

Initial Trace of Solutions of Some Quasilinear Parabolic Equations with Absorption

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Communicated by H. Brezis

Received October 3, 2001; accepted October 10, 2001

We study the existence of an initial trace of nonnegative solutions of the problem

$$\partial_t u - \nabla \cdot (|\nabla u|^{p-2} \nabla u) + u^q = 0 \quad \text{in } Q_T = \Omega \times (0, T).$$

We prove that the initial trace is an outer regular Borel measure *which may not be locally bounded for some values of the parameters p and q* . We study also the corresponding Cauchy problems with a given generalized Borel measure as initial data. © 2002 Elsevier Science (USA)

Key Words: P -Laplace; initial trace; Borel measures; entropy solution; generalized measure.

AMS Classification: 35K15; 35K55.

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1. INTRODUCTION

Let Ω be a domain in \mathbb{R}^N ($N \geq 1$), possibly unbounded. The aim of this article is to investigate the initial trace problem for the following class of quasilinear equations with absorption:

$$\partial_t u - \nabla \cdot (|\nabla u|^{p-2} \nabla u) + u^q = 0 \quad \text{in } Q_T = \Omega \times (0, T), \quad (1.1)$$

in the range $q > 0$, $p > 1$. We prove the existence of an initial trace in the class $\mathcal{B}_+^{\text{reg}}(\Omega)$ of outer regular positive Borel measures in Ω , *not necessarily locally bounded*. Conversely, given a measure $\nu \in \mathcal{B}_+^{\text{reg}}(\Omega)$, we study the initial-boundary value problem for Eq. (1.3) with initial data ν .

The initial trace problem for a nonnegative solution u of the mere heat equation ($p = 2$, no absorption) in Q_T is classical. It is easy to establish that there exists a unique positive Radon measure μ in Ω which is the initial trace of u , in the sense that

$$\lim_{t \rightarrow 0} \int_{\Omega} u(x, t) \zeta(x) dx = \int_{\Omega} \zeta(x) d\mu(x), \quad \forall \zeta \in C_c(\Omega).$$

Moreover, when $\Omega = \mathbb{R}^N$, the initial trace μ satisfies the following growth condition:

$$\int_{\mathbb{R}^N} e^{-|x|^2/4T} d\mu(x) < \infty. \quad (1.2)$$

Conversely, if μ is a nonnegative Radon measure in \mathbb{R}^N satisfying (1.2), there exists a unique solution of the heat equation in $\mathbb{R}^N \times (0, T)$ with initial trace μ .

Concerning the semilinear heat equation with absorption

$$\partial_t u - \Delta u + u^q = 0, \quad (1.3)$$

a rather complete picture of the initial trace problem is provided by Marcus and Véron [33] who prove that the initial trace is well defined in the class $\mathcal{B}_+^{\text{reg}}(\Omega)$. Their first result asserts that there exists a relatively closed subset $\mathcal{S} \subseteq \Omega$ and a nonnegative Radon measure μ on $\mathcal{R} = \Omega \setminus \mathcal{S}$ such that

$$\lim_{t \rightarrow 0} \int_{\mathcal{R}} u(x, t) \zeta(x) dx = \int_{\mathcal{R}} \zeta(x) d\mu(x), \quad \forall \zeta \in C_c(\mathcal{R}), \quad (1.4)$$

and

$$\lim_{t \rightarrow 0} \int_V u(x, t) dx = \infty, \quad \forall V \text{ open s.t. } \mathcal{S} \cap V \neq \emptyset. \quad (1.5)$$

Moreover, \mathcal{S} and μ are uniquely determined. They define in a unique way an outer regular Borel measure ν which is the initial trace of u :

$$tr_\Omega(u) = (\mathcal{S}, \mu) \approx \nu.$$

The reverse problem is to reconstruct the solution from a given outer regular Borel measure $\nu = (\mathcal{S}, \mu) \in \mathcal{B}_+^{\text{reg}}(\Omega)$. The role of the critical exponent

$$q_* = 1 + 2/N \quad (1.6)$$

is pointed out by the fact that if $1 < q < q_*$ there exist no removable singularity for Eq. (1.3), in the sense that there exist nontrivial solutions of (1.3) continuous in $\bar{Q} \setminus \{(a, 0)\}$ (where $a \in \Omega$) and vanishing on $\bar{Q} \setminus \{a\} \times \{0\} \cup \partial\Omega \times [0, T)$. From this result follows that, always in the subcritical case $1 < q < q_*$, any outer regular Borel measure ν is eligible for being the initial trace of a nonnegative solution of (1.3). Moreover, under minor additional assumptions, uniqueness of the solution of the generalized Cauchy–Dirichlet problem

$$\begin{cases} \partial_t u - \Delta u + u^q = 0 & \text{in } Q_T, \\ u = f \in L_+^1(\partial\Omega \times (0, T)), \\ tr_\Omega(u) = \nu, \end{cases} \quad (1.7)$$

is obtained. On the opposite, if $q \geq q_*$ the Cauchy–Dirichlet problem (1.7) becomes much more difficult: necessary and sufficient conditions on the concentration of the Radon part μ and the singular part \mathcal{S} of ν are expressed in terms of Bessel capacities $C_{2/q, q^*}$. When these conditions are fulfilled existence of a maximal solution follows, but usually uniqueness does not hold.

When the heat equation is replaced by the porous medium equation

$$\partial_t u - \Delta u^m = 0 \quad \text{in } \mathbb{R}^N \times (0, T), \quad (1.8)$$

the existence of an initial trace μ in the set of positive Radon measures in \mathbb{R}^N is obtained by Aronson and Caffarelli [2]. They also prove that μ has to satisfy some polynomial growth at infinity. The corresponding initial value problem is studied in [10]. The trace question for the equation with absorption

$$\partial_t u - \Delta u^m + u^q = 0 \quad \text{in } Q_T \quad (1.9)$$

is considered by the author in [20, 21] in the range $1 < m \leq q$. If u is a nonnegative solution of (1.9) there exists an outer regular Borel measure $\nu = (\mathcal{S}, \mu)$ such that properties (1.4) and (1.5) hold in the same way as above. This defines the initial trace $tr_\Omega(u)$. There are two critical values

$$q = m \quad \text{and} \quad q = q_m = m + 2/N. \quad (1.10)$$

If $q = m$ the diffusion is comparable to the absorption, which is superlinear. Therefore there are only two possibilities:

- (i) either $\mathcal{S} = \emptyset$ and ν is reduced to its regular part which is a Radon measure with some growth at infinity,
- (ii) or $\mathcal{S} = \Omega$ and u is the flat solution, that is

$$u(x, t) = \left(\frac{1}{t(m-1)} \right)^{1/(m-1)}.$$

If $m < q$ the absorption dominates the diffusion but the situation differs according to $m < q < q_m$ or $q \geq q_m$. If $m < q < q_m$ any outer regular Borel measure ν is eligible for being the initial trace of a nonnegative solution u of (1.9). An existence result of a solution of the generalized Cauchy–Dirichlet problem

$$\begin{cases} \partial_t u - \Delta u^m + u^q = 0 & \text{in } Q_T, \\ u = f \in L_+^m(\partial\Omega \times (0, T)), \\ tr_\Omega(u) = \nu \in \mathcal{B}_+^{\text{reg}}(\Omega) \end{cases} \quad (1.11)$$

also holds, but uniqueness is not proved except in the case where ν has no singular part and $f = 0$. If $q \geq q_m$, the situation is again more difficult as a result of the fact that isolated points at $t = 0$ are removable singularities for solutions of (1.9). Some sufficient conditions for the existence of a solution to (1.11) exist, however up to now they have not been proved to be necessary.

