\frac{a}{b}\right)^{1/a} u_{[u_0,0]}^{b/a},$$

hence $(u_{[u_0,0]}, v_{[u_0,0]})$ is a translated of (u^*, v^*) given by (2.4). It has a compact support. If for example $C > 0$, then u is still given by (2.8). It has a compact support from (2.1).

(v) Let $a \leqslant 0$. Then v is positive, since $\ell > 1$. Assume moreover that $k < 1$, thus $b > 0$. If $a < 0$, then (2.6) takes the form

$$|a|u^b + \frac{b}{v^{|a|}} = C, \tag{2.9}$$

hence $C > 0$, and u is given by

$$\int_u^{u_0} \frac{\mathrm{d}u}{u^k(C - |a|u^b)^{p/a}} = b^{-p/a} t; \tag{2.10}$$

if $a = 0 < b$, then $u,\ v$ are given by

$$\int_u^{u_0} \frac{\mathrm{d}u}{u^k\, \mathrm{e}^{pu^b/b}} = v_0^p\, \mathrm{e}^{-pu_0^b/b} t, \qquad v = v_0\, \mathrm{e}^{(u^b - u_0^b)/b}.$$

In any case u has a compact support. $\square$

Remark 2.1. Assume that $u_0, v_0 > 0$, and $a, b > 0$. Observe that condition $k < 1$ does not imply that u has a compact support: for example if $bv_0^a < au_0^b$, then u stays positive. Notice also that condition $\delta < 0$ implies that at least one of the solutions has a compact support. Condition $\delta \geqslant 0$ implies that the sum $u + v$ stays positive.

Remark 2.2. Assume that $u_0, v_0 > 0$ and $ab > 0$, and for example $au_0^b \leqslant bv_0^a$. Then $C \geqslant 0$ in (2.6), so that in any case $v \geqslant (a/b)^{1/a} u^{b/a}$. Thus u, v are respectively subsolution and supersolution of ordinary differential equations:

$$u_t + \left(\frac{a}{b}\right)^{p/a} u^{\mathbf{Q}} \leqslant 0, \qquad v_t + \left(\frac{a}{b}\right)^{-q/b} v^{\mathbf{P}} \geqslant 0,$$

where $\mathbf{P},\ \mathbf{Q}$ are defined at (1.6). We will extend these properties to system (1.1) in Section 4.

3. Existence and first properties

3.1. Some useful formulas on heat equation

Here we briefly mention some well-known results. Let $Q_T = \Omega \times [0, T]$, for any $T > 0$. We denote by $C_0(\overline{\Omega})$ the set of functions $w \in C(\overline{\Omega})$ which vanish on $\partial\Omega$.

For any $(y_0, F) \in L^1(\Omega) \times L^1(Q_T)$ there exists a unique function $y \in C([0,T], L^1(\Omega))$, such that $y \in L^1((0,T), W_0^{1,1}(\Omega))$ (resp. $L^1((0,T), W^{1,1}(\Omega))$), solution of problem

$$\begin{cases} y_t - \Delta y + F = 0, & \text{in } Q_T, \\ y = 0 \text{ (resp. } \partial y/\partial\nu = 0), & \text{on } \partial\Omega \times (0,T), \\ y(0) = y_0, & \text{in } \Omega, \end{cases} \tag{3.1}$$

in the weak sense

$$\int_0^T \int_\Omega (-y\varphi_t - y\Delta\varphi + F\varphi)\, dx\, dt = 0, \tag{3.2}$$

for any $\varphi \in \mathcal{D}(Q_T)$ (resp. $\varphi \in C^\infty(Q_T)$ with compact support in $\overline{\Omega} \times (0,T)$, such that $\partial\varphi/\partial\nu = 0$). And y is given by

$$y(t) = S(t)y_0 - \int_0^t S(t-s)F(s)\, ds. \tag{3.3}$$

The mapping $\mathcal{S} : (y_0, F) \mapsto y$ is compact from $L^1(\Omega) \times L^1(Q_T)$ into $L^r(Q_T)$ for $1 \leqslant r < (N+2)/N$, see [4]. Also it is continuous from $L^1(\Omega) \times L^1(Q_T)$ into $C([0,T], L^1(\Omega))$ and into $L^s((0,T), W_0^{1,\rho}(\Omega))$ (resp. $L^s((0,T), W^{1,\rho}(\Omega))$) for $2/s + N/\rho > N+1$. More generally, if $y_0 \in \mathcal{M}_b(\Omega)$, problem (3.1) admits a unique weak solution $y \in L^1(Q_T)$, such that $y(t) \to y_0$, weakly in $\mathcal{M}_b(\Omega)$, as $t \to 0$. And y is the only solution in $L^1(Q_T)$ of problem

$$\int_0^T \int_\Omega (-y\psi_t - y\Delta\psi + F\psi)\, dx\, dt = \int_\Omega \psi(0)\, dy_0, \tag{3.4}$$

for any $\psi \in C^\infty(Q_T)$ with compact support in $\Omega \times [0,T)$ (resp. in $\overline{\Omega} \times [0,T)$, such that $\partial\varphi/\partial\nu = 0$).

Moreover, the semigroups S_d and S_n share some regularizing properties: for any $y_0 \in L^\theta(\Omega)$ and $1 \leqslant \theta \leqslant \tau \leqslant \infty$,

$$\begin{cases} \|S_d(t)y_0\|_{L^\tau(\Omega)} \leqslant Ct^{-(1/\theta - 1/\tau)N/2} \|y_0\|_{L^\theta(\Omega)}, \\ \|S_n(t)y_0\|_{L^\tau(\Omega)} \leqslant C\big(1 + t^{-(1/\theta - 1/\tau)N/2}\big) \|y_0\|_{L^\theta(\Omega)}, \end{cases} \tag{3.5}$$

see, for example, [26]. In particular $S(\cdot)y_0 \in L^\rho(Q_T)$ for $1 \leqslant \rho < \theta(N+2)/N$. Also for any $y_0 \in C_0(\overline{\Omega})$,

$$\|S_d(t)y_0\|_{L^\infty(\Omega)} \leqslant C\, e^{-\lambda_1 t} \|y_0\|_{L^\infty(\Omega)},$$

where λ_1 is the first eigenvalue of $-\Delta$ in Ω. In the sequel we will denote by φ_1 the eigenfunction associated to λ_1 such that $\varphi_1 > 0$ and $\|\varphi_1\|_{L^\infty(\Omega)} = 1$.

Next, we recall some formulations of parabolic Kato inequality: let $(y_0, F) \in L^1(\Omega) \times L^1(Q_T)$ and y be the solution of (3.1), with Dirichlet or Neuman data. Then

$$|y|_t - \Delta|y| + F \operatorname{sign}_0 y \leqslant 0, \tag{3.6}$$

in $\mathcal{D}'(Q_T)$, and more precisely for any $t \in [0, T]$ and a.e. in Ω,

$$|y(t)| + \int_0^t S(t-s)F(s)\,\mathrm{sign}_0\, y(s)\,\mathrm{d}s \leqslant S(t)|y_0|. \tag{3.7}$$

In particular if $F \cdot y \geqslant 0$, a.e. in Q_T, then for any $t \in [0, T]$, $|y(t)| \leqslant S(t)|y_0|$, a.e. in Ω. If moreover $y_0 \geqslant 0$, a.e. in Ω, then $y(t) \geqslant 0$, a.e. in Ω.

3.2. Formulation of the Cauchy problem for the system

Let us come to system (1.1). Notice that it always admits $(0, 0)$ as a solution, and also solutions of the form $(0, v)$ with v solution of the heat equation, and $(u, 0)$, with u solution of the heat equation. We consider the Cauchy problem with Dirichlet or Neuman data (1.7).

First assume $u_0, v_0 \in L^1(\Omega)$. By solution (u, v) of (1.7), we mean any couple of functions $u, v \in C([0, \infty), L^1(\Omega))$ such that $|v|^p|u|^k, |u|^q|v|^\ell \in L^1_{\mathrm{loc}}(\overline{Q_\infty})$, $u(0) = u_0$, $v(0) = v_0$, which are weak solutions of their respective equations in the sense of (3.2). It can be expressed in an equivalent way by

$$\begin{cases} u(t) = S(t)u_0 - \displaystyle\int_0^t S(t-s)\big|v(s)\big|^p\big|u(s)\big|^{k-1}u(s)\,\mathrm{d}s, \\ v(t) = S(t)v_0 - \displaystyle\int_0^t S(t-s)\big|u(s)\big|^q\big|v(s)\big|^{\ell-1}v(s)\,\mathrm{d}s, \end{cases} \tag{3.8}$$

for any $t \geqslant 0$, with $S = S_d$ or $S = S_n$. Assume now that $u_0, v_0 \in \mathcal{M}_b(\Omega)$. By solution of (1.7), we mean any couple of functions $u, v \in L^1_{\mathrm{loc}}(\overline{Q_\infty})$, such that $|v|^p|u|^k, |u|^q|v|^\ell \in L^1_{\mathrm{loc}}(\overline{Q_\infty})$, and are weak solutions, and

$$u(t) \to u_0, \quad v(t) \to v_0 \quad \text{weakly in } \mathcal{M}_b(\Omega), \text{ as } t \to 0. \tag{3.9}$$

It can be expressed equivalently as in (3.4). As a direct consequence of Kato inequality, we deduce the following:

Lemma 3.1. *For any $u_0, v_0 \in L^1(\Omega)$, any weak solution (u, v) of problem* (1.7) *satisfies* (1.9)*. Moreover if $u_0 \geqslant 0$ (resp. $v_0 \geqslant 0$), then, for any $t \in [0, \infty)$, $u(t) \geqslant 0$ (resp. $v(t) \geqslant 0$), a.e. in Ω.*

Remark 3.1. In particular if for example $u_0 \equiv 0$, then uniqueness holds: indeed $u \equiv 0$ and $v(\cdot) = S(\cdot)v_0$.

Remark 3.2. Assume that $u_0, v_0 \in L^1(\Omega)$. From (1.9) and the regularizing effect of the semi-group (3.5), any solution (u, v) of (1.7), if it exists, is locally bounded in Q_∞. From the standard regularity theory, u, v lie in $W^{2,1,m}_{\mathrm{loc}}(Q_\infty)$ for any $m > 1$, hence in $C^{1,0}_{\mathrm{loc}}(Q_\infty)$, and in $C^{2,1}_{\mathrm{loc}}(Q_\infty)$ if $k, \ell \neq 0$, and in fact in Hölder functions spaces. Thus the equations are satisfied a.e. in Q_∞. From (3.5), for any $1 \leqslant \theta \leqslant \sigma \leqslant \infty$ and $1 \leqslant \lambda \leqslant \tau \leqslant \infty$,

$$\begin{aligned} &\big\|u(t)\big\|_{L^\sigma(\Omega)} \leqslant C\big(1 + t^{-(1/\theta - 1/\sigma)N/2}\big)\|u_0\|_{L^\theta(\Omega)}, \\ &\big\|v(t)\big\|_{L^\tau(\Omega)} \leqslant Ct^{-(1/\lambda - 1/\tau)N/2}\|v_0\|_{L^\lambda(\Omega)}. \end{aligned} \tag{3.10}$$

If $u_0, v_0 \in L^\infty(\Omega)$, then $u, v \in L^\infty(Q_\infty)$, hence $u, v \in W^{2,1,m}(Q_T)$ for any $1 \leqslant m < \infty$ and any $T > 0$, in particular $u, v \in C([0,\infty), L^m(\Omega))$. Moreover, if $u_0, v_0 \in C_0(\overline{\Omega})$ for the Dirichlet problem (resp. $u, v \in C(\overline{\Omega})$ for the Neuman problem), then $u, v \in C(\overline{Q_\infty})$.

3.3. Proofs of existence

First we prove Theorem 1.1.

Lemma 3.2. *Let $u_0, v_0 \in L^\infty(\Omega)$ and $F_1,\ F_2$ locally Lipschitz from $\mathbb{R}^2$ into $\mathbb{R}$, such that*

$$F_1(0,0) = F_2(0,0) = 0 \quad \textit{and} \quad F_1(r,s)r \geqslant 0, \quad F_2(r,s)s \geqslant 0, \quad \forall r, s \in \mathbb{R}.$$

Then there exist $u, v \in C([0,\infty), L^1(\Omega))$, unique, such that

$$\begin{cases} u_t - \Delta u + F_1(u,v) = 0, & \textit{in } Q_\infty, \\ v_t - \Delta v + F_2(u,v) = 0, & \textit{in } Q_\infty, \\ u = v = 0 \ (\textit{resp. } \partial u/\partial\nu = \partial v/\partial\nu = 0), & \textit{on } \partial\Omega \times [0,\infty), \\ u(0) = u_0, \quad v(0) = v_0, & \textit{in } \Omega. \end{cases} \tag{3.11}$$

And $u, v \in L^\infty(Q_\infty) \cap C([0,\infty),\ L^m(\Omega))$ for any $m \geqslant 1$.

Proof. Let $U_{0,m}, V_{0,m} \in \mathcal{D}(\Omega)$, uniformly bounded in Ω, such that $U_{0,m} \to u_0$ and $V_{0,m} \to v_0$ in $L^1(\Omega)$ as $m \to \infty$. Then there exists a unique solution u_m, v_m of the problem

$$\begin{cases} u_{m,t} - \Delta u_m + F_1(u_m, v_m) = 0, & \text{in } \Omega \times [0, T_m), \\ v_{m,t} - \Delta v_m + F_2(u_m, v_m) = 0, & \text{in } \Omega \times [0, T_m), \\ u_m = v_m = 0 \ (\text{resp. } \partial u_m/\partial\nu = \partial v_m/\partial\nu = 0), & \text{on } \partial\Omega \times [0, T_m), \\ u_m(0) = U_{0,m}, \quad v_{n,m}(0) = V_{0,m}, & \text{in } \Omega, \end{cases}$$

defined on a maximal interval $[0, T_m)$. From the Kato inequality,

$$\big|u_m(t)\big| \leqslant S(t)|U_{0,m}|, \quad \big|v_m(t)\big| \leqslant S(t)|V_{0,m}|, \quad \text{in } \Omega,$$

so that $u_m(t),\ v_m(t)$ are bounded in $L^\infty(Q_\infty)$, which implies $T_m = \infty$; and $F_1(u_m, v_m),\ F_2(u_m, v_m)$ are bounded in $L^\infty(Q_\infty)$. Now

$$\begin{cases} u_m(t) = S(t)U_{0,m} - \displaystyle\int_0^t S(t-s)F_1\big(u_m(s), v_m(s)\big)\,\mathrm{d}s, \\ v_m(t) = S(t)V_{0,m} - \displaystyle\int_0^t S(t-s)F_2\big(u_m(s), v_m(s)\big)\,\mathrm{d}s. \end{cases} \tag{3.12}$$

From the compactness properties of $\mathcal{S}$, up to a subsequence, u_m and v_m converge a.e. in Q_∞ and in $L^r(Q_T)$ for $1 \leqslant r < (N+2)/N$ and any $T > 0$ to some u and v. Then $F_1(u_m, v_m)$ and $F_2(u_m, v_m)$ converge to $F_1(u,v)$ and $F_2(u,v)$, strongly in $L^1(Q_T)$. Then we can go to the limit in (3.12) as $m \to \infty$, thus (u,v) is a solution of (3.11). And $u,v \in L^\infty(Q_\infty)$, hence $F_1(u,v), F_2(u,v) \in L^1((0,T), L^\infty(\Omega))$, and $\int_0^t S(t-s)F_i(u(s),v(s))\,\mathrm{d}s \in C([0,\infty), L^\infty(\Omega))$ for $i = 1,2$. Moreover $S(t)u_0, S(t)v_0 \in C([0,\infty), L^m(\Omega))$ for any $m \geqslant 1$, thus also by addition $u,v \in C([0,\infty), L^m(\Omega))$. Uniqueness holds because $F_1,\ F_2$ are locally Lipschitz continuous. □

Proof of Theorem 1.1. For any real $\eta \geqslant 0$, and any $n \in \mathbb{N}$, let $g_{n,\eta}(r)$ be an odd monotone locally Lipschitz approximation of $r \mapsto |r|^{\eta-1}r$, such that $|g_{n,\eta}(r)| \leqslant |r|^\eta$ for any $r \in \mathbb{R}$. Let $u_{0,n},\ v_{0,n}$ be the truncatures of $u_0,\ v_0$ by $\pm n$. From Lemma 3.2, there exists a unique solution (u_n,v_n) of the problem

$$\begin{cases} u_{n,t} - \Delta u_n + g_{n,p}(|v_n|)g_{n,k}(u_n) = 0, & \text{in } Q_\infty,\\ v_{n,t} - \Delta v_n + g_{n,q}(|u_n|)g_{n,\ell}(v_n) = 0, & \text{in } Q_\infty,\\ u_n = v_n = 0 \text{ (resp. } \partial u_n/\partial\nu = \partial v_n/\partial\nu = 0), & \text{on } \partial\Omega \times [0,\infty),\\ u_n(0) = u_{0,n}, \quad v_n(0) = v_{0,n}, & \text{in } \Omega, \end{cases} \tag{3.13}$$

and

$$|u_n(t)| \leqslant S(t)|u_{0,n}| \leqslant S(t)|u_0|, \quad |v_n(t)| \leqslant S(t)|v_{0,n}| \leqslant S(t)|v_0| \quad \text{in } \Omega.$$

Then u_n, v_n are bounded in $L^\infty_{\mathrm{loc}}(\overline{\Omega} \times (0,\infty))$. Moreover,

$$\begin{cases} u_n(t) = S(t)(u_{0,n}) - \displaystyle\int_0^t S(t-s)g_{n,p}(|v_n(s)|)g_{n,k}(u_n(s))\,\mathrm{d}s,\\ v_n(t) = S(t)(v_{0,n}) - \displaystyle\int_0^t S(t-s)g_{n,q}(|u_n(s)|)g_{n,\ell}(v_n(s))\,\mathrm{d}s, \end{cases} \tag{3.14}$$

and

$$\begin{cases} |g_{n,p}(|v_n|)g_{n,k}(u_n)| \leqslant g_{n,p}(S(\cdot)|v_0|)g_{n,k}(S(\cdot)|u_0|) \leqslant |S(\cdot)|u_0||^k |S(\cdot)|v_0||^p,\\ |g_{n,q}(|u_n|)g_{n,\ell}(v_n)| \leqslant g_{n,q}(S(\cdot)|u_0|)g_{n,k}(S(\cdot)|v_0|) \leqslant |S(\cdot)|u_0||^q |S(\cdot)|v_0||^\ell, \end{cases}$$

and for any $T > 0$, $|S(\cdot)|u_0||^k|S(\cdot)|v_0||^p, |S(\cdot)|u_0||^q|S(\cdot)|v_0||^\ell \in L^1(Q_T)$ from (1.8). Up to a subsequence, (u_n,v_n) converge to some (u,v), strongly in $(L^r(Q_T))^2$ for $1 \leqslant r < (N+2)/N$, and a.e. in Q_∞.

