

Removable singularities and existence for a quasilinear equation with absorption or source term and measure data

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Abstract

Here we study the solutions of equations with absorption or source term

$$-\Delta_p u = \pm |u|^{q-1} u + \mu$$

in a domain Ω of \mathbb{R}^N , where $1 < p < N$, $q > p - 1$, and μ is a Radon measure on Ω . We introduce a notion of local entropy solution, and give necessary conditions on μ for the existence of solutions in terms of capacity. We study the question of removability sets, and prove some stability results. Finally we give existence results in \mathbb{R}^N for the case of absorption.

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1 Introduction

Let Ω be a regular domain in \mathbb{R}^N , which may be unbounded, and μ be a Radon measure on Ω . Here we consider the elliptic problem with an absorption term:

$$-\Delta_p u + |u|^{q-1} u = \mu, \quad \text{in } \Omega, \quad (1.1)$$

and also the problem with a source term

$$-\Delta_p u = |u|^{q-1} u + \mu, \quad \text{in } \Omega, \quad (1.2)$$

where $1 < p < N$, $q > p - 1$, and

$$u \longmapsto \Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$$

is the p -Laplace operator. We study the existence of local or global solutions and the question of removable sets $E \subset \Omega$ in terms of capacity conditions on μ and E . This leads us to come back to the problem without perturbation term,

$$-\Delta_p u = \mu, \quad \text{in } \Omega, \quad (1.3)$$

for which we define a notion of *local entropy solution*, and give convergence properties, essential to our proofs.

We denote by $\mathcal{M}(\Omega)$ the set of all the Radon measures in Ω (resp. $\mathcal{M}_b(\Omega)$ the subspace of bounded Radon measures in Ω) and $\mathcal{M}^+(\Omega)$ (resp. $\mathcal{M}_b^+(\Omega)$) the subset of nonnegative ones. The capacity $\operatorname{cap}_{m,r}$ associated to $W_0^{m,r}(\Omega)$, for any $m \geq 1, r > 1$ is defined by

$$\operatorname{cap}_{m,r}(K, \Omega) = \inf \left\{ \|\psi\|_{W_0^{m,r}(\Omega)}^r : \psi \in \mathcal{D}(\Omega), 0 \leq \psi \leq 1, \psi = 1 \text{ on } K \right\},$$

for any compact set $K \subset \Omega$. In the sequel we set

$$q^* = q/(q - p + 1), \quad (1.4)$$

so that $q^* = q' = q/(q - 1)$ when $p = 2$.

The first question is to find conditions on the measure μ which ensure the existence of a solution. In the case $p = 2$, a necessary and sufficient condition was found in [3] for the problem with absorption with Dirichlet data on $\partial\Omega$: for any $\mu \in \mathcal{M}_b(\Omega)$, problem

$$\begin{cases} -\Delta u + |u|^{q-1} u = \mu, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.5)$$

has a (weak) solution if and only if μ does not charge the sets E such that $\text{cap}_{2,q'}(E, \mathbb{R}^N) = 0$. In the case of the problem with a source term, this condition is also necessary. A precise necessary and sufficient condition was given in [4] for the existence of (integral) solutions of problem

$$\begin{cases} -\Delta u = (u^+)^q + \mu, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

When μ has a compact support and $\mu \geq 0$, it is equivalent to the existence of a constant $C > 0$, such that

$$\int_K d\mu \leq C \text{cap}_{2,q'}(K, \mathbb{R}^N), \quad \text{for any compact set } K \subset \Omega,$$

see [2]. It implies in particular a limitation of the size of the measure. In the case $p \neq 2$, the question becomes more difficult, because the full duality argument used in [3] and [4] is no more available. Concerning problem (1.1) with Dirichlet data, it was recently shown in [21] that if μ charges the sets E such that $\text{cap}_{1,R}(E, \mathbb{R}^N) = 0$, for some $R > pq^*$, then sequences of approximate solutions do not converge to a "reasonable" solution. This suggested that in some sense problem (1.1) might have no solution. Using our notion of local entropy solution, we show that the result is true, local, and much more general:

Theorem 1.1 *Let $\mu \in \mathcal{M}(\Omega)$, and $q > p - 1$. Suppose that μ charges some set E such that $\text{cap}_{1,R}(E, \mathbb{R}^N) = 0$ for some $R > pq^*$. Then problems (1.1) and (1.2) admit no local entropy solution. More generally, there exists no local entropy solution of problem (1.3) such that $|u|^q \in L^1_{loc}(\Omega)$.*

Notice that we have no restriction of the sign of u and μ , which is unusual in the case of source term. This result concerns the supercritical case $q \geq \bar{P}$, where

$$\bar{P} = \frac{N(p-1)}{N-p}$$

is the first critical exponent. Indeed in the subcritical case $q < \bar{P}$, any nonempty set $E \subset \Omega$ satisfies $\text{cap}_{1,R}(E, \mathbb{R}^N) > 0$. In particular, problems (1.1) or (1.2) have no solution if μ charges the points and $q > \bar{P}$. Recall that when $p = 2$, any set E such that $\text{cap}_{1,R}(E, \mathbb{R}^N) = 0$ for some $R > 2q'$ satisfies $\text{cap}_{2,q'}(E, \mathbb{R}^N) = 0$, from [1]. We have a stronger result for the problem with source term when u and μ are nonnegative, which has to be compared to the one of [4]:

Theorem 1.2 *Let $\mu \in \mathcal{M}^+(\Omega)$. If problem*

$$-\Delta_p u = u^q + \mu, \quad \text{in } \Omega \quad (1.6)$$

has a nonnegative local entropy solution, then for any $R > pq^$,*

$$\int_K d\mu \leq C (\text{cap}_{1,R}(K, \Omega))^{pq^*/R}, \quad \text{for any compact set } K \subset \Omega, \quad (1.7)$$

where $C = C(N, p, q, R, \Omega)$.

Now we come to the second question, namely the characterization of removable sets. When $p = 2$ it was shown in [3] that they are exactly the sets E with $\text{cap}_{2,q'}(E, \mathbb{R}^N) = 0$. In case $p \neq 2$, a recent result of [22] for problem (1.1) suggested that the compact sets K such that $\text{cap}_{1,R}(K, \mathbb{R}^N) = 0$, for some $R > pq^*$, are in some sense removable. We show that it is true:

Theorem 1.3 *Let F be a relatively closed set in Ω , such that $\text{cap}_{1,R}(F, \mathbb{R}^N) = 0$ for some $R > pq^*$. Let $\mu \in \mathcal{M}(\Omega)$ such that μ does not charge the set F . Then F is removable: any local entropy solution of problem*

$$-\Delta_p u + |u|^{q-1} u = \mu, \quad \text{in } \Omega \setminus F$$

is a local entropy solution of

$$-\Delta_p u + |u|^{q-1} u = \mu, \quad \text{in } \Omega.$$

In particular any point is removable when $q > \bar{p}$. It applies also to problem (1.6):

Theorem 1.4 *Let F be a relatively closed set in Ω , such that $\text{cap}_{1,R}(F, \mathbb{R}^N) = 0$ for some $R > pq^*$. Let $\mu \in \mathcal{M}^+(\Omega)$ such that μ does not charge the set F . Then any local nonnegative entropy solution of problem*

$$-\Delta_p u = u^q + \mu, \quad \text{in } \Omega \setminus F,$$

is a local entropy solution of

$$-\Delta_p u = u^q + \mu, \quad \text{in } \Omega.$$

These results are based on local a priori estimates of the solutions, given in Theorems 4.1 and 4.2. We also prove a convergence theorem for the case of absorption:

Theorem 1.5 *Let F be a relatively closed set in Ω , such that $\text{cap}_{1,R}(F, \mathbb{R}^N) = 0$ for some $R > pq^*$. Let $f_\nu, f \in L^1_{\text{loc}}(\Omega)$ such that f_ν converges strongly to f in $L^1_{\text{loc}}(\Omega \setminus F)$. Let u_ν be a local entropy solution of problem*

$$-\Delta_p u_\nu + |u_\nu|^{q-1} u_\nu = f_\nu, \quad \text{in } \Omega \setminus F.$$

Then up to a subsequence, u_ν converges to a local entropy solution u of

$$-\Delta_p u + |u|^{q-1} u = f, \quad \text{in } \Omega.$$

We also show similar results for global solutions when Ω is bounded, see Theorem 7.3. Thus we improve a result of [16] given for $p = 2$, and the result of [22] quoted above. The proof is based upon a local stability result for the problem (1.3) following the global stability result of [18], see Theorem 3.3. Notice that Theorem 1.5 has no corresponding in the case of source term, due to the fact that problem (1.6) can admit no solution in the supercritical case for some measures $\mu \in L^1(\Omega)$, independently of their size.

Another consequence of the local a priori estimates and stability is an existence result in whole \mathbb{R}^N for the problem with absorption without growth conditions on the data, which improves the results of [14]:

Theorem 1.6 *Assume $q > p - 1 > 0$. Then for every $f \in L^1_{\text{loc}}(\mathbb{R}^N)$, there exists a local entropy solution u of problem*

$$-\Delta_p u + |u|^{q-1} u = f, \quad \text{in } \mathbb{R}^N. \quad (1.8)$$

And $u \geq 0$ if $f \geq 0$.

We also give existence results in the subcritical case $q < \bar{P}$ with a measure data and Ω bounded, or $\Omega = \mathbb{R}^N$, see Theorems 8.2 and 8.3. For the problems with a source term (1.2) and (1.6), the existence was proved in [20] when $q < \bar{P}$, at least when $p > P_0$, and Ω, μ are bounded, and the size of μ is small enough. The problem is open in the case $q \geq \bar{P}$, even when $\mu \in L^s(\Omega)$ with s large enough.

For simplification all our results are given for the p -Laplace operator, and the nonlinear term is $|u|^{q-1} u$, but they can be extended to elliptic operators $A(x, \nabla u)$, with power growth in $|\nabla u|$ of the order $p - 1$, and a perturbation term $\pm g(x, u)$ such that $g(x, u)u \geq 0$ and which grows in u like $|u|^q$.

2 Global and local entropy solutions

First recall some well-known results concerning the problem

$$\begin{cases} -\Delta_p u = \mu, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (2.1)$$

with $\mu \in \mathcal{M}_b(\Omega)$ and Ω bounded. We set

$$P_0 = \frac{2N}{N+1}, \quad P_1 = 2 - \frac{1}{N},$$

so that $1 < P_0 < P_1$, and $p > P_0 \iff \bar{P} > 1$. When $p > P_1$, problem (2.1) admits at least a solution such that $u \in W_0^{1,m}(\Omega)$ for any $1 \leq m < (p-1)N/(N-1)$, in the sense of distributions. Hence the gradient is well defined in $L^1(\Omega)$. In the general case one needs to define a notion of entropy solutions, or renormalized solutions, which can be done in four equivalent ways, as shown in [18], and allows to give a sense to the gradient. They are solutions such that $\nabla T_k(u) \in L^p(\Omega)$ for any $k > 0$, where

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k \operatorname{sign} s & \text{if } |s| > k, \end{cases} \quad (2.2)$$

and the gradient of u , denoted by $y = \nabla u$ is defined by

$$\nabla(T_k(u)) = y \times 1_{\{|u| \leq k\}} \quad \text{a.e. in } \Omega. \quad (2.3)$$

Such solutions u may not be in $L^1(\Omega)$ when $p \leq P_0$. For any $p > 1$ there exists at least a solution of (2.1), and it is unique if $\mu \in L^1(\Omega)$. Moreover any entropy solution satisfies the equation in the sense of distributions.

Now we mention the usual definitions of entropy solutions and above all define a notion of local entropy solution. We call $\mathcal{M}_0(\Omega)$, the set of measures $\mu_0 \in \mathcal{M}(\Omega)$ such that

$$\mu_0(B) = 0 \quad \text{for any Borel set } B \subset \Omega \text{ such that } \operatorname{cap}_{1,p}(B, \Omega) = 0. \quad (2.4)$$

First recall that any measure $\mu \in \mathcal{M}(\Omega)$ admits a unique decomposition as

$$\mu = \mu_0 + \mu_s^+ - \mu_s^- \quad (2.5)$$

where $\mu_0 \in \mathcal{M}_0(\Omega)$ and μ_s^\pm are nonnegative and singular, concentrated on sets E^\pm with $\operatorname{cap}_{1,p}(E^\pm, \mathbb{R}^N) = 0$, see [19]. Moreover μ_0 is nonnegative, and $\mu_s^- = 0$, if μ is nonnegative. Notice also that

$$\mu_s^+ \leq \mu^+, \quad \mu_s^- \leq \mu^- \quad \text{and} \quad |\mu_0| \leq |\mu|. \quad (2.6)$$

For any $n, k > 0$ and any $r \in \mathbb{R}$, we set

$$\begin{aligned} S_{n,k}(r) &= \min((|r| - n)^+/k, 1) \operatorname{sign} r, \\ h_{n,k}(r) &= \min((r - n)^+/k, 1), \quad H_{n,k}(r) = 1 - \min((|r| - n)^+/k, 1). \end{aligned} \quad (2.7)$$

1) Global entropy solutions.

Here Ω is bounded, and $\mu = \mu_0 + \mu_s^+ - \mu_s^- \in \mathcal{M}_b(\Omega)$. Following [18], we will say that u is an (global) entropy solution of problem (2.1) if

$$u \text{ is measurable and finite a.e. in } \Omega, \quad (2.8)$$

$$T_k(u) \in W_0^{1,p}(\Omega) \quad \text{for every } k > 0, \quad (2.9)$$

$$|\nabla u|^{p-1} \in L^r(\Omega), \quad \text{for any } 1 \leq r < N/(N-1), \quad (2.10)$$

where the gradient is defined by (2.3), and u satisfies

D1: for any $h \in W^{1,\infty}(\mathbb{R})$ such that h' has a compact support, and any $\varphi \in W^{1,m}(\Omega)$ for some $m > N$, such that $h(u)\varphi \in W_0^{1,p}(\Omega)$,

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (h(u)\varphi) dx = \int_{\Omega} h(u)\varphi d\mu_0 + h(+\infty) \int_{\Omega} \varphi d\mu_s^+ - h(-\infty) \int_{\Omega} \varphi d\mu_s^-. \quad (2.11)$$

Three other definitions are equivalent:

Theorem 2.1 ([18]) *Let u be a function such that (2.8), (2.9) and (2.10) hold. Then **D1** is equivalent to any of the conditions*

D2: if $\omega \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and if there exists $k > 0$ and $\omega^+, \omega^- \in W^{1,r}(\Omega) \cap L^\infty(\Omega)$ with $r > N$, such that $\omega = \omega^+$ a.e. on the set $\{u > k\}$ and $\omega = \omega^-$ a.e. on the set $\{u < -k\}$, then

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \omega dx = \int_{\Omega} \omega d\mu_0 + \int_{\Omega} \omega^+ d\mu_s^+ - \int_{\Omega} \omega^- d\mu_s^-. \quad (2.12)$$

D3: for any $k > 0$, there exist $\alpha_k, \beta_k \in \mathcal{M}_0(\Omega) \cap \mathcal{M}_b^+(\Omega)$, concentrated on the sets $\{u = k\}$ and $\{u = -k\}$ respectively, converging weakly to μ_s^+, μ_s^- such that

$$\int_{\Omega} |\nabla u_k|^{p-2} \nabla u_k \cdot \nabla \psi dx = \int_{\{|u| \leq k\}} \psi d\mu_0 + \int_{\Omega} \psi d\alpha_k - \int_{\Omega} \psi d\beta_k, \quad (2.13)$$

for any $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

D4: for any $h \in W^{1,\infty}(\mathbb{R})$ with compact support, and $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $h(u)\varphi \in W_0^{1,p}(\Omega)$,

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (h(u)\varphi) dx = \int_{\Omega} h(u)\varphi d\mu_0, \quad (2.14)$$

and for any $\varphi \in C(\Omega)$ and bounded,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\{n \leq u \leq 2n\}} |\nabla u|^p \varphi dx = \int_{\Omega} \varphi d\mu_s^+, \quad \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\{-2n \leq u \leq -n\}} |\nabla u|^p \varphi dx = \int_{\Omega} \varphi d\mu_s^-. \quad (2.15)$$

Remark 2.1. In the definition of global entropy solution, (2.10) can be weakened in

$$|\nabla u|^{p-1} \in L^1(\Omega), \quad (2.16)$$

and then

$$|u|^{p-1} \in L^s(\Omega), \quad \text{for any } 1 \leq s < N/(N-p), \quad (2.17)$$

from [5, Lemma 4.1]. Also condition (2.11) has to be satisfied only for any $\varphi \in \mathcal{D}^+(\Omega)$. Indeed this implies that **D3** holds for such φ , and by density for any nonnegative $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, then for any $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, hence **D3** holds, hence also **D1**.