The situation for the p -Laplacian-type equation is in some sense more difficult than for the porous-medium-type equation. One of the reasons is that the full duality argument in a L^1 framework, which was at the core of Marcus and Véron or Chasseigne's results, does not fit with the p -Laplace operator. As a consequence the proof of the existence of an initial trace is much more difficult to obtain. The notion of weak solution is not at all straightforward, even in the case of the equation without absorption

$$\partial_t u - \nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0 \quad \text{in } Q_T = \Omega \times (0, T). \quad (1.12)$$

Two critical values for p exist for expressing the notion of weak solution to (1.12) with a measure or an integrable function as initial data,

$$p_0 = \frac{2N}{N+1} \quad \text{and} \quad p_1 = \frac{2N+1}{N+1}. \quad (1.13)$$

If $1 < p \leq p_1$ the gradient is not well defined, in particular is not an integrable function. This is why the different authors who studied as well stationary or time-dependent equations involving the p -Laplacian were led to introduce the notion of *renormalized* or *entropy solutions* which are solutions such that $\nabla T_k(u) \in L^1_{\text{loc}}(Q_T)$ for any $k > 0$, where

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k \operatorname{sign}(s) & \text{if } |s| > k. \end{cases} \quad (1.14)$$

Using this definition, Di Benedetto and Herrero proved in [7, 8] that any nonnegative solution of (1.12) in $\mathbb{R}^N \times (0, T)$ admits an initial trace μ which is a nonnegative Radon measure, and the proof heavily relies on the parabolic Harnack inequality. When $p > 2$, it is also derived that the measure μ has to satisfy a precise polynomial growth at infinity. Moreover, the same authors study the corresponding initial value problem with an initial data $\mu \in L^1_{\text{loc}}(\mathbb{R}^N)$ and obtain existence and uniqueness results. If Ω is bounded, the initial value problem is considered in [3, 16] when $p > p_1$ and in [14] (see also [15]) in the general case. When μ is a Radon measure, new difficulties appear in the range $1 < p \leq p_0$. The a priori estimates show that u may not be an integrable function, and some questions dealing with the concentration of the measure are unavoidable.

The same kind of difficulties are present in Eq. (1.1). Since it is only assumed that q is positive, three critical values for q appear

$$q = 1, \quad q = p - 1, \quad q = q_c = p - 1 + \frac{p}{N}.$$

The case $q > p - 1$ is the analogous of the superlinearity case in the case $p = 2$. It means that the absorption term is dominant with respect to the diffusion operator. The exponent $q = q_c$ is the mere extension of the critical values q_* and q_m for Eqs. (1.3) and (1.9). When $q > 1$, the absorption term is dominant with respect to the diffusion operator, and (1.1) admits a particular singular solution in $\mathbb{R}^N \times (0, \infty)$, called the *flat solution*, which is the function

$$(x, t) \mapsto W(x, t) = \left(\frac{1}{t(q-1)} \right)^{1/(q-1)}. \quad (1.15)$$

This particular solution plays an important role since it dominates any nonnegative solution of (1.1) which is locally bounded in $(0, \infty) \times \mathbb{R}^N$.

1.1. Existence Results for the Initial Trace

Our main result which states the basis of the definition of the initial trace of nonnegative solution of (1.1) is the following:

• Let u be a nonnegative weak solution of (1.1) in Q_T . Then for any $y \in \Omega$ the following alternative occurs:

(i) either for any open subset $U \subset \Omega$ containing y

$$\lim_{t \rightarrow 0} \int_U u(x, t) dx = \infty,$$

(ii) or there exist an open neighborhood $U \subset \Omega$ of y and a Radon measure ℓ_U on U such that for any $\zeta \in C_c(U)$,

$$\lim_{t \rightarrow 0} \int_U u(x, t) \zeta(x) dx = \ell_{U^*}(\zeta).$$

Owing to this result it follows the definition of the initial trace of u as an outer regular Borel measure $\nu = (\mathcal{S}, \mu) \in \mathcal{B}_+^{\text{reg}}(\Omega)$. Notice that this result also applies to Eq. (1.12) with $\nu = (\emptyset, \mu)$, and our proof does not require the Harnack inequality, which, among others, brings some simplification to the proofs in [7, 8].

The first case in which the singular set \mathcal{S} is empty is the following:

• When $0 < q \leq 1$, and $p < 2$ the initial trace is reduced to a Radon measure.

If $0 < q \leq p - 1$ and $p > 2$, the absorption term is dominated and the initial trace derives from the diffusion operator. However, the situation differs completely according to $0 < q \leq 1$ or $q > 1$, and Ω is bounded or \mathbb{R}^N .

• When $0 < q \leq 1$, $p > 2$, $\Omega = \mathbb{R}^N$, and u is continuous in Q_T , the initial trace is a Radon measure.

The proof of the last result is based upon the notion of concentration of mass on a single point.

• When $0 < q < 1$, $p > 2$, and Ω is bounded, then

(i) either the initial trace of u is a Radon measure in Ω ,
(ii) or

$$\liminf_{t \rightarrow 0} t^{1/(p-2)} u(x, t) \geq \nu_\Omega(x),$$

where $v = v_\Omega$ is the unique nonnegative solution of

$$\begin{cases} -\nabla \cdot (|\nabla v|^{p-2} \nabla v) = \frac{1}{p-2} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Notice that in that case the function $V_\Omega(x, t) = t^{-1/(p-2)} v_\Omega(x)$ is a particular singular solution of

$$\begin{cases} \partial_t u - \nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0 & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } \partial\Omega \times (0, \infty). \end{cases}$$

When $q = p - 1$ the same dichotomy holds with v_Ω replaced by $w = w_\Omega$ unique nonnegative solution of

$$\begin{cases} -\nabla \cdot (|\nabla w|^{p-2} \nabla w) + w^{p-1} = \frac{1}{p-2} w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

When $1 < q \leq p - 1$ the absorption term is still dominated but the existence of the particular flat singular solution W creates a more complicated situation. In the case of \mathbb{R}^N we prove the following.

- When $1 < q \leq p - 1$ and $\Omega = \mathbb{R}^N$, then
 - (i) either the initial trace of u is a Radon measure in \mathbb{R}^N ,
 - (ii) or

$$\liminf_{t \rightarrow 0} t^{1/(q-1)} u(x, t) \geq \left(\frac{1}{(q-1)} \right)^{1/(q-1)}.$$

If $u(\cdot, t)$ is bounded for any $t > 0$, assertion (ii) becomes
(ii) or

$$u(x, t) \equiv W(x, t).$$

When Ω is bounded, the situation is very intricate because of the existence of many different types of singular solutions, and the role of the boundary conditions, if any, may be dominant.

1.2. The Generalized Initial Value Problem

The reverse problem is to construct a solution u of (1.1) with a given initial trace $v = (\mathcal{S}, \mu)$.