First assume $k,\ell \neq 0$. Then $g_{n,p}(|v_n|)g_{n,k}(u_n)$ converges to $|v|^p|u|^{k-1}u$ and $g_{n,q}(|u_n|)g_{n,\ell}(v_n)$ converges to $|u|^q|v|^{\ell-1}v$, a.e. in Q_∞ and in $L^1(Q_T)$. We can pass to the limit in (3.14) by Lebesgue theorem, then (u,v) satisfies (3.8), thus it is a solution of problem.

Now assume for example that $k = 0$. Then

$$|g_{n,p}(|v_n|)g_{n,0}(u_n)| \leqslant |S(\cdot)|v_0||^p,$$

and from (1.8), $|S(\cdot)|v_0||^p \in L^s(Q_T)$ for some $s > 1$, and any $T > 0$. Moreover $u_n,\ v_n$ are bounded in $L^\infty_{\mathrm{loc}}(Q_T)$, hence also $g_{n,p}(|v_n|)g_{n,0}(u_n) = \Delta u_n - u_{n,t}$. This implies that u_n is bounded in $W^{2,1,m}_{\mathrm{loc}}(Q_\infty)$ for any $m > 1$, from Remark 3.2. Hence Δu_n and $u_{n,t}$ are bounded in $L^m_{\mathrm{loc}}(Q_\infty)$. Then for fixed T, after extraction of a subsequence (depending eventually on T), $g_{n,p}(|v_n|)g_{n,0}(u_n)$ converges weakly in $L^s(Q_T)$ to some function Φ. Therefore u satisfies

$$u_t - \Delta u + \Phi = 0$$

in $\mathcal{D}'(Q_T)$. From the compactness properties of the semi-group, we get

$$u(t) = S(t)u_0 - \int_0^t S(t-s)\Phi(s)\,\mathrm{d}s.$$

From the chain rule, we deduce that $\Phi = 0$, a.e. on the set $\{u = 0\}$. And $h_{n,p}(v_n)g_{n,0}(u_n)$ converges to $|v|^p$, a.e. on the set $\{u \neq 0\}$, hence $\Phi = |v|^p$, a.e. on this set. This shows that $\Phi = |v|^p \operatorname{sign}_0 u$, and there holds

$$u(t) = S(t)(u_0) - \int_0^t S(t-s)v^p(s) \operatorname{sign}_0 u(s)\,\mathrm{d}s. \qquad \square$$

Now we prove Corollary 1.1. More precisely we get the following:

Corollary 3.1. *Assume that $u_0 \in L^{\theta_1}(\Omega)$ and $v_0 \in L^{\theta_2}(\Omega)$ for some $1 \leqslant \theta_1, \theta_2 \leqslant \infty$, and* (1.10) *holds. Then there exists a solution (u, v) of problem* (1.7)*. Moreover any solution satisfies $|v|^p|u|^k \in L^{s_1}(Q_T)$, $|u|^q|v|^\ell \in L^{s_2}(Q_T)$, for any*

$$1 \leqslant s_1 < (N+2)\Big/N\left(\frac{k}{\theta_1} + \frac{p}{\theta_2}\right), \qquad 1 \leqslant s_2 < (N+2)\Big/N\left(\frac{q}{\theta_1} + \frac{\ell}{\theta_2}\right). \tag{3.15}$$

Also $|v|^p|u|^k \in L^1((0,T), L^{S_1}(\Omega))$, $|u|^q|v|^\ell \in L^1((0,T), L^{S_2}(\Omega))$, for any

$$1 \leqslant S_1 < 1\Big/\left(\frac{k}{\theta_1} + \frac{p}{\theta_2} - \frac{2}{N}\right)^+, \qquad 1 \leqslant S_1 < \infty \quad \textit{if } \frac{k}{\theta_1} + \frac{p}{\theta_2} < \frac{2}{N}, \tag{3.16}$$

$$1 \leqslant S_2 < 1\Big/\left(\frac{q}{\theta_1} + \frac{\ell}{\theta_2} - \frac{2}{N}\right)^+, \qquad 1 \leqslant S_2 < \infty \quad \textit{if } \frac{q}{\theta_1} + \frac{\ell}{\theta_2} < \frac{2}{N}. \tag{3.17}$$

Thus, if $\theta_1 < \infty$, then $u \in C([0,\infty), L^{m_1}(\Omega))$, with $m_1 = \min(\theta_1, S_1)$; if $\theta_2 < \infty$, then $v \in C([0,\infty), L^{m_2}(\Omega))$, with $m_2 = \min(\theta_2, S_2)$.

Proof. First observe that if $\theta_1, \theta_2 = \infty$, the result follows directly from Theorem 1.1, since $S(\cdot)(|u_0| + |v_0|) \in L^\infty(Q_\infty)$. It also follows if $\theta_1, \theta_2 = 1$ under condition (1.10), which reduces to (1.11). Indeed for any $T > 0$, and any $r, s > 1$,

$$\begin{aligned}
&\int_0^T\!\!\int_\Omega \big(S(t)|u_0|\big)^{ks}\big(S(t)|v_0|\big)^{ps}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant \left(\int_0^T\!\!\int_\Omega \big(S(t)|u_0|\big)^{ksr}\,\mathrm{d}x\,\mathrm{d}t\right)^{1/r}\left(\int_0^T\!\!\int_\Omega \big(S(t)|v_0|\big)^{psr'}\,\mathrm{d}x\,\mathrm{d}t\right)^{1/r'}.
\end{aligned} \tag{3.18}$$

Now $S(\cdot)(|u_0|+|v_0|) \in L^\rho(Q_T)$ for any $1 \leqslant \rho < (N+2)/N$. Taking $r = (k+p)/k$, we have $ksr = psr' = (k+p)s$, and from (1.11) we can choose $1 \leqslant s < (N+2)/N(k+p)$, so that the right-hand side is finite. Hence (1.8) holds.

Let us come to the general case $1 \leqslant \theta_1, \theta_2 \leqslant \infty$. From (3.5), $(S(\cdot)|u_0|)^\sigma \in L^1(Q_T)$ for any $0 < \sigma < \theta_1(N+2)/N$, and $(S(\cdot)|v_0|)^\tau \in L^1(Q_T)$ for any $0 < \tau < \theta_2(N+2)/N$, since Ω is bounded. For any $r, s > 1$, there still holds (3.18). Let

$$c = \frac{Nk}{(N+2)\theta_1}, \qquad d = \frac{Np}{(N+2)\theta_2},$$

with the convention $c = 0$ if $\theta_1 = \infty$, $d = 0$ if $\theta_2 = \infty$. Then $c + d < 1$ from (1.10). We choose $s > 1$ such that $(c+d)s < 1$, and $r > 1$ such that $cs < 1/r < 1 - ds$, so that

$$ksr < \theta_1(N+2)/N \quad \text{and} \quad psr' < \theta_2(N+2)/N.$$

Then (1.8) holds, which proves the existence; and $|v|^p|u|^k \in L^{s_1}(Q_T)$, for any s_1 given by (3.15).

Also consider s and r as above, with moreover $N/(N+2) \leqslant (c+d)s$ and $1 - ds(N+2)/N \leqslant 1/r \leqslant cs(N+2)/N$. Then $ksr \geqslant \theta_1 \geqslant 1$ and $psr' \geqslant \theta_2 \geqslant 1$, and

$$\int_\Omega \left(S(t)|u_0|\right)^{ks}\left(S(t)|v_0|\right)^{ps} \mathrm{d}x\, \mathrm{d}t \leqslant \left(\int_\Omega \left(S(t)|u_0|\right)^{ksr} \mathrm{d}x\, \mathrm{d}t\right)^{1/r}\left(\int_\Omega \left(S(t)|v_0|\right)^{psr'} \mathrm{d}x\, \mathrm{d}t\right)^{1/r'}$$

for any $t > 0$. From (3.10), we derive

$$\begin{aligned}\left\|\left(S(t)|u_0|\right)^k\left(S(t)|v_0|\right)^p\right\|_{L^s(\Omega)} &\leqslant \left\|S(t)|u_0|\right\|^k_{L^{ksr}(\Omega)}\left\|S(t)|v_0|\right\|^p_{L^{psr'}(\Omega)} \\ &\leqslant t^{-(\alpha k+\beta p)}\|u_0\|^k_{L^{\theta_1}(\Omega)}\|v_0\|^p_{L^{\theta_2}(\Omega)},\end{aligned} \tag{3.19}$$

where $\alpha = (1/\theta_1 - 1/ksr)N/2$ and $\beta = (1/\theta_2 - 1/psr')N/2$. Thus $|v|^p|u|^k \in L^1((0,T), L^s(\Omega))$ for any $s \geqslant 1/(k/\theta_1 + p/\theta_2)$, such that $\alpha k + \beta p < 1$, that means $k/\theta_1 + p/\theta_2 - 2/N < 1/s$; hence for any $s \leqslant S_1$ satisfying (3.16), since Ω is bounded. Moreover $S(t)u_0 \in C([0,\infty), L^{\theta_1}(\Omega))$, if $\theta_1 < \infty$, and $u - S(t)u_0 \in C([0,\infty), L^{S_1}(\Omega))$, hence $u \in C([0,\infty), L^{m_1}(\Omega))$; and similarly for v. $\square$

Finally we come to the case of measures as initial data: $u_0, v_0 \in \mathcal{M}_b(\Omega)$.

Proof of Theorem 1.2. Let $u_0, v_0 \in \mathcal{M}_b(\Omega)$. Let $u_{0,n}, v_{0,n} \in L^\infty(\Omega)$ such that $u_{0,n},\ v_{0,n}$ are bounded in $L^1(\Omega)$ and converge weakly to $u_0,\ v_0$ in $\mathcal{M}_b(\Omega)$. As above problem (3.13) admits a solution (u_n, v_n). And

$$\left|u_n(t)\right| \leqslant S(t)|u_{0,n}|, \quad \left|v_n(t)\right| \leqslant S(t)|v_{0,n}| \quad \text{in } \Omega.$$

Hence $u_n(t),\ v_n(t)$ are bounded in $L^\infty_{\mathrm{loc}}(\overline{\Omega} \times (0,\infty))$. And

$$\begin{cases} \left|g_{n,p}\left(|v_n|\right)g_{n,k}(u_n)\right| \leqslant g_{n,p}\left(S(\cdot)|v_{0,n}|\right)g_{n,k}\left(S(\cdot)|u_{0,n}|\right) \leqslant \left|S(\cdot)|u_{0,n}|\right|^k\left|S(\cdot)|v_{0,n}|\right|^p, \\ \left|g_{n,q}\left(|u_n|\right)g_{n,\ell}(v_n)\right| \leqslant g_{n,q}\left(S(\cdot)|u_{0,n}|\right)g_{n,k}\left(S(\cdot)|v_{0,n}|\right) \leqslant \left|S(\cdot)|u_{0,n}|\right|^q\left|S(\cdot)|v_{0,n}|\right|^\ell. \end{cases}$$

Moreover for $s > 1$ small enough such that $(k+p)s < (N+2)/N$, and for any $T > 0$, we find

$$\|(S(\cdot)|u_{0,n}|)^k (S(\cdot)|v_{0,n}|)^p\|_{L^s(Q_T)} \leqslant \|S(\cdot)|u_{0,n}|\|^k_{L^{(k+p)s}(Q_T)} \|S(\cdot)|v_{0,n}|\|^p_{L^{(k+p)s}(Q_T)}$$

from (3.18); and $S(\cdot)(|u_{0,n}| + |v_{0,n}|)$ is bounded in $L^{(k+p)s}(Q_T)$, since $u_{0,n},\ v_{0,n}$ are bounded in $L^1(\Omega)$. Then $|S(\cdot)|u_{0,n}||^k |S(\cdot)|v_{0,n}||^p$ is bounded in $L^s(Q_T)$. Up to a subsequence, $u_n,\ v_n$ converge strongly in $L^r(Q_T)$ for $1 \leqslant r < (N+2)/N$, and a.e. in Q_∞ to some $u,\ v$. Moreover u_n satisfies

$$\int_0^T\int_\Omega \big(-u_n\psi_t - u_n\Delta\psi + g_{n,p}(|v_n|)g_{n,k}(u_n)\psi\big)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \psi(0)u_{0,n}\,\mathrm{d}x, \tag{3.20}$$

for any $\psi \in C^\infty(Q_T)$ with compact support in $\Omega \times [0,T)$ (resp. in $\overline{\Omega} \times [0,T)$, such that $\partial\varphi/\partial\nu = 0$). If $k \neq 0$, then $g_{n,p}(|v_n|)g_{n,k}(u_n)$ converges to $|v|^p|u|^{k-1}$, a.e. in Q_∞ and in $L^1(Q_T)$. Thus we can go to the limit in (3.20) and deduce

$$\int_0^T\int_\Omega \big(-u\psi_t - u\Delta\psi + |v|^p|u|^{k-1}u\psi\big)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \psi(0)\,\mathrm{d}u_0. \tag{3.21}$$

If $k = 0$, then, as in Theorem 3, $g_{n,p}(|v_n|)g_{n,0}(u_n)$ converges weakly in $L^s(Q_T)$ to $\Phi = |v|^p \operatorname{sign}_0 u$, since $s > 1$, hence (3.21) is still valid. Similarly for v, thus (u,v) is a solution of (1.7). □

Remark 3.3. In the scalar case of Eq. (1.3) with Dirichlet (or Neuman conditions), if $w(0) \in L^1(\Omega)$, no condition on the power Q is required for existence, see [11, Remark 5]. Condition $Q < (N+2)/N$, analogous to (1.11) is only required when $w(0) \in \mathcal{M}_b(\Omega)$. The proof of the existence when $w(0) \in L^1(\Omega)$ lies essentially on the monotonicity of the nonlinear term, and cannot be extended to system (1.1).