Remark 2.2 The different notions of entropy solutions are given in general for bounded measures, since it appears to be a good frame for existence theorems. Notice however that when $p = 2$, the problem

$$\begin{cases} -\Delta u = \mu, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

is well posed for possibly unbounded measures μ , such that $\int_{\Omega} \rho(x) d\mu(x) < \infty$, where $\rho(x)$ is the distance from x to $\partial\Omega$. In the following we are interested by local solutions as well as global ones with Dirichlet data. Thus we will not always require, when it is possible, that the measures are bounded, contrarily to most of the litterature on the subject.

2) Local entropy solutions

Here Ω and $\mu \in \mathcal{M}(\Omega)$ are possibly unbounded. We will say that u is a local entropy solution of problem (1.3) if u satisfies (2.8),

$$T_k(u) \in W_{loc}^{1,p}(\Omega) \quad \text{for any } k > 0, \quad (2.18)$$

$$|u|^{p-1} \in L_{loc}^s(\Omega), \quad \text{for any } 1 < s < N/(N-p), \quad (2.19)$$

$$|\nabla u|^{p-1} \in L_{loc}^r(\Omega), \quad \text{for any } 1 \leq r < N/(N-1), \quad (2.20)$$

and

D1loc: for any $h \in W^{1,\infty}(\mathbb{R})$ such that h' has a compact support, and $\varphi \in W^{1,m}(\Omega)$ for some $m > N$, with compact support, such that $h(u)\varphi \in W^{1,p}(\Omega)$,

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (h(u)\varphi) dx = \int_{\Omega} h(u)\varphi d\mu_0 + h(+\infty) \int_{\Omega} \varphi d\mu_s^+ - h(-\infty) \int_{\Omega} \varphi d\mu_s^-. \quad (2.21)$$

Here also we will use equivalent definitions.

Theorem 2.2 Let u be a function such that (2.8), (2.19), (2.18) and (2.20) hold. Then **D2loc** is equivalent to one of the conditions

D2loc: if $\omega \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with compact support in Ω , and if there exists $k > 0$ and $\omega^+, \omega^- \in W^{1,r}(\Omega) \cap L^\infty(\Omega)$ with $r > N$, such that $\omega = \omega^+$ a.e. on the set $\{u > k\}$ and $\omega = \omega^-$ a.e. on the set $\{u < -k\}$, then

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \omega dx = \int_{\Omega} \omega d\mu_0 + \int_{\Omega} \omega^+ d\mu_s^+ - \int_{\Omega} \omega^- d\mu_s^-. \quad (2.22)$$

D3loc: there exist $\alpha_k, \beta_k \in \mathcal{M}_0(\Omega) \cap \mathcal{M}^+(\Omega)$, concentrated on the sets $\{u = k\}$ and $\{u = -k\}$ respectively, converging weakly to μ_s^+, μ_s^- such that

$$\int_{\Omega} |\nabla u_k|^{p-2} \nabla u_k \cdot \nabla \psi dx = \int_{\{|u| \leq k\}} \psi d\mu_0 + \int_{\Omega} \psi d\alpha_k - \int_{\Omega} \psi d\beta_k, \quad (2.23)$$

for any $\psi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with compact support in Ω .

D4loc: for any $h \in W^{1,\infty}(\mathbb{R})$ with compact support, and $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with compact support in Ω , such that $h(u)\varphi \in W^{1,p}(\Omega)$,

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (h(u)\varphi) dx = \int_{\Omega} h(u)\varphi d\mu_0, \quad (2.24)$$

and for any $\varphi \in C(\Omega)$ with compact support in Ω ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\{n \leq u \leq 2n\}} |\nabla u|^p \varphi dx = \int_{\Omega} \varphi d\mu_s^+, \quad \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\{-2n \leq u \leq -n\}} |\nabla u|^p \varphi dx = \int_{\Omega} \varphi d\mu_s^-. \quad (2.25)$$

The proof follows the one of [18], with some modifications due to the fact that Lemma 4.3 of [17] does not apply. It is given in Appendix A for a better comprehension.

3) Solutions with a perturbation term:

Let f be a Caratheodory function on $\Omega \times \mathbb{R}$. For bounded Ω , and $\mu \in \mathcal{M}_b(\Omega)$, an entropy solution u of problem

$$\begin{cases} -\Delta_p u = f(x, u) + \mu, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

will be a function u such that $f(x, u) \in L^1(\Omega)$ and u is an entropy solution of problem in the sense above. For general Ω and $\mu \in \mathcal{M}(\Omega)$, a local entropy solution of problem

$$-\Delta_p u = f(x, u) + \mu, \quad \text{in } \Omega,$$

will be a function u such that $f(x, u) \in L^1_{loc}(\Omega)$ and u is a local entropy solution of problem in the sense above.

3 Local solutions without absorption

Here also we show that the assumptions on the local entropy solutions of problem without perturbation (1.3) can be weakened. *This is the key point for all the sequel.* Recall that Ω may be unbounded.

Theorem 3.1 *Let u be a function satisfying (2.8), (2.18), and*

$$|u|^q \in L^1_{loc}(\Omega), \text{ for some } q > p - 1, \quad (3.1)$$

$$|\nabla u|^{p-1} \in L^1_{loc}(\Omega), \quad (3.2)$$

and **D1loc**. Then

$$(|u| + 1)^{\alpha-1} |\nabla u|^p \in L^1_{loc}(\Omega), \quad \text{for any } \alpha < 0, \quad (3.3)$$

and u satisfies (2.19) and (2.20). Moreover

$$|\nabla u|^{p-1} \in L^\sigma_{loc}(\Omega), \quad \text{for any } 1 \leq \sigma < p'q/(q+1). \quad (3.4)$$

Proof. Step 1: Estimate (3.3). Let $\alpha < 0$. We set $u_k = T_k(u)$, for any $k > 0$. We take

$$h_k(u) = (1 - (|u_k| + 1)^\alpha) \text{ sign } u \quad (3.5)$$

in (2.21), and get for any $\varphi \in \mathcal{D}^+(\Omega)$,

$$\begin{aligned} & |\alpha| \int_{\Omega} (|u_k| + 1)^{\alpha-1} |\nabla u_k|^p \varphi dx \\ &= - \int_{\Omega} h_k(u) |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx + \int_{\Omega} h_k(u) \varphi d\mu_0 + (1 - (k+1)^\alpha) \int_{\Omega} \varphi d|\mu_s| \\ &\leq \int_{\Omega} |\nabla u_k|^{p-1} |\nabla \varphi| dx + \int_{\{|u|>k\}} |\nabla u|^{p-1} |\nabla \varphi| dx + \int_{\Omega} \varphi d|\mu|. \end{aligned} \quad (3.6)$$

Now from Hölder inequality,

$$\begin{aligned}
& \int_{\Omega} |\nabla u_k|^{p-1} |\nabla \varphi| dx \\
&= \int_{\Omega} (|u_k| + 1)^{(\alpha-1)/p'} |\nabla u_k|^{p-1} (|u_k| + 1)^{(1-\alpha)/p'} |\nabla \varphi| dx \\
&\leq \left(\int_{\Omega} (|u_k| + 1)^{\alpha-1} |\nabla u_k|^p \varphi dx \right)^{1/p'} \left(\int_{\Omega} (|u_k| + 1)^{(1-\alpha)(p-1)} \varphi^{1-p} |\nabla \varphi|^p dx \right)^{1/p}. \quad (3.7)
\end{aligned}$$

Since u satisfies (3.1), we can fix α such that

$$\tau = q/(p-1)(1-\alpha) > 1. \quad (3.8)$$

Then we get

$$\begin{aligned}
& \int_{\Omega} |\nabla u_k|^{p-1} |\nabla \varphi| dx \\
&\leq \left(\int_{\Omega} (|u_k| + 1)^{\alpha-1} |\nabla u_k|^p \varphi dx \right)^{1/p'} \left(\int_{\Omega} (|u_k| + 1)^q \varphi dx \right)^{1/\tau p} \left(\int_{\Omega} \varphi^{1-\tau'p} |\nabla \varphi|^{\tau'p} dx \right)^{1/\tau'p}; \quad (3.9)
\end{aligned}$$

hence

$$\begin{aligned}
\int_{\Omega} |\nabla u_k|^{p-1} |\nabla \varphi| dx &\leq \frac{|\alpha|}{2} \int_{\Omega} (|u_k| + 1)^{\alpha-1} |\nabla u_k|^p \varphi dx \\
&\quad + C \left(\int_{\Omega} (|u| + 1)^q \varphi dx \right)^{1/\tau} \times \left(\int_{\Omega} \varphi^{1-\tau'p} |\nabla \varphi|^{\tau'p} dx \right)^{1/\tau'}, \quad (3.10)
\end{aligned}$$

where $C = C(\alpha)$. Reporting (3.10) into (3.6), it comes

$$\begin{aligned}
\frac{|\alpha|}{2} \int_{\Omega} (|u_k| + 1)^{\alpha-1} |\nabla u_k|^p \varphi dx &\leq \int_{\{|u|>k\}} |\nabla u|^{p-1} |\nabla \varphi| dx + \int_{\Omega} \varphi d|\mu| \\
&\quad + C(\alpha) \left(\int_{\Omega} (|u| + 1)^q \varphi dx \right)^{1/\tau} \times \left(\int_{\Omega} \varphi^{1-\tau'p} |\nabla \varphi|^{\tau'p} dx \right)^{1/\tau'}. \quad (3.11)
\end{aligned}$$

Going to the limit as $k \rightarrow \infty$ from (3.2), we deduce that

$$\begin{aligned}
\frac{|\alpha|}{2} \int_{\Omega} (|u| + 1)^{\alpha-1} |\nabla u|^p \varphi dx &\leq C \left(\int_{\Omega} (|u| + 1)^q \varphi dx \right)^{1/\tau} \left(\int_{\Omega} \varphi^{1-\tau'p} |\nabla \varphi|^{\tau'p} dx \right)^{1/\tau'} \\
&\quad + \int_{\Omega} \varphi d|\mu| \quad (3.12)
\end{aligned}$$

And from (3.9),

$$\begin{aligned} & \int_{\Omega} |\nabla u|^{p-1} |\nabla \varphi| dx \\ & \leq \left(\int_{\Omega} (|u| + 1)^{\alpha-1} |\nabla u|^p \varphi dx \right)^{1/p'} \left(\int_{\Omega} (|u| + 1)^q \varphi dx \right)^{1/\tau p} \left(\int_{\Omega} \varphi^{1-\tau'p} |\nabla \varphi|^{\tau'p} dx \right)^{1/\tau'p}, \end{aligned} \quad (3.13)$$

where the gradient is defined in (2.3). Taking

$$\varphi = \zeta^{p\tau'}, \quad \text{with } \zeta \in \mathcal{D}^+(\Omega), \quad (3.14)$$

so that $\varphi^{1-\tau'p} |\nabla \varphi|^{\tau'p} \in L^1(\Omega)$, we deduce that $(|u| + 1)^{\alpha-1} |\nabla u|^p \in L^1_{loc}(\Omega)$ for $|\alpha|$ small enough, which implies (3.3).

Step 2: other estimates. Let U be any domain such that $U \subset\subset \Omega$. For any $v \in L^1_{loc}(\Omega)$ we set

$$\bar{v} = |U|^{-1} \int_U v dx.$$

Recall the Galliaro-Nirenberg estimate: for any $\lambda \geq 1$, for any $v \in W^{1,p}(U) \cap L^\lambda(U)$,

$$\|v - \bar{v}\|_{L^\gamma(\Omega)} \leq c \|\nabla v\|_{L^p(\Omega)}^\theta \|v - \bar{v}\|_{L^\lambda(\Omega)}^{1-\theta}, \quad (3.15)$$

for any $\gamma \in [1, +\infty)$ and $\theta \in [0, 1]$ such that

$$\frac{1}{\gamma} = \theta \left(\frac{1}{p} - \frac{1}{N} \right) + \frac{1-\theta}{\lambda}, \quad (3.16)$$

where $c = c(N, p, \lambda, \theta, U)$. Let us take $\alpha \in (1-p, 0)$ and

$$v = (1 + |u|)^\beta, \quad \beta = (\alpha + p - 1)/p, \quad \lambda = (p-1)/\beta. \quad (3.17)$$

Then $v \in L^p(U)$, since $\beta p < p-1 < q$ and $u \in L^q(U)$. Moreover $u_k = T_k(u) \in W^{1,p}(U)$, hence from the chain rule, $v_{k+1} = T_{k+1}(v) = (1 + |u_k|)^\beta \in W^{1,p}(U)$ and

$$\nabla v_{k+1} = \beta(|u_k| + 1)^{\beta-1} \nabla u_k \operatorname{sign} u_k.$$

By definition of the gradient, we have $\nabla v_{k+1} = \nabla v \times 1_{\{|u| \leq k\}}$, hence

$$\int_{U \cap \{|u| \leq k\}} |\nabla v|^p dx = \int_U |\nabla v_{k+1}|^p dx \leq \beta^p \int_U (|u_k| + 1)^{\alpha-1} |\nabla u_k|^p dx.$$

And $(u + 1)^{\alpha-1} |\nabla u|^p \in L^1(U)$ from (3.3). Thus we can go to the limit as $k \rightarrow \infty$, and get $|\nabla v| \in L^p(U)$. This implies $v \in W^{1,p}(U)$ and the gradient above coincides with the distributional gradient of v , see [18, Remark 2.10]. And $u \in L^{\beta\gamma}(U)$, since

$$\begin{aligned} & \int_U (|u| + 1)^\beta - \bar{v}^\gamma \, dx \\ & \leq c^\gamma \left(\int_U (|u| + 1)^{\alpha-1} |\nabla u|^p \, dx \right)^{\theta\gamma/p} \left(\int_U (|u| + 1)^\beta - \bar{v}^{(p-1)/\beta} \, dx \right)^{(1-\theta)\gamma\beta/(p-1)}, \end{aligned} \quad (3.18)$$

and $|u|^{p-1} \in L^1(U)$. Chosing θ close to 1, and α close to 0, we deduce (2.19). Moreover for any $0 < \eta < p$, we find

$$\int_\Omega |\nabla u|^\eta \, dx \leq \left(\int_U (|u| + 1)^{\alpha-1} |\nabla u|^p \, dx \right)^{\eta/p} \left(\int_U (|u| + 1)^{(1-\alpha)\eta/(p-\eta)} \, dx \right)^{(p-\eta)/p}, \quad (3.19)$$

from Hölder inequality. Choosing again α close to 0, the left-hand side is finite for any η such that $\eta/(p-\eta) < (p-1)N/(N-p)$, that means $0 < \eta < N(p-1)/(N-1)$. Hence (2.20) holds. Finally for any $1 \leq \sigma < p'$,

$$\int_U |\nabla u|^{(p-1)\sigma} \varphi \, dx \leq \left(\int_U (|u| + 1)^{\alpha-1} |\nabla u|^p \varphi \, dx \right)^{\sigma/p'} \times \left(\int_U (|u| + 1)^{(1-\alpha)/(p'/\sigma-1)} \varphi \, dx \right)^{1-\sigma/p'}, \quad (3.20)$$

and the left-hand side is finite if $1/(p'/\sigma - 1) < q$, that means $\sigma < p'q/(q+1)$, which proves (3.4). \blacksquare

Remark 3.1 Estimate (3.4) was first observed in [14] for problem (1.1) in \mathbb{R}^N . It is stronger than (2.20) whenever $q > \bar{P}$.