We begin with the case where the initial trace is a Radon measure, i.e. $\mathcal{S} = \emptyset$. Similarly to the case $p = 2$, q_c is a critical exponent. If $q \geq q_c$ and $q > 1$, Eq. (1.1) admits no solution with isolated singularities, see [27] when $p > 2$, and [23] when $p < 2$. If $1 < q < q_c$, for any $k > 0$, there exists a unique

solution \tilde{w}_k of problem

$$\begin{cases} \partial_t \tilde{w}_k - \nabla \cdot (|\nabla \tilde{w}_k|^{p-2} \nabla \tilde{w}_k) + \tilde{w}_k^q = 0 & \text{in } \mathbb{R}^N \times (0, \infty), \\ \tilde{w}_k(\cdot, 0) = k\delta_0 & \text{in } \mathbb{R}^N, \end{cases} \quad (1.16)$$

from [32] when $p > 2$ and [23] when $p < 2$. Our existence result is the following:

- Assume $q < q_c$, $p > p_0$, and $p - 1 < q$, or $p < 2$. For any nonnegative Radon measure μ on \mathbb{R}^N , there exists a weak solution u of (1.1) in $\mathbb{R}^N \times (0, \infty)$ with initial trace μ .

Notice that no growth condition is assumed, since $p - 1 < q$, or $p < 2$. In the case where Ω is bounded, we also extend the results of [3] from $p > p_1$ to $p > p_0$.

Next, we come to the general problem of existence of a solution for a given outer regular nonnegative Borel measure $\nu = (\mathcal{S}, \mu)$ as initial trace. Since in the case $q \leq p - 1$, where the diffusion term is dominant, either the solutions are almost explicit and $\mathcal{S} = \mathbb{R}^N$, or the initial trace is a Radon measure and $\mathcal{S} = \emptyset$, we concentrate on the case

$$q > \max(1, p - 1), \quad (1.17)$$

where the absorption is dominant. In the subcritical case $q < q_c$, we construct solutions by approximating ν by a sequence of Radon measures $\{\mu_k\}$. Since $q > 1$, the corresponding sequence of solutions $\{u_k\}$ is dominated by the flat solution W . Hence $\{u_k\}$ remains locally uniformly bounded and the truncation plays no role. Moreover, since $q > p - 1$, no growth condition is needed. Nevertheless, the problem is still complicated by the possible nonuniqueness of the u_k , and we show that we can construct a nondecreasing sequence of such solutions, that we call *constructive*. We get the following.

- When $\max(1, p - 1) < q < q_c$, for any given nonnegative Borel measure ν in \mathbb{R}^N there exists a nonnegative solution u of (1.1) in $\mathbb{R}^N \times (0, \infty)$ with initial trace ν .

In the supercritical case $q \geq q_c$, the situation is much more delicate and we give a sufficient condition on ν to be an initial trace. This condition is the natural extension of the one discovered by Marcus and Véron [33] in the case $p = 2$. In their case this condition is expressed in terms of Bessel capacity.

2. PRELIMINARIES

In this section, we assume that Ω is an any domain in \mathbb{R}^N , $T > 0$, and set

$$Q_T = \Omega \times (0, T).$$

We denote by $\mathcal{M}^+(\Omega)$ the set of nonnegative Radon measures on Ω . In the sequel, for any open set U , we write $U \subset \subset \Omega$ whenever U has a compact closure in Ω .

We consider a more general equation than (1.1), namely

$$\partial_t u - \nabla \cdot (|\nabla u|^{p-2} \nabla u) + a(x)u^q = 0, \quad (2.1)$$

where $a \in L^\infty_{\text{loc}}(\Omega)$.

DEFINITION 2.1. A nonnegative function u is said to be a *weak solution* of (2.1) in Q_T , if

$$u \in L^1_{\text{loc}}(Q_T), \quad au^q \in L^1_{\text{loc}}(Q_T), \quad \nabla u \in L^p_{\text{loc}}(Q_T)$$

and

$$\int_0^T \int_\Omega (-H(u)\partial_t \varphi + |\nabla u|^{p-2} \nabla u \cdot \nabla (h(u)\varphi) + h(u)u^q \varphi) dx dt = 0 \quad (2.2)$$

for any $\varphi \in C^\infty_c(Q_T)$ and any function $h \in C(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ where $H'(r) = h(r)$.

It follows from the definition that for any $\eta \in C_c(\Omega)$,

$$t \mapsto \int_\Omega H(u(x, t))\eta(x) dx,$$

is continuous and there holds

$$\begin{aligned} & \int_t^\theta \int_\Omega (-H(u)\partial_t \varphi + |\nabla u|^{p-2} \nabla u \cdot \nabla (h(u)\varphi) + h(u)au^q \varphi) dx dt \\ &= \int_\Omega H(u(x, t))\varphi(x, t) dx - \int_\Omega H(u(x, \theta))\varphi(x, \theta) dx \end{aligned} \quad (2.3)$$

for any $0 < t < \theta < T$ and $\varphi \in C^\infty_c(\Omega \times [0, T])$.

As a consequence of (2.3) there holds

$$\begin{aligned} & \int_t^\theta \int_\Omega (h(u)|\nabla u|^{p-2} \nabla u \cdot \nabla \zeta + h'(u)|\nabla u|^p \zeta + h(u)au^q \zeta) dx dt \\ &= \int_\Omega H(u(x, t))\zeta(x) dx - \int_\Omega H(u(x, \theta))\zeta(x) dx \end{aligned} \quad (2.4)$$

for any $0 < t < \theta < T$ and $\zeta \in C_c^\infty(\Omega)$.

Remark 2.1. In the definition of the weak solution, we can impose to the function h to be constant outside $[-k, k]$ for some $k > 0$. Therefore the term $|\nabla u|^p$ in identity (2.2) is only considered on the set

$$\{(x, t) \in Q_T : |u(x, t)| \leq k\}.$$

The assumption $\nabla u \in L_{\text{loc}}^p(Q_T)$ can be replaced by

$$|\nabla u|^{p-1} \in L_{\text{loc}}^1(Q_T), \quad \nabla T_k(u) \in L_{\text{loc}}^p(Q_T),$$

where T_k is defined in (1.14), and the gradient of u , denoted by $y = \nabla u$, is defined by

$$\nabla(T_k(u)) = y \times 1_{\{|u| \leq k\}} \quad \text{a.e. in } Q_T.$$

We will again say that u is a *weak solution*. This class corresponds to the classes of entropy or renormalized solutions defined in [1, 44], and is closely related to the ones of [8, 14, 15, 39, 40], see Remark 2.4.

2.1. Integral Estimates

We give below some integral estimates valid for the solutions of Eq. (1.1). The method employed here is formally settled upon multiplying successively the equation by $(1 + u)^\alpha \eta$ with $\alpha < 0$ and $\eta \in C_c^\infty(\Omega)$ and by η . This technique has been used in [8] for proving the Harnack inequality in the case of the mere diffusion equation

$$\partial_t u - \nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0, \quad (2.5)$$

in the case $1 < p < 2$. It has also been employed by Bidaut-Véron and Pohozaev [13] and Mitidieri and Pohozaev [37] for getting estimates and nonexistence results of solutions of the stationary equation

$$-\nabla \cdot (|\nabla u|^{p-2} \nabla u) + u^q = 0. \quad (2.6)$$

PROPOSITION 2.1. *Let $\alpha < 0$, $\alpha \neq -1$, and $0 < t < \theta < T$. Let u be a nonnegative weak solution of (2.1) in Q_T . For any nonnegative function $\zeta \in C_c^\infty(\Omega)$,*