Remark 3.4. Let $x_0, y_0 \in \Omega$ be fixed. Consider Dirac masses $\delta_{x_0},\ \delta_{y_0}$ at these points. From Theorem 1.2, if $\max(k+p, q+\ell) < (N+2)/N$, then problem (1.7), with initial data

$$u_0 = U_0 + \alpha_0\delta_{x_0}, \qquad v_0 = V_0 + \beta_0\delta_{y_0}, \tag{3.22}$$

has a solution for any real numbers $\alpha_0,\ \beta_0$ and any $U_0, V_0 \in L^1(\Omega)$. If the two Dirac masses are not at the same point, that is if $x_0 \neq y_0$, and for example $U_0, V_0 \in L^\infty(\Omega)$ (in particular $U_0 = V_0 = 0$), we can improve this result: in that case we have existence whenever

$$\max(p,q,k,\ell) < (N+2)/N. \tag{3.23}$$

Indeed in the proof of Theorem 1.2, we can approximate $u_0,\ v_0$ by

$$u_{0,n} = U_0 + \alpha_0\rho_n(\cdot - x_0), \qquad v_{0,n} = V_0 + \beta_0\rho_n(\cdot - y_0),$$

where ρ_n is a regularizing sequence with support in $B(0,m)$, with $m = |x_0 - y_0|/2$. As above, problem (3.13) admits a solution (u_n, v_n). And there exists $C > 0$, such that, for any $t \geqslant 0$,

$$S(t)|u_{0,n}| \leqslant C + |\alpha_0|S(t)\rho_n(\cdot - x_0), \quad S(t)|v_{0,n}| \leqslant C + |\beta_0|S(t)\rho_n(\cdot - y_0), \quad \text{a.e. in } \Omega.$$

Then for fixed $T > 0$, $S(\cdot)|v_{0,n}|$ is bounded in $(\Omega\backslash B(y_0, m) \times [0, T]$, and $S(\cdot)|u_{0,n}|$ is bounded in $(\Omega\backslash B(x_0, m) \times [0, T]$. Thus $|S(\cdot)|u_{0,n}||^k |S(\cdot)|v_{0,n}||^p$ is bounded in $L^s(Q_T)$ for some $s > 1$, since $k, p < (N+2)/N$. The same property holds for $|S(\cdot)|u_{0,n}||^q |S(\cdot)|v_{0,n}||^\ell$, since $q, \ell < (N+2)/N$. More generally, we have existence in Theorem 1.2 under the assumption (3.23), as soon as $u_0, v_0 \in \mathcal{M}_b(\Omega)$ with compact disjoint supports.

Remark 3.5. We have supposed that $p, q > 0$ in order that the solutions u, v are actually coupled. In fact the existence theorems are still available when $p = 0$ or $q = 0$. In the same way, we get an existence result for a scalar equation:

Proposition 3.1. *For any $k \geqslant 0$, any $u_0 \in L^1(\Omega)$ and any measurable function $W \geqslant 0$ on Q_∞ such that*

$$\bigl(S(\cdot)|u_0|\bigr)^k W(\cdot) \in L^s_{\mathrm{loc}}\bigl(\overline{Q_\infty}\bigr), \quad \textit{for some } s \geqslant 1 \ (s > 1 \textit{ if } k = 0), \tag{3.24}$$

there exists a unique weak solution u of problem

$$\begin{cases} u_t - \Delta u + W|u|^{k-1}u = 0, & \textit{in } Q_\infty, \\ u = 0 \ (\textit{resp. } \partial u/\partial \nu = 0), & \textit{on } \partial\Omega \times (0, \infty), \\ u(0) = u_0, & \textit{in } \Omega. \end{cases} \tag{3.25}$$

It is also the case when $u_0 \in \mathcal{M}_b(\Omega)$ and

$$W \in L^\eta_{\mathrm{loc}}\bigl(\overline{Q_\infty}\bigr), \quad \textit{for some } \eta > 1, \textit{ such that } k\eta' < (N+2)/N. \tag{3.26}$$

Moreover if u is a subsolution, with $u \in L^1_{\mathrm{loc}}(\overline{Q_\infty})$ and $u_t - \Delta u \in L^1_{\mathrm{loc}}(\overline{Q_\infty})$, and u' is a supersolution, then $u \leqslant u'$ a.e. in Q_∞.

Proof. First assume that $u_0 \in L^1(\Omega)$. Let W_n, $u_{0,n}$ be the truncatures of W, u_0 by $\pm n$ and u_n be the solution of the approximate problem

$$\begin{cases} u_{n,t} - \Delta u_n + W_n g_{n,k}(u_n) = 0, & \text{in } Q_\infty, \\ u_n = 0 \ (\text{resp. } \partial u_n/\partial \nu = 0), & \text{on } \partial\Omega \times (0, \infty), \\ u_n(0) = u_{0,n}, & \text{in } \Omega. \end{cases} \tag{3.27}$$

Then

$$\bigl|W_n g_{n,k}(u_n)\bigr| \leqslant W\bigl(S(\cdot)|u_0|\bigr)^k,$$

hence we obtain the existence of a solution of (3.25) from (3.24), as in Theorem 1.1. Now assume that $u_0 \in \mathcal{M}_b(\Omega)$, and consider the approximate problem (3.27), where $u_{0,n}$ is given as in Theorem 1.2. If $k \neq 0$, then $W_n|u_n|^k$ is bounded in $L^s(Q_T)$, for $s > 1$ small enough. Indeed for any $r > 1$,

$$\bigl\|\bigl(S(\cdot)|u_{0,n}|\bigr)^k W_n\bigr\|_{L^s(Q_T)} \leqslant \bigl\|\bigl(S(\cdot)|u_0|\bigr)\bigr\|^k_{L^{ksr}(Q_T)} \|W\|_{L^{sr'}(Q_T)};$$

Since $k\eta' < (N+2)/N$, we can take $1 \leqslant s < \eta/(1+k\eta N/(N+2))$, and $r' = \eta/s$; thus $ksr < (N+2)/N$ and the right-hand side is finite. If $k = 0$, then $W|u_n|^k = W \in L^\eta(Q_T)$ and the conclusion follows again. The existence of a solution of (3.25) follows as in Theorem 1.2. Moreover the solution is unique, from monotonicity, and the comparison principle holds. □

3.4. *A principle of cross comparison*

Now we consider the case of nonnegative initial data, and we look for some kind of comparison principles. Notice a simple property of contravariance. Assume that (u, v) and (u', v') are two nonnegative solutions of problem (1.7), and (1.8) holds; then

$$v \leqslant v', \text{ in } Q_\infty \quad \Longrightarrow \quad u \geqslant u', \text{ in } Q_\infty; \tag{3.28}$$

indeed

$$u_t - \Delta u + v'^p u^k \geqslant 0 \quad \text{and} \quad u'_t - \Delta u' + v'^p u'^k = 0,$$

and $v'^p u^k \in L^1(Q_T)$, for any $T > 0$, from (1.8); then (3.28) follows from the usual comparison principle. Thus we cannot expect that the conditions $u(0) \leqslant u'(0)$ and $v(0) \leqslant v'(0)$ imply $u \leqslant u'$ and $v \leqslant v'$, in Q_∞. Using this idea we can prove the existence of minimal–maximal solution and a maximal–minimal one:

Proof of Theorem 1.3. From Remark 3.1, we can suppose that $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$. First assume that $k \geqslant 0$, and $\ell > 0$. Let w be the solution of heat equation with initial data u_0:

$$\begin{cases} w_t - \Delta w = 0, & \text{in } Q_\infty, \\ w = 0 \text{ (resp. } \partial w/\partial\nu = 0), & \text{on } \partial\Omega \times (0, \infty), \\ w(0) = u_0. \end{cases}$$

We construct a first sequence of approximate solutions (u_n, v_n), such that for any $n \geqslant 1$,

$$\begin{cases} u_n = v_n = 0 \text{ (resp. } \partial u_n/\partial\nu = \partial v_n/\partial\nu = 0), & \text{on } \partial\Omega \times (0, \infty), \\ u_n(0) = u_0, \quad v_n(0) = v_0, & \text{in } \Omega. \end{cases} \tag{3.29}$$

We take for v_1 the solution of heat equation

$$v_{1t} - \Delta v_1 = 0, \quad \text{in } Q_\infty,$$

hence $v_1 > 0$, and $v \leqslant v_1$ in Q_∞ from the maximum principle. Then we define a nonnegative function $u_1 \geqslant 0$ by

$$u_{1t} - \Delta u_1 + v_1^p |u_1|^{k-1} u_1 = 0, \quad \text{in } Q_\infty.$$

Such a solution exists from Proposition 3.1 with $W = v_1^p$. Indeed if $u_0 \in L^1(\Omega)$, then $v_1 = S(\cdot)v_0$, and (3.24) holds from (1.8); if $u_0 \in \mathcal{M}_b(\Omega)$, then (3.26) holds with condition (1.11), with $\eta = 1 + k/p$ if

$k \neq 0$, and any $\eta \in (1, (N+2)/Np)$ if $k = 0$. And $v^p|u_1|^{k-1}u_1 \in L^1_{\mathrm{loc}}(\overline{Q_\infty})$, moreover $u_{1t} - \Delta u_1 + v^p|u_1|^{k-1}u_1 \leqslant 0$, hence $u_1 \leqslant w$. In the same way we define a unique $v_2 \geqslant 0$ by

$$v_{2t} - \Delta v_2 + u_1^q|v_2|^{\ell-1}v_2 = 0, \quad \text{in } Q_\infty,$$

and $v_2 \leqslant v_1$. We define u_2 by

$$u_{2t} - \Delta u_2 + v_2^p|u_2|^{k-1}u_2 = 0, \quad \text{in } Q_\infty,$$

and $u_2 \geqslant u_1$. By induction we define $v_n \geqslant 0$ and $u_n \geqslant 0$, for any $n \geqslant 2$, by

$$\begin{cases} v_{nt} - \Delta v_n + u_{n-1}^q|v_n|^{\ell-1}v_n = 0, & \text{in } Q_\infty, \\ u_{nt} - \Delta u_n + v_n^p|u_n|^{k-1}u_n = 0, & \text{in } Q_\infty; \end{cases}$$

and we find

$$0 \leqslant u_1 \leqslant u_2 \leqslant \cdots \leqslant u_n \leqslant u_{n+1} \leqslant \cdots \leqslant w, \qquad 0 \leqslant \cdots \leqslant v_{n+1} \leqslant v_n \leqslant \cdots \leqslant v_2 \leqslant v_1. \tag{3.30}$$

Then (u_n, v_n) converges, a.e. in Q_∞, to some (U, V), from monotonicity. And $v_n^p|u_n|^{k-1}u_n \to V^p|U|^{k-1}U$, even if $k = 0$, since (u_n) is nondecreasing. The convergence holds in $L^1(Q_T)$, from (1.8) or (1.11), since $v_n^p u_n^k \leqslant v_1^p w^k$. If $\ell > 0$, then $u_{n-1}^q|v_n|^{\ell-1}v_n \to U^q|V|^{\ell-1}V$, a.e. in Q_∞ and in $L^1(Q_T)$. If $\ell = 0$, $\operatorname{sign}_0 v_n$ does not converge in general to $\operatorname{sign}_0 v$, because the sequence (v_n) is nonincreasing. As in Theorem 1.1 after exchanging u and v, we deduce that $u_{n-1}^q \operatorname{sign}_0 v_n$ converges to $\Psi = U^q \operatorname{sign}_0 V$, weakly in $L^s_{\mathrm{loc}}(\overline{Q_\infty})$ for some $s > 1$. Thus (U, V) is a solution of the system. Let (u, v) be any other solution. Then $u_1 \leqslant u$, and $v \leqslant v_2$, since

$$u_{1t} - \Delta u_1 + v^p|u_1|^{k-1}u_1 \leqslant 0, \qquad v_{2t} - \Delta v_2 + u^q|v_2|^{\ell-1}v_2 \geqslant 0.$$

By induction, $u_1 \leqslant u$ and $v \leqslant v_n$ for any $n \geqslant 1$, hence $U \leqslant u$ and $v \leqslant V$.

Exchanging the two equations, we define other sequence $(\tilde{u}_n, \tilde{v}_n)$ satisfying also (3.29), with initial data (u_0, v_0), such that

$$\begin{cases} \tilde{u}_{1t} - \Delta \tilde{u}_1 = 0, & \text{in } Q_\infty, \\ \tilde{v}_{1t} - \Delta \tilde{v}_1 + \tilde{u}_1^q|\tilde{v}_1|^{\ell-1}\tilde{v}_1 = 0, & \text{in } Q_\infty, \end{cases}$$

and

$$\begin{cases} \tilde{u}_{nt} - \Delta \tilde{u}_n + \tilde{v}_{n-1}^p|\tilde{u}_n|^{k-1}\tilde{u}_n = 0, & \text{in } Q_\infty, \\ \tilde{v}_{nt} - \Delta \tilde{v}_n + \tilde{u}_n^q|\tilde{v}_n|^{\ell-1}\tilde{v}_n = 0, & \text{in } Q_\infty \end{cases}$$

for any $n \geqslant 2$; then

$$\tilde{v}_1 \leqslant \tilde{v}_2 \leqslant \cdots \leqslant \tilde{v}_n \leqslant \tilde{v}_{n+1} \leqslant .. \leqslant v, \qquad u \leqslant \cdots \leqslant \tilde{u}_{n+1} \leqslant \tilde{u}_n \leqslant \cdots \leqslant \tilde{u}_2 \leqslant \tilde{u}_1,$$

and $(\tilde{u}_n, \tilde{v}_n)$ converges to a solution $(\widetilde{U}, \widetilde{V})$ of the system, and $u \leqslant \widetilde{U}$ and $\widetilde{V} \leqslant v$.

Now assume $0 \leqslant u_0 \leqslant u_0'$ and $0 \leqslant v_0' \leqslant v_0$. By induction we find easily that $u_n \leqslant u_n'$, $v_n' \leqslant v_n$, and $\tilde{u}_n \leqslant \tilde{u}_n'$, $\tilde{v}_n' \leqslant \tilde{v}_n$, with obvious notations, hence (1.13) follows. □

Remark 3.6. Consider problem (1.7) with Neuman conditions, and assume that $u_0, v_0 \in L^\infty(\Omega)$, $u_0, v_0 \geqslant 0$, and $\inf_{x\in\Omega} v_0(x) > 0$. Let (u', v') be the unique solution of the o.d.e. problem (1.5) such that $u'(0) = \|u_0\|_{L^\infty(\Omega)}$ and $v'(0) = \inf_{x\in\Omega} v_0(x)$. Then any solution (u, v) of (1.7) satisfies

$$u \leqslant u' \quad \text{and} \quad v' \leqslant v, \quad \text{in } Q_\infty.$$

Indeed taking $u'_0 = \|u_0\|_{L^\infty(\Omega)}$ and $v'_0 = \inf_{x\in\Omega} v_0(x)$, the sequences (u'_n) and (v'_n) constructed above only depend on t, and converge respectively to u' and v'.

4. Links between u and v

Here we study the question of invariant regions whenever $u_0, v_0 \in L^\infty(\Omega)$ and $ab > 0$, and prove Theorem 1.4. Our method is based on the ideas of [6] for elliptic systems, also used in [9], namely a comparison between two powers of u and v, chosen in a suitable way. First let us give a scheme of the proof of the main points of Theorem 1.4. The basic idea is to consider the function

$$Y = v - \lambda^* u^{b/a},$$

where

$$\lambda^* = (a/b)^{1/a}. \tag{4.1}$$

When $b/a \geqslant 1$, the formal computation of ΔY leads to an equation of the form

$$Y_t - \Delta Y + K = M,$$

where $M = cu^{b/a-2}|\nabla u|^2$, with $c = \lambda^*(b/a)(b/a - 1) \geqslant 0$, and K has the sign of Y if $a \leqslant 1$, and the opposite sign if $a > 1$. Formally, if $a \leqslant 1$, if $Y(0) \geqslant 0$, then $Y \geqslant 0$ from the maximum principle. Moreover, if $\delta \geqslant 0$, we prove that $K \leqslant CY$ for some $C > 0$, hence $Y > 0$ or $Y \equiv 0$ from the strict maximum principle. If $a > 1$, and $\delta \geqslant 0$, then we prove that $K \leqslant CY^-$, hence again $Y(0) \geqslant 0$, then $Y \geqslant 0$ from the maximum principle. Technically, one has to justify the use of these maximum principles, because function Y is not regular enough: in particular, ΔY is not defined a.e., due of the term M. This is the purpose of next lemma.