In the same way we deduce convergence properties.

Theorem 3.2 *Let (μ_ν) be a sequence of Radon measures in Ω , uniformly locally bounded. Let (u_ν) be a sequence of local entropy solutions of*

$$-\Delta_p u_\nu = \mu_\nu, \quad \text{in } \Omega,$$

such that $(|u_\nu|^q)$ is bounded in $L^1_{loc}(\Omega)$, for some $q > p-1$. Then

$$(T_k(u_\nu)) \quad \text{is bounded in } W^{1,p}_{loc}(\Omega) \quad \text{for any } k > 0, \quad (3.21)$$

$$(|u_\nu|^{p-1}) \quad \text{is bounded in } L^s_{loc}(\Omega) \quad \text{for any } 1 \leq s < N/(N-p), \quad (3.22)$$

$$(|\nabla u_\nu|^{p-1}) \quad \text{is bounded in } L^r_{loc}(\Omega) \quad \text{for any } 1 \leq r < \max(N/(N-1), p'q/(q+1)). \quad (3.23)$$

And up to a subsequence, μ_ν converges weakly to a measure μ , and (u_ν) converges locally in measure in Ω and a.e. in Ω to some function u . And u satisfies (2.8), (2.19), (2.18) and (2.20), (3.4), and (∇u_ν) converges to ∇u locally in measure in Ω .

Proof. Step 1: a priori estimates. Taking the same notations as above, and the test function φ defined in (3.14), we have from (3.12) and (3.13),

$$\frac{|\alpha|}{2} \int_{\Omega} (|u_\nu| + 1)^{\alpha-1} |\nabla u|^p \varphi dx \leq \int_{\Omega} \varphi d|\mu_\nu| + C \left(\int_{\Omega} (|u_\nu| + 1)^q \varphi dx \right)^{1/\tau},$$

with a constant $C = C(\alpha, \varphi) > 0$. Hence for any domain $U \subset\subset \Omega$,

$$\int_U (|u_\nu| + 1)^{\alpha-1} |\nabla u|^p dx \quad \text{is bounded.} \quad (3.24)$$

In particular $(u_{\nu,k})$ is bounded in $W^{1,p}(U)$, and (3.21) holds. Then (3.22) and (3.23) follow from (3.18), (3.19) and (3.20) applied to (u_ν) , after noticing that the sequence (v_ν) defined by

$$v_\nu = (|u_\nu| + 1)^\beta, \quad \beta = (\alpha + p - 1)/p,$$

is bounded in $L^1_{loc}(\Omega)$; hence the sequence (\bar{v}_ν) of its mean values on U is bounded. Moreover considering the two functions

$$v'_\nu = (u_\nu^+ + 1)^\beta, \quad v''_\nu = (u_\nu^- + 1)^\beta,$$

one has

$$|\nabla v'_\nu| \leq \beta(|u_\nu| + 1)^{(\alpha-1)/p} |\nabla u_\nu|, \quad |\nabla v''_\nu| \leq \beta(|u_\nu| + 1)^{(\alpha-1)/p} |\nabla u_\nu|,$$

from the chain rule applied to the truncatures $u_{\nu,k}$, and by definition of the gradient. Hence $(|\nabla v'_\nu|)$ and $(|\nabla v''_\nu|)$ are bounded in $L^p_{loc}(\Omega)$, so that

$$(v'_\nu) \text{ and } (v''_\nu) \quad \text{are bounded in } W^{1,p}_{loc}(\Omega). \quad (3.25)$$

Up to a subsequence, (μ_ν) converges weakly to a measure μ , and v'_ν and v''_ν converge weakly in $W^{1,p}_{loc}(\Omega)$, strongly in $L^p_{loc}(\Omega)$, and $v'_\nu \rightarrow v'$, $v''_\nu \rightarrow v''$ a.e. in Ω . Then u_ν converges locally in measure and a.e. in Ω to a function u ; and $|u|^q \in L^1_{loc}(\Omega)$, and u satisfies (2.19) from the Fatou lemma. For fixed $k > 0$, $(u_{\nu,k})$ is bounded in $W^{1,p}_{loc}(\Omega)$, and converges a.e. to (u_k) . We can extract a subsequence (depending on k) converging weakly in $W^{1,p}_{loc}(\Omega)$ and a.e. in Ω , and necessarily to u_k . Then $u_k \in W^{1,p}_{loc}(\Omega)$, and the whole sequence $(u_{\nu,k})$ converges weakly to u_k in $W^{1,p}_{loc}(\Omega)$.

Step 2: convergence of the gradients. We set

$$\mu_\nu = \mu_{\nu,0} + \mu_{\nu,s}^+ - \mu_{\nu,s}^-,$$

where $\mu_{\nu,0} \in \mathcal{M}_0(\Omega)$ and $\mu_{\nu,s}^\pm$ are nonnegative and singular. Here we use the equation satisfied by the truncated functions $u_{\nu,k}$ as in [18]. From Theorem 2.2, there exist $\alpha_{\nu,k}, \beta_{\nu,k} \in \mathcal{M}_0(\Omega) \cap \mathcal{M}^+(\Omega)$, concentrated on the sets $\{u_\nu = k\}, \{u_\nu = -k\}$, such that

$$\int_{\Omega} |\nabla u_{\nu,k}|^{p-2} \nabla u_{\nu,k} \cdot \nabla \psi dx = \int_{\{|u_\nu| \leq k\}} \psi d\mu_{\nu,0} + \int_{\Omega} \psi d\alpha_{\nu,k} - \int_{\Omega} \psi d\beta_{\nu,k},$$

for any $\psi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with compact support in Ω . Taking $\psi = u_{\nu,k} \varphi$ with $\varphi \in \mathcal{D}^+(\Omega)$, we get

$$\begin{aligned} \int_{\Omega} |\nabla u_{\nu,k}|^p \varphi dx + \int_{\Omega} u_{\nu,k} |\nabla u_{\nu,k}|^{p-2} \nabla u_{\nu,k} \cdot \nabla \varphi dx &= \int_{\{|u_\nu| \leq k\}} u_{\nu,k} \varphi d\mu_{\nu,0} \\ &+ k \left(\int_{\Omega} \varphi d\alpha_{\nu,k} + \int_{\Omega} \varphi d\beta_{\nu,k} \right). \end{aligned}$$

And $|u_{\nu,k}| \leq k$, a.e. in Ω , and $\mu_{\nu,0}$ - a.e. in Ω , hence

$$\int_{\Omega} \varphi d\alpha_{\nu,k} + \int_{\Omega} \varphi d\beta_{\nu,k} \leq \int_{\Omega} \varphi d|\mu_{\nu,0}| + \frac{1}{k} \int_{\Omega} |\nabla u_{\nu,k}|^p \varphi dx + \int_{\Omega} |\nabla u_{\nu,k}|^{p-1} |\nabla \varphi| dx.$$

Taking φ such that $\varphi = 1$ on $U \subset\subset \Omega$, we get, from (3.24),

$$\int_U d\alpha_{\nu,k} + \int_U d\beta_{\nu,k} \leq C (1 + k^{|\alpha|})$$

for any $k > 1$, where $C > 0$ does not depend on ν , since $|\mu_{\nu,0}| \leq |\mu_\nu|$ from (2.6). Then $\alpha_{\nu,k}, \beta_{\nu,k}$ are locally bounded independently on ν for fixed k , hence also

$$\mu_{\nu,k} = \chi_{\{|u_\nu| \leq k\}} \mu_{\nu,0} + \alpha_{\nu,k} - \beta_{\nu,k}.$$

And $u_{\nu,k}$ is a solution in the sense of $\mathcal{D}'(\Omega)$ of equation

$$-\Delta_p u_{\nu,k} = \mu_{\nu,k},$$

with $u_{\nu,k}$ bounded in $W_{loc}^{1,p}(\Omega)$ and in $L^\infty(\Omega)$. Following the proof of [15, Theorem 2.1], we deduce that, after extracting a diagonal subsequence, which depends on k , $\nabla u_{\nu,k}$ converges

a.e. in Ω and strongly to ∇u_k in $(L_{loc}^\lambda(\Omega))^N$ for any $1 \leq \lambda < p$, hence also the whole sequence converges. But for any $\varepsilon > 0$, and any $\nu, \rho \in \mathbb{N}$,

$$\{|\nabla u_\nu - \nabla u_\rho| > \varepsilon\} \subset \{|u_\nu| > k\} \cup \{|u_\rho| > k\} \cup \{|\nabla u_{\nu,k} - \nabla u_{\rho,k}| > \varepsilon\},$$

and $|u_\nu|^q$ is bounded in $L_{loc}^1(\Omega)$. Hence (∇u_ν) is a locally a Cauchy sequence in measure. Up to a subsequence, ∇u_ν converges a.e. in Ω , and necessarily to ∇u , by definition of the gradient. Then ∇u satisfies (2.20) and (3.4). And $|\nabla u_\nu|^{p-2} \nabla u_\nu$ converges strongly to $|\nabla u|^{p-2} \nabla u$ in $L_{loc}^r(\Omega)$ for any $1 \leq r < N/(N-1)$, from the Vitali theorem. \blacksquare

Now we give a local stability result, following the global result of [18, Theorem 3.4]. We have not searched to extend it completely to the local problem, because it was not needed in our situations, where the perturbation term in fact requires stronger convergences properties. We give the proof in Appendix B.

Theorem 3.3 *Let $\lambda \in \mathcal{M}(\Omega)$. Let $f_\nu, f \in L_{loc}^1(\Omega)$, such that f_ν converges weakly to f in $L_{loc}^1(\Omega)$. Let u_ν be a local entropy solution of problem*

$$-\Delta_p u_\nu = f_\nu + \lambda, \quad \text{in } \Omega, \quad (3.26)$$

such that $(|u_\nu|^q)$ is bounded in $L_{loc}^1(\Omega)$, for some $q > p-1$. Then up to a subsequence, u_ν converges a.e. in Ω to a local entropy solution u of

$$-\Delta_p u = f + \lambda, \quad \text{in } \Omega. \quad (3.27)$$

4 Estimates for absorption or source term

Here we give universal a priori estimates for problems (1.1) and (1.6). First consider the case of absorption :

Theorem 4.1 *Let u be any local entropy solution of problem (1.1). Then for any $R > pq^*$, there exists $C = C(N, p, q, R, \Omega)$ such that for any $\zeta \in \mathcal{D}^+(\Omega)$,*

$$\int_{\Omega} (|u| + 1)^q \zeta^R dx \leq C \left(\int_{\Omega} \zeta^R dx + \int_{\Omega} \zeta^R d|\mu| + \int_{\Omega} |\nabla \zeta|^R dx \right). \quad (4.1)$$

And for any $\alpha < 0$, there exists $C = C(\alpha, N, p, q, R, \Omega)$ such that

$$\int_{\Omega} (|u| + 1)^{\alpha-1} |\nabla u|^p \zeta^R dx \leq C \left(\int_{\Omega} \zeta^R dx + \int_{\Omega} \zeta^R d|\mu| + \int_{\Omega} |\nabla \zeta|^R dx \right). \quad (4.2)$$

Proof. Let $R > pq^*$. By hypothesis $|u|^q \in L^1_{loc}(\Omega)$, and $q > p - 1$, hence Theorem 3.1 applies. We take the test functions h_k defined in (3.5), and get, for any $\varphi \in \mathcal{D}^+(\Omega)$,

$$\begin{aligned} & |\alpha| \int_{\Omega} (|u_k| + 1)^{\alpha-1} |\nabla u_k|^p \varphi dx + \int_{\Omega} |u|^{q-1} u h_k(u) \varphi dx \\ &= - \int_{\Omega} h_k(u) |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx + \int_{\Omega} h_k(u) \varphi d\mu_0 + (1 - (k+1)^\alpha) \int_{\Omega} \varphi d|\mu_s| \\ &\leq \int_{\Omega} |\nabla u_k|^{p-1} |\nabla \varphi| dx + \int_{\{|u|>k\}} |\nabla u|^{p-1} |\nabla \varphi| dx + \int_{\Omega} \varphi d|\mu|. \end{aligned}$$

This estimate is similar to (3.6), with an additional nonnegative term $\int_{\Omega} |u|^{q-1} u h_k(u) \varphi dx$ in the left-hand side, since $|u|^{q-1} u$ is an absorption term. Applying the Fatou lemma, we find, as in (3.12),

$$\begin{aligned} & \frac{|\alpha|}{2} \int_{\Omega} (|u| + 1)^{\alpha-1} |\nabla u|^p \varphi dx + \int_{\Omega} |u|^q (1 - (|u| + 1)^\alpha) \varphi dx \\ &\leq \int_{\Omega} \varphi d|\mu| + C \left(\int_{\Omega} (|u| + 1)^q \varphi dx \right)^{1/\tau} \times \left(\int_{\Omega} \varphi^{1-\tau'p} |\nabla \varphi|^{\tau'p} dx \right)^{1/\tau'}, \end{aligned}$$

where τ is defined in (3.8). And $1 - (|u| + 1)^\alpha \geq 1 - 2^\alpha > 0$ on the set $\{|u| \geq 1\}$. We can choose $\alpha < 0$ small enough such that

$$\tau'p = R,$$

and take $\varphi = \zeta^R$, where $\zeta \in \mathcal{D}^+(\Omega)$. Defining

$$X = \int_{\Omega} (|u| + 1)^q \zeta^R dx, \quad Y = \int_{\Omega} (|u| + 1)^{\alpha-1} |\nabla u|^p \zeta^R dx,$$

we find

$$X + Y \leq C \left(\int_{\Omega} \zeta^R dx + \int_{\Omega} \zeta^R d|\mu| + C(\zeta) X^{1/\tau} \right) \quad (4.3)$$

with $C(\varphi) = \left(\int_{\Omega} |\nabla \zeta|^R dx \right)^{p/R}$ and $C = C(\alpha, q)$. Then from Young inequality, with a new $C = C(\alpha, p, q)$,

$$X + Y \leq C \left(\int_{\Omega} \zeta^R dx + \int_{\Omega} \zeta^R d|\mu| + \int_{\Omega} |\nabla \zeta|^R dx \right). \quad (4.4)$$

Hence (4.1) follows for such α , from (4.3), and then (4.2) follows for any $\alpha < 0$. \blacksquare

Now we consider the problem with a source term. In case of nonnegative μ and u , we have more precise results.

Theorem 4.2 *Let $\mu \in \mathcal{M}^+(\Omega)$ and u be any nonnegative local entropy solution of problem (1.6). Then for any $R > pq^*$, there exists $C = C(N, p, q, R, \Omega) > 0$ such that for any $\zeta \in \mathcal{D}^+(\Omega)$, with $0 \leq \zeta \leq 1$ in Ω ,*

$$\int_{\Omega} \zeta^R d\mu + \int_{\Omega} u^q \zeta^R dx \leq C \left(\int_{\Omega} |\nabla \zeta|^R dx \right)^{pq^*/R}. \quad (4.5)$$

And for any $\alpha < 0$, there exists $C = C(\alpha, N, p, q, R, \Omega) > 0$ such that

$$\int_{\Omega} (u+1)^{\alpha-1} |\nabla u|^p \zeta^R dx \leq C \left(1 + \int_{\Omega} u^q \zeta^R dx \right) \left(\int_{\Omega} |\nabla \zeta|^R dx \right)^{p/R}. \quad (4.6)$$

Proof. It has been given in [3] in case of global entropy solutions of the problem with Dirichlet data, but it is still valid for local solutions. It is based on the use of test functions

$$\tilde{h}_k(r) = (T_k(r^+) + \varepsilon)^\alpha, \quad k > 0, \varepsilon > 0, \quad (4.7)$$

which are nondecreasing, contrarily to the ones defined in (3.5), and then of the test function $h(r) = 1$. ■

Remark 4.1 These ideas were already used in [10] to obtain upper estimates for more general problems with possible singularities, and also in [9] to study the initial trace problem for a parabolic equation with absorption.