and any $\tau > p$,

$$\begin{aligned}
& \frac{1}{\alpha+1} \int_{\Omega} (1+u(x,t))^{\alpha+1} \zeta^{\tau}(x) dx + \frac{|\alpha|}{2} \int_t^{\theta} \int_{\Omega} (1+u)^{\alpha-1} |\nabla u|^p \zeta^{\tau} dx dt \\
& \leq \frac{1}{\alpha+1} \int_{\Omega} (1+u(x,\theta))^{\alpha+1} \zeta^{\tau}(x) dx + C \int_t^{\theta} \int_{\Omega} (1+u)^{q+\alpha} \zeta^{\tau} dx dt \\
& \quad + C \int_t^{\theta} \int_{\Omega} (1+u)^{\alpha+p-1} \zeta^{\tau-p} |\nabla \zeta|^p dx dt
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
\int_{\Omega} (1+u(x,t)) \zeta^{\tau}(x) dx & \leq \int_{\Omega} (1+u(x,\theta)) \zeta^{\tau}(x) dx + C \int_t^{\theta} \int_{\Omega} (1+u)^q \zeta^{\tau} dx dt \\
& \quad + C \int_t^{\theta} \int_{\Omega} (1+u)^{\alpha-1} |\nabla u|^p \zeta^{\tau} dx dt \\
& \quad + C \int_t^{\theta} \int_{\Omega} |\nabla \zeta|^p (1+u)^{(1-\alpha)(p-1)} \zeta^{\tau-p} dx dt,
\end{aligned} \tag{2.8}$$

where $C = C(\alpha, p, q, \tau, \|a\|_{L^{\infty}(\Omega)})$. Conversely, if $a(x) \geq 0$ a.e. in Ω , then

$$\begin{aligned}
& \frac{1}{4} \int_{\Omega} (1+u(x,\theta)) \zeta^{\tau}(x) dx + \frac{1}{2} \int_t^{\theta} \int_{\Omega} a u^q \zeta^{\tau} dx dt \\
& \leq \int_{\Omega} (1+u(x,t)) \zeta^{\tau}(x) dx + C \int_t^{\theta} \int_{\Omega} (1+u)^{\alpha-1} |\nabla u|^p \zeta^{\tau} dx dt \\
& \quad + C \int_t^{\theta} \int_{\Omega} |\nabla \zeta|^p (1+u)^{(1-\alpha)(p-1)} \zeta^{\tau-p} dx dt,
\end{aligned} \tag{2.9}$$

where $C = C(\alpha, p, q, \tau)$.

Proof. We set for any $\alpha \leq 0$, $\alpha \neq -1$,

$$g_{\alpha}(u) = (1+u)^{\alpha}, \quad G_{\alpha}(u) = \frac{(1+u)^{1+\alpha}}{1+\alpha}$$

and take

$$\eta = \zeta^{\tau} \quad \text{and} \quad H(u) = G_{\alpha}(u) \tag{2.10}$$

for test functions in (2.4). Then

$$\begin{aligned} & \int_t^\theta \int_\Omega (\tau g_\alpha(u) \zeta^{\tau-1} |\nabla u|^{p-2} \nabla u \cdot \nabla \zeta + g'_\alpha(u) \zeta^\tau |\nabla u|^p + g_\alpha(u) \zeta^\tau a u^q) dx dt \\ &= \int_\Omega G_\alpha(u(x, t)) \zeta^\tau(x) dx - \int_\Omega G_\alpha(u(x, \theta)) \zeta^\tau(x) dx, \end{aligned}$$

or, equivalently,

$$\begin{aligned} & \frac{1}{\alpha+1} \int_\Omega (1+u(x, t))^{\alpha+1} \zeta^\tau(x) dx + |\alpha| \int_t^\theta \int_\Omega (1+u)^{\alpha-1} \zeta^\tau |\nabla u|^p dx dt \\ &= \frac{1}{\alpha+1} \int_\Omega (1+u(x, \theta))^{\alpha+1} \zeta^\tau(x) dx + \int_t^\theta \int_\Omega (1+u)^\alpha \zeta^\tau a u^q dx dt \\ &+ \tau \int_t^\theta \int_\Omega (1+u)^\alpha \zeta^{\tau-1} |\nabla u|^{p-2} \nabla u \cdot \nabla \zeta dx dt. \end{aligned} \quad (2.11)$$

Writing

$$|\nabla u|^{p-1} \zeta^{\tau-1} |\nabla \zeta| = |\nabla u|^{p-1} (1+u)^{(\alpha-1)/p'} (1+u)^{(1-\alpha)/p'} \zeta^{\tau-1} |\nabla \zeta|, \quad (2.12)$$

we derive for $\alpha < 0$,

$$\begin{aligned} & \tau \int_t^\theta \int_\Omega (1+u)^\alpha \zeta^{\tau-1} |\nabla u|^{p-2} \nabla u \cdot \nabla \zeta dx dt \\ & \leq \frac{|\alpha|}{2} \int_t^\theta \int_\Omega (1+u)^{\alpha-1} |\nabla u|^p \zeta^\tau dx dt \\ & + C \int_t^\theta \int_\Omega (1+u)^{\alpha+p-1} \zeta^{\tau-p} |\nabla \zeta|^p dx dt. \end{aligned} \quad (2.13)$$

Hence (2.7) holds.

As a particular case of (2.11) (with $\alpha_0 = 0$),

$$\begin{aligned} \int_\Omega (1+u(x, t)) \zeta^\tau(x) dx &= \int_\Omega (1+u(x, \theta)) \zeta^\tau(x) dx + \int_t^\theta \int_\Omega \zeta^\tau a u^q dx dt \\ &+ \tau \int_t^\theta \int_\Omega \zeta^{\tau-1} |\nabla u|^{p-2} \nabla u \cdot \nabla \zeta dx dt. \end{aligned} \quad (2.14)$$

From (2.12) we derive, for any $\alpha < 0$,

$$\begin{aligned} & \int_t^\theta \int_\Omega |\nabla u|^{p-1} \zeta^{\tau-1} |\nabla \zeta| dx dt \\ & \leq \int_t^\theta \int_\Omega |\nabla u|^p (1+u)^{\alpha-1} \zeta^\tau dx dt \\ & \quad + \int_t^\theta \int_\Omega |\nabla \zeta|^p (1+u)^{(1-\alpha)(p-1)} \zeta^{\tau-p} dx dt. \end{aligned} \quad (2.15)$$

Estimate (2.8) follows from (2.14), (2.15) and (2.7).

Next, we set

$$d_\alpha(u) = 1 - g_\alpha(u) = 1 - (1+u)^\alpha \quad \text{and} \quad D_\alpha = (1+u) - G_\alpha.$$

Subtracting (2.11) from (2.14) infers

$$\begin{aligned} & \int_\Omega D_\alpha(u(x, \theta)) \zeta^\tau(x) dx + |\alpha| \int_t^\theta \int_\Omega (1+u)^{\alpha-1} |\nabla u|^p \zeta^\tau dx dt \\ & \quad + \int_t^\theta \int_\Omega a u^q d_\alpha(u) \zeta^\tau dx dt \\ & = \int_\Omega D_\alpha(u(x, t)) \zeta^\tau(x) dx - \tau \int_t^\theta \int_\Omega d_\alpha(u) \zeta^{\tau-1} |\nabla u|^{p-2} \nabla u \cdot \nabla \zeta dx dt. \end{aligned} \quad (2.16)$$

But $d_\alpha(u) > 0$ and $1+u \geq D_\alpha(u)$. Moreover,

$$D_\alpha(u) \geq u/4 \quad \text{and} \quad d_\alpha(u) \geq 1/2 \quad \text{on} \quad \{(x, t) \in Q_T : u(x, t) \geq 2(2^{-1/\alpha} - 1)\}.$$

Henceforth, since a is nonnegative,

$$\begin{aligned} & \frac{1}{4} \int_\Omega u(x, \theta) \zeta^\tau(x) dx + \frac{1}{2} \int_t^\theta \int_\Omega a u^q \zeta^\tau dx dt \\ & \leq C + \int_\Omega u(x, t) \zeta^\tau(x) dx + \tau \int_t^\theta \int_\Omega \zeta^{\tau-1} |\nabla u|^{p-1} |\nabla \zeta| dx dt. \end{aligned} \quad (2.17)$$

Thus (2.9) follows from (2.15) and (2.17). ■

Remark 2.2. In case $\alpha = -1$, the term $\frac{1}{\alpha+1} \int_\Omega (1+u(x, t))^{\alpha+1} \zeta^\tau(x) dx$ at time t in (2.7) has to be replaced by $\int_\Omega \ln(1+u(x, t)) \zeta^\tau(x) dx$, similarly for the term at time θ .