Lemma 4.1. *Let (u, v) be any solution of (1.7), with $u_0, v_0 \in L^\infty(\Omega)$, $u_0, v_0 \geqslant 0$. Let $\alpha \geqslant 1 \geqslant \beta > 0$, such that $\beta - 1 + \ell \geqslant 0$, and*

$$Y = v^\beta - \lambda u^\alpha, \quad \lambda > 0, \tag{4.2}$$

$$K = \beta u^q |v|^{\beta-2+\ell} v - \alpha\lambda v^p |u|^{\alpha-2+k} u, \tag{4.3}$$

with the convention (1.2). Then for any real $C > 0$, and any $t > 0$,

$$\mathrm{e}^{-Ct} \int_\Omega (Y^-)^2(t)\,\mathrm{d}x \leqslant \int_\Omega (Y^-)^2(0)\,\mathrm{d}x + \int_0^t \int_\Omega \mathrm{e}^{-C\tau} \bigl(2K(\tau) - CY^-(\tau)\bigr) Y^-(\tau)\,\mathrm{d}x\,\mathrm{d}\tau. \tag{4.4}$$

Moreover $W(t) = Y(t) + S(t)Y(0)^-$ *satisfies*

$$\int_\Omega (W^-)^2(t)\,\mathrm{d}x \leqslant \int_0^t\int_\Omega 2K(\tau)W^-(\tau)\,\mathrm{d}x\,\mathrm{d}\tau. \tag{4.5}$$

Also for any $C' > 0$, $Z(t) = \mathrm{e}^{C't}Y(t) - S(t)Y(0)$ *satisfies*

$$\int_\Omega (Z^-)^2(t)\,\mathrm{d}x \leqslant \int_0^t\int_\Omega \bigl(2K(\tau) - C'Y(\tau)\bigr)Z^-(\tau)\,\mathrm{d}x\,\mathrm{d}\tau. \tag{4.6}$$

Proof. Let $T > 0$ be fixed, and

$$F = v^p|u|^{k-1}u, \qquad G = u^q|v|^{\ell-1}v,$$

hence $F, G \in L^\infty(Q_T)$. Let $(u_{0,n}, F_n), (v_{0,n}, G_n) \in \mathcal{D}(\Omega) \times \mathcal{D}(Q_T)$ with $F_n, G_n \geqslant 0$, converging respectively to (u_0, F), (v_0, G) in $L^2(\Omega) \times L^2(Q_T)$, with $u_{0,n}, v_{0,n}$ bounded in $L^\infty(\Omega)$ and F_n, G_n bounded in $L^\infty(Q_T)$. Then there exist unique classical solutions u_n, v_n of problems

$$\begin{cases} u_{n,t} - \Delta u_n + F_n = 0, \quad v_{n,t} - \Delta v_n + G_n = 0, & \text{in } Q_T,\\ u_n = v_n = 0 \text{ (resp. } \partial u_n/\partial\nu = \partial v_n/\partial\nu = 0), & \text{on } \partial\Omega \times (0,T),\\ u_n(0) = u_{0,n}, \quad v_n(0) = v_{0,n}, & \text{in } \Omega. \end{cases} \tag{4.7}$$

And u_n, v_n are bounded in $L^\infty(Q_T)$ and converge to u, v, strongly in $C([0,T], L^2(\Omega))$, and a.e. in Q_T, and in $L^2((0,T), W^{1,2}(\Omega))$. Let $\varepsilon > 0$, and $\varepsilon' = (\lambda\varepsilon^\alpha)^{1/\beta}$. Let us define

$$f_{n,\varepsilon} = (u_n + \varepsilon)^\alpha, \qquad g_{n,\varepsilon'} = (v_n + \varepsilon')^\beta, \qquad Y_{n,\varepsilon} = g_{n,\varepsilon'} - \lambda f_{n,\varepsilon}.$$

Then in the classical sense

$$\begin{cases} (f_{n,\varepsilon})_t - \Delta f_{n,\varepsilon} + \alpha(u_n + \varepsilon)^{\alpha-1}F_n = -\alpha(\alpha-1)(u_n+\varepsilon)^{\alpha-2}|\nabla u_n|^2 \leqslant 0,\\ (g_{n,\varepsilon'})_t - \Delta g_{n,\varepsilon'} + \beta(v_n + \varepsilon')^{\beta-1}G_n = \beta(1-\beta)(v_n+\varepsilon')^{\beta-2}|\nabla v_n|^2 \geqslant 0. \end{cases}$$

Thus $Y_{n,\varepsilon}$ satisfies the equation

$$(Y_{n,\varepsilon})_t - \Delta Y_{n,\varepsilon} + K_{n,\varepsilon} = M_{n,\varepsilon}, \tag{4.8}$$

where $K_{n,\varepsilon}, M_{n,\varepsilon} \in L^\infty(Q_T)$ are defined by

$$\begin{cases} K_{n,\varepsilon} = \beta(v_n + \varepsilon')^{\beta-1}G_n - \alpha\lambda(u_n+\varepsilon)^{\alpha-1}F_n,\\ M_{n,\varepsilon} = \beta(1-\beta)(v_n+\varepsilon')^{\beta-2}|\nabla v_n|^2 + \alpha(\alpha-1)\lambda(u_n+\varepsilon)^{\alpha-2}|\nabla u_n|^2 \geqslant 0. \end{cases}$$

For the Dirichlet problem, there holds $Y_{n,\varepsilon} = 0$ on $\partial\Omega \times (0,T)$, from the choice of ε'. For the Neuman problem, we find $\partial Y_{n,\varepsilon}/\partial\nu = \beta(v_n+\varepsilon')^{\beta-1}\partial v_n/\partial\nu - \alpha\lambda(u_n+\varepsilon)^{\alpha-1}\partial u_n/\partial\nu = 0$. In any case

$$Y_{n,\varepsilon}(t) = S(t)\bigl(Y_{n,\varepsilon}(0)\bigr) + \int_0^t S(t-s)(M_{n,\varepsilon} - K_{n,\varepsilon})(s)\,\mathrm{d}s.$$

Now for any $C > 0$, multiplying (4.8) by $\mathrm{e}^{-Ct}Y_{n,\varepsilon}^-$, and integrating over Ω, we find

$$\mathrm{e}^{Ct}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-Ct}\int_\Omega (Y_{n,\varepsilon}^-)^2\,\mathrm{d}x\right) \leqslant \int_\Omega \big(2K_{n,\varepsilon}(t) - CY_{n,\varepsilon}^-(t)\big)Y_{n,\varepsilon}^-(t)\,\mathrm{d}x.$$

We derive, for any $t \in [0,T]$,

$$\mathrm{e}^{-Ct}\int_\Omega (Y_{n,\varepsilon}^-)^2(t)\,\mathrm{d}x \leqslant \int_\Omega (Y_{n,\varepsilon}^-)^2(0)\,\mathrm{d}x + \int_0^t\int_\Omega \mathrm{e}^{-C\tau}\big(2K_{n,\varepsilon}(\tau) - CY_{n,\varepsilon}^-(\tau)\big)Y_{n,\varepsilon}^-(\tau)\,\mathrm{d}x\,\mathrm{d}\tau. \quad (4.9)$$

In the same way, let

$$W_{n,\varepsilon}(t) = Y_{n,\varepsilon}(t) + S(t)\big(Y_{n,\varepsilon}(0)^-\big);$$

then $W_{n,\varepsilon}$ satisfies the same equation (4.8) as $Y_{n,\varepsilon}$, and multiplying by $W_{n,\varepsilon}^-$, we obtain

$$\int_\Omega (W_{n,\varepsilon}^-)^2(t)\,\mathrm{d}x \leqslant 2\int_0^t\int_\Omega K_{n,\varepsilon}(\tau)W_{n,\varepsilon}^-(\tau)\,\mathrm{d}x\,d\tau. \quad (4.10)$$

Also, for any $C' > 0$,

$$Z_{n,\varepsilon}(t) = \mathrm{e}^{C't}Y_{n,\varepsilon}(t) - S(t)Y_{n,\varepsilon}(0)$$

satisfies the equation

$$(Z_{n,\varepsilon})_t - \Delta Z_{n,\varepsilon} = \mathrm{e}^{C't}(M_{n,\varepsilon} + C'Y_{n,\varepsilon} - K_{n,\varepsilon}).$$

Multiplying by $Z_{n,\varepsilon}^-$, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega (Z_{n,\varepsilon}^-)^2\,\mathrm{d}x\right) \leqslant \int_\Omega \big(2K_{n,\varepsilon}(t) - C'Y_{n,\varepsilon}(t)\big)Z_{n,\varepsilon}^-(t)\,\mathrm{d}x,$$

hence

$$\int_\Omega (Z_{n,\varepsilon}^-)^2(t)\,\mathrm{d}x \leqslant \int_0^t\int_\Omega \big(2K_{n,\varepsilon}(\tau) - C'Y_{n,\varepsilon}(\tau)\big)Z_{n,\varepsilon}^-(\tau)\,\mathrm{d}x\,\mathrm{d}\tau. \quad (4.11)$$

For fixed ε, $K_{n,\varepsilon}$ is bounded in $L^\infty(Q_T)$, and $M_{n,\varepsilon}$ is bounded in $L^1(Q_T)$. Hence as $n \to \infty$, $Y_{n,\varepsilon}$ converges to

$$Y_\varepsilon = g_{\varepsilon'} - \lambda f_\varepsilon = (v + \varepsilon')^\beta - \lambda(u+\varepsilon)^\alpha,$$

strongly in $C([0,T], L^1(\Omega))$, and a.e. in Q_T; and in $L^m(Q_T)$ for any $m > 1$, since $Y_{n,\varepsilon}$ is bounded in $L^\infty(Q_T)$. And $K_{n,\varepsilon}$ converges to

$$K_\varepsilon = \beta(v+\varepsilon')^{\beta-1}u^q|v|^{\ell-1}v - \alpha(u+\varepsilon)^{\alpha-1}v^p|u|^{k-1}u$$

in $L^m(Q_T)$ and a.e. in Q_T. Taking $m = 2$, we can pass to the limit in (4.9), and obtain

$$\mathrm{e}^{-Ct}\int_\Omega \left(Y_\varepsilon^-\right)^2(t)\,\mathrm{d}x \leqslant \int_\Omega \left(Y_\varepsilon^-\right)^2(0)\,\mathrm{d}x + \int_0^t\int_\Omega \mathrm{e}^{-C\tau}\left(2K_\varepsilon(\tau) - CY_\varepsilon^-(\tau)\right)Y_\varepsilon^-(\tau)\,\mathrm{d}x\,\mathrm{d}\tau. \tag{4.12}$$

Now let us go to the limit as $\varepsilon \to 0$. Then K_ε converges to function K defined by (4.3) (even when $\ell = 0$, since in that case $\beta = 1$), a.e. in Q_T, and strongly in $L^2(Q_T)$. Indeed

$$(v+\varepsilon')^{\beta-1}u^q|v|^{\ell-1}v \leqslant u^q v^{\beta-1+\ell},$$

and $u^q v^{\beta-1+\ell}$ is bounded in Q_T, from the assumption $\beta - 1 + \ell \geqslant 0$. And Y_ε converges to Y defined by (4.2), a.e. in Q_T, and strongly in $L^2(Q_T)$. And $Y_\varepsilon(0)$ converges to $v_0^\beta - \lambda u_0^\alpha$, strongly in $L^2(\Omega)$. Then we can go to the limit in (4.12), and deduce (4.4). Similarly (4.5) and (4.6) follow from (4.10) and (4.11). $\square$

Remark 4.1. For studying the invariance property of the region $\{au^b \leqslant bv^a\}$, the simplest choice would be to compare u^b and v^a, that means to choose $\alpha = b$ and $\beta = a$ in Lemma 4.1, and $\lambda = a/b$. It works in a restrictive case

$$0 < a \leqslant 1 \leqslant b, \tag{4.13}$$

that means $\ell - 1 < p \leqslant \ell$ and $k \leqslant q$. Indeed in that case

$$K = 0, \qquad Y = v^a - (a/b)u^b.$$

Applying (4.6) with $C' = 0$, we deduce that

$$v^a(t) - (a/b)u^b(t) \geqslant S(t)\left(v_0^a - (a/b)u^b\right). \tag{4.14}$$

Therefore, if $au_0^b \leqslant bv_0^a$, a.e. in Ω, then $au^b \leqslant bv^a$ in Q_∞. Moreover if $au_0^b \not\equiv bv_0^a$, then $au^b < bv^a$ in Q_∞. Thus Theorem 1.4 is proved in that special case. A first step had been done in that case in $\mathbb{R}^N$ by Kalashnikov in [21], with assumption (4.13) and moreover $k, \ell < 1$, in order to get strict positivity or extinction properties. Also his method only proved the existence of invariant regions of the form $\{au^b + \varepsilon \leqslant bv^a\}$ for some $\varepsilon > 0$.

Now we prove the comparison theorem in the general case:

Proof of Theorem 1.4. (1) Under our assumptions, we have $ab > 0$, and $b/a \geqslant 1$. First take $\alpha = b/a$ and $\beta = 1$ in Lemma 4.1 and $\lambda = \lambda^*$, where λ^* is defined in (4.1), hence

$$Y = v - \lambda^* u^{b/a}, \qquad K = u^q v^\ell \operatorname{sign}_0 v - \frac{b}{a}\lambda^* u^{b/a-1} v^p u^k \operatorname{sign}_0 u.$$

We can write K under the forms

$$K = u^q v^p\left(v^{1-a} - \left(\lambda^* u^{b/a}\right)^{1-a}\right), \quad \text{if } a \leqslant 1, \tag{4.15}$$

$$K = \left(\lambda^*\right)^{1-a} u^d v^\ell\left(\left(\lambda^* u^{b/a}\right)^{a-1} - v^{a-1}\right), \quad \text{if } a > 1, \tag{4.16}$$

where

$$d = q + \frac{b}{a}(1-a) = \frac{b}{a} + k - 1 \geqslant k \geqslant 0;$$

in particular $K = 0$ if $a = 1$. When $a \leqslant 1$, notice that $\ell \neq 0$, and (4.15) holds even when $k = 0$, since in that case $b > a$.

(i) First suppose that $0 < a \leqslant b$.

• If $a \leqslant 1$, then $K \leqslant 0$ on the set $\{Y \leqslant 0\} = \{v \leqslant \lambda^* u^{b/a}\}$, from (4.15). Taking $C = 0$ in (4.4), we deduce that

$$\int_\Omega (Y^-)^2(t)\,\mathrm{d}x \leqslant \int_\Omega (Y^-)^2(0)\,\mathrm{d}x.$$

Thus the region $\{au^b \leqslant bv^a\} = \{\lambda^* u^{b/a} \leqslant v\} = \{Y \geqslant 0\}$ is invariant: if $Y(0) \geqslant 0$, then $Y \geqslant 0$ in Q_∞. Assume moreover that $\delta \geqslant 0$. Since $\lambda^* u^{b/a} \leqslant v$, we can write

$$\begin{aligned} K &= u^q v^p \big(v^{1-a} - (\lambda^* u^{b/a})^{1-a}\big) \leqslant u^q v^{p-a}\big(v - \lambda^* u^{b/a}\big) \\ &\leqslant (\lambda^*)^{-qa/b} v^{qa/b+p-a}\big(v - \lambda^* u^{b/a}\big) = (\lambda^*)^{-qa/b} v^{\delta/b}\big(v - \lambda^* u^{b/a}\big). \end{aligned}$$

Now $(\lambda^*)^{qa/b} v^{\delta/b}$ is bounded by a constant $C' > 0$, hence $K \leqslant C'Y$ in Q_∞. From (4.6), $Z(t) = \mathrm{e}^{C't} Y(t) - S(t)Y(0)$ satisfies $Z^-(t) = 0$ for any $t > 0$, that means

$$Y(t) \geqslant \mathrm{e}^{-C't} S(t)Y(0). \tag{4.17}$$

As a consequence, if $Y(0) \not\equiv 0$, then $Y > 0$ in Q_∞. This is also true when $1 \leqslant b$, from Remark 4.1.

• If $a > 1$, then K has the opposite sign of Y, from (4.16). On the set $\{Y \leqslant 0\}$, there holds

$$(\lambda^* u^{b/a})^{a-1} - v^{a-1} \leqslant c_a (\lambda^* u^{b/a})^{a-2} (\lambda^* u^{b/a} - v),$$

at any point where $u \neq 0$, with $c_a = \max(1, a-1)$. Assume moreover that $\delta \geqslant 0$. Observing that $d + (a - 2 + \ell)b/a = \delta/a \geqslant 0$, we find

$$K \leqslant c_a (\lambda^*)^{\ell-1} u^{\delta/a} (\lambda^* u^{b/a} - v),$$

on the set $\{Y \leqslant 0\} = \{v \leqslant \lambda^* u^{b/a}\}$, from (4.16), even at the points where $u = 0$. Since $u^{\delta/a}$ is bounded, it follows that $K \leqslant CY^-$ for some $C > 0$. We can apply (4.4) with this value of C. As a consequence, if $Y(0) \geqslant 0$, then $Y \geqslant 0$ in Q_∞, that means the region $\{au^b \leqslant bv^a\}$ is still invariant. Taking $C' = 0$ in Lemma 4.1, the function $t \mapsto Z(t) = Y(t) - S(t)Y(0)$ satisfies

$$\int_\Omega (Z^-)^2(t)\,\mathrm{d}x \leqslant \int_0^t \int_\Omega K(\tau) Z^-(\tau)\,\mathrm{d}x\,\mathrm{d}\tau \leqslant 0,$$

hence $Y(t) \geqslant S(t)Y(0)$, that means

$$v(t) - \lambda^* u^{b/a}(t) \geqslant S(t)\big(v_0 - \lambda^* u_0^{b/a}\big); \tag{4.18}$$

thus if $Y(0) \not\equiv 0$, then $Y > 0$, in Q_∞.