5 Necessary conditions of existence

Here we prove our general necessary conditions of existence:

Proof of Theorem 1.1. It is enough to consider a solution a local entropy solution u of (1.3), such that $|u|^q \in L^1_{loc}(\Omega)$. Hence Theorem 3.1 applies. Let $R > pq^*$, then we still can choose $\alpha < 0$ such that τ given by (3.8) satisfies

$$p\tau' = R.$$

Let E be a Borel set such that $\text{cap}_{1,R}(E, \mathbb{R}^N) = 0$. There exist two measurable disjoint sets A, B such that $\Omega = A \cup B$ and $\mu^+(B) = \mu^-(A) = 0$. Let us show that

$$\mu^+(A \cap E) = \mu^-(B \cap E) = 0.$$

Let K be any fixed compact set in $A \cap E$. Since $\mu^-(K) = 0$, there exists an open set $\omega \subset\subset \Omega$ containing K , such that $\mu^-(\omega) < \varepsilon$. From [3, Lemma 2.1], there exists $\zeta_n \in \mathcal{D}(\omega)$ such that

$0 \leq \zeta_n \leq 1$, and $\zeta_n = 1$ on a neighborhood of K contained in ω , and $\zeta_n \rightarrow 0$ in $W^{1,R}(\mathbb{R}^N)$. We choose $\varphi = \varphi_n = \zeta_n^R$ in (2.21). From (3.12) we have

$$\frac{|\alpha|}{2} \int_{\Omega} (|u| + 1)^{\alpha-1} |\nabla u|^p \varphi_n dx \leq \int_{\Omega} \varphi_n d|\mu| + C \left(\int_{\Omega} (|u| + 1)^q \varphi_n dx \right)^{1/\tau} \left(\int_{\Omega} |\nabla \zeta_n|^R dx \right)^{p/R}.$$

where $C = C(\alpha, R)$. But

$$\begin{aligned} \int_{\Omega} \varphi_n d|\mu| &\leq \int_{\text{supp} \Psi_K} d|\mu|, \\ \lim_{n \rightarrow \infty} \int_{\Omega} (|u| + 1)^q \varphi_n dx &= 0, \end{aligned} \tag{5.1}$$

since $u \in L_{loc}^q(\Omega)$. Then

$$\int_{\Omega} (|u| + 1)^{\alpha-1} |\nabla u|^p \varphi_n dx \text{ is bounded.} \tag{5.2}$$

And from (3.13),

$$\begin{aligned} \int_{\Omega} |\nabla u|^{p-1} |\nabla \varphi_n| dx &\leq C \left(\int_{\Omega} (|u| + 1)^{\alpha-1} |\nabla u|^p \varphi_n dx \right)^{1/p'} \left(\int_{\Omega} (|u| + 1)^q \varphi_n dx \right)^{1/\tau p} \\ &\times \left(\int_{\Omega} |\nabla \zeta_n|^R dx \right)^{1/R}, \end{aligned}$$

hence

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla u|^{p-1} |\nabla \varphi_n| dx = 0.$$

Let us apply (2.21) with now $h(u) = 1$ and the same φ_n . We find

$$\int_{\Omega} \varphi_n d\mu = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi_n dx$$

hence $\lim_{n \rightarrow \infty} \int_{\Omega} \varphi_n d\mu = 0$. And

$$\mu^+(K) \leq \int_{\omega} \varphi_n d\mu^+ = \int_{\omega} \varphi_n d\mu + \int_{\omega} \varphi_n d\mu^-,$$

hence $\mu^+(K) \leq \varepsilon$, for any $\varepsilon > 0$, hence $\mu^+(A \cap E) = 0$, and similarly $\mu^-(B \cap E) = 0$. \blacksquare

Proof of Theorem 1.2. Here we apply the estimates of Theorem 4.2. Let $R > pq^*$ and K be a compact set contained in Ω . Let $\psi_n \in \mathcal{D}(\Omega)$ such that

$$0 \leq \psi_n \leq 1, \psi_n \geq \chi_K \quad \text{and} \quad \|\psi_n\|_{W^{1,R}(\Omega)}^R \rightarrow \text{cap}_{1,R}(K, \Omega).$$

Choosing $\varphi = \psi_n^R$ in (4.5), we deduce that, with new constants $C = C(N, p, q, R, \Omega)$,

$$\int_K d\mu \leq C \left(\int_{\Omega} |\nabla \psi_n|^R dx \right)^{pq^*/R} \leq C \|\psi_n\|_{W^{1,R}(\Omega)}^{pq^*},$$

and (1.7) follows. ■

6 Removable singularities

Now we consider the question of removable sets. Here the results are based on the estimates of Section 4.

Proof of Theorem 1.3. By hypothesis, u is measurable and finite *a.e.* in $\Omega \setminus F$, hence in Ω , since F has a Lebesgue measure zero, and $|u|^q \in L_{loc}^1(\Omega \setminus F)$, $T_k(u) \in W_{loc}^{1,p}(\Omega \setminus F)$ for every $k > 0$, and $|\nabla u|^{p-1} \in L_{loc}^1(\Omega \setminus F)$. And u satisfies

$$\begin{aligned} \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (h(u)\varphi) dx + \int_{\Omega} |u|^{q-1} u h(u) \varphi dx &= \int_{\Omega} h(u) \varphi d\mu_0 \\ &+ h(+\infty) \int_{\Omega} \varphi d\mu_s^+ - h(-\infty) \int_{\Omega} \varphi d\mu_s^- \end{aligned} \quad (6.1)$$

for any $h \in W^{1,\infty}(\mathbb{R})$ and h' has a compact support, and $\varphi \in \mathcal{D}^+(\Omega \setminus F)$.

Step 1: $|u|^q \in L_{loc}^1(\Omega)$. From Theorem 4.1 in $\Omega \setminus F$, for any $\phi \in \mathcal{D}^+(\Omega \setminus F)$,

$$\int_{\Omega} (|u| + 1)^q \phi^R dx \leq C \left(\int_{\Omega} \phi^R dx + \int_{\Omega} \phi^R d|\mu| + \int_{\Omega} |\nabla \phi|^R dx \right) \quad (6.2)$$

where $C = C(N, p, q, R, \Omega, F)$, and for any $\alpha < 0$,

$$\int_{\Omega} (|u| + 1)^{\alpha-1} |\nabla u|^p \phi^R dx \leq C \left(\int_{\Omega} \phi^R dx + \int_{\Omega} \phi^R d|\mu| + \int_{\Omega} |\nabla \phi|^R dx \right), \quad (6.3)$$

with $C = C(\alpha, N, p, q, R, \Omega, F)$. Let $\zeta \in \mathcal{D}^+(\Omega)$. Let $K = F \cap \text{supp} \zeta$, hence K is compact with $\text{cap}_{1,R}(K, \mathbb{R}^N) = 0$. Let $\xi_n \in \mathcal{D}(\mathbb{R}^N)$ such that

$$0 \leq \xi_n \leq 1 \quad \text{in } \mathbb{R}^N, \quad \xi_n = 1 \text{ in a neighborhood of } K,$$

and $\xi_n \rightarrow 0$ in $W^{1,R}(\mathbb{R}^N)$ and everywhere on $\mathbb{R}^N \setminus N$, where $\text{cap}_{1,R}(N, \mathbb{R}^N) = 0$, see [3]. We take

$$\phi = \zeta_n = \zeta(1 - \xi_n), \quad (6.4)$$

and get

$$\begin{aligned} \int_{\Omega} (|u| + 1)^q \zeta_n^R dx &\leq C \left(\int_{\Omega} \zeta_n^R dx + \int_{\Omega} \zeta_n^R d|\mu| + \int_{\Omega} |\nabla \zeta_n|^R dx \right) \\ &\leq C \left(\int_{\Omega} \zeta^R dx + \int_{\Omega} \zeta^R d|\mu| + \int_{\Omega} |\nabla \zeta|^R dx \right). \end{aligned}$$

But $|\nabla \zeta_n|$ is bounded in $L^R(\mathbb{R}^N)$, and $\varphi_n \rightarrow \zeta^R$ a.e. in Ω . Hence from the Fatou lemma,

$$\int_{\Omega} (|u| + 1)^q \zeta^R dx < \infty,$$

so that $|u|^q \in L^1_{loc}(\Omega)$.

Step 2: $|\nabla u|^{p-1} \in L^1_{loc}(\Omega)$. As above, from (6.3),

$$\int_{\Omega} (|u| + 1)^{\alpha-1} |\nabla u|^p \zeta_n^R dx \quad \text{is bounded.} \quad (6.5)$$

Now

$$\int_{\Omega} |\nabla u|^{p-1} \zeta_n^R dx \leq \left(\int_{\Omega} (|u| + 1)^{\alpha-1} |\nabla u|^p \zeta_n^R dx \right)^{1/p'} \int_{\Omega} (|u| + 1)^{(1-\alpha)(p-1)} \zeta_n^R dx,$$

from Hölder inequality. Since $(1 - \alpha)(p - 1) < q$, we deduce that

$$\int_{\Omega} |\nabla u|^{p-1} \zeta_n^R dx \quad \text{is bounded,}$$

hence the conclusion.

Step 3: $u_k = T_k(u) \in W^{1,p}_{loc}(\Omega)$ and $|\nabla u|^{p-1} \in L^1_{loc}(\Omega)$. Let $k > 0$ be fixed. From (6.5),

$$\int_{\Omega} |\nabla u_k|^p \zeta_n^R dx \quad \text{is bounded.} \quad (6.6)$$

Let us set for fixed $k > 0$,

$$w_n = u_k \zeta_n^{R/p} \in W^{1,p}(\Omega).$$

We have $|w_n| \leq k\zeta^{R/p}$, hence w_n is bounded in $L^\infty(\Omega)$. And $w_n \rightarrow u_k\zeta^{R/p}$ *a.e.* in Ω . Now

$$\nabla w_n = \zeta_n^{R/p} \nabla u_k + \frac{R}{p} u_k \zeta_n^{R/p-1} \nabla \zeta_n;$$

and in particular $\nabla \zeta_n$ is bounded in $L^p_{loc}(\mathbb{R}^N)$, hence ∇w_n is bounded in $L^p(\Omega)$, from (6.6). It follows that $u_k \zeta^{R/p} \in W^{1,p}(\Omega)$, hence $u_k \in W^{1,p}_{loc}(\Omega)$. Now the gradient of u has a sense in Ω ; it is defined by (2.3), hence it coincides with the gradient still defined *a.e.* in $\Omega \setminus F$.

Step 4: $|\nabla u|^{p-1} \in L^{R'}_{loc}(\Omega)$. For any $1 \leq \sigma < p'$,

$$\int_U |\nabla u|^{(p-1)\sigma} \zeta_n^R dx \leq \left(\int_U (|u| + 1)^{\alpha-1} |\nabla u|^p \zeta_n^R dx \right)^{\sigma/p'} \left(\int_U (|u| + 1)^{(1-\alpha)/(p'/\sigma-1)} \zeta_n^R dx \right)^{1-\sigma/p'},$$

hence as in (3.20),

$$\int_\Omega |\nabla u|^{(p-1)\sigma} \zeta_n^R dx \quad \text{is bounded,}$$

for any $\sigma < p'q/(q+1)$, hence $|\nabla u|^{p-1} \in L^\sigma_{loc}(\Omega)$, in particular for $\sigma = R'$.

Step 5: u is a local entropy solution in Ω . Let $h \in W^{1,\infty}(\mathbb{R})$ such that h' has a compact support. Let $\psi \in \mathcal{D}^+(\Omega)$ be fixed. Taking now the test function

$$\psi_n = \psi(1 - \xi_n)^R,$$

we have

$$\begin{aligned} & \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla (h(u) \psi_n) dx + \int_\Omega |u|^{q-1} u h(u) \psi_n dx \\ &= \int_\Omega h(u) \psi_n d\mu_0 + h(+\infty) \int_\Omega \psi_n d\mu_s^+ - h(-\infty) \int_\Omega \psi_n d\mu_s^-. \end{aligned} \quad (6.7)$$

And $\xi_n \rightarrow 0$ everywhere on $\mathbb{R}^N \setminus N$, but μ does not charge N , hence

$$\int_\Omega \psi_n d|\mu_0| \rightarrow \int_\Omega \psi d|\mu_0|, \quad \int_\Omega \psi_n d\mu_s^\pm \rightarrow \int_\Omega \psi d\mu_s^\pm.$$

And

$$\left| \int_\Omega h(u) (\psi - \psi_n) d\mu_0 \right| \leq \|h\|_{L^\infty(\mathbb{R})} \int_\Omega (\psi - \psi_n) d|\mu_0|,$$

hence

$$\int_\Omega h(u) \psi_n d\mu_0 \rightarrow \int_\Omega h(u) \psi d\mu_0.$$

Also $|u|^q \in L^1_{loc}(\Omega)$, hence from the Lebesgue theorem,

$$\int_{\Omega} |u|^{q-1} u h(u) \psi_n dx \rightarrow \int_{\Omega} |u|^{q-1} u h(u) \psi dx.$$

And $|\nabla u|^{p-1} \in L^{R'}_{loc}(\Omega)$, thus

$$\left| \int_{\Omega} h(u) |\nabla u|^{p-2} \nabla u \cdot \nabla (\psi - \psi_n) dx \right| \leq C \|\nabla u\|^{p-1}_{L^{R'}(\Omega')} \|\nabla (\psi - \psi_n)\|_{L^R(\Omega')},$$

where Ω' contains $\text{supp } \psi$. But

$$\nabla(\psi - \psi_n) = (1 - (1 - \xi_n)^R) \nabla \psi - R \psi (1 - \xi_n)^{R-1} \nabla \xi_n,$$

hence $\nabla(\psi - \psi_n) \rightarrow 0$ a.e. in Ω , and

$$|\nabla(\psi - \psi_n)|^R \leq C(|\nabla \psi|^R + |\nabla \xi_n|^R)$$

where $C > 0$ does not depend on n . Then from the Lebesgue theorem, $\nabla(\psi - \psi_n) \rightarrow 0$ strongly in $L^R(\Omega)$, and

$$\int_{\Omega} h(u) |\nabla u|^{p-2} \nabla u \cdot \nabla \psi_n dx \rightarrow \int_{\Omega} h(u) |\nabla u|^{p-2} \nabla u \cdot \nabla \psi dx.$$

At last

$$\left| \int_{\Omega} h'(u) |\nabla u|^p (\psi - \psi_n) dx \right| \rightarrow 0$$

from the Lebesgue theorem, since $|\nabla u_k| \in L^p_{loc}(\Omega)$ for any k , and h' has a compact support. Then u satisfies the equation

$$\begin{aligned} & \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (h(u) \psi) dx + \int_{\Omega} |u|^{q-1} u h(u) \psi dx \\ &= \int_{\Omega} h(u) \psi d\mu_0 + h(+\infty) \int_{\Omega} \psi d\mu_s^+ - h(-\infty) \int_{\Omega} \psi d\mu_s^-, \end{aligned}$$

for any $\psi \in \mathcal{D}^+(\Omega)$, hence u is a local entropy solution of the problem in Ω . ■

Proof of Theorem 1.4. From (4.5), for any $\phi \in \mathcal{D}^+(\Omega \setminus F)$, with $0 < \phi \leq 1$ in Ω ,

$$\int_{\Omega} \phi^R d\mu + \int_{\Omega} u^q \phi^R dx \leq C \left(\int_{\Omega} |\nabla \phi|^R dx \right)^{pq^*/R},$$

with $C = C(N, p, q, R, \Omega, F)$, and for any $\alpha < 0$, there exists > 0 such that

$$\int_{\Omega} (u+1)^{\alpha-1} |\nabla u|^p \phi^R dx \leq C \left(1 + \int_{\Omega} u^q \phi^R dx \right) \left(\int_{\Omega} |\nabla \phi|^R dx \right)^{p/R},$$

with $C = C(\alpha, N, p, q, R, \Omega, F)$. Taking $\phi^R = \zeta_n$ defined in (6.4), we deduce that

$$\int_{\Omega} \zeta_n^R d\mu + \int_{\Omega} u^q \zeta_n^R dx + \int_{\Omega} (u+1)^{\alpha-1} |\nabla u|^p \zeta_n^R dx \quad \text{is bounded,}$$

and the proof follows as above, after minor change due to the signs. ■

7 Stability properties

Let us recall a well-known stability property for global solutions:

Theorem 7.1 ([5]) *Let $f_{\nu}, f \in L^1(\Omega)$, with Ω bounded, such that f_{ν} converges strongly to f in $L^1(\Omega)$. Let u_{ν} be the unique entropy solution of problem*

$$\begin{cases} -\Delta_p u_{\nu} + |u_{\nu}|^{q-1} u_{\nu} = f_{\nu}, & \text{in } \Omega, \\ u_{\nu} = 0, & \text{on } \partial\Omega. \end{cases}$$

Then u_{ν} converges a.e. in Ω to the unique entropy solution u of

$$\begin{cases} -\Delta_p u + |u|^{q-1} u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Now we prove analogous stability properties for the local entropy solutions, which we need for proving Theorem 1.5. Here Ω may be unbounded.