Remark 2.3. If we consider the notion of weak solutions as it is defined in Remark 2.1, we have to replace the test function g_α with $\alpha \leq 0$ by $g_{\alpha,k} = g_\alpha \circ T_k$. Up to this change, relations (2.11), (2.14), (2.15) and estimates

(2.7), (2.8) remain valid with u replaced by $T_k(u)$. The term u^q remaining unchanged.

2.2. Regularity Properties

Here we derive regularity properties of the solutions under assumptions of boundedness for some integrals of u . We assume that Ω is an any domain in \mathbb{R}^N , and recall that $Q_T = \Omega \times (0, T)$.

PROPOSITION 2.2. *Let u be a nonnegative solution of (2.1) in Q_T , with $a \in L_{\text{loc}}^\infty(\Omega)$. Let $0 < \theta < T$. Assume that two of the three following conditions hold, for any open set $U \subset \subset \Omega$:*

$$\sup_{t \in (0, \theta]} \int_U u(x, t) \, dx < \infty, \quad (2.18)$$

$$\int_0^\theta \int_U (|a|u^q + u^{p-1}) \, dx \, dt < \infty, \quad (2.19)$$

$$\int_0^\theta \int_U |\nabla u|^{p-1} \, dx \, dt < \infty. \quad (2.20)$$

Then the third one holds for any $U \subset \subset \Omega$. Moreover,

$$\int_0^\theta \int_U u^\sigma \, dx \, dt < \infty, \quad \forall \sigma \in (0, q_c) \quad (2.21)$$

and

$$\int_0^\theta \int_U |\nabla u|^r \, dx \, dt < \infty, \quad \forall r \in \left(0, \frac{N}{N+1}q_c\right) = \left(0, p - \frac{N}{N+1}\right). \quad (2.22)$$

Finally, there exists a Radon measure $\ell \in \mathcal{M}^+(\Omega)$ such that for any $\xi \in C_c^\infty(\Omega)$,

$$\lim_{t \rightarrow 0} \int_\Omega u(x, t) \xi(x) \, dx = \ell(\xi) \quad (2.23)$$

and u satisfies

$$\begin{aligned} & \int_0^\theta \int_\Omega (-u \partial_t \varphi + |\nabla u|^{p-2} \nabla u \cdot \nabla u \varphi) + a u^q \varphi \, dx \, dt \\ &= \int_\Omega \varphi(x, 0) \, d\ell(x) - \int_\Omega u(x, \theta) \varphi(x, \theta) \, dx, \end{aligned} \quad (2.24)$$

for any $0 < \theta < T$ and $\varphi \in C_c^\infty(\Omega \times [0, T])$.

Proof. (i) Assume that (2.18) and (2.20) hold. Let ζ and τ be as in Proposition 2.1. From (2.14) it follows that $au^q \in L^1((0, \theta), L^1_{\text{loc}}(\Omega))$. Now let us prove that $u^{p-1} \in L^1((0, \theta), L^1_{\text{loc}}(\Omega))$. Let U be any open subset such that $U \subset \subset \Omega$. If $p \leq 2$, taking $\zeta \equiv 1$ in U , we observe that

$$\int_t^\theta \int_U (1+u)^{p-1} dx dt \leq \int_t^\theta \int_U (1+u)\zeta^\tau dx dt \leq C.$$

If $p > 2$, we use Poincaré's inequality,

$$\int_U |u(x, t) - \bar{u}(t)|^{p-1} dx \leq C \int_U |\nabla u(x, t)|^{p-1} dx,$$

where, for any $v \in L^1(U)$, we have set $\bar{v} = |U|^{-1} \int_U v dx$. Hence

$$\int_t^\theta \int_U |u(x, t)|^{p-1} dx \leq C \int_t^\theta \int_U |\nabla u(x, t)|^{p-1} dx + C$$

follows from (2.18), and (2.19) holds.

(ii) Assume (2.19) and (2.20). Then (2.18) holds: indeed, we can choose $\theta > 0$ such that $\int_\Omega u(x, \theta)\zeta^\tau(x) dx$ is finite, since $u \in L^1_{\text{loc}}(Q_T)$.

(iii) Assume that (2.18) and (2.19) hold. Let $\alpha \in (\max(1-p, -1), 0)$ be fixed. From (2.7) we get for any $0 < t < \theta$,

$$\begin{aligned} & \frac{|\alpha|}{2} \int_t^\theta \int_\Omega (1+u)^{\alpha-1} |\nabla u|^p \zeta^\tau dx dt \\ & \leq \frac{1}{\alpha+1} \int_\Omega (1+u(x, \theta))^{\alpha+1} \zeta^\tau(x) dx + C \int_t^\theta \int_\Omega (1+u)^{p-1+\alpha} \zeta^\tau dx dt \\ & + C \int_t^\theta \int_\Omega (1+u)^{q+\alpha} \zeta^\tau dx dt. \end{aligned} \tag{2.25}$$

Since $(1+u)^{q+\alpha} \leq (1+u)^q$, and $(1+u)^{p-1+\alpha} \leq (1+u)^{p-1}$, we find

$$\int_t^\theta \int_\Omega (1+u)^{\alpha-1} |\nabla u|^p \zeta^\tau dx dt \leq C$$

(and in fact for any $\alpha < 0$), hence

$$\int_0^\theta \int_U (1+u)^{\alpha-1} |\nabla u|^p dx dt < \infty. \tag{2.26}$$

We recall the Gagliardo–Nirenberg estimate: for any $p, m \geq 1$, $\gamma \in [1, +\infty)$ and $s \in [0, 1]$ such that

$$\frac{1}{\gamma} = s \left(\frac{1}{p} - \frac{1}{N} \right) + \frac{1-s}{m}, \quad (2.27)$$

there exists $c = c(N, p, m, s, \Omega) > 0$ such that for any $v \in W^{1,p}(U) \cap L^m(U)$,

$$\|v - \bar{v}\|_{L^\gamma(U)} \leq c \|\nabla v\|_{L^p(U)}^s \|v - \bar{v}\|_{L^m(U)}^{1-s}. \quad (2.28)$$

We apply it to v where

$$v(x, t) = (1 + u(x, t))^\beta, \quad \beta = (\alpha + p - 1)/p. \quad (2.29)$$