(ii) Assume that $b \leqslant a < 0$. Then K has the sign of Y since $a < 1$, hence as above the region

$$\{\lambda^* u^{b/a} \leqslant v\} = \{au^b \leqslant bv^a\} = \{|b|u^{|b|} \leqslant |a|v^{|a|}\}$$

is invariant. Moreover $1 - a > 1$, hence from (4.15),

$$K \leqslant C'(v - \lambda^* u^{b/a}),$$

for some $C' > 0$, so that (4.17) holds again; then if $Y(0) \not\equiv 0$, then $Y > 0$ in Q_∞.

(2) Here $1 < a \leqslant b$ and $u, v \in C(\overline{Q_\infty})$, and $au_0^b < bv_0^a$ in Ω. The function K has the opposite sign of Y, and $Y \in C(\overline{Q_\infty})$, and for any ball $B = B(x_0, r)$ such that $\overline{B} \subset \Omega$, we have $\min_{\overline{B}} Y(0) = m_B > 0$. Hence there exists $\tau > 0$ such that $\min_{\overline{B}} Y(t) > 0$ for any $t \in [0, \tau)$. Let

$$\tau_B = \sup\Big\{\tau > 0\colon \min_{\overline{B}} Y(t) > 0,\ \forall t \in [0, \tau)\Big\}.$$

If $\tau_B < \infty$, there exists $x \in \overline{B}$ such that $Y(x, \tau_B) = 0$. But $Y \geqslant 0$ on $\partial B \times (0, T)$, and consequently $Y(t) \geqslant S_B(t) m_B$, where S_B is the semi-group in B with Dirichlet conditions; hence $Y > 0$ in $[0, \tau] \times \overline{B}$, and we reach a contradiction. Then $\tau_B = \infty$, so that $Y > 0$ in Q_∞, that means $au^b < bv^a$ in Q_∞, and (4.18) holds again. □

Remark 4.2. If $0 < a \leqslant b$ and $a \leqslant 1$, or if $b \leqslant a \leqslant 0$, we can compare u and v at any time $t > 0$, *without comparison assumptions on the initial data* u_0, v_0: we claim that

$$v(t) - \lambda^* u^{b/a}(t) \geqslant -S(t)\big(v - \lambda^* u_0^{b/a}\big)^-, \quad \text{in } \Omega, \tag{4.19}$$

for any $t > 0$. Indeed taking again $\alpha = b/a$ and $\beta = 1$ in Lemma 4.1, and $Y = v - \lambda^* u^{b/a}$, the function $W = Y + S(\cdot)Y(0)^-$ satisfies, from (4.5),

$$\int_\Omega \big(W^-\big)^2(t)\,\mathrm{d}x \leqslant \int_0^t \int_\Omega K(\tau) W^-(\tau)\,\mathrm{d}x\,\mathrm{d}\tau \leqslant 0,$$

since $Y \leqslant 0$ on the set $\{W \leqslant 0\}$, and $K \leqslant 0$ on the set $\{W \leqslant 0\}$ from (4.15). Then $W \geqslant 0$, in Q_∞, which proves (4.19).

Remark 4.3. Under any of the assumptions (i) or (ii) of Theorem 1.4, one can easily show that the region $\{\lambda u^{b/a} \leqslant v\}$ is invariant for any $\lambda \geqslant \lambda^*$.

Remark 4.4. Consider the Neumann problem. If ($0 < a \leqslant b$ and $a \leqslant 1$ and $\delta \geqslant 0$) or $b \leqslant a < 0$, and moreover

$$\lambda^* u_0^{b/a} + \varepsilon \leqslant v_0, \quad \text{a.e. in } \Omega,$$

for some $\varepsilon > 0$, then from (4.17), there exists $C' > 0$ such that

$$\lambda^* u(t)^{b/a} + \varepsilon\,\mathrm{e}^{-C't} \leqslant v(t), \quad \text{in } Q_\infty.$$

If $a \leqslant 1 \leqslant b$, then from (4.14) the region of $\mathbb{R}^2$

$$\{au^b + \varepsilon \leqslant bv^a\}$$

is invariant. If $1 \leqslant a \leqslant b$ and ($\delta \geqslant 0$,or $u, v \in C(\overline{Q_\infty})$), then from (4.18) the region

$$\{\lambda^* u^{b/a} + \varepsilon \leqslant v\}$$

is invariant.

Remark 4.5. Under any of the assumptions of Theorem 1.4, we can extend Remark 2.2 to system (1.1). We have $\lambda^* u^{b/a} \leqslant v$ in Q_∞, hence $0 \leqslant (\lambda^*)^p u^{1+\delta/a} \leqslant v^p |u|^{k-1} u$. Thus u satisfies the scalar inequality

$$u_t - \Delta u + \left(\frac{a}{b}\right)^{p/a} u^{\mathbf{Q}} \leqslant 0 \tag{4.20}$$

in $\mathcal{D}'(Q_\infty)$, where $\mathbf{Q}$ is defined at (1.6), and more precisely

$$u(t) \leqslant S(t)u_0 - \left(\frac{a}{b}\right)^{p/a} \int_0^t S(t-s) u^{\mathbf{Q}}(s)\,\mathrm{d}s. \tag{4.21}$$

Similarly $u^q |v|^{\ell-1} v \leqslant (\lambda^*)^{-aq/b} v^{1+\delta/b}$, and $v^{1+\delta/b} = v^{\mathbf{P}} \in L^1(Q_T)$, hence

$$v_t - \Delta v + \left(\frac{a}{b}\right)^{-q/b} v^{\mathbf{P}} \geqslant 0 \tag{4.22}$$

in $\mathcal{D}'(Q_\infty)$, and

$$v(t) \geqslant S(t)u_0 - \left(\frac{a}{b}\right)^{-q/b} \int_0^t S(t-s) v^{\mathbf{P}}(s)\,\mathrm{d}s. \tag{4.23}$$

Thus $u,\ v$ are respectively subsolution and supersolution of scalar equations of type (1.3) with the exponents $\mathbf{Q} > 1$ and $\mathbf{P} > 1$.

5. New existence results

Using the comparison properties, we can improve the existence result of Theorem 1.1 in case of nonnegative solutions:

Proof of Theorem 1.5. We set

$$u_{0,n} = \min(u_0, n), \qquad w_{0,n} = \min\bigl(v_0, \lambda^* n^{b/a}\bigr),$$

and consider $\mu_{0,n} \in L^1(\Omega)$, $\mu_{0,n} \geqslant 0$, bounded in $L^1(\Omega)$, such that $\mu_{0,n}$ converges weakly to μ_0 in $\mathcal{M}_b(\Omega)$, and define $v_{0,n} = w_{0,n} + \mu_{0,n}$. Then $u_{0,n}, v_{0,n} \in L^\infty(\Omega)$, and $\lambda^* u_{0,n}^{b/a} \leqslant w_{0,n} \leqslant v_{0,n}$, a.e. in Ω. From Theorem 1.1, there exists at least one nonnegative solution (u_n, v_n) of problem

$$\begin{cases} u_{n,t} - \Delta u_n + v_n^p |u_n|^{k-1} u_n = 0, & \text{in } Q_\infty, \\ v_{n,t} - \Delta v_n + u_n^q |v_n|^{\ell-1} v_n = 0, & \text{in } Q_\infty, \\ u_n = v_n = 0 \text{ (resp. } \partial u_n/\partial\nu = \partial v_n/\partial\nu = 0), & \text{on } \partial\Omega \times (0,T), \\ u_n(0) = u_{0,n}, \quad v_n(0) = v_{0,n}, & \text{on } \Omega. \end{cases}$$

From Theorem 1.4, one has $\lambda^* u_n^{b/a} \leqslant v_n$ in Q_∞. Hence

$$\begin{cases} 0 \leqslant v_n^p |u_n|^{k-1} u_n \leqslant (\lambda^*)^{-ak/b} v_n^{p+ka/b} = (\lambda^*)^{-ak/b} v_n^{1+\delta/b} = (\lambda^*)^{-ak/b} v_n^{\mathbf{P}}, \\ 0 \leqslant u_n^q |v_n|^{\ell-1} v_n \leqslant (\lambda^*)^{-aq/b} v_n^{\ell+qa/b} = (\lambda^*)^{-aq/b} v_n^{1+\delta/b} = (\lambda^*)^{-aq/b} v_n^{\mathbf{P}}. \end{cases}$$

Now $v_n(t) \leqslant S(t)v_{0,n}$, and $(v_{0,n})$ is bounded in $L^1(\Omega)$, hence $S(t)v_{0,n}$ is bounded in $L^r(Q_T)$ for any $T > 0$ and $1 \leqslant r < (N+2)/N$, in particular for some $r > \mathbf{P}$. As in the proof of Theorem 1.2, up to a subsequence, (u_n, v_n) converge to some (u, v), a.e. in Q_∞ and strongly in $L^r(Q_T)$ for $1 \leqslant r < (N+2)/N$, and any $T > 0$, and (u, v) is a solution of problem. □

Remark 5.1. Assumption (1.18) is an improvement of (1.11): indeed if $a \leqslant b$, then $p+k \leqslant q+\ell$, hence (1.11) implies $q + \ell < (N+2)/N$, and it is easy to verify that $\mathbf{P} \leqslant q + \ell$. Notice that (1.18) can be written under the form

$$\max\left(\frac{a}{\delta}, \frac{b}{\delta}\right) = \max\left(\frac{p+1-\ell}{\delta}, \frac{q+1-k}{\delta}\right) > \frac{N}{2}.$$

It defines a condition which is well adapted to the system, linked to the particular solutions defined in (2.3), (2.4). It appears also in the study of the stationary solutions of system (1.1) with $N/2$ replaced by $(N-2)/2$, see [8, Theorem 7.1]. When $k = \ell = 0$, the region in $\mathbb{R}^2$ defined by the relation

$$\max\left(\frac{p+1}{pq-1}, \frac{q+1}{pq-1}\right) > \frac{N}{2}$$

is delimited by to arcs of hyperbolas, intersecting at point $(p,q) = ((N+2)/N, (N+2)/N)$.

Remark 5.2. It seems difficult to prove that the existence results of Theorems 1.2 and 1.5 are optimal. Indeed the first question is the possible nonuniqueness of the solutions. For proving nonexistence when (1.11) or (1.18) does not hold, we have to restrict ourselves to the "reachable" solutions, that means those which can be obtained by limits of solutions of regular problems. Then nonexistence results can follow from suitable upper-estimates for the solutions. We give some results in that sense in [7].

6. Positivity or extinction properties

In this section we give sufficient conditions for positivity of one of the components u, v of the solutions of system (1.1), according to the values of p, q, k, ℓ, when the initial data are bounded. Our first result extends the result of Proposition 2.1(i) for the o.d.e. system (1.5):

Theorem 6.1. *Suppose that* $u_0, v_0 \in L^1(\Omega)$, *and* (1.8) *holds. If* $u_0 \geqslant 0,\ u_0 \not\equiv 0$, *and* $k \geqslant 1$, *then for any solution* (u, v) *of* (1.7),

$$u > 0, \quad \text{in } Q_\infty.$$

Similarly if $v_0 \geqslant 0,\ v_0 \not\equiv 0$, *and* $\ell \geqslant 1$, *then* $v > 0$ *in* Q_∞.

Proof. Let (u, v) be any solution of the problem. Since $u_0 \geqslant 0$, we know that $u \geqslant 0$ in Q_∞, from Lemma 3.1. From (1.8) and Remark 3.5, there exists a weak nonnegative solution of problem

$$\begin{cases} \tilde{u}_t - \Delta\tilde{u} + \big(S(\cdot)|v_0|\big)^p \tilde{u}^k = 0, & \text{in } Q_\infty, \\ \tilde{u} = 0 \text{ (resp. } \partial\tilde{u}/\partial\nu = 0), & \text{on } \partial\Omega \times (0, \infty), \\ \tilde{u}(0) = u_0, & \text{in } \Omega. \end{cases}$$

Moreover $(S(\cdot)|v_0|)^p u^k \in L^1_{\mathrm{loc}}(\overline{Q_\infty})$ from (1.8), and

$$u_t - \Delta u + \big(S(\cdot)|v_0|\big)^p u^k \geqslant 0, \quad \text{in } Q_\infty,$$

hence $u \geqslant \tilde{u}$ in Q_∞ from Proposition 3.1. Since $u_0 \not\equiv 0$ and $\tilde{u} \in C([0, \infty), L^1(\Omega)) \cap C(Q_\infty)$, for any $\varepsilon > 0$ sufficiently small, there exists a ball B_ε or radius ε, such that $\overline{B_\varepsilon} \subset \Omega$, and $\inf_{\overline{B_\varepsilon}} \tilde{u}(\varepsilon) > 0$. Otherwise $(S(t)|v_0|)^p$ is bounded on $\Omega \times [\varepsilon, \infty)$ by a constant C_ε, and $C_\varepsilon \tilde{u}^k \in L^\infty(\Omega \times [\varepsilon, \infty))$, and

$$\tilde{u}_t - \Delta\tilde{u} + C_\varepsilon \tilde{u}^k \geqslant 0, \quad \text{in } \Omega \times [\varepsilon, \infty).$$

Since $\tilde{u}(\varepsilon) \in L^\infty(\Omega)$, there exists a solution ξ of problem

$$\begin{cases} \xi_t - \Delta\xi + C_\varepsilon \xi^k = 0, & \text{in } \Omega \times (\varepsilon, \infty), \\ \xi = 0 \text{ (resp. } \partial\xi/\partial\nu = 0), & \text{on } \partial\Omega \times (\varepsilon, \infty), \\ \xi(\varepsilon) = \tilde{u}(\varepsilon), & \text{in } \Omega, \end{cases}$$

and $\xi > 0$ by the strict maximum principle, since $k \geqslant 1$. This implies $\tilde{u} \geqslant \xi > 0$, in $\Omega \times (\varepsilon, \infty)$, by the usual comparison principle. Letting ε tend to 0, it follows that $u > 0$ in Q_∞. □

If the initial data are bounded, we obtain more precise lower estimates:

Proposition 6.1. *Assume that* $u_0, v_0 \in L^\infty(\Omega),\ u_0 \geqslant 0,\ u_0 \not\equiv 0$ *and* $k \geqslant 1$. *Then for any solution* (u, v) *of* (1.7),

(i) *there exists a constant* $C > 0$ *such that, for any* $t > 0$,

$$\mathrm{e}^{-Ct} S(t)u_0 \leqslant u(t) \leqslant S(t)u_0, \quad \textit{in } \Omega; \tag{6.1}$$

(ii) *for the Dirichlet problem, there exists a constant* $C' > 0$ *such that, for any* $t > 0$,

$$C' S_d(t)u_0 \leqslant u(t) \leqslant S_d(t)u_0, \quad \textit{in } \Omega; \tag{6.2}$$

(iii) *for the Neuman problem, if* $\inf_{x\in\Omega} u_0(x) = \delta > 0$, *there exists* $C > 0$ *such that, for any* $t > 0$,

$$u(t) \geqslant \left(\delta^{1-k} + C(k-1)t\right)^{1/(1-k)}, \quad \textit{in } \Omega.$$

Proof. (i) Under our assumptions, $(S(\cdot)|v_0|)^p (S(\cdot)u_0)^{k-1}$ is bounded on Q_∞ by some constant $C > 0$, hence

$$u_t - \Delta u + Cu \geqslant 0, \quad \text{in } Q_\infty.$$

Let z be the solution of problem

$$\begin{cases} z_t - \Delta z + Cz = 0, & \text{in } Q_\infty, \\ z = 0 \text{ (resp. } \partial z/\partial\nu = 0), & \text{on } \partial\Omega \times (0,\infty), \\ z(0) = u_0, & \text{in } \Omega. \end{cases} \tag{6.3}$$

Then $u(t) \geqslant z(t) = \mathrm{e}^{-Ct} S(t)u_0$, in Ω, for any $t \geqslant 0$, which proves (6.1).