Theorem 7.2 *Let $f_{\nu}, f \in L^1_{loc}(\Omega)$, such that f_{ν} converges strongly to f in $L^1_{loc}(\Omega)$. Let u_{ν} be local entropy solution of problem*

$$-\Delta_p u_{\nu} + |u_{\nu}|^{q-1} u_{\nu} = f_{\nu}, \quad \text{in } \Omega. \tag{7.1}$$

Then up to a subsequence, u_{ν} converges a.e. in Ω to a local entropy solution u of

$$-\Delta_p u + |u|^{q-1} u = f, \quad \text{in } \Omega. \tag{7.2}$$

Proof. Step 1: A priori estimates. Let u_ν be a local entropy solution of (7.1). From Theorem 4.1, for any $R > pq^*$, there exists $C = C(N, p, q, R, \Omega)$ such that, for any $\zeta \in \mathcal{D}^+(\Omega)$,

$$\int_{\Omega} (|u_\nu| + 1)^q \zeta^R dx \leq C \left(\int_{\Omega} \zeta^R dx + \int_{\Omega} \zeta^R |f_\nu| dx + \int_{\Omega} |\nabla \zeta|^R dx \right) \quad (7.3)$$

Hence $(|u_\nu|^q)$ is bounded in $L^1_{loc}(\Omega)$. Writing the equation under the form

$$-\Delta_p u_\nu = \mu_\nu, \quad \text{where } \mu_\nu = f_\nu - |u_\nu|^{q-1} u_\nu, \quad (7.4)$$

then after an extraction (u_ν) converges to a function u satisfying conclusions of Theorem 3.2.

Step 2: Convergence of the nonlinear term. Following [5] and [14] we prove the local equiintegrability of $(|u_\nu|^q)$: for any domain $U \subset\subset \Omega$ and any $\varepsilon > 0$, any subset $A \subset U$ such that $\text{meas} A \leq \varepsilon(k+1)^{-q}$, we have

$$\int_A |u_\nu|^q dx \leq \varepsilon + \int_{A \cap \{|u_\nu| \geq k+1\}} |u_\nu|^q dx,$$

for any $k \geq 0$. Now from **D1loc**,

$$\int_{\Omega} |\nabla u_\nu|^{p-2} \nabla u_\nu \cdot \nabla (h(u_\nu) \varphi) dx + \int_{\Omega} |u_\nu|^{q-1} u_\nu \varphi dx = \int_{\Omega} h(u_\nu) f_\nu \varphi dx,$$

for any $h \in W^{1,\infty}(\mathbb{R})$ such that h' has a compact support, and any $\varphi \in \mathcal{D}(\Omega)$, such that $h(u_\nu) \varphi \in W^{1,p}(\Omega)$. Taking $\varphi \in \mathcal{D}^+(\Omega)$ such that $\varphi = 1$ on U and $h = S_{k,k}$ defined in (2.7), we get

$$\begin{aligned} \int_{U \cap \{k \leq |u_\nu| \leq k+1\}} |\nabla u_\nu|^p \varphi dx + \int_{U \cap \{|u_\nu| \geq k+1\}} |u_\nu|^q \varphi dx &\leq \int_{\{|u_\nu| \geq k\}} |f_\nu| \varphi dx \\ &+ \int_{\{|u_\nu| \geq k\}} |\nabla u_\nu|^{p-1} |\nabla \varphi| dx. \end{aligned}$$

Since f_ν and $|\nabla u_\nu|^{p-1}$ converge strongly in $L^1_{loc}(\Omega)$, there exists $g \in L^1(U)$ such that

$$\int_{U \cap \{|u_\nu| \geq k+1\}} |u_\nu|^q dx \leq \int_{U \cap \{|u_\nu| \geq k\}} g dx.$$

Now $(|u_\nu|^q)$ is bounded in $L^1_{loc}(\Omega)$, hence $\text{meas} \{|u_\nu| \geq k\} \leq Ck^{-q}$, where $C > 0$ does not depend on ν . Hence

$$\int_{A \cap \{|u_\nu| \geq k+1\}} |u_\nu|^q dx \leq \varepsilon,$$

for k large enough. Then $|u_\nu|^{q-1}u_\nu$ converges strongly in $L^1_{loc}(\Omega)$ to $|u|^{q-1}u$.

Step 3. Conclusion. Let us apply Theorem 3.3 with f_ν replaced by μ_ν , and $\mu = 0$. Since μ_ν converges (strongly) in $L^1_{loc}(\Omega)$ to $f - |u|^{q-1}u$, we deduce that u is a local entropy solution of equation

$$-\Delta_p u = f - |u|^{q-1}u, \quad \text{in } \Omega,$$

that means a solution of (7.2). ■

Theorem 1.5 follows as a direct consequence of Theorem 7.2:

Proof of Theorem 1.5. From Theorem 7.2, up to a subsequence, u_ν converges to a local entropy solution u of

$$-\Delta_p u + |u|^{q-1}u = f, \quad \text{in } \Omega \setminus F.$$

From Theorem 1.3, it is a solution in Ω , since f does not charge F , since $\text{meas} F = 0$. ■

This implies also a global result:

Theorem 7.3 *Assume that Ω is bounded. Let K be a compact set in Ω , with $\text{cap}_{1,R}(K, \mathbb{R}^N) = 0$ for some $R > pq^*$. Let $f_\nu, f \in L^1(\Omega)$ such that f_ν converges strongly to f in $L^1_{loc}(\overline{\Omega} \setminus K)$. Let u_ν be the entropy solution of problem*

$$\begin{cases} -\Delta_p u_\nu + |u_\nu|^{q-1}u_\nu = f_\nu, & \text{in } \Omega, \\ u_\nu = 0, & \text{on } \partial\Omega. \end{cases} \quad (7.5)$$

Then u_ν converges to the entropy solution u of

$$\begin{cases} -\Delta_p u + |u|^{q-1}u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (7.6)$$

Proof. From Theorem 1.5, up to a subsequence, (u_ν) converges *a.e.* in Ω to a local entropy solution u of the problem. Let U, U', U'' be any regular bounded open sets such that $K \subset U \subset\subset U' \subset\subset U'' \subset\subset \Omega$. By hypothesis $u_{\nu,k} = T_k(u_\nu) \in W^{1,p}_0(\Omega)$ for any $k > 0$, and from **D1**,

$$\int_{\Omega} |\nabla u_\nu|^{p-2} \nabla u_\nu \cdot \nabla (h(u_\nu)\varphi) dx + \int_{\Omega} |u_\nu|^{q-1} u_\nu h(u_\nu) \varphi dx = \int_{\Omega} h(u_\nu) f_\nu \varphi dx, \quad (7.7)$$

for any $h \in W^{1,\infty}(\mathbb{R})$ and h' has a compact support, and $\varphi \in W^{1,m}(\Omega)$ for some $m > N$, such that $h(u)\varphi \in W^{1,p}_0(\Omega)$. We can take $h = T_k$, and $\varphi \in C^1(\overline{\Omega})$ with values in $[0, 1]$, such that $\varphi = 1$ on $\Omega \setminus U'$, with support in $\Omega \setminus U$. Denoting $\Omega' = \Omega \setminus \overline{U'}$, we obtain

$$\int_{\Omega} |\nabla u_{\nu,k}|^p \varphi dx + \int_{\Omega} |u_\nu|^{q-1} u_\nu u_{\nu,k} \varphi dx = \int_{\Omega} u_{\nu,k} |\nabla u_\nu|^{p-2} \nabla u_\nu \cdot \nabla \varphi dx + \int_{\Omega} u_{\nu,k} f_\nu \varphi dx,$$

hence

$$\frac{1}{k} \int_{\Omega'} |\nabla u_{\nu,k}|^p dx \leq \int_{\Omega \setminus U} \varphi |f_\nu| dx + \int_{U' \setminus U} |\nabla u_\nu|^{p-1} |\nabla \varphi| dx \leq C \quad (7.8)$$

where $C > 0$ does not depend on ν and k . Indeed $\overline{U'} \setminus U$ is compact, contained in $\Omega \setminus K$, and $|\nabla u_\nu|^{p-1}$ is bounded in $L^1_{loc}(\Omega \setminus K)$ from Theorem 3.2 applied in $\Omega \setminus K$; and f_ν converges to f in $L^1(\Omega \setminus U)$. Then for fixed $k > 0$, $(u_{\nu,k})$ is bounded in $W^{1,p}_0(\Omega')$. But for any $k > 0$, $u_{\nu,k}$ converges *a.e.* to u_k , hence $u_k \in W^{1,p}_0(\Omega') \cap W^{1,p}_{loc}(\Omega)$, that means $u_k \in W^{1,p}_0(\Omega)$, and u satisfies (2.9). Also going to the limit in (7.8), we have

$$\int_{\Omega'} |\nabla u_k|^p dx \leq Ck.$$

But u is a local entropy solution in Ω , hence $(|u| + 1)^{\alpha-1} |\nabla u|^p \in L^1_{loc}(\Omega)$ for any $\alpha < 0$ from Theorem 3.1. As a consequence,

$$\int_{U''} |\nabla u_k|^p dx \leq C(1+k)^{1+|\alpha|},$$

with another $C > 0$ depending on U'' , α , but not on k . Hence

$$\int_{\Omega} |\nabla u_k|^p dx \leq C(1+k)^{1+|\alpha|}.$$

This estimate implies (2.10) and (2.17), see for example [6, Lemma 6.2]. And u is a local solution of the problem, hence from Remark 2.1, it is the unique entropy solution of (7.6), and the whole sequence converges to u . \blacksquare

Remark 7.1 Let us recall the result of [22]: if Ω is bounded and K_1, K_2 are two disjoint compact sets with $cap_{1,R}(K_i, \mathbb{R}^N) = 0$ for some $R > pq^*$, and if $f \in L^1(\Omega)$ and $f_{1,\nu}, f_{2,\nu} \in L^\infty(\Omega)$ are nonnegative and $f_{1,\nu}$ converge strongly to f^+ in $L^1_{loc}(\overline{\Omega} \setminus K_1)$, and $f_{2,\nu}$ converge strongly to f^- in $L^1_{loc}(\overline{\Omega} \setminus K_1)$, the entropy solution u_ν of problem

$$\begin{cases} -\Delta_p u_\nu + |u_\nu|^{q-1} u_\nu = f_{1,\nu} - f_{2,\nu}, & \text{in } \Omega, \\ u_\nu = 0, & \text{on } \partial\Omega, \end{cases}$$

converges to a solution, in the sense of distributions, of problem

$$\begin{cases} -\Delta_p u + |u|^{q-1} u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Corollary 7.3 improves this result since it does not require sign conditions on the approximating sequence, and proves that the limit u is an entropy solution.

As a consequence we deduce the following, which extends to the case $p \neq 2$ the result of Brézis [16], but for the critical case $q = \overline{P}$:

Corollary 7.4 *Assume that Ω is bounded, and $q > \overline{P}$. Let $a \in \Omega$, and $f_\nu, f \in L^1(\Omega)$ such that f_ν converges strongly to f in $L^1_{loc}(\overline{\Omega} \setminus \{a\})$. Let u_ν be the entropy solution of problem (7.5). Then u_ν converges to the entropy solution of problem (7.6).*

Remark 7.2 Concerning the problem with source term (1.6), there is no global (or local) stability result, even if $p = 2$. Indeed when $q \geq N/(N - 2)$, there exists a nonnegative function $f \in L^1(\Omega)$, with compact support, such that problem

$$\begin{cases} -\Delta u = u^q + \lambda f, & \text{in } \Omega, \\ u_\nu = 0, & \text{on } \partial\Omega, \end{cases}$$

has no solution, for any $\lambda > 0$, from [4, Corollary 3.3]. And there exists $f_\nu, f \in L^1(\Omega)$, such that f_ν converges strongly to f in $L^1(\Omega)$, such that for $\lambda > 0$ small enough, problem

$$\begin{cases} -\Delta u_\nu = u_\nu^q + \lambda f_\nu, & \text{in } \Omega, \\ u_\nu = 0, & \text{on } \partial\Omega, \end{cases} \quad (7.9)$$

has a solution for any $\nu \in \mathbb{N}$. Indeed if (ρ_ν) is a sequence of mollifiers, we can take $f_\nu = \rho_\nu * f$. From [4, Corollary 3.2], there exists $\lambda_\nu > 0$ such that problem (7.9) admits a solution for $\lambda = \lambda_\nu$. Then from [11, Remark 3.4] it has a solution for any $\lambda > 0$ such that $\lambda \|f_\nu\|_{L^1(\Omega)} \leq C = C(N, q, \Omega)$ independent on ν , hence for $\lambda < C / \|f\|_{L^1(\Omega)}$.

8 Existence results

In the case of absorption the a priori estimates allow to give existence results in \mathbb{R}^N for data in $L^1_{loc}(\mathbb{R}^N)$, without growth conditions. This was noted first in [14], where the case $p > P_1$ was solved:

Theorem 8.1 ([14]) *Assume $p > P_1$ and $q > p - 1$. Then for every $f \in L^1_{loc}(\mathbb{R}^N)$, there exists a function $u \in W^{1,r}_{loc}(\mathbb{R}^N)$ for any $r \in [1, \overline{P})$ solution of*

$$-\Delta_p u + |u|^{q-1} u = f, \quad \text{in } \mathcal{D}'(\mathbb{R}^N),$$

and $u \geq 0$ if $f \geq 0$.

They gave also an existence result when $p \leq P_1$ but $q > 1/(p - 1)$, so that the gradient is well defined in $L^1_{loc}(\Omega)$ from (3.4). With our stability result, we now prove Theorem 1.6, which does not assume the condition on p or q , and shows the existence of a local entropy solution.

Proof of Theorem 1.6. Let $B_r = \{|x| < r\}$ for any $r > 0$. For any $\nu \in \mathbb{N}$, there exists a unique global entropy solution of problem

$$\begin{cases} -\Delta_p u_\nu + |u_\nu|^{q-1} u_\nu = f, & \text{in } B_\nu, \\ u_\nu = 0, & \text{on } \partial B_\nu, \end{cases}$$

since $f \in L^1(B_r)$, from [5].