Choosing

$$\gamma = p + \frac{p}{N\beta}, \quad s = \frac{p}{\gamma}, \quad m = \frac{1}{\beta}, \quad (2.30)$$

for which (2.27) holds, then

$$\|\bar{v}(\cdot)\|_{L^\infty((0,\theta))} \leq C \quad (2.31)$$

is derived from (2.18) and the Hölder inequality, because $\beta \in (0, 1)$. Therefore, for almost all $t \in (0, \theta)$,

$$\begin{aligned} \int_U |(1 + u(x, t))^\beta - \bar{v}(t)|^\gamma dx &\leq C \left(\int_U (1 + u(x, t))^{\alpha-1} |\nabla u(x, t)|^p dx \right)^{s\gamma/p} \\ &\quad \times \left(\int_U |(1 + u(x, t))^\beta - \bar{v}(t)|^{1/\beta} dx \right)^{(1-s)\gamma\beta} \\ &\leq C \left(\int_U (1 + u(x, t))^{\alpha-1} |\nabla u(x, t)|^p dx \right) \\ &\quad \times \left(\int_U |(1 + u(x, t))^\beta - \bar{v}(t)|^{1/\beta} dx \right)^{(1-s)\gamma\beta} \end{aligned}$$

and

$$\int_U |(1 + u(x, t))^\beta - \bar{v}(t)|^{1/\beta} dx \leq C, \quad \int_U \bar{v}(t)^\gamma dx = |U| \overline{v_n}(t)^\gamma \leq C,$$

from (2.18) and (2.31). Therefore

$$\int_U (1 + u(x, t))^{\beta\gamma} dx \leq C \int_U (1 + u(x, t))^{\alpha-1} |\nabla u(x, t)|^p dx + C.$$

Integrating on $(0, \theta)$ we get

$$\int_0^\theta \int_U (1 + u(t))^{\beta\gamma} dx \leq C \int_0^\theta \int_U (1 + u)^{\alpha-1} |\nabla u|^p dx dt + C\theta \leq C,$$

from (2.26). If we set $\sigma = \beta\gamma$, this means

$$\int_0^\theta \int_U (1 + u(t))^\sigma dx \leq C,$$

for any σ such that $p/N + \max(p-2, 0) < \sigma < p-1 + p/N = q_c$, and for any $0 < \sigma < q_c$ by the Hölder inequality. This proves (2.21).

Next for any $0 < r < p$, and any $\alpha < 0$, we find

$$\begin{aligned} \int_0^\theta \int_U |\nabla u|^r dx &\leq \left(\int_0^\theta \int_U (1 + u)^{\alpha-1} |\nabla u|^p dx dt \right)^{r/p} \\ &\quad \times \left(\int_0^\theta \int_U (1 + u)^{(1-\alpha)r/(p-r)} dx \right)^{(p-r)/p}. \end{aligned} \quad (2.32)$$

Hence

$$\int_0^\theta \int_U |\nabla u|^r dx \leq C,$$

holds if r is such that $0 < r < Nq_c/(N+1)$. This proves (2.22), which implies (2.20) in particular, since $p-1 < p-N/(N+1)$. Now from (2.4) with $h = 1$, for any $\xi \in C_c^\infty(\Omega)$ and any $0 < t < \theta < T$,

$$\begin{aligned} \int_\Omega u(x, t) \xi(x) dx &= \int_\Omega u(x, \theta) \xi(x) dx \\ &\quad + \int_t^\theta \int_\Omega (|\nabla u|^{p-2} \nabla u \cdot \nabla \xi + u^q \xi) dx dt. \end{aligned} \quad (2.33)$$

Because the right-hand side of (2.11) has a finite limit when $t \rightarrow 0$, the same holds with $t \mapsto \int_\Omega u(x, t) \xi(x) dx$. The mapping $\xi \mapsto \lim_{t \rightarrow 0} \int_\Omega u(x, t) \xi(x) dx$ is clearly a positive linear functional over the space $C_c^\infty(\Omega)$. It can be extended in a unique way as a Radon measure ℓ on Ω , and (2.23) holds in Ω .

Finally, let $0 < t < \theta$ be fixed. Taking $h = 1$ in (2.3) with $\varphi \in C_c^\infty(\Omega \times [0, T))$, infers

$$\begin{aligned} & \int_t^\theta \int_\Omega (-u \partial_t \varphi + |\nabla u|^{p-2} \nabla u \cdot \nabla u \varphi + u^q \varphi) dx dt \\ &= \int_\Omega u(x, t) \varphi(x, t) dx - \int_\Omega u(x, \theta) \varphi(x, \theta) dx. \end{aligned}$$

Letting t go to 0 in the left-hand side of the above equality and using (2.18)–(2.20) yields to

$$\left| \int_\Omega u(x, t) (\varphi(x, t) - \varphi(x, 0)) dx \right| \leq Ct \int_\Omega u(x, t) dx.$$

Therefore,

$$\int_\Omega u(x, t) \varphi(x, t) dx \rightarrow \int_\Omega \varphi(x, 0) d\ell(x),$$

from (2.23). This proves (2.24), and the proof is complete. ■

Remark 2.4. Estimates (2.21) and (2.22) are still valid for the weak solutions defined in Remark 2.1, by considering $T_k(u)$ and going to the limit as $k \rightarrow \infty$, and using the definition of the gradient. Next, we consider the class Σ of solutions introduced in [8] with $a \equiv 0$ and $p < 2$, defined by the conditions

$$u \in C((0, T), L_{\text{loc}}^1(\Omega)), \quad \nabla T_k(u) \in L_{\text{loc}}^p(Q_T), \quad \partial_t(T_k(u)) \in L_{\text{loc}}^1(Q_T)$$

and satisfying

$$\int_0^T \int_\Omega ((\varphi - u)_+ \partial_t u + |\nabla u|^{p-2} \nabla u \cdot \nabla ((\varphi - u)_+)) dx dt = 0 \quad (2.34)$$

for any $\varphi \in C_c^\infty(Q_T)$. By a straightforward argument, it is shown in [8, Lemma 3.2] that any $u \in \Sigma$ satisfies $(1 + u)^{\alpha-1} |\nabla u|^p \in L_{\text{loc}}^1(Q_T)$ for any $\alpha \in (1 - p, 0)$. Then, by the Gagliardo–Nirenberg estimate applied on (t, θ) instead of $(0, \theta)$, we deduce that $u^\sigma \in L_{\text{loc}}^1(Q_T)$ for any $\sigma \in (0, q_c)$, and $|\nabla u|^r \in L_{\text{loc}}^1(Q_T)$ for any $r \in (0, Nq_c/(N + 1))$. In particular $|\nabla u|^{p-1} \in L_{\text{loc}}^1(Q_T)$. By [8, Proposition 2.1],

$$\int_0^T \int_\Omega (\varphi \partial_t(H(u)) + |\nabla u|^{p-2} \nabla u \cdot \nabla (h(u)\varphi)) dx dt = 0$$

for any $\varphi \in C_c^\infty(Q_T)$ and any function $h \in C_c^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ where $H'(r) = h(r)$. Consequently,

$$\int_0^T \int_\Omega (-H(u) \partial_t \varphi + |\nabla u|^{p-2} \nabla u \cdot \nabla (h(u) \varphi)) \, dx \, dt = 0. \quad (2.35)$$

In order to prove uniqueness of solutions with a given initial data $\mu \in L_{\text{loc}}^1(\mathbb{R}^N)$, they introduced in [8, Proposition 2.1] a class $\Sigma^* \subset \Sigma$ of functions which satisfies

$$\lim_{k \rightarrow \infty} \left(\int_0^T \int_{K \cap \{k < u < k+1\}} |\nabla u|^p \, dx \, dt \right) = 0,$$

for any compact $K \subset Q_T$. Since it can be proved that (2.35) remains true for any $h \in C(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ constant outside $[-k, k]$ for some $k > 0$, any $u \in \Sigma^*$ is a weak solution.