(ii) With Dirichlet conditions, there exists $C'' > 0$ such that for any $t \geqslant 1$,

$$\left\| S_d(t)|v_0| \right\|^p_{L^\infty(\Omega)} \left\| S_d(t)|u_0| \right\|^{k-1}_{L^\infty(\Omega)} \leqslant C'' \,\mathrm{e}^{-\lambda_1 (p+k-1)(t-1)},$$

from (3.5) between 1 and t. Then $u \geqslant Z$ in $\Omega \times [1,\infty)$, where Z is the solution of problem

$$\begin{cases} Z_t - \Delta Z + C'' \,\mathrm{e}^{-\lambda_1 (p+k-1)(t-1)} Z = 0, & \text{in } \Omega \times (1,\infty), \\ Z = 0, & \text{on } \partial\Omega \times (1,\infty), \\ Z(1) = u(1), & \text{in } \Omega. \end{cases}$$

By computation, for any $t \geqslant 1$,

$$Z(t) \geqslant \exp\left(-\int_1^\infty \mathrm{e}^{-\lambda_1 (p+k-1)s}\, \mathrm{d}s\right) S_d(t-1)u(1) \geqslant \exp\left(-C - \int_1^\infty \mathrm{e}^{-\lambda_1 (p+k-1)s}\, \mathrm{d}s\right) S_d(t)u_0$$

from (6.1), hence (6.2) holds, by considering separately $[1,\infty)$ and $[0,1]$.

(iii) Here $(S(t)|v_0|)^p$ is bounded on Q_∞ by a constant C, hence $u \geqslant \zeta$, where ζ is the solution of the o.d.e. problem

$$\begin{cases} \zeta_t + C\zeta^k = 0, & \text{in } (0,\infty), \\ \xi(0) = \delta, \end{cases}$$

which is given by $\zeta(t) = (\delta^{1-k} + C(k-1)t)^{1/(1-k)}$. $\square$

Next we apply the results of Section 4 to deduce supplementary positivity or extinction results, under comparison assumptions on the initial data.

Theorem 6.2. *Assume that* $0 < a \leqslant b$, *and* $u_0, v_0 \in L^\infty(\Omega)$, $u_0, v_0 \geqslant 0$ *such that*

$$au_0^b \leqslant bv_0^a, \quad \text{a.e. in } \Omega, \quad \text{and} \quad au_0^b \not\equiv bv_0^a.$$

Let (u, v) *be any solution of* (1.7).

(i) *If* $\delta \geqslant 0$ *or* $a \leqslant 1 \leqslant b$, *then* $v > 0$ *in* Q_∞.
(ii) *If* $\delta < 0$, *and* $a \leqslant 1$ *or* $(1 < a$ *and* $au_0^b < bv_0^a$, *a.e. in* Ω, *with* $u, v \in C(\overline{Q_\infty}))$, *then* $u(\cdot, t) \equiv 0$ *for* t *large enough.*

Proof. (i) From Theorem 1.4, in any case u, v satisfy $au^b < bv^a$ in Q_∞, and the result follows. If $\delta \geqslant 0$, we can also conclude from Remark 4.5, since v is a supersolution of the scalar equation

$$w_t - \Delta w + \left(\frac{a}{b}\right)^{-q/b} w^{\mathbf{P}} = 0,$$

with the exponent $\mathbf{P} = 1 + \delta/b \geqslant 1$.

(ii) In any case there holds $au^b \leqslant bv^a$ in Q_∞, hence from Remark 4.5, u is a subsolution of the scalar equation

$$w_t - \Delta w + \left(\frac{a}{b}\right)^{p/a} w^{\mathbf{Q}} = 0,$$

and the exponent $\mathbf{Q} = 1 + \delta/a \in (0, 1)$; indeed $\delta < 0$ implies $\mathbf{Q} < 1$, and also $k, \ell < 1$ from (2.1), hence $a + \delta = p(q + 1) + k(1 - \ell) > 0$. Then u has a compact support in t. $\square$

Concerning the Neuman problem, we can extend some other results of Proposition 2.1 by using Remarks 3.6 and 4.4:

Theorem 6.3. *Let* $u_0, v_0 \in L^\infty(\Omega)$, $u_0, v_0 \geqslant 0$, *and for example* $\inf_{x\in\Omega} v_0(x) > 0$. *Let* (u, v) *be any solution of* (1.7), *with Neuman conditions.*

(i) *Assume that* $a \leqslant 0$. *Then* $v > 0$ *in* Q_∞; *if moreover* $k < 1$, *then* $u(\cdot, t) \equiv 0$ *for large* t.
(ii) *Assume that* $0 < a, b$ (*and not necessarily* $a \leqslant b$) *and*

$$b \sup_{x\in\Omega} u_0^b(x) < a \inf_{x\in\Omega} v_0^a(x). \tag{6.4}$$

Then $v > 0$ *in* Q_∞; *if moreover* $k < 1$, *then* $u(\cdot, t) \equiv 0$ *for large* t.

(iii) *Assume that* $a < 0 < b$. *Then*

$$|a|u^b + \frac{b}{v^{|a|}} \leqslant |a| \sup_{x\in\Omega} u_0^b(x) + \frac{b}{\inf_{x\in\Omega} v_0^{|a|}(x)}, \quad \text{in } Q_\infty, \tag{6.5}$$

and $\inf_{(x,t)\in Q_\infty} v > 0$.

(iv) *Assume that* $0 < a \leqslant 1 \leqslant b$, *or* $1 \leqslant a \leqslant b$ *and* ($\delta \geqslant 0$ *or* $u_0, v_0 \in C(\overline{\Omega})$), *and that*

$$au_0^b + \varepsilon \leqslant v_0^a, \quad a.e.\ in\ \Omega,$$

for some $\varepsilon > 0$. *If* $k < 1$, *then* $u(\cdot, t) \equiv 0$ *for large* t.

Proof. From Remark 3.6, we know that $u \leqslant u'$ and $v' \leqslant v$ in Q_∞, where (u', v') be the unique solution of the o.d.e. problem (1.5) such that $u'(0) = \|u_0\|_{L^\infty(\Omega)}$ and $v'(0) = \inf_{x \in \Omega} v_0(x)$.

(i) Since $a \leqslant 0$, we have $\ell \geqslant 1$; then v is positive, from Theorem 6.1. Moreover u' has a compact support, from Proposition 2.1(v), hence also u.

(ii) From (6.4) and (2.6),

$$bv'^a - au'^b = bv_0'^a - au_0'^b = C' > 0.$$

Thus v' is positive, and u' has a compact support in t if $k < 1$, from Proposition 2.1(iii), (iv), hence the conclusions hold.

(iii) Here we obtain, from (2.6),

$$|a|u^b + \frac{b}{v^{|a|}} \leqslant |a|u'^b + \frac{b}{v'^{|a|}} = |a|u_0'^b + \frac{b}{v_0'^{|a|}} = C' > 0,$$

and $v \geqslant v' \geqslant (b/C')^{1/|a|}$.

(iv) From Theorem 1.4 and Remark 4.4, there exists $C > 0$ such that $v \geqslant C$ in $\overline{Q_\infty}$. Then u is a subsolution of the scalar equation

$$w_t - \Delta w + C^p w^k = 0,$$

with the exponent $k < 1$, hence it has a compact support in t. $\square$

We end this paragraph with some properties of positivity of the sum $u + v$. We already know that it remains positive if k or $\ell \geqslant 1$ from Theorem 6.1. Thus we consider the case $k, \ell < 1$.

Theorem 6.4. *Suppose that* $u_0, v_0 \in L^\infty(\Omega)$, $u_0, v_0 \geqslant 0$, *and* $u_0 + v_0 \not\equiv 0$. *Assume* $k, \ell < 1$, *and*

$$\min(p, q, pq) \geqslant (1 - k)(1 - \ell). \tag{6.6}$$

Let u, v *be any solutions of* (1.7). *Then* $u + v$ *remains positive in* Q_∞.

Proof. First assume that $p \geqslant 1 - k$ and $q \geqslant 1 - \ell$: in that case

$$(u + v)_t - \Delta(u + v) + v^p u^k + u^q v^\ell \geqslant 0,$$

and u, v are bounded, hence there exists $C > 0$ such that $u + v$ is a supersolution of the equation

$$w_t - \Delta w + Cw^Q = 0,$$

where $Q = \min(p + k, q + \ell) \geqslant 1$. Thus $u + v$ remains positive.

In the general case we prove that an expression of the form $u^\alpha + v^\beta$ is a supersolution of an equation of the same type, for some $\alpha, \beta \leqslant 1$. Here again we use the approximations u_n, v_n of u, v defined by (4.7), and we put

$$f_{n,\varepsilon} = (u_n + \varepsilon)^\alpha - \varepsilon^\alpha, \qquad g_{n,\varepsilon} = (v_n + \varepsilon)^\beta - \varepsilon^\beta,$$

for $0 < \varepsilon \leqslant 1$, and $0 < \alpha, \beta \leqslant 1$. Then

$$\begin{cases} (f_{n,\varepsilon})_t - \Delta f_{n,\varepsilon} + \alpha(u_n + \varepsilon)^{\alpha-1} F_n = \alpha(1-\alpha)(u_n+\varepsilon)^{\beta-2}|\nabla u_n|^2 \geqslant 0, \\ (g_{n,\varepsilon})_t - \Delta g_{n,\varepsilon} + \beta(v_n + \varepsilon)^{\beta-1} G_n = \beta(1-\beta)(v_n+\varepsilon)^{\beta-2}|\nabla v_n|^2 \geqslant 0. \end{cases}$$

Consider the sum

$$Y_{n,\varepsilon} = f_{n,\varepsilon} + g_{n,\varepsilon}.$$

It also satisfies Dirichlet or Neuman conditions, and an equation of the form

$$(Y_{n,\varepsilon})_t - \Delta Y_{n,\varepsilon} + H_{n,\varepsilon} = L_{n,\varepsilon},$$

where $H_{n,\varepsilon}, L_{n,\varepsilon} \in L^\infty(Q_T)$ are defined by

$$\begin{cases} H_{n,\varepsilon} = \alpha(u_n + \varepsilon)^{\alpha-1} F_n + \beta(v_n + \varepsilon)^{\beta-1} G_n, \\ L_{n,\varepsilon} = \alpha(1-\alpha)(u_n+\varepsilon)^{\alpha-2}|\nabla v_n|^2 + \beta(1-\beta)(v_n+\varepsilon)^{\beta-2}|\nabla v_n|^2 \geqslant 0. \end{cases}$$

We can go to the limit as $n \to \infty$ for fixed ε, as in the proof of Lemma 4.1. Then $Y_{n,\varepsilon}$ converges to

$$Y_\varepsilon = (u+\varepsilon)^\alpha + (v+\varepsilon)^\beta - \varepsilon^\alpha - \varepsilon^\beta,$$

a.e. in Q_∞, and $H_{n,\varepsilon}$ converges to

$$H_\varepsilon = \alpha v^p (u+\varepsilon)^{\alpha-1}|u|^{k-1}u + \beta u^q (v+\varepsilon)^{\beta-1}|v|^{\ell-1}v \leqslant v^p(u+\varepsilon)^{\alpha-1+k} + u^q(v+\varepsilon)^{\beta-1+\ell}.$$

From assumption (6.6), we can find $\alpha \in [1-k, 1]$ and $\beta \in [1-\ell, 1]$ such that

$$\frac{1-k}{p} \leqslant \frac{\alpha}{\beta} \leqslant \frac{q}{1-\ell}. \tag{6.7}$$

Then

$$H_\varepsilon \leqslant \left(Y_\varepsilon + \varepsilon^\alpha + \varepsilon^\beta\right)^{p/\beta + (\alpha-1+k)/\alpha} + \left(Y_\varepsilon + \varepsilon^\alpha + \varepsilon^\beta\right)^{q/\alpha + (\beta-1+\ell)/\beta} \leqslant C\left(Y_\varepsilon + \varepsilon^\alpha + \varepsilon^\beta\right), \tag{6.8}$$

for some $C > 0$. The constant C does not depend on ε, since the two exponents in (6.8) are bigger than 1 and Y_ε is bounded independently on ε. The function $t \mapsto Z_{n,\varepsilon}(t) = e^{Ct} Y_{n,\varepsilon}(t) - S(t)Y_{n,\varepsilon}(0)$ satisfies

$$(Z_{n,\varepsilon})_t - \Delta Z_{n,\varepsilon} = \mathrm{e}^{Ct}(L_{n,\varepsilon} + CY_{n,\varepsilon} - H_{n,\varepsilon}).$$

Multiplying by $Z_{n,\varepsilon}^-$, we obtain

$$\int_\Omega (Z_{n,\varepsilon}^-)^2(t)\,\mathrm{d}x \leqslant \int_0^t\int_\Omega \big(H_{n,\varepsilon}(\tau) - CY_{n,\varepsilon}(\tau)\big) Z_{n,\varepsilon}^-(\tau)\,\mathrm{d}x\,\mathrm{d}\tau.$$

Going to the limit as $n \to \infty$, and setting $\gamma = \min(\alpha, \beta)$, the function $t \mapsto Z_\varepsilon(t) = \mathrm{e}^{Ct} Y_\varepsilon(t) - S(t)Y_\varepsilon(0)$ satisfies

$$\int_\Omega (Z_\varepsilon^-)^2(t)\,\mathrm{d}x \leqslant \int_0^t\int_\Omega \big(H_\varepsilon(\tau) - CY_\varepsilon(\tau)\big) Z_\varepsilon^-(\tau)\,\mathrm{d}x\,\mathrm{d}\tau \leqslant 2C\varepsilon^\gamma \int_0^t\int_\Omega Z_\varepsilon^-(\tau)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\leqslant C\varepsilon^\gamma \left(1 + \int_0^t\int_\Omega (Z_\varepsilon^-)^2(\tau)\,\mathrm{d}x\,\mathrm{d}\tau\right).$$

By the Gronwall lemma, for any $T > 0$ and any $t \in [0, T]$,

$$\int_\Omega (Z_\varepsilon^-)^2(t)\,\mathrm{d}x \leqslant C\varepsilon^\gamma\,\mathrm{e}^{C\varepsilon^\gamma t} \leqslant C\,\mathrm{e}^{CT}\varepsilon^\gamma.$$

Going to the limit as $\varepsilon \to 0$, and denoting $Y = u^\alpha + v^\beta$, it follows that

$$\mathrm{e}^{Ct} Y(t) - S(t)Y(0) \geqslant 0, \quad \text{in } \Omega,$$

for any $t \in [0, T]$; in particular Y remains positive in Q_T, since $Y(0) \not\equiv 0$, from the strict maximum principle, hence also in Q_∞. □

Remark 6.1. It would be interesting to know if the result of Proposition 6.4 is still valid under the only condition $pq \geqslant (1-k)(1-\ell)$, that means $\delta \geqslant 0$, as it is the case for system (1.5), from Remark 2.1.

7. Uniqueness results

In the scalar case of Eq. (1.3), uniqueness comes from monotonicity, for any $Q > 0$, for any initial data in $L^1(\Omega)$. In the case of system (1.1), the problem is much harder. First recall that uniqueness holds if $u_0 \equiv 0$ or $v_0 \equiv 0$, from Remark 3.1. Thus we can assume that $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$.