Let $r > 0$ be fixed. Then for any $\nu > 2r$, u_ν is a local entropy solution in B_{2r} . Hence from Theorem 4.1, as in (7.3), $(|u_\nu|^q)_{\nu > 2r}$ is bounded in $L^1(B_r)$. Then from Theorem 3.2, $(u_\nu)_{\nu > 2r}$ satisfies (3.21), (3.22), and (3.23) in B_r and one can extract a subsequence converging locally in measure and *a.e.* in B_r .

Therefore, we can extract a diagonal subsequence (u_ρ) , such that u_ρ converges locally in measure and *a.e.* in \mathbb{R}^N to a function u . And u satisfies (2.8), (2.18), (2.19) and (2.20), (3.4) in \mathbb{R}^N , and ∇u_ρ converges locally in measure to ∇u . But u_ρ is a local entropy solution in B_r for $\rho > 2r$, hence from Theorem 7.2 applied with $f_\nu = f$, there exists a subsequence, depending on r , converging *a.e.* to a local entropy solution of the equation in B_r . Necessarily it coincides with u , and the whole sequence (u_ρ) converges to u . Then u is a local entropy solution in any ball B_r , hence it is a local entropy solution of problem (1.8) in \mathbb{R}^N . In particular it satisfies the equation

$$-\Delta_p u + |u|^{q-1} u = f, \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

If $f \geq 0$, then $u_n \geq 0$ from [5, Theorem 7.1], hence $u \geq 0$ *a.e.* in \mathbb{R}^N . ■

Now we assume that q is subcritical. We first give a global existence result for measures when Ω is bounded, by using the global stability properties shown in [18, Theorem 3.4]. Recall that the first results were given in [12] when $P > p_1$. We get the following:

Theorem 8.2 *Assume that Ω is bounded, and $p - 1 < q < \overline{P}$. Then for any $\mu \in \mathcal{M}_b(\Omega)$, there exists an entropy solution of problem*

$$\begin{cases} -\Delta_p u + |u|^{q-1} u = \mu, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (8.1)$$

Proof. From [13], μ can be decomposed as

$$\mu = f - \operatorname{div} g + \mu_s^+ - \mu_s^-,$$

with $f \in L^1(\Omega)$, $g \in (L^{p'}(\Omega))^N$, and $\mu_s^+, \mu_s^- \in \mathcal{M}_b^+(\Omega)$ are singular. And there exists an approximation of μ by a sequence (μ_ν) such that

$$\mu_\nu = f_\nu - \operatorname{div} g + \lambda_\nu - \eta_\nu, \quad f_\nu, \lambda_\nu, \eta_\nu \in L^{p'}(\Omega),$$

hence $\mu_\nu \in W^{-1,p'}(\Omega) \cap \mathcal{M}_b(\Omega) \cap \mathcal{M}_0(\Omega)$, and f_ν converges weakly in $L^1(\Omega)$ to f , and λ_ν (resp. η_ν) converges to μ_s^+ (resp. μ_s^-) in the narrow topology. Then there exists a (unique) weak solution u_ν of problem

$$\begin{cases} -\Delta_p u_\nu + |u_\nu|^{q-1} u_\nu = \mu_\nu, & \text{in } \Omega, \\ u_\nu = 0, & \text{on } \partial\Omega, \end{cases}$$

such that $u_\nu \in W_0^{1,p}(\Omega) \cap L^{q+1}(\Omega)$. It is also an entropy solution, since it satisfies **D2**. From Theorem 4.1, $(|u_\nu|^q)$ is bounded in $L_{loc}^1(\Omega)$. From Theorem 3.2, up to a subsequence, (u_ν) converges locally in measure in Ω and *a.e.* in Ω to some function u , and $(|u_\nu|^{p-1})$ is bounded in $L_{loc}^s(\Omega)$, for $1 < s < N/(N-p)$. In fact the estimate is global. Indeed, by hypothesis,

$$\int_{\Omega} |\nabla u_\nu|^{p-2} \nabla u_\nu \cdot \nabla (h(u_\nu) \varphi) dx + \int_{\Omega} |u_\nu|^{q-1} u_\nu h(u_\nu) \varphi dx = \int_{\Omega} h(u_\nu) \varphi d\mu_\nu \quad (8.2)$$

for any $h \in W^{1,\infty}(\mathbb{R})$ and h' has a compact support, and $\varphi \in W^{1,m}(\Omega)$ for some $m > N$, such that $h(u) \varphi \in W_0^{1,p}(\Omega)$. Taking $h = T_k$ for any $k > 0$, and $\varphi = 1$, one gets

$$\frac{1}{k} \int_{\{|u_\nu| < k\}} |\nabla u_\nu|^p dx \leq \int_{\Omega} u_{\nu,k} d\mu_\nu \leq \int_{\Omega} d|\mu_\nu| \leq C,$$

where $C > 0$ does not depend on ν and k . Then $(|u_\nu|^{p-1})$ is bounded in $L^s(\Omega)$, for $1 < s < N/(N-p)$, from [5, Lemma 4.1]. Therefore $|u_\nu|^{q-1} u_\nu$ converges strongly in $L^1(\Omega)$ to $|u|^{q-1} u$, since $q < \bar{P}$. From [18, Theorem 3.4] applied with $f_\nu - |u_\nu|^{q-1} u_\nu$ instead of f_ν , up to a subsequence, u_ν converges *a.e.* in Ω to an entropy solution w of

$$\begin{cases} -\Delta_p w = \mu - |u|^{q-1} u, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases}$$

Necessarily $w = u$, and u is an entropy solution of problem (1.1). ■

In turn using this result and the local stability result of Theorem 3.3, we can show an existence result in \mathbb{R}^N :

Theorem 8.3 *Assume $p-1 < q < \bar{P}$. Then for any $\mu \in \mathcal{M}(\mathbb{R}^N)$, there exists a local entropy solution of problem (1.1).*

Proof. For any $\nu \in \mathbb{N}$, let $\mu_\nu B_\nu$ be the restriction of μ to B_ν . Then there exists an entropy solution u_ν of problem

$$\begin{cases} -\Delta_p u_\nu + |u_\nu|^{q-1} u_\nu = \mu_\nu B_\nu, & \text{in } B_\nu, \\ u_\nu = 0, & \text{on } \partial B_\nu, \end{cases}$$

since $\mu \llcorner B_\nu \in \mathcal{M}_b(B_\nu)$, from Theorem 8.2. And from Theorem 4.1, (u_ν) satisfies the same estimates as in Theorem 1.6. Hence up to a subsequence, u_ν converges *a.e.* in \mathbb{R}^N to some function u . For $\nu > 2r$, u_ν is a local solution in B_r of

$$-\Delta_p u_\nu + |u_\nu|^{q-1} u_\nu = \mu \llcorner B_r.$$

Thus we can extract a diagonal subsequence, (u_ρ) , such that u_ρ converges locally in measure and *a.e.* in \mathbb{R}^N to a function u satisfying the conclusions of Theorem 3.2 in \mathbb{R}^N . As above, $|u_\rho|^{q-1} u_\rho$ converges strongly in $L^1_{loc}(\mathbb{R}^N)$ to $|u|^{q-1} u$, since q is subcritical. Now we apply Theorem 3.3 in B_r with second member $-|u_\rho|^{q-1} u_\rho + \mu \llcorner B_r$. We can extract a subsequence which depends on r , converging *a.e.* in B_r , to a local entropy solution w of

$$-\Delta_p w + |w|^{q-1} w = \mu \llcorner B_r.$$

Necessarily $w = u$, and the whole sequence converges, and u is a local entropy solution in \mathbb{R}^N . ■

9 Appendix A: Proof of Theorem 2.2

Step 1: D1loc \implies D3loc. Let u be a local entropy solution. Consider any domain $U \subset\subset \Omega$. Taking in (2.21) $h = h_{k,\varepsilon}$ defined in (2.7) with $0 < \varepsilon < k$, and $\varphi_U \in \mathcal{D}^+(\Omega)$ such that $\varphi_U = 1$ on U , we get

$$\begin{aligned} \frac{1}{\varepsilon} \int_{U \cap \{k < u < k+\varepsilon\}} |\nabla u|^p dx &\leq \int_{\Omega} h_{k,\varepsilon}(u) \varphi_U d\mu_0 + \int_{\Omega} \varphi_U d\mu_s^+ - \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi_U h_{k,\varepsilon}(u) dx \\ &\leq C(U) \end{aligned}$$

where $C(U)$ does not depend on ε, k , from (2.20); and similarly

$$\frac{1}{\varepsilon} \int_{U \cap \{k < |u| < k+\varepsilon\}} |\nabla u|^p dx \leq C(U). \quad (9.1)$$

Taking now $h = H_{k,\varepsilon}$ defined in (2.7) and $\varphi \in \mathcal{D}(U)$ in (2.21), we deduce that

$$\begin{aligned} \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi H_{k,\varepsilon}(u) dx &= \frac{1}{\varepsilon} \int_{\{k < u < k+\varepsilon\}} |\nabla u|^p \varphi dx - \frac{1}{\varepsilon} \int_{\{-k-\varepsilon < u < -k\}} |\nabla u|^p \varphi dx \\ &\quad + \int_{\Omega} H_k(u) \varphi d\mu_0 \leq C(U) \|\varphi\|_{L^\infty(U)}. \end{aligned}$$

And $H_{k,\varepsilon}(u) \rightarrow \chi_{\{|u| \leq k\}}$ a.e. in Ω . Hence from (2.20), for any $\varphi \in \mathcal{D}(U)$,

$$\left| \int_{\Omega} |\nabla u_k|^{p-2} \nabla u_k \cdot \nabla \varphi dx \right| \leq C(U) \|\varphi\|_{L^\infty(U)}.$$

As a consequence, there exists a measure $\mu_k \in \mathcal{M}(\Omega)$, such that

$$\int_{\Omega} |\nabla u_k|^{p-2} \nabla u_k \cdot \nabla \psi dx = \int_{\Omega} \psi d\mu_k, \quad (9.2)$$

for any $\psi \in \mathcal{D}(\Omega)$, hence in particular $\mu_k \in W_0^{-1,p'}(U)$. By density (9.2) holds for any $\psi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with compact support. And (μ_k) is uniformly bounded in U . Hence there exists a sequence $(k_n) \rightarrow \infty$ such that μ_{k_n} converges weakly to a measure $\lambda \in \mathcal{M}(\Omega)$. And for any $\psi \in \mathcal{D}(\Omega)$,

$$\lim \int_{\Omega} \psi d\mu_{k_n} = \lim \int_{\Omega} |\nabla u_{k_n}|^{p-2} \nabla u_{k_n} \cdot \nabla \psi dx = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \psi dx = \int_{\Omega} \psi d\mu,$$

from (2.21) with $h(r) = 1$, hence $\lambda = \mu$. Following the proof of [17, Lemma 5.1] in U and using a partition of unity, there exists a measure $\nu \in \mathcal{M}_0(\Omega)$ such that the restrictions $\nu_{\perp}\{|u| = k\}$ and $\mu_{\ell\perp}\{|u| = k\}$ coincide for any $\ell \geq k > 0$. Hence $\nu_{\perp}\{|u| = k\} = \mu_{\perp}\{|u| = k\}$. Considering $\mu_{k\perp}U$ and following [17, Lemma 5.3], we find that

$$\int_{\{|u| > k\}} d|\mu_k| = 0, \quad \text{for any } k > 0.$$

Defining

$$\alpha_k = \mu_{k\perp}\{u = k\}, \quad \beta_k = -\mu_{k\perp}\{u = -k\}$$

and taking $\psi = h_{k-\varepsilon,\varepsilon}(u)\phi$ in (9.2), where $\phi \in \mathcal{D}(\Omega)$, we get

$$\frac{1}{\varepsilon} \int_{\{k-\varepsilon < u < k\}} |\nabla u|^p \phi dx = \int_{\Omega} h_{k-\varepsilon,\varepsilon}(u) \phi d\mu_k - \int_{\{k-\varepsilon < u < k\}} h_{k-\varepsilon,\varepsilon}(u) |\nabla u|^{p-2} \nabla u \cdot \nabla \phi dx.$$

Hence

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\{k-\varepsilon < u < k\}} |\nabla u|^p \phi dx = \int_{\{u \geq k\}} \phi d\mu_k = \int_{\Omega} \phi d\alpha_k,$$

so that $\alpha_k \geq 0$, and $\beta_k \geq 0$ in the same way. Taking now $h = H_{k-\varepsilon,\varepsilon}$ defined in (2.7) for $0 < \varepsilon < k$, and $\varphi \in \mathcal{D}(U)$ in (2.21), we obtain from (9.1),

$$\begin{aligned} \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi H_{k-\varepsilon,\varepsilon}(u) dx &= \frac{1}{\varepsilon} \int_{\{k-\varepsilon < u < k\}} |\nabla u|^p \varphi dx \\ &- \frac{1}{\varepsilon} \int_{\{-k < u < -k+\varepsilon\}} |\nabla u|^p \varphi dx + \int_{\Omega} H_{k-\varepsilon,\varepsilon}(u) \varphi d\mu_0. \end{aligned}$$

Then as $\varepsilon \rightarrow 0$,

$$\int_{\Omega} |\nabla u_k|^{p-2} \nabla u_k \cdot \nabla \varphi dx = \int_{\Omega} \varphi d\alpha_k - \int_{\Omega} \varphi d\beta_k + \int_{\{|u| \leq k\}} \varphi d\mu_0.$$

By density it holds for any $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with compact support in Ω , so that (2.23) holds. Also $\alpha_k \leq |\mu_k|$, $\beta_k \leq |\mu_k|$, hence using a partition of unity, there exists a sequence $(k_n) \rightarrow \infty$ such that $\alpha_{k_n}, \beta_{k_n}$ converge weakly respectively to some measures $\alpha, \beta \in \mathcal{M}^+(\Omega)$. Changing k into $2k$ in (2.23) and taking $\psi = h_{k,k}(u)\phi$, with $\phi \in \mathcal{D}(\Omega)$, we find

$$\begin{aligned} & \int_{\{0 < u < 2k\}} h_{k,k}(u) |\nabla u|^{p-2} \nabla u \cdot \nabla \phi dx + \frac{1}{k} \int_{\{k < u < 2k\}} |\nabla u|^p \phi dx \\ &= \int_{\Omega} h_{k,k}(u) \phi d\alpha_k - \int_{\Omega} h_{k,k}(u) \phi d\beta_k + \int_{\{|u| \leq 2k\}} h_{k,k}(u) \phi d\mu_0 \\ &= \int_{\Omega} \phi d\alpha_k + \int_{\{|u| \leq 2k\}} h_{k,k}(u) \phi d\mu_0. \end{aligned}$$

And $\lim \int_{\{|u| \leq 2k\}} h_{k,k}(u) \phi d\mu_0 = 0$, since $h_{k,k}(u) \rightarrow 0$, μ_0 -a.e. in Ω . But from (2.21) we get

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi h_{k,k}(u) dx + \frac{1}{k} \int_{\{k < u < 2k\}} |\nabla u|^p \phi dx = \int_{\Omega} h_{k,k}(u) \phi d\mu_0 + \int_{\Omega} \phi d\mu_s^+,$$

hence

$$\lim \int_{\Omega} \phi d\alpha_k = \int_{\Omega} \phi d\mu_s^+,$$

that means $\alpha = \mu_s^+$ and $\beta = \mu_s^-$.

Step 2: D3loc \implies D2loc \implies D4loc \implies D1loc The proofs are the same as in [18], after choosing test functions with compact support in Ω .