Next, we consider the class of solutions introduced in [15] for initial data $\mu \in L^1(\Omega)$ for bounded Ω . They satisfy $u \in C([0, T], L^1(\Omega))$ and (2.35) for any $\varphi \in C_c^\infty(Q_T)$ and any function $h \in C_c^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ where $H'(r) = h(r)$, with

$$\lim_{k \rightarrow \infty} \left(\int_0^T \int_{\{k < u < k+1\}} |\nabla u|^p \, dx \, dt \right) = 0.$$

Because these above integrals are bounded, for any $\alpha < 0$, there holds

$$\begin{aligned} & \int_0^\theta \int_U (1+u)^{\alpha-1} |\nabla u|^p \, dx \, dt \\ & \leq \sum_{k=0}^\infty (1+k)^{\alpha-1} \int_0^\theta \int_{\{k < u < k+1\}} |\nabla u|^p \, dx \, dt \leq C \sum_{k=0}^\infty (1+k)^{\alpha-1}. \end{aligned}$$

By (2.32) in the proof of Proposition 2.2, it follows that (2.20) is valid, hence also (2.19), and u is a weak solution of the problem.

3. EXISTENCE OF THE INITIAL TRACE

The main result of this section which settled the basis of the definition of the initial trace of a nonnegative solution of (1.1) is the following.

THEOREM 3.1. *Let u be a nonnegative weak solution of (1.1) in Q_T . Then for any $y \in \Omega$ the following alternative holds:*

(i) either for any open subset $U \subset \Omega$ containing y

$$\lim_{t \rightarrow 0} \int_U u(x, t) dx = \infty, \quad (3.1)$$

(ii) or there exist an open neighborhood $U^* \subset \Omega$ of y and a nonnegative Radon measure $\ell_{U^*} \in \mathcal{M}^+(U^*)$ such that for any $\xi \in C_c(U^*)$,

$$\lim_{t \rightarrow 0} \int_{U^*} u(x, t) \xi(x) dx = \ell_{U^*}(\xi), \quad (3.2)$$

and (2.21) and (2.22) hold in any open set $U \subset \subset U^*$.

It will be proved below and precised according to the different values of q with respect to $p - 1$. With Theorem 3.1 we can define a set \mathcal{R} (depending on u) by

$$\mathcal{R} = \left\{ y \in \Omega : \exists U \text{ open } \subset \Omega, \quad y \in U, \quad \limsup_{t \rightarrow 0} \int_U u(x, t) dx < \infty \right\}. \quad (3.3)$$

Then \mathcal{R} is an open subset of Ω and there exists a unique Radon measure $\mu \in \mathcal{M}^+(\mathcal{R})$ such that

$$\lim_{t \rightarrow 0} \int_{\mathcal{R}} u(x, t) \xi(x) dx = \int_{\mathcal{R}} \xi(x) d\mu(x), \quad \forall \xi \in C_c(\mathcal{R}). \quad (3.4)$$

By Proposition 2.2, u satisfies

$$\begin{aligned} & \int_0^\theta \int_{\mathcal{R}} (-u \partial_t \varphi + |\nabla u|^{p-2} \nabla u \cdot \nabla u \varphi) + u^q \varphi dx dt \\ &= \int_{\mathcal{R}} \varphi(x, 0) d\mu(x) - \int_{\mathcal{R}} u(x, \theta) \varphi(x, \theta) dx, \end{aligned} \quad (3.5)$$

for any $0 < \theta < T$ and $\varphi \in C_c^\infty(\mathcal{R} \times [0, T))$, and (2.21) and (2.22) hold in any open set $U \subset \subset \mathcal{R}$.

The next definition is parallel to the one introduced by Marcus and Véron [33–35].

DEFINITION 3.1. Let u be a nonnegative weak solution of (1.1) in Q_T . A point $y \in \Omega$ is called a *regular point* if $y \in \mathcal{R}$. Otherwise it is called a *singular point*. The set of singular points is denoted $\mathcal{S} = \Omega \setminus \mathcal{R}$; it is a relatively closed subset of Ω . We shall denote

$$tr_\Omega(u) = (\mathcal{S}, \mu),$$

where μ is the Radon measure in (3.4) and call it the *initial trace* of u at $t = 0$.

The initial trace can also be represented in a unique way by a positive, outer regular Borel measure ν with the following values on any Borel subset A :

$$\nu(A) = \begin{cases} \infty & \text{if } A \cap \mathcal{S} \neq \emptyset, \\ \mu(A) & \text{if } A \subset \mathcal{R}. \end{cases} \quad (3.6)$$

It is known that there is a one-to-one correspondence between the sets

$$\mathcal{B}_{\text{reg}}^+(\Omega) = \{\nu \text{ outer regular Borel measure on } \Omega, \nu \geq 0\}$$

and

$$CM^+(\Omega) = \{(\mathcal{S}, \mu) : \mathcal{S} \text{ relatively closed in } \Omega, \mu \in \mathcal{M}^+(\Omega \setminus \mathcal{S})\}.$$

When considering the initial trace of u as an element of $\mathcal{B}_{\text{reg}}^+(\Omega)$, we shall denote it by $Tr_{\Omega}(u) = \nu$. If $\mathcal{S} = \emptyset$, we will say that u admits for initial trace the Radon measure μ .

3.1. The Case $q > p - 1 > 0$

In this case the proof of Theorem 3.1 is based upon the following lemma.

LEMMA 3.2. *Let $q > p - 1 > 0$. Under the assumptions of Theorem 3.1, let $\tau > pq/(q - p - 1)$. For any nonnegative function $\zeta \in C_c^\infty(\Omega)$ the following dichotomy occurs:*

(i) *either $\int_0^T \int_{\Omega} u^q \zeta^\tau dx dt < \infty$, then*

$$t \mapsto \int_{\Omega} u(x, t) \zeta^\tau(x) dx$$

remains bounded near $t = 0$.

(ii) *or $\int_0^T \int_{\Omega} u^q \zeta^\tau dx dt = \infty$, then*

$$\lim_{t \rightarrow 0} \int_{\Omega} u(x, t) \zeta^\tau(x) dx = \infty.$$

Proof. **Step 1.** Let $\alpha \in (\max((1 - p), -\frac{1}{2}), 0)$ and $0 < \theta < T$. We start from (2.7). Since $d = (q + \alpha)/(\alpha + p - 1) > 1$ and $\tau > pd' = pd/(d - 1)$, it follows

by Hölder's inequality

$$\begin{aligned}
& \int_t^\theta \int_\Omega (1+u)^{\alpha+p-1} \zeta^{\tau-p} |\nabla \zeta|^p dx dt \\
& \leq \int_t^\theta \int_\Omega ((1+u)^{q+\alpha} \zeta^\tau + \zeta^{\tau-pd'} |\nabla \zeta|^{pd'}) dx dt \\
& \leq C + \int_t^\theta \int_\Omega (1+u)^{q+\alpha} \zeta^\tau dx dt,
\end{aligned} \tag{3.7}$$

where $C = C(\alpha, p, q, \tau, \zeta)$. Therefore from (2.7), since $\alpha > -1$,

$$\begin{aligned}
& \frac{|\alpha|}{2} \int_t^\theta \int_\Omega (1+u)^{\alpha-1} |\nabla u|^p \zeta^\tau dx dt \\
& \leq \frac{1}{\alpha+1} \int_\Omega (1+u(x, \theta))^{\alpha+1} \zeta^\tau(x) dx + C + C \int_t^\theta \int_\Omega (1+u)^{q+\alpha} \zeta^\tau dx dt
\end{aligned} \tag{3.8}$$

with another $C = C(\alpha, p, q, \tau, \zeta)$.