7.1. The case $p, q \geqslant 1$

If $p, q, k, \ell \geqslant 1$, the function $(u, v) \mapsto (|v|^p|u|^{k-1}u, |u|^q|v|^{k-1}v)$ is locally Lipschitz continuous, hence uniqueness follows from Lemma 3.2 when $u_0, v_0 \in L^\infty(\Omega)$. This result can be improved: in fact, using the monotonicity of the terms $|u|^{k-1}u$ and $|v|^{\ell-1}v$, we prove Theorem 1.6, which requires only $p, q \geqslant 1$, and does not assume that the initial data are bounded:

Proof of Theorem 1.6. Assume $p, q \geqslant 1$, and $u_0 \in L^{\theta_1}(\Omega)$, $v_0 \in L^{\theta_2}(\Omega)$, satisfying (1.19). Let $(u, v), (\hat{u}, \hat{v})$ be two solutions of the system. By difference we find

$$\begin{cases} (\hat{u} - u)_t - \Delta(\hat{u} - u) + |\hat{v}|^p|\hat{u}|^{k-1}\hat{u} - |v|^p|u|^{k-1}u = 0, & \text{in } Q_\infty, \\ (\hat{v} - v)_t - \Delta(\hat{v} - v) + |\hat{u}|^q|\hat{v}|^{\ell-1}\hat{v} - |u|^q|v|^{\ell-1}v = 0, & \text{in } Q_\infty, \\ (\hat{u} - u)(0) = 0, \quad (\hat{v} - v)(0) = 0, & \text{in } \Omega. \end{cases} \tag{7.1}$$

We can write the two equations under the form

$$\begin{cases} (\hat{u}-u)_t - \Delta(\hat{u}-u) + |\hat{v}|^p\big(|\hat{u}|^{k-1}\hat{u} - |u|^{k-1}u\big) = \big(|v|^p - |\hat{v}|^p\big)|u|^{k-1}u, \\ (\hat{v}-v)_t - \Delta(\hat{v}-v) + |\hat{u}|^q\big(|\hat{v}|^{\ell-1}\hat{v} - |v|^{\ell-1}v\big) = \big(|u|^q - |\hat{u}|^q\big)|v|^{\ell-1}v. \end{cases}$$

Then, from the Kato inequality,

$$\begin{aligned} &\big|(\hat{u}-u)(t)\big| + \int_0^t S(t-s)\big(|\hat{v}|^p\big(|\hat{u}|^{k-1}\hat{u} - |u|^{k-1}u\big)\,\mathrm{sign}_0(\hat{u}-u)\big)(s)\,\mathrm{d}s \\ &\quad\leqslant \int_0^t S(t-s)\big(\big(|v|^p - |\hat{v}|^p\big)|u|^{k-1}u\,\mathrm{sign}_0(\hat{u}-u)\big)(s)\,\mathrm{d}s. \end{aligned}$$

Using the monotonicity of function $r \mapsto |r|^{k-1}r$, we derive

$$\big|(\hat{u}-u)(t)\big| \leqslant \int_0^t S(t-s)\big(|u|^k\big||\hat{v}|^p - |v|^p\big|\big)(s)\,\mathrm{d}s, \tag{7.2}$$

and similarly

$$\big|(\hat{v}-v)(t)\big| \leqslant \int_0^t S(t-s)\big(|v|^\ell\big||\hat{u}|^q - |u|^q\big|\big)(s)\,\mathrm{d}s. \tag{7.3}$$

First suppose that $u_0, v_0 \in L^\infty(\Omega)$. Then $u, v \in L^\infty(Q_\infty)$, hence there exists $C > 0$ such that, for any $t \geqslant 0$,

$$\begin{cases} \big|(\hat{u}-u)(t)\big| \leqslant C\displaystyle\int_0^t S(t-s)\big(|\hat{v}-v|\big)(s)\,\mathrm{d}s, \\ \big|(\hat{v}-v)(t)\big| \leqslant C\displaystyle\int_0^t S(t-s)\big(|\hat{u}-u|\big)(s)\,\mathrm{d}s. \end{cases}$$

Setting $f = \|\hat{u}-u\|_{L^1(\Omega)} + \|\hat{v}-v\|_{L^1(\Omega)} \geqslant 0$, we obtain, by addition,

$$f(t) \leqslant C\int_0^t f(s)\,\mathrm{d}s.$$

Then $\varphi(t) = \sup_{[0,t]} f(s)$ satisfies $\varphi(t) \leqslant Ct\varphi(t)$, hence $\varphi(t) = 0$ on $[0, 1/C]$; and by induction on $[0,\infty)$, thus $\hat{u} = u$ and $\hat{v} = v$.

Now consider the general case $u_0 \in L^{\theta_1}(\Omega)$, $v_0 \in L^{\theta_2}(\Omega)$. Existence follows from Corollary 3.1. Indeed (1.19) implies (1.10), since

$$\frac{k}{\theta_1} + \frac{p}{\theta_2} < \frac{1}{\theta_1} + \frac{2}{N} \leqslant \frac{N+2}{N}.$$

Moreover (1.20) follows from (3.16) and (3.17) with $S_1 = \theta_1$ and $S_2 = \theta_2$. We claim that there exists a constant $C > 0$ and real numbers $\mu, \gamma \geqslant 0$, such that $\mu + \gamma < 1$, and

$$\big\|(\hat{u}-u)(t)\big\|_{L^{\theta_1}(\Omega)} \leqslant C\int_0^t \big(1 + (t-s)^{-\mu}\big)\big(1 + s^{-\gamma}\big)\big\|(\hat{v}-v)(s)\big\|_{L^{\theta_2}(\Omega)}\,\mathrm{d}s \tag{7.4}$$

for any $t \geqslant 0$. The proof is divided in two cases.

(i) Case $\theta_1 \geqslant \theta_2$. Here we use in (7.2) the regularizing effect (3.5), successively from $L^{\theta_2}(\Omega)$ to $L^{\theta_1}(\Omega)$, and from $L^{\theta_1}(\Omega), L^{\theta_2}(\Omega)$ to $L^{\infty}(\Omega)$. Defining $\mu = (1/\theta_2 - 1/\theta_1)N/2$ and $\gamma = (k/\theta_1 + (p-1)/\theta_2)N/2$, we obtain

$$\begin{aligned}
\frac{1}{p}\|(\hat{u}-u)(t)\|_{L^{\theta_1}(\Omega)} &\leqslant \int_0^t \left(1+(t-s)^{-\mu}\right)\left\||u|^k\left(|\hat{v}|^{p-1}+|v|^{p-1}\right)|\hat{v}-v|(s)\right\|_{L^{\theta_2}(\Omega)}\,\mathrm{d}s \\
&\leqslant \int_0^t \left(1+(t-s)^{-\mu}\right)\|u(s)\|^k_{L^{\infty}(\Omega)}\left(\|v(s)\|^{p-1}_{L^{\infty}(\Omega)}+\|\hat{v}(s)\|^{p-1}_{L^{\infty}(\Omega)}\right)\|(\hat{v}-v)(s)\|_{L^{\theta_2}(\Omega)}\,\mathrm{d}s \\
&\leqslant C\|u_0\|^k_{L^{\theta_1}(\Omega)}\|v_0\|^{p-1}_{L^{\theta_2}(\Omega)}\int_0^t \left(1+(t-s)^{-\mu}\right)\left(1+s^{-\gamma}\right)\|(\hat{v}-v)(s)\|_{L^{\theta_2}(\Omega)}\,\mathrm{d}s,
\end{aligned}$$

for some $C > 0$. This proves (7.4), and $\mu + \gamma < 1$, from (1.19).

(ii) Case $\theta_1 < \theta_2$. Since $S(\cdot)$ is a contraction in $L^{\theta_1}(\Omega)$, we have

$$\frac{1}{p}\|(\hat{u}-u)(t)\|_{L^{\theta_1}(\Omega)} \leqslant \int_0^t \left\|\left(|u|^k\left(|\hat{v}|^{p-1}+|v|^{p-1}\right)|\hat{v}-v|\right)(s)\right\|_{L^{\theta_1}(\Omega)}\,\mathrm{d}s.$$

Let $\eta, r \geqslant 1$ be two parameters, with $\eta > 1$ if $k > 0$ and $\eta = 1$ if $k = 0$, and $r = 1$ if $p = 1$. We derive successively

$$\begin{aligned}
&\left\||u|^k\left(|\hat{v}|^{p-1}+|v|^{p-1}\right)|\hat{v}-v|(s)\right\|_{L^{\theta_1}(\Omega)} \\
&\quad\leqslant \left(\int_\Omega u(s)^{k\eta'\theta_1}\right)^{1/\eta'\theta_1}\left\|\left(|\hat{v}|^{p-1}+|v|^{p-1}\right)|\hat{v}-v|(s)\right\|_{L^{\eta\theta_1}(\Omega)} \\
&\quad\leqslant \left(\int_\Omega u(s)^{k\eta'\theta_1}\right)^{1/\eta'\theta_1}\left\||\hat{v}-v|(s)\right\|_{L^{r\eta\theta_1}(\Omega)}\left\|\left(|\hat{v}|^{p-1}+|v|^{p-1}\right)(s)\right\|_{L^{r'\eta\theta_1}(\Omega)},
\end{aligned}$$

where by convention $(\int_\Omega u(s)^{k\eta'\theta_1})^{1/\eta'\theta_1} = 1$ if $k = 0$, and $\|(|\hat{v}|^{p-1}+|v|^{p-1})(s)\|_{L^{r'\eta\theta_1}(\Omega)} = 1$ if $p = 1$.

• First suppose that

$$(1-k)\theta_2 \leqslant p\theta_1 \tag{7.5}$$

(which holds in particular when $k \geqslant 1$). If $k > 0$ and $p > 1$, we can choose η such that $k\eta' \geqslant 1$ and $\theta_2 \leqslant p\eta\theta_1 < p\theta_2$; then we choose $r = \theta_2/\eta\theta_1$, hence $r' = \theta_2/(\theta_2 - \eta\theta_1)$. Setting $h = (p-1)r'\eta\theta_1 = (p-1)\eta\theta_1\theta_2/(\theta_2 - \eta\theta_1)$, there holds $h \geqslant \theta_2$ from (7.5), and

$$\begin{aligned}
&\left\||u|^k\left(|\hat{v}|^{p-1}+|v|^{p-1}\right)|\hat{v}-v|(s)\right\|_{L^{\theta_1}(\Omega)} \\
&\quad\leqslant C\|u(s)\|^k_{L^{k\eta'\theta_1}(\Omega)}\left\|\left(|\hat{v}|+|v|\right)(s)\right\|^{p-1}_{L^{h}(\Omega)}\|(\hat{v}-v)(s)\|_{L^{\theta_2}(\Omega)}.
\end{aligned}$$

Using the regularizing effect, we obtain

$$\|(\hat{u}-u)(t)\|_{L^{\theta_1}(\Omega)} \leqslant C\|u_0\|^k_{L^{\theta_1}(\Omega)}\int_0^t \left(1+s^{-(\alpha k+\beta(p-1))}\right)\|(\hat{v}-v)(s)\|_{L^{\theta_2}(\Omega)}\,\mathrm{d}s,$$

where $\alpha = (1 - 1/k\eta')N/2\theta_1$ and $\beta = (\theta_2 - 1/h)N/2$. After computation,

$$\alpha k + \beta(p-1) = \frac{N}{2}\left(\frac{k-1}{\theta_1} + \frac{p}{\theta_2}\right) < 1$$

from our assumption. Thus there exists $\lambda \in [0,1)$ such that

$$\|(\hat{u} - u)(t)\|_{L^{\theta_1}(\Omega)} \leqslant C \int_0^t (1 + s^{-\lambda}) \|(\hat{v} - v)(s)\|_{L^{\theta_2}(\Omega)} \, ds. \tag{7.6}$$

This is also true in the limiting cases: $k = 0$, $p > 1$, since taking $r = \theta_2/\theta_1$, we still have $h = (p-1)\theta_1\theta_2/(\theta_2 - \theta_1) \geqslant \theta_2$; also $k > 0$, $p = 1$, since choosing $\eta = \theta_2/\theta_1$ we still have $k\eta' \geqslant 1$; and finally $k = 0$, $p = 1$, by choosing again $\eta = \theta_2/\theta_1$. Thus (7.4) holds with $\mu = 0$ and $\gamma = \lambda$.

• Now assume that

$$(1-k)\theta_2 > p\theta_1, \tag{7.7}$$

then $k < 1$. We choose $\eta = 1/(1-k)$ if $k > 0$ and $r = p$ if $p > 1$. Let us set $\widetilde{\theta_2} = p\theta_1/(1-k)$. In any case $k \geqslant 0$ and $p \geqslant 1$, we deduce

$$\begin{aligned} &\big\| |u|^k \big(|\hat{v}|^{p-1} + |v|^{p-1}\big) |\hat{v} - v|(s) \big\|_{L^{\theta_1}(\Omega)} \\ &\quad \leqslant \|u(s)\|^k_{L^{\theta_1}(\Omega)} \big\| \big(|\hat{v}| + |v|\big)(s) \big\|^{p-1}_{L^{\widetilde{\theta_2}}(\Omega)} \|(\hat{v} - v)(s)\|_{L^{\widetilde{\theta_2}}(\Omega)} \\ &\quad \leqslant \|u_0\|^k_{L^{\theta_1}(\Omega)} \big\| \big(|\hat{v}_0| + |v_0|\big)(s) \big\|^{p-1}_{L^{\widetilde{\theta_2}}(\Omega)} \|(\hat{v} - v)(s)\|_{L^{\widetilde{\theta_2}}(\Omega)}, \end{aligned}$$

with the conventions $\|u_0\|^k_{L^{\theta_1}(\Omega)} = 1$ if $k = 0$ and$\|(|\hat{v}| + |v|)(s)\|^{p-1}_{L^{\widetilde{\theta_2}}(\Omega)} = 1$ if $p = 1$. Then (7.6) holds again with $\lambda = 0$, since $L^{\theta_2}(\Omega) \subset L^{\widetilde{\theta_2}}(\Omega)$. Thus (7.4) holds with $\mu = \gamma = 0$.

In any case claim (7.4) is proved. In the same way, there exist $\tilde{\mu}, \tilde{\gamma} \geqslant 0$ such that $\tilde{\mu} + \tilde{\gamma} < 1$ and $\widetilde{C} > 0$ such that for any $t \geqslant 0$,

$$\|(\hat{v} - v)(t)\|_{L^{\theta_2}(\Omega)} \leqslant \widetilde{C} \int_0^t \big(1 + (t-s)^{-\tilde{\mu}}\big)\big(1 + s^{-\tilde{\gamma}}\big) \|(\hat{v} - v)(s)\|_{L^{\theta_2}(\Omega)} \, ds. \tag{7.8}$$

Defining

$$\varphi(t) = \sup_{s \in [0,t]} \big(\|(\hat{u} - u)(s)\|_{L^{\theta_1}(\Omega)} + \|(\hat{v} - v)(s)\|_{L^{\theta_2}(\Omega)} \big),$$

we find, for any $t \in [0,1]$,

$$\|(\hat{u} - u)(t)\|_{L^{\theta_1}(\Omega)} + \|(\hat{v} - v)(t)\|_{L^{\theta_2}(\Omega)} \leqslant 2\varphi(t)\left(C \int_0^t (t-s)^{-\mu} s^{-\gamma} \, ds + \widetilde{C} \int_0^t (t-s)^{-\tilde{\mu}} s^{-\tilde{\gamma}} \, ds \right),$$

therefore

$$\varphi(t) \leqslant 4\varphi(t)\left(C \int_0^1 s^{-(\mu+\gamma)} \, ds + \widetilde{C} \int_0^1 s^{-(\tilde{\mu}+\tilde{\gamma})} \, ds \right),$$

so that $\varphi(t) = 0$ on some $[0, T_0]$ with $T_0 \leqslant 1$ small enough, and on $[0, \infty)$ by induction. $\square$

Remark 7.1. In particular, if $p, q \geqslant 1$, and $u_0, v_0 \in L^\theta(\Omega)$ with

$$\theta > \frac{N}{2}\max(k+p-1, q+\ell-1),$$

then problem (1.7) admits a unique solution (u, v), and $u, v \in C([0,\infty), L^\theta(\Omega))$. This result has to be compared to the result of [10, Theorem 1] for the scalar equation with source term

$$w_t - \Delta w = w^Q,$$

with $w(0) \in L^\theta(\Omega)$, and $\theta > N(Q-1)/2$.

7.2. The case $u_0, v_0 \geqslant 0$

In the proof of Theorem 1.6 we have only used the inequalities (7.2), (7.3), issued from (7.1) by applying Kato's inequality. In this way we have neglected the parameters k, ℓ. By taking them into account, we can improve the results when the initial data are nonnegative and bounded.