10 Appendix B: Proof of Theorem 3.3

We essentially follow the proof of [18, Theorem 3.2] and adapt it to the local case. Let

$$\lambda = \lambda_0 + \lambda_s^+ - \lambda_s^-$$

be the decomposition of λ given by (2.5), and E^+, E^- be the disjoint sets where λ_s^+, λ_s^- are concentrated. Let u_ν be any solution of (3.26). By definition, if $\omega \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with

compact support in Ω , and if there exists $k > 0$ and $\omega^+, \omega^- \in W^{1,r}(\Omega) \cap L^\infty(\Omega)$ with $r > N$, such that $\omega = \omega^+$ a.e. on the set $\{u_\nu > k\}$ and $\omega = \omega^-$ a.e. on the set $\{u_\nu < -k\}$, then

$$\int_{\Omega} |\nabla u_\nu|^{p-2} \nabla u_\nu \cdot \nabla \omega dx = \int_{\Omega} \omega f_\nu dx + \int_{\Omega} \omega d\lambda_0 + \int_{\Omega} \omega^+ d\lambda_s^+ - \int_{\Omega} \omega^- d\lambda_s^-. \quad (10.1)$$

Assuming that $|u_\nu|^q$ is bounded in $L^1_{loc}(\Omega)$, then Theorem 3.2 applies, hence (3.21), (3.22), (3.23) hold, and after an extraction we can assume that (u_ν) converges a.e. to some function u . Let Ω', Ω'' be two fixed regular domains such that $\Omega' \subset \subset \Omega'' \subset \subset \Omega$, and

$$E''^+ = E^+ \cap \Omega'', \quad E''^- = E^- \cap \Omega''.$$

Let $\varphi, \Psi \in \mathcal{D}(\Omega)$ be a fixed functions with values in $[0, 1]$, such that $\varphi \equiv 1$ on Ω' , with support in Ω'' , and $\Psi \equiv 1$ in Ω'' .

For any $\delta, \eta > 0$ and $n, \nu \in \mathbb{N}$, we denote by $\varpi(\eta, \delta, n, \nu)$ any quantity such that

$$\lim_{\eta \rightarrow 0} \limsup_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \limsup_{\nu \rightarrow \infty} |\varpi(\eta, \delta, n, \nu)| = 0.$$

Let $\delta, \eta > 0$. From [18, Lemma 5.1], we can define two compacts sets K_δ^+, K_δ^- such that

$$\lambda_s^+(E''^+ \setminus K_\delta^+) \leq \delta, \quad \lambda_s^-(E''^- \setminus K_\delta^-) \leq \delta,$$

and $\psi_\delta^+, \psi_\delta^- \in \mathcal{D}(\Omega'')$ with values in $[0, 1]$ and disjoint supports, such that $\psi_\delta^+ = 1$ on K_δ^+ , $\psi_\delta^- = 1$ on K_δ^- , and

$$\int_{\Omega''} |\nabla \psi_\delta^+|^p dx + \int_{\Omega''} |\nabla \psi_\delta^-|^p dx + \int_{\Omega''} \psi_\delta^- d\lambda_s^+ + \int_{\Omega''} \psi_\delta^+ d\lambda_s^- \leq \delta, \quad (10.2)$$

$$\int_{\Omega''} (1 - \psi_\delta^+ \psi_\eta^+) d\lambda_s^+ + \int_{\Omega''} (1 - \psi_\delta^- \psi_\eta^-) d\lambda_s^- \leq \delta + \eta. \quad (10.3)$$

Step 1. Behaviour near E'' . Let $\phi_\eta^+, \phi_\eta^- \in W^{1,\infty}(\Omega)$, with values in $[0, 1]$, and compact support in Ω'' , such that

$$\int_{\Omega''} \phi_\eta^- d\lambda_s^+ + \int_{\Omega''} \phi_\eta^+ d\lambda_s^- \leq \eta. \quad (10.4)$$

First we extend [18, Lemma 6.1], showing that

$$\frac{1}{n} \left(\int_{\{n \leq u_\nu \leq 2n\}} |\nabla u_\nu|^p \phi_\eta^- dx + \int_{\{-2n \leq u_\nu \leq -n\}} |\nabla u_\nu|^p \phi_\eta^+ dx \right) \leq \varpi_\eta(n, \nu) + \eta. \quad (10.5)$$

Indeed we first take in (10.1) $\omega = h_{n,n}(u_\nu)\Psi$, where $h_{n,n}$ is defined in (2.7) and obtain

$$\begin{aligned} \frac{1}{n} \int_{\{n < u_\nu < 2n\}} |\nabla u_\nu|^p \Psi dx &= \int_{\Omega} f_\nu h_{n,n}(u_\nu) \Psi dx + \int_{\Omega} h_{n,n}(u_\nu) \Psi d\lambda_0 \\ &\quad - \int_{\Omega} h_{n,n}(u_\nu) |\nabla u_\nu|^{p-2} \nabla u_\nu \cdot \nabla \Psi dx + \int_{\Omega} \Psi d\lambda_s^+. \end{aligned}$$

Hence from (3.23),

$$\frac{1}{n} \int_{\{n < u_\nu < 2n\}} |\nabla u_\nu|^p \Psi dx \leq \|f_\nu \Psi\|_{L^1(\Omega)} + \int_{\Omega} \Psi d|\lambda| + \int_{\text{supp } \Psi} |\nabla u_\nu|^{p-1} dx \leq C$$

where C depends on λ, Ψ, f , but not on n and ν . Then

$$\frac{1}{n} \int_{\Omega'' \cap \{n < u_\nu < 2n\}} |\nabla u_\nu|^p dx = \int_{\Omega''} |\nabla h_{n,n}(u_\nu)|^p dx \leq Cn^{1-p}. \quad (10.6)$$

As $\nu \rightarrow \infty$, $h_{n,n}(u_\nu) \rightarrow h_{n,n}(u)$ *a.e.* in Ω , is bounded in $L^\infty(\Omega)$, hence converges strongly in $W^{1,p}(\Omega'')$ from (10.6), so that

$$\int_{\Omega''} |\nabla h_{n,n}(u)|^p dx \leq Cn^{1-p}. \quad (10.7)$$

As $n \rightarrow \infty$, $h_{n,n}(u) \rightarrow 0$ *a.e.* in Ω , is bounded in $L^\infty(\Omega)$, hence converges strongly in $W^{1,p}(\Omega'')$ from (10.7).

Then we take in (10.1) $\omega = h_{n,n}(u_\nu)\phi_\eta^-$, and get

$$\begin{aligned} \frac{1}{n} \int_{\{n < u_\nu < 2n\}} |\nabla u_\nu|^p \phi_\eta^- dx &= \int_{\Omega} f_\nu h_{n,n}(u_\nu) \phi_\eta^- dx + \int_{\Omega} h_{n,n}(u_\nu) \phi_\eta^- d\lambda_0 \\ &\quad - \int_{\Omega} h_{n,n}(u_\nu) |\nabla u_\nu|^{p-2} \nabla u_\nu \cdot \nabla \phi_\eta^- dx + \int_{\Omega} \phi_\eta^- d\lambda_s^+. \end{aligned}$$

And $|\nabla u_\nu|^{p-2} \nabla u_\nu$ converges strongly in $(L^r(\Omega''))^N$ for any $1 \leq r < N/(N-1)$. Then

$$\begin{aligned} \int_{\Omega} h_{n,n}(u_\nu) |\nabla u_\nu|^{p-2} \nabla u_\nu \cdot \nabla \phi_\eta^- dx &= \int_{\Omega} h_{n,n}(u) |\nabla u|^{p-2} \nabla u \cdot \nabla \phi_\eta^- dx + \varpi_{\eta,n}(\nu) = \varpi_\eta(n, \nu), \\ \int_{\Omega} h_{n,n}(u_\nu) \phi_\eta^- d\lambda_0 &= \int_{\Omega} h_{n,n}(u) \phi_\eta^- d\lambda_0 + \varpi_{\eta,n}(\nu) = \varpi_\eta(n, \nu). \end{aligned}$$

Also from [18, Proposition 2.8],

$$\int_{\Omega} f_{\nu} h_{n,n}(u_{\nu}) \phi_{\eta}^{-} dx = \int_{\Omega} f h_{n,n}(u) \phi_{\eta}^{-} dx + \varpi_{\eta,n}(\nu) = \varpi_{\eta}(n, \nu).$$

hence from (10.2),

$$\frac{1}{n} \int_{\{n \leq u_{\nu} \leq 2n\}} |\nabla u_{\nu}|^p \phi_{\eta}^{-} dx \leq \varpi_{\eta}(n, \nu) + \eta,$$

and (10.5) follows, after exchanging ϕ_{η}^{-} into ϕ_{η}^{+} and $h_{n,n}(r)$ into $h_{n,n}(-r)$.

Now we extend [18, Lemma 6.3], showing that for fixed $k > 0$, and for any $\eta, \delta > 0$,

$$\int_{\Omega} |\nabla u_{\nu,k}|^p \psi_{\delta}^{+} \psi_{\eta}^{+} dx + \int_{\Omega} |\nabla u_{\nu,k}|^p \psi_{\delta}^{-} \psi_{\eta}^{-} dx = \varpi(\eta, \delta, \nu). \quad (10.8)$$

We take $\omega = H_{n,n}(u_{\nu})(k - u_{\nu,k}) \psi_{\delta}^{+} \psi_{\eta}^{+}$ in (10.1), where $H_{n,n}$ is defined in (2.7), and obtain

$$\begin{aligned} & - \int_{\Omega} |\nabla u_{\nu,k}|^p H_{n,n}(u_{\nu}) \psi_{\delta}^{+} \psi_{\eta}^{+} dx + \int_{\Omega} |\nabla u_{\nu}|^{p-2} \nabla u_{\nu} \cdot \nabla \psi_{\delta}^{+} H_{n,n}(u_{\nu})(k - u_{\nu,k}) \psi_{\eta}^{+} dx \\ & + \int_{\Omega} |\nabla u_{\nu}|^{p-2} \nabla u_{\nu} \cdot \nabla \psi_{\eta}^{+} H_{n,n}(u_{\nu})(k - u_{\nu,k}) \psi_{\delta}^{+} dx + \int_{\Omega} |\nabla u_{\nu}|^p H'_{n,n}(u_{\nu})(k - u_{\nu,k}) \psi_{\delta}^{+} \psi_{\eta}^{+} dx \\ & = \int_{\Omega} f_{\nu}(k - u_{\nu,k}) H_{n,n}(u_{\nu}) \psi_{\delta}^{+} \psi_{\eta}^{+} dx + \int_{\Omega} (k - u_{\nu,k}) H_{n,n}(u_{\nu}) \psi_{\delta}^{+} \psi_{\eta}^{+} d\lambda_0. \end{aligned} \quad (10.9)$$

As $\nu \rightarrow \infty$, $H_{n,n}(u_{\nu}) \rightarrow H_{n,n}(u)$ and *a.e.* in Ω , is bounded, and strongly in $W^{1,p}(\Omega'')$ from (10.6) and the corresponding estimate on $\Omega'' \cap \{-2n < u_{\nu} < -n\}$. From (10.5) applied with $\phi_{\eta}^{+} = \psi_{\eta}^{+}$,

$$\begin{aligned} \int_{\Omega} |\nabla u_{\nu}|^p H'_{n,n}(u_{\nu})(k - u_{\nu,k}) \psi_{\delta}^{+} \psi_{\eta}^{+} dx &= \frac{2k}{n} \int_{\{-2n < u_{\nu} \leq -n\}} |\nabla u_{\nu}|^p \psi_{\delta}^{+} \psi_{\eta}^{+} dx \\ &\leq \frac{2k}{n} \int_{\{-2n < u_{\nu} \leq -n\}} |\nabla u_{\nu}|^p \psi_{\eta}^{+} dx \\ &\leq \varpi_{\eta}(n, \nu) + \eta = \varpi(\eta, n, \nu). \end{aligned}$$

As in the proof of [18, Lemma 6.3], since ψ_{η}^{+} has a compact support, we get successively

$$\begin{aligned} \int_{\Omega} |\nabla u_{\nu}|^{p-2} \nabla u_{\nu} \cdot \nabla \psi_{\delta}^{+} H_{n,n}(u_{\nu})(k - u_{\nu,k}) \psi_{\eta}^{+} dx &= \varpi_{\eta,n}(\delta, \nu), \\ \int_{\Omega} |\nabla u_{\nu}|^{p-2} \nabla u_{\nu} \cdot \nabla \psi_{\eta}^{+} H_{n,n}(u_{\nu})(k - u_{\nu,k}) \psi_{\delta}^{+} dx &= \varpi_{\eta,n}(\delta, \nu), \\ \int_{\Omega} f_{\nu}(k - u_{\nu,k}) H_{n,n}(u_{\nu}) \psi_{\delta}^{+} \psi_{\eta}^{+} dx &= \varpi_{\eta,n}(\delta, \nu), \end{aligned}$$

and

$$\int_{\Omega} (k - u_{\nu,k}) H_{n,n}(u_{\nu}) \psi_{\delta}^{+} \psi_{\eta}^{+} d\lambda_0 = \int_{\Omega} (k - u_k) H_{n,n}(u) \psi_{\delta}^{+} \psi_{\eta}^{+} d\lambda_0 + \varpi_{\eta,n,\delta}(\nu) = \varpi_{\eta,n}(\delta, \nu).$$

Indeed $\lambda_0 \in \mathcal{M}_0(\Omega)$ and $(k - u_{\nu,k}) H_{n,n}(u_{\nu}) \psi_{\delta}^{+} \psi_{\eta}^{+}$ converges weakly in $W_0^{1,p}(\Omega'')$ to $(k - u_{\nu,k}) H_{n,n}(u_{\nu}) \psi_{\delta}^{+} \psi_{\eta}^{+}$ as $\nu \rightarrow \infty$, because $u_{\nu,k}$ converges weakly to u_k in $W^{1,p}(\Omega'')$ from (3.21); also $(k - u_k) H_{n,n}(u) \psi_{\delta}^{+} \psi_{\eta}^{+}$ converges strongly to 0 in $W_0^{1,p}(\Omega'')$ from (10.2) as $\delta \rightarrow 0$. Hence from (10.9),

$$\int_{\Omega} |\nabla u_{\nu,k}|^p H_{n,n}(u_{\nu}) \psi_{\delta}^{+} \psi_{\eta}^{+} dx = \varpi(\eta, n, \delta, \nu),$$

and more precisely

$$\int_{\Omega} |\nabla u_{\nu,k}|^p h_{n,n}(u_{\nu}) \psi_{\delta}^{+} \psi_{\eta}^{+} dx = \int_{\Omega} |\nabla u_{\nu,k}|^p \psi_{\delta}^{+} \psi_{\eta}^{+} dx = \varpi(\eta, \delta, \nu)$$

since $n > k$. We deduce (10.8) after replacing the test function by $\omega = H_{n,n}(u_{\nu})(k + u_{\nu,k}) \psi_{\delta}^{-} \psi_{\eta}^{-}$.

Step 2. Behaviour far from E'' . Now we define

$$\Phi_{\delta,\eta} = \psi_{\delta}^{+} \psi_{\eta}^{+} + \psi_{\delta}^{-} \psi_{\eta}^{-}$$

and following [18, Lemma 7.1], we show that for fixed $k > 0$,

$$\int_{\Omega} (|\nabla u_{\nu,k}|^p - |\nabla u_k|^p) (1 - \Phi_{\delta,\eta}) \varphi dx = \varpi(\eta, \delta, \nu). \quad (10.10)$$

In that aim we first prove as in [18, Lemma 7.3] that

$$\begin{aligned} & \int_{\Omega} |\nabla u_{\nu,k}|^p (1 - \Phi_{\delta,\eta}) \varphi dx - \int_{\Omega} u_k \varphi |\nabla u_k|^{p-2} \nabla u_k \cdot \nabla \Phi_{\delta,\eta} dx + \int_{\Omega} \Phi_{\delta,\eta} u_k |\nabla u_k|^{p-2} \nabla u_k \cdot \nabla \varphi dx \\ &= \int_{\Omega} f(1 - \Phi_{\delta,\eta}) u_k \varphi dx + \int_{\Omega} (1 - \Phi_{\delta,\eta}) u_k \varphi d\lambda_0 + \varpi(\eta, \delta, \nu). \end{aligned} \quad (10.11)$$

Indeed we choose $\omega = (1 - \Phi_{\delta,\eta}) u_{\nu,k} \varphi$ as test function, and obtain

$$\begin{aligned} & \int_{\Omega} |\nabla u_{\nu,k}|^p (1 - \Phi_{\delta,\eta}) \varphi dx - \int_{\Omega} u_{\nu,k} \varphi |\nabla u_{\nu,k}|^{p-2} \nabla u_{\nu,k} \cdot \nabla \Phi_{\delta,\eta} dx \\ &+ \int_{\Omega} \Phi_{\delta,\eta} u_{\nu,k} |\nabla u_{\nu,k}|^{p-2} \nabla u_{\nu,k} \cdot \nabla \varphi dx \\ &= \int_{\Omega} f_{\nu} (1 - \Phi_{\delta,\eta}) u_{\nu,k} \varphi dx + \int_{\Omega} (1 - \Phi_{\delta,\eta}) u_{\nu,k} \varphi d\lambda_0 \\ &+ k \int_{\Omega} (1 - \Phi_{\delta,\eta}) \varphi d\lambda_s^{+} - k \int_{\Omega} (1 - \Phi_{\delta,\eta}) \varphi d\lambda_s^{-}; \end{aligned}$$

we get successively, as in [18, Lemma 7.3],

$$\begin{aligned}
\int_{\Omega} u_{\nu,k} \varphi |\nabla u_{\nu,k}|^{p-2} \nabla u_{\nu,k} \cdot \nabla \Phi_{\delta,\eta} dx &= \int_{\Omega} u_k \varphi |\nabla u_k|^{p-2} \nabla u_k \cdot \nabla \Phi_{\delta,\eta} dx + \omega_{\eta,\delta}(\nu); \\
\int_{\Omega} \Phi_{\delta,\eta} u_{\nu,k} |\nabla u_{\nu,k}|^{p-2} \nabla u_{\nu,k} \cdot \nabla \varphi dx &= \int_{\Omega} \Phi_{\delta,\eta} u_k |\nabla u_k|^{p-2} \nabla u_k \cdot \nabla \varphi dx + \omega_{\eta,\delta}(\nu); \\
\int_{\Omega} f_{\nu} (1 - \Phi_{\delta,\eta}) u_{\nu,k} \varphi dx &= \int_{\Omega} f (1 - \Phi_{\delta,\eta}) u_k \varphi dx + \varpi_{\eta,\delta}(\nu); \\
\int_{\Omega} (1 - \Phi_{\delta,\eta}) u_{\nu,k} \varphi d\lambda_0 &= \int_{\Omega} (1 - \Phi_{\delta,\eta}) u_k \varphi d\lambda_0 + \varpi_{\eta,\delta}(\nu);
\end{aligned}$$

and from (10.3), since φ has a compact support in Ω'' , and values in $[0, 1]$,

$$\int_{\Omega} (1 - \psi_{\delta}^+ \psi_{\eta}^+) \varphi d\lambda_s^+ + \int_{\Omega''} (1 - \psi_{\delta}^- \psi_{\eta}^-) \varphi d\lambda_s^- \leq \delta + \eta,$$

hence, since k is fixed,

$$k \int_{\Omega} (1 - \Phi_{\delta,\eta}) \varphi d\lambda_s^+ - k \int_{\Omega} (1 - \Phi_{\delta,\eta}) \varphi d\lambda_s^- = \omega(\delta, \eta),$$

and this shows (10.11).

Then as in [18, Lemma 7.4], we show that

$$\frac{1}{n} \int_{\{n < |u_{\nu}| < 2n\}} |\nabla u_{\nu}|^p (1 - \Phi_{\delta,\eta}) \varphi dx = \varpi(\eta, \delta, n, \nu). \quad (10.12)$$

Indeed we have

$$\begin{aligned}
&\frac{1}{n} \int_{\{n < |u_{\nu}| < 2n\}} |\nabla u_{\nu}|^p (1 - \Phi_{\delta,\eta}) \varphi dx \\
&= \frac{1}{n} \int_{\{n < u_{\nu} < 2n\}} |\nabla u_{\nu}|^p (1 - \psi_{\delta}^+ \psi_{\eta}^+) \varphi dx - \frac{1}{n} \int_{\{n < u_{\nu} < 2n\}} |\nabla u_{\nu}|^p \psi_{\delta}^+ \psi_{\eta}^+ \varphi dx \\
&+ \frac{1}{n} \int_{\{-2n < u_{\nu} < -n\}} |\nabla u_{\nu}|^p (1 - \psi_{\delta}^- \psi_{\eta}^-) \varphi dx - \frac{1}{n} \int_{\{-2n < u_{\nu} < -n\}} |\nabla u_{\nu}|^p \psi_{\delta}^+ \psi_{\eta}^+ \varphi dx.
\end{aligned}$$

Using the fact that φ has values in $[0, 1]$, we conclude to (10.12) by applying (10.5) to $\phi_{\delta+\eta}^- = (1 - \psi_{\delta}^+ \psi_{\eta}^+) \varphi$ and $\phi_{\delta+\eta}^+ = (1 - \psi_{\delta}^- \psi_{\eta}^-) \varphi$, since (10.3) holds; and then to $\phi_{\delta,\eta}^- = \psi_{\delta}^- \psi_{\eta}^- \varphi$ and $\phi_{\delta,\eta}^+ = \psi_{\delta}^+ \psi_{\eta}^+ \varphi$, since (10.2) holds for $\psi_{\eta}^-, \psi_{\eta}^+$ and $\psi_{\delta}^+, \psi_{\delta}^-$ take their values in $[0, 1]$.

Finally as in [18, Lemma 7.5] we show that

$$\begin{aligned}
& \int_{\Omega} |\nabla u_k|^p (1 - \Phi_{\delta,\eta}) \varphi dx - \int_{\Omega} u_k \varphi |\nabla u|^{p-2} \nabla u \cdot \nabla \Phi_{\delta,\eta} dx + \int_{\Omega} \Phi_{\delta,\eta} u_k |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx \\
&= \int_{\Omega} f(1 - \Phi_{\delta,\eta}) u_k \varphi dx + \int_{\Omega} (1 - \Phi_{\delta,\eta}) u_k \varphi d\lambda_0 + \varpi(\eta, \delta).
\end{aligned} \tag{10.13}$$

Indeed first observe that $H_{n,n}(u_\nu) \rightarrow H_{n,n}(u)$ *a.e.* in Ω , is bounded in $L^\infty(\Omega)$, and converges strongly in $W^{1,p}(\Omega'')$ as $\nu \rightarrow \infty$. Also $H_{n,n}(u) \rightarrow 1$ *a.e.* in Ω , is bounded in $L^\infty(\Omega)$, and converges to 1 strongly in $W^{1,p}(\Omega'')$, as $n \rightarrow \infty$. Choosing $\omega = (1 - \Phi_{\delta,\eta}) u_k H_{n,n}(u_\nu) \varphi$ as test function in (10.1) with $n > k$, we obtain

$$\begin{aligned}
& \int_{\Omega} (1 - \Phi_{\delta,\eta}) H_{n,n}(u_\nu) \varphi |\nabla u_\nu|^{p-2} \nabla u_\nu \cdot \nabla u_k dx \\
& - \int_{\Omega} u_k H_{n,n}(u_\nu) \varphi |\nabla u_k|^{p-2} \nabla u_k \cdot \nabla \Phi_{\delta,\eta} dx + \int_{\Omega} (1 - \Phi_{\delta,\eta}) H_{n,n}(u_\nu) u_k |\nabla u_\nu|^{p-2} \cdot \nabla \varphi dx \\
& + \int_{\Omega} |\nabla u_\nu|^p u_k (1 - \Phi_{\delta,\eta}) H'_{n,n}(u_\nu) \varphi dx \\
&= \int_{\Omega} f(1 - \Phi_{\delta,\eta}) u_k H_{n,n}(u_\nu) \varphi dx + \int_{\Omega} (1 - \Phi_{\delta,\eta}) u_k H_{n,n}(u_\nu) \varphi d\lambda_0.
\end{aligned}$$

As in [18, Lemma 7.5], we deduce

$$\begin{aligned}
& \int_{\Omega} (1 - \Phi_{\delta,\eta}) H_{n,n}(u_\nu) \varphi |\nabla u_\nu|^{p-2} \cdot \nabla u_k dx \\
&= \int_{\Omega} (1 - \Phi_{\delta,\eta}) H_{n,n}(u) \varphi |\nabla u_{2n}|^{p-2} \cdot \nabla u_k dx + \omega_{\eta,\delta,n}(\nu) \\
&= \int_{\Omega} |\nabla u_k|^p (1 - \Phi_{\delta,\eta}) \varphi dx + \omega_{\eta,\delta}(n, \nu),
\end{aligned}$$

$$\begin{aligned}
\int_{\Omega} u_k H_{n,n}(u_\nu) \varphi |\nabla u_k|^{p-2} \nabla u_k \cdot \nabla \Phi_{\delta,\eta} dx &= \int_{\Omega} u_k H_{n,n}(u) \varphi |\nabla u|^{p-2} \nabla u \cdot \nabla \Phi_{\delta,\eta} dx + \varpi_{\eta,\delta,n}(\nu) \\
&= \int_{\Omega} u_k \varphi |\nabla u|^{p-2} \nabla u \cdot \nabla \Phi_{\delta,\eta} dx + \varpi_{\eta,\delta}(n, \nu);
\end{aligned}$$

$$\begin{aligned}
\int_{\Omega} (1 - \Phi_{\delta,\eta}) H_{n,n}(u_\nu) u_k |\nabla u_\nu|^{p-2} \cdot \nabla \varphi dx &= \int_{\Omega} (1 - \Phi_{\delta,\eta}) u_k |\nabla u|^{p-2} \cdot \nabla \varphi h_{n,n}(u) dx + \varpi_{\eta,\delta,n}(\nu) \\
&= \int_{\Omega} (1 - \Phi_{\delta,\eta}) u_k |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx + \varpi_{\eta,\delta}(n, \nu);
\end{aligned}$$

and also

$$\left| \int_{\Omega} |\nabla u_{\nu}|^p u_k (1 - \Phi_{\delta,\eta}) H'_{n,n}(u_{\nu}) \varphi dx \right| \leq \varpi(\eta, \delta, n, \nu)$$

from (10.12), since k is fixed. And

$$\begin{aligned} \int_{\Omega} f(1 - \Phi_{\delta,\eta}) u_k H_{n,n}(u_{\nu}) \varphi dx &= \int_{\Omega} f(1 - \Phi_{\delta,\eta}) u_k H_{n,n}(u) \varphi dx + \varpi_{\eta,\delta,n}(\nu) \\ &= \int_{\Omega} f(1 - \Phi_{\delta,\eta}) u_k \varphi dx + \omega_{\eta,\delta}(n, \nu); \end{aligned}$$

$$\begin{aligned} \int_{\Omega} (1 - \Phi_{\delta,\eta}) u_k H_{n,n}(u_{\nu}) \varphi d\lambda_0 &= \int_{\Omega} (1 - \Phi_{\delta,\eta}) u_k H_{n,n}(u) \varphi d\lambda_0 + \varpi_{\eta,\delta,n}(\nu) \\ &= \int_{\Omega} (1 - \Phi_{\delta,\eta}) u_k \varphi d\lambda_0 + \varpi_{\eta,\delta}(n, \nu). \end{aligned}$$

Hence (10.13) holds, because all the terms do not depend on n or ν . At last (10.10) follows from (10.11) and (10.13).

Step 3. Strong convergence of truncates in $W_{loc}^{1,p}(\Omega)$.

We consider the difference

$$\begin{aligned} &\int_{\Omega} (|\nabla u_{\nu,k}|^p - |\nabla u_k|^p) \varphi dx \\ &= \int_{\Omega} (|\nabla u_{\nu,k}|^p - |\nabla u_k|^p) (1 - \Phi_{\delta,\eta}) \varphi dx + \int_{\Omega} |\nabla u_{\nu,k}|^p \Phi_{\delta,\eta} \varphi dx - \int_{\Omega} |\nabla u_k|^p \Phi_{\delta,\eta} \varphi dx. \end{aligned}$$

From (10.8), we have

$$\int_{\Omega} |\nabla u_{\nu,k}|^p \Phi_{\delta,\eta} \varphi dx = \varpi(\eta, \delta, \nu).$$

Since $|\nabla u_k|^p \in L^1(\Omega'')$ and $\Phi_{\delta,\eta}$ converges to 0 *a.e.* in Ω and is bounded in $L^\infty(\Omega'')$, we have

$$\int_{\Omega} |\nabla u_k|^p \Phi_{\delta,\eta} \varphi dx = \varpi(\eta, \delta)$$

hence from (10.10), we deduce that

$$\int_{\Omega} (|\nabla u_{\nu,k}|^p - |\nabla u_k|^p) \varphi dx = \varpi(\eta, \delta, \nu) = \varpi(\nu);$$

now $|\nabla u_{\nu,k}| \nabla u_{\nu,k}$ converges *a.e* in Ω , hence strongly in $L^1(\Omega')$, hence in $L^1_{loc}(\Omega)$ and $\nabla u_{\nu,k}$ converges strongly in $L^p(\Omega')$, hence in $L^p_{loc}(\Omega)$.

Step 4: u is a local entropy solution of (3.27). We have, from **D1loc**,

$$\begin{aligned} \int_{\Omega} |\nabla u_{\nu}|^{p-2} \nabla u_{\nu} \cdot \nabla (h(u_{\nu}) \psi) dx &= \int_{\Omega} h(u_{\nu}) f_{\nu} \psi dx + \int_{\Omega} h(u_{\nu}) \psi d\lambda_0 \\ &\quad + h(+\infty) \int_{\Omega} \psi d\lambda_s^+ - h(-\infty) \int_{\Omega} \psi d\lambda_s^- \end{aligned} \quad (10.14)$$

for any $h \in W^{1,\infty}(\mathbb{R})$ such that h' has a support in some interval $[-k, k]$, and $\psi \in \mathcal{D}^+(\Omega)$. And $h(u_{\nu})$ converges to $h(u)$ *a.e.* in Ω and strongly in $W^{1,p}_{loc}(\Omega)$, hence λ_0 -*a.e.* in Ω and $h(u_{\nu})$ is bounded in $L^\infty(\Omega)$, and $|\nabla u_{\nu}|^{p-2} \nabla u_{\nu}$ converges strongly in $L^1_{loc}(\Omega)$, hence

$$\begin{aligned} \lim \int_{\Omega} h(u_{\nu}) |\nabla u_{\nu}|^{p-2} \nabla u_{\nu} \cdot \nabla \psi dx &= \int_{\Omega} h(u) |\nabla u|^{p-2} \nabla u \cdot \nabla \psi dx, \\ \lim \int_{\Omega} h(u_{\nu}) f_{\nu} \psi dx &= \int_{\Omega} h(u) f \psi dx, \\ \lim \int_{\Omega} h(u_{\nu}) \psi d\lambda_0 &= \int_{\Omega} h(u) \psi d\lambda_0, \end{aligned}$$

and $h'(u_{\nu})$ converges to $h'(u)$ *a.e.* in Ω and is bounded in $L^\infty(\Omega)$, hence from Step 3,

$$\lim \int_{\Omega} h'(u_{\nu}) |\nabla u_{\nu}|^p \psi dx = \lim \int_{\Omega} h'(u_{\nu,k}) |\nabla u_{\nu,k}|^p \psi dx = \int_{\Omega} h'(u) |\nabla u|^p \psi dx.$$

Thus

$$\begin{aligned} \int_{\Omega} h'(u) |\nabla u|^p \psi dx &= \int_{\Omega} h(u) f \psi dx - \int_{\Omega} h(u) |\nabla u|^{p-2} \nabla u \cdot \nabla \psi dx \\ &\quad + \int_{\Omega} h(u) \psi d\lambda_0 + h(+\infty) \int_{\Omega} \psi d\lambda_s^+ - h(-\infty) \int_{\Omega} \psi d\lambda_s^-, \end{aligned}$$

and the conclusion follows.

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