Proof of Theorem 1.7. We exclude the case $p, q \geqslant 1$, still treated in Theorem 1.6. Any of our other assumptions implies $k, \ell > 0$. Here we consider the solutions (U, V) and $(\widetilde{U}, \widetilde{V})$ defined in Theorem 1.3 and prove that they coincide. Recall that $\widetilde{U} - U \geqslant 0$ and $V - \widetilde{V} \geqslant 0$. They satisfy

$$\begin{cases} (\widetilde{U} - U)_t - \Delta(\widetilde{U} - U) + \widetilde{V}^p\widetilde{U}^k - V^pU^k = 0, \\ (V - \widetilde{V})_t - \Delta(V - \widetilde{V}) + U^qV^\ell - \widetilde{U}^q\widetilde{V}^\ell = 0. \end{cases} \tag{7.9}$$

Let $A > 0$ be a parameter. Let

$$\Phi = A(\widetilde{U} - U) + V - \widetilde{V},$$
$$G(u, v) = Av^pu^k - u^qv^\ell, \quad \forall u, v \geqslant 0.$$

Then $\Phi \geqslant 0$, $\Phi(0) = 0$, and

$$\Phi_t - \Delta\Phi = G(U, V) - G(\widetilde{U}, \widetilde{V}) = G(U, V) - G(\widetilde{U}, V) + G(\widetilde{U}, V) - G(\widetilde{U}, \widetilde{V}).$$

At any point where $U > 0$, there exists $\eta \in (U, \widetilde{U})$ such that

$$G(U, V) - G(\widetilde{U}, V) = -\frac{\partial G}{\partial U}(\eta, V)(\widetilde{U} - U) = (q\eta^{q-1}V^\ell - AkV^p\eta^{k-1})(\widetilde{U} - U).$$

At any point where $U = 0$, we find

$$\begin{aligned} G(U, V) - G(\widetilde{U}, V) = -G(\widetilde{U}, V) &= \widetilde{U}^qV^\ell - AV^p\widetilde{U}^k \\ &= \begin{cases} (\widetilde{U}^{q-1}V^\ell - AV^p\widetilde{U}^{k-1})(\widetilde{U} - U), & \text{if } \widetilde{U} \neq 0, \\ 0, & \text{if } \widetilde{U} = 0. \end{cases} \end{aligned}$$

Then, there exists $C > 0$ such that

$$G(U,V) - G(\widetilde{U},V) \leqslant C(\widetilde{U} - U),$$

whenever for any $\xi \in (U,\widetilde{U}]$,

$$\xi^{q-1}V^{\ell} - mAV^p\xi^{k-1} \leqslant C, \tag{7.10}$$

where $m = \min(1, k/q)$. In the same way

$$G(U,V) - G(\widetilde{U},V) \leqslant C(V - U)$$

whenever for any $\zeta \in (\widetilde{V}, V]$,

$$\zeta^{p-1}\widetilde{U}^k - \frac{m'}{A}\widetilde{U}^q\zeta^{\ell-1} \leqslant C, \tag{7.11}$$

where $m' = \min(1, \ell/p)$. If, for fixed $T > 0$, we can find A and C, possibly depending on T, such that (7.10) and (7.11) hold in Q_T, then

$$\Phi_t - \Delta\Phi = G(U,V) - G(\widetilde{U},\widetilde{V}) \leqslant C\Phi,$$

and uniqueness in Q_T follows from the maximum principle, since $G(U,V) - G(\widetilde{U},V)$ and $\Phi \in L^\infty(Q_\infty)$. Consider for example (7.10). It is satisfied in any of the following cases:

- When $q \geqslant 1$: since $\xi^{q-1}V^\ell \leqslant \widetilde{U}^{q-1}V^\ell$, and $\widetilde{U}^{q-1}V^\ell$ is bounded.
- When

$$0 < k \leqslant q < 1 \quad \text{and} \quad p(1-q) \leqslant \ell(1-k).$$

Indeed from Young inequality, for any $s > 1$,

$$\xi^{q-1}V^\ell = \left(\xi^{k-1}V^p\right)^{1/s}\left(\xi^{q-1+(1-k)/s}V^{\ell-p/s}\right) \leqslant \xi^{k-1}V^p + \left(\xi^{q-1+(1-k)/s}V^{\ell-p/s}\right)^{s'}.$$

If $q > k$, we can choose s such that

$$\frac{p}{\ell} \leqslant s \leqslant \frac{1-k}{1-q},$$

then $\xi^{q-1+(1-k)/s}V^{\ell-p/s} \leqslant \widetilde{U}^{q-1+(1-k)/s}V^{\ell-p/s}$, and this term is bounded; thus (7.10) holds for A large enough. If $q = k$, then $p \leqslant \ell$, hence $\xi^{q-1}(V^\ell - mAV^p) \leqslant 0$, for A large enough.

- When $1 - \ell \leqslant q < 1 \leqslant k$ and (1.22) holds. Since $q < 1 \leqslant k$, for any $T > 0$, there exists $C_T > 0$ such that for any $(x,t) \in \Omega \times (0,T]$ and any $\xi \in (U(t)(x), \widetilde{U}(t)(x)]$,

$$\xi^{q-1}V^\ell \leqslant C_T^{q-1}\left(\left(S(t)u_0\right)(x)\right)^{q-1}\left(\left(S(t)v_0\right)(x)\right)^\ell$$

in $\Omega \times (0, T]$, from Proposition 6.1(i). From (1.22) and the maximum principle, we have

$$S(t)v_0 \leqslant c^{1/\ell} S(t) u_0^{(1-q)/\ell}.$$

One easily proves that

$$\big(S(t)w\big)^r \leqslant S(t)w^r, \quad \text{for any } r \geqslant 1, \text{ and any } w \in L^\infty(\Omega), w \geqslant 0, \tag{7.12}$$

see, for example, [14]. Taking $r = \ell/(1-q)$, we deduce that

$$S(t)v_0 \leqslant c^{1/\ell} \big(S(t)u_0\big)^{(1-q)/\ell},$$

thus $\xi^{q-1}V^\ell$ is still bounded.

Similar sufficient conditions hold for (7.11). Conclusions follow by considering all the possible combinations of such conditions. $\square$

Remark 7.2. Condition (1.22) is clearly satisfied when $q \geqslant 1$. When $q < 1$, it holds as soon as $\inf_{x\in\Omega} u_0(x) > 0$. Also for the Dirichlet problem it holds when

$$C_1 v_0(x) \leqslant \varphi_1(x) \leqslant C_2 u_0(x), \quad \text{a.e. in } \Omega, \text{ for some } C_1, C_2 > 0;$$

in particular when

$$u_0, v_0 \in C_0(\overline{\Omega}) \cap C^1(\overline{\Omega}) \quad \text{and} \quad u_0 > 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u_0}{\partial \nu} < 0 \quad \text{on } \partial\Omega, \tag{7.13}$$

from the Hopf lemma. Similarly for (1.23) after exchanging u and v.

Remark 7.3. The proof used for Theorem 1.7 consists into showing that $V - AU = \widetilde{V} - A\widetilde{U}$, where $A > 0$ is a suitable parameter. Thus it is linked to the uniqueness of the function $v - Au$. Notice that in any case, uniqueness of $v - Au$ implies uniqueness of (u, v). Indeed assume that (u, v) and $(\hat{u}, \hat{v})$ be two solutions such that $v - Au = \hat{v} - A\hat{u} = y$. Then u and $\hat{u}$ are two solutions of the scalar equation

$$u_t - \Delta u + u^q (Au + y)^p = 0.$$

We can observe that the solution is unique, since the function $r \mapsto r^q (r + y(x))^p$ defined for $r \geqslant -y(x)$ is nondecreasing; then $u = \hat{u}$, and $v = \hat{v}$.

In the particular case $p = \ell$ and $q = k$, taking $A = 1$, we get $G = 0$, hence uniqueness follows immediately. We can find it in another way: the function $y = v - u$, satisfies precisely $y_t - \Delta y = 0$, hence y is unique: $y(t) = S(t)(v_0 - u_0)$, hence again uniqueness follows.

In the case of Neuman problem, we can improve again the results:

Theorem 7.1. *Assume that $u_0, v_0 \in L^\infty(\Omega)$ and $u_0, v_0 \geqslant 0$. Then problem* (1.7) *with Neuman data has a unique solution in any of the following cases*:

(i) $p, q \geqslant 1$;
(ii) $0 < k \leqslant q < 1,\ 0 < \ell \leqslant p < 1$ *and* $p(1-q) \leqslant \ell(1-k)$, *and* $q(1-p) \leqslant k(1-\ell)$;
(iii) $k, \ell \geqslant 1$ *and* $\inf_{x\in\Omega} u_0(x) > 0,\ \inf_{x\in\Omega} v_0(x) > 0$;
(iv) $p, \ell \geqslant 1$ *and* $0 < k \leqslant q$, *and* $\inf_{x\in\Omega} v_0(x) > 0$;
(v) $q, k \geqslant 1$ *and* $0 < \ell \leqslant p$, *and* $\inf_{x\in\Omega} u_0(x) > 0$.

Proof. (i) and (ii) are given in Theorem 1.7.

(iii) Since $k \geqslant 1$, and $\inf_{x\in\Omega} u_0(x) > 0$, for any $T > 0$, there exists $c_T > 0$ such that $U \geqslant c_T$ in Q_T, from Proposition 6.1. Then $\xi^{q-1}V^\ell \leqslant \widetilde{U}^{q-1}V^\ell$, and $\widetilde{U}^{q-1}V^\ell$ is bounded in Q_T, thus (7.10) holds. Similarly since $\ell \geqslant 1$, and $\inf_{x\in\Omega} v_0(x) > 0$, (7.11) holds.

(iv) In the same way, there exists $c_T > 0$ such that $V \geqslant c_T > 0$ in Q_T, since $\ell \geqslant 1$, and $\inf_{x\in\Omega} v_0(x) > 0$, and

$$\xi^{q-1}V^\ell - mAV^p\xi^{k-1} = \xi^{k-1}\left(\xi^{q-k}V^\ell - mAV^p\right) \leqslant U^{k-1}\left(\widetilde{U}^{q-k}V^\ell - mAV^p\right) \leqslant 0$$

for large A, since $k \leqslant q$. Thus (7.10) holds. Otherwise (7.11) holds because $p \geqslant 1$.

(v) Follows by symmetry. □

We deduce also uniqueness results from the comparison properties:

Theorem 7.2. *Assume that $u, v \in C(\overline{Q_\infty})$, and that any of the following conditions holds*:

(i) $p \geqslant 1,\ 0 < k \leqslant q < 1$, *and* $0 < a \leqslant b$, *with* $au_0^b < bv_0^a$ *in* Ω.
(ii) $0 < a \leqslant b,\ 0 < k \leqslant q$, *with Neuman conditions, and* $au_0^b + \varepsilon \leqslant bv_0^a$ *in* Ω, *for some* $\varepsilon > 0$.

Then there exists a unique solution of the problem (1.7).

Proof. (i) Since $p \geqslant 1$, condition (7.11) is satisfied. We know that any solution satisfies $au^b \leqslant bv^a$ in Q_∞, from Theorem 1.4. Let us verify (7.10). Since $k \leqslant q$, for any $\xi \in (U, \widetilde{U})$,

$$\xi^{q-k} \leqslant \widetilde{U}^{q-k} \leqslant \left(\lambda^*\right)^{a(q-k)/b}\widetilde{V}^{a(q-k)/b} \leqslant \left(\lambda^*\right)^{a(q-k)/b}V^{a(q-k)/b} \leqslant mAV^{p-\ell}$$

for A large enough. Indeed $V^{a(q-k)/b+\ell-p}$ is bounded, because

$$a(q-k) + b(\ell-p) = a(b-1) - b(a-1) = b - a \geqslant 0.$$

Then (7.10) follows.

(ii) Notice that $1 \leqslant b$. From Remark 4.4, for any $T > 0$, there exists $c_T > 0$, such that $V \geqslant \widetilde{V} \geqslant c_T$ in Q_T. Since $V, \widetilde{U}^k$ are bounded in Q_T, condition (7.11) holds. And condition (7.10) is also satisfied, because $\widetilde{U}^{q-k}$ is bounded. □

7.3. Estimates and comments

Finally we give upper estimates of the difference between two (possibly signed) solutions, in case of nonuniqueness.

Proposition 7.1. *Assume $u_0, v_0 \in L^\infty(\Omega)$. Let $(u, v), (\hat{u}, \hat{v})$ be two solutions of problem* (1.7)*. If $p, q < 1$, there exists $C > 0$ such that*

$$\left|\hat{u}(t) - u(t)\right| \leqslant Ct^{(p+1)/(1-pq)}, \qquad \left|\hat{v}(t) - v(t)\right| \leqslant Ct^{(q+1)/(1-pq)}, \tag{7.14}$$

in Q_∞. Moreover if $u_0, v_0 \geqslant 0$, with Dirichlet conditions, then there exists $C > 0$ such that

$$|\hat{u} - u| \leqslant \mathbf{u} \quad \textit{and} \quad |\hat{v} - v| \leqslant \mathbf{v}, \quad \textit{in } Q_\infty,$$

where $(\mathbf{u}, \mathbf{v})$ is the solution of the system with source terms

$$\begin{cases} \mathbf{u}_t - \Delta\mathbf{u} = C\mathbf{v}^p, \\ \mathbf{v}_t - \Delta\mathbf{v} = C\mathbf{u}^q, \end{cases} \tag{7.15}$$

such that $\mathbf{u}(0) = \mathbf{v}(0) = 0$ and $\mathbf{u} > 0$, $\mathbf{v} > 0$ in Q_∞.

If $p < 1 \leqslant q$, the same results hold with q replaced by 1.

Proof. (i) Assume $p < 1$. Since u is bounded, we obtain, from (7.2),

$$|\hat{u} - u|(t) \leqslant C\int_0^t S(t-s)\big||\hat{v}|^p - |v|^p\big|(s)\,\mathrm{d}s \leqslant C\int_0^t S(t-s)|\hat{v} - v|^p(s)\,\mathrm{d}s,$$

for some $C > 0$ independent of t. Now using (7.12) with $r = 1/p$, and Hölder inequality, we derive

$$\begin{aligned} |\hat{u} - u|(t) &\leqslant C\int_0^t \left(S(t-s)|\hat{v} - v|(s)\right)^p \mathrm{d}s \leqslant Ct^{1-p}\left(\int_0^t S(t-s)|\hat{v} - v|(s)\,\mathrm{d}s\right)^p \\ &\leqslant Ct^{1-p}\left(\int_0^t S(t-s)\left(\int_0^s S(s-\sigma)|\hat{u} - u|^q(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}s\right)^p. \end{aligned}$$

If $p, q < 1$, denoting $\psi(t) = \sup_{\Omega\times[0,t]}|\hat{u} - u|$, we deduce

$$\psi(t) \leqslant Ct^{(1-p)}\left(\int_0^t S(t-s)\left(\int_0^s S(s-\sigma)\psi(\sigma)\,\mathrm{d}\sigma\right)^q \mathrm{d}s\right)^p \leqslant Ct^{p+1}\psi^{pq}(t),$$

with another $C > 0$. This proves the first estimate of (7.14), and the second one follows by symmetry. If $p < 1 \leqslant q$, then with new constants $C > 0$,

$$|\hat{v} - v|(t) \leqslant C\int_0^t S(t-s)\big||\hat{u}|^q - |u|^q\big|(s)\,\mathrm{d}s \leqslant C\int_0^t S(t-s)|\hat{u} - u|(s)\,\mathrm{d}s$$

and we deduce the estimates

$$\left|\hat{u}(t) - u(t)\right| \leqslant Ct^{(p+1)/(1-p)}, \qquad \left|\hat{v}(t) - v(t)\right| \leqslant Ct^{2/(1-p)},$$

relative to p and 1.

(ii) Assume $u_0, v_0 \geqslant 0$, with Dirichlet conditions. We consider the solutions (U, V) and $(\widetilde{U}, \widetilde{V})$, defined in Theorem 1.3. Recall that $\widetilde{U} - U \geqslant 0$ and $V - \widetilde{V} \geqslant 0$, and $U \leqslant u \leqslant \widetilde{U},\ U \leqslant \hat{u} \leqslant \widetilde{U},\ \widetilde{V} \leqslant v \leqslant V,\ \widetilde{V} \leqslant \hat{v} \leqslant V$. If $p, q < 1$, then

$$\begin{cases} (\widetilde{U} - U)_t - \Delta(\widetilde{U} - U) \leqslant C(\widetilde{V} - V)^p, \\ (\widetilde{V} - V)_t - \Delta(\widetilde{V} - V) \leqslant C(\widetilde{U} - U)^q, \end{cases}$$

for some $C > 0$. As in [14, Lemma 2.5], [12], it follows that

$$|\hat{u} - u| \leqslant \widetilde{U} - U \leqslant \mathbf{u} \quad \text{and} \quad |\hat{v} - v| \leqslant V - \widetilde{V} \leqslant \mathbf{v},$$

where $(\mathbf{u}, \mathbf{v})$ is the solution of (7.15) which uniqueness is proved in [14, Lemma 3.1]. Recall that (7.15) admits precisely the solutions $(0, 0)$, $(\mathbf{u}, \mathbf{v})$ and its translated in time $(\mathbf{u}(t-t_0)^+, \mathbf{v}(t-t_0)^+)$, where $t_0 > 0$ is arbitrary. Similar results hold when $p < 1 \leqslant q$, after replacing q by 1 in system (7.15). □

Remark 7.4. In conclusion, some questions arise. Does uniqueness requires the assumptions $p, q \geqslant 1$ in the case where u_0, v_0 are signed functions? Does it hold for any $p, q, k, \ell > 0$ when $u_0, v_0 \geqslant 0$? What happens when $k = 0$ or $\ell = 0$? The result of Theorem (1.6) has also to be compared with the one of [5] for the (cooperative) system

$$\begin{cases} u_t - \Delta u + u^k = v^p, \\ v_t - \Delta v + v^\ell = u^q, \end{cases} \tag{7.16}$$

which holds under the same conditions $p, q \geqslant 1,\ k, \ell \geqslant 0$. The problem of uniqueness for system (7.16) is also open when $p < 1$ or $q < 1$.

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