Step 2. Next we use (2.15). Since $q > p-1$, we can choose α such that $\delta = q/(1-\alpha)(p-1) > 1$, and $|\alpha|$ small enough in order to have $\tau > p\delta' \geq pd'$ (remember that from assumption $\tau > pq/(q-p+1)$, and notice that $\delta' \geq d'$ is equivalent to $pq+1 > p-\alpha p+\alpha$ since $\alpha < 0$). Therefore, by Hölder's inequality,

$$\begin{aligned}
& \int_t^\theta \int_\Omega |\nabla \zeta|^p (1+u)^{(1-\alpha)(p-1)} \zeta^{\tau-p} dx dt \\
& \leq \int_t^\theta \int_\Omega (1+u)^q \zeta^\tau dx dt + \int_t^\theta \int_\Omega |\nabla \zeta|^{p\delta'} \zeta^{\tau-p\delta'} dx dt.
\end{aligned} \tag{3.9}$$

Combining this estimate with (2.8) and (3.7), (3.8) yields

$$\int_\Omega u(x, t) \zeta^\tau(x) dx \leq C + \int_\Omega u(x, \theta) \zeta^\tau(x) dx + C \int_t^\theta \int_\Omega u^q \zeta^\tau dx dt, \tag{3.10}$$

where $C = C(\tau, p, q, \alpha, \zeta)$.

Step 3. We start from (3.8) and the converse estimate (2.9). Using (3.8) and Hölder's inequality as in (2.15), (3.9), for any $\varepsilon > 0$,

$$\begin{aligned}
& \tau \int_t^\theta \int_\Omega \zeta^{\tau-1} |\nabla u|^{p-1} |\nabla \zeta| dx dt \\
& \leq \varepsilon \frac{|\alpha|}{2} \int_t^\theta \int_\Omega (1+u)^{\alpha-1} |\nabla u|^p \zeta^\tau dx dt + \varepsilon \int_t^\theta \int_\Omega u^q \zeta^\tau dx dt + C_\varepsilon, \\
& \leq 2\varepsilon \int_\Omega (u(x, \theta)) \zeta^\tau(x) dx + \varepsilon(1+C) \int_t^\theta \int_\Omega u^q \zeta^\tau dx dt + C_\varepsilon,
\end{aligned}$$

where $C_\varepsilon = C_\varepsilon(\tau, p, q, \alpha, \zeta, \varepsilon)$. If we choose ε such that $\frac{1}{4} - 2\varepsilon > \frac{1}{8}$ and $\frac{1}{2} - \varepsilon(1 + C) > \frac{1}{4}$, and combine this inequality with (2.9), we obtain

$$\int_{\Omega} u(x, \theta) \zeta^\tau(x) dx + \int_t^\theta \int_{\Omega} u^q \zeta^\tau dx dt \leq 8 \int_{\Omega} u(x, t) \zeta^\tau(x) dx + C, \quad (3.11)$$

where $C = C(\tau, p, q, \alpha, \zeta)$.

Step 4. Assume first that $\int_0^T \int_{\Omega} u^q \zeta^\tau dx dt < \infty$, and let $0 < \theta < T$ such that $\int_{\Omega} u(x, \theta) \zeta^\tau dx < \infty$. Then

$$\int_{\Omega} u(x, t) \zeta^\tau(x) dx < \infty, \quad (3.12)$$

for almost all $0 < t < \theta$, from (3.10). Since θ can be taken arbitrarily close to T , (2.11) holds a.e. on $(0, T)$. Suppose now that $\int_0^T \int_{\Omega} u^q \zeta^\tau dx dt = \infty$, and that there exists a sequence $\{t_n\}$ converging to 0 such that $\int_{\Omega} u(x, t_n) \zeta^\tau \times (x) dx \leq M$ for some constant M . Taking $t = t_n$ and $T = \theta$ in (2.11) yields a contradiction. ■

Remark 3.1. The result is still valid for the weaker solutions defined in Remark 2.2. Indeed, estimates (3.8) and (3.9) remain valid with u replaced by $T_k(u)$, the term u^q remaining unchanged. Therefore (3.10) holds under the form

$$\begin{aligned} & \int_{\Omega} T_k(u(x, t)) \zeta^\tau(x) dx \\ & \leq C + C \int_{\Omega} T_k(u(x, \theta)) \zeta^\tau(x) dx + C \int_t^\theta \int_{\Omega} u^q \zeta^\tau dx dt. \end{aligned} \quad (3.13)$$

In estimate (2.16) u^q can be replaced by $T_k(u)^q$ which is smaller. Therefore (2.17) holds with u replaced by $T_k(u)$. Since (3.9) holds with $(1 + u)^q$ replaced by $(1 + T_k(u))^q$, (2.11) is valid under the form

$$\begin{aligned} & \frac{1}{8} \int_{\Omega} T_k(u(x, \theta)) \zeta^\tau(x) dx + \frac{1}{4} \int_t^\theta \int_{\Omega} (T_k(u))^q \zeta^\tau dx dt \\ & \leq \int_{\Omega} T_k(u(x, t)) \zeta^\tau(x) dx + C. \end{aligned} \quad (3.14)$$

Letting $k \rightarrow \infty$ in (3.13) and (2.11) implies that (3.10) and (2.11) are still valid and the conclusion of Lemma 3.2 follows.

Remark 3.2. The results make use essentially of the term u^q in Eq. (1.1). They extend to Eq. (2.1) under the condition $a(x) \geq C > 0$ a.e. in Ω .

Proof of Theorem 3.1. Case $q > p - 1 > 0$. We first assume that for any open subset U of Ω containing y and any nonnegative $\zeta \in C_c^\infty(U)$ with value 1 in a neighborhood of y , and $\tau > pq/(q - p - 1)$,

$$\int_0^T \int_U u^q \zeta^\tau dx dt = \infty.$$

Then (3.1) holds from Lemma 3.2.

Assume now that there exists an open neighborhood $\tilde{U} \subset \Omega$ of y and a $C_c^\infty(\tilde{U})$ -function ζ with value in $[0, 1]$ and value 1 in a neighborhood $U_\zeta = U^*$ of y such that

$$\int_0^T \int_{\tilde{U}} u^q \zeta^\tau dx dt < \infty.$$

Then

$$t \mapsto \int_{U^*} u(x, t) dx$$

remains bounded in a neighborhood of $t = 0$ from Lemma 3.2. Moreover, we have

$$\int_0^T \int_{U^*} |\nabla u|^{p-1} dx dt < \infty. \quad (3.15)$$

Indeed from (2.12),

$$\begin{aligned} \int_t^\theta \int_\Omega |\nabla u|^{p-1} \zeta^\tau dx dt &\leq \int_t^\theta \int_\Omega |\nabla u|^p (1+u)^{\alpha-1} \zeta^\tau dx dt \\ &\quad + \int_t^\theta \int_\Omega (1+u)^{(1-\alpha)(p-1)} \zeta^\tau dx dt \end{aligned} \quad (3.16)$$

and, without using Proposition 2.2 but the assumption $q > p - 1$,

$$\int_t^\theta \int_\Omega (1+u)^{(1-\alpha)(p-1)} \zeta^\tau dx dt \leq \int_t^\theta \int_\Omega (1+u)^q \zeta^\tau dx dt$$

by choosing α such that $(1-\alpha)(p-1) \leq q$. Then (3.15) follows from (3.8). Thus we can apply Proposition 2.2 in U^* , and deduce (3.2) and the regularity results. ■

3.2. The Case $q \leq 1$ and $p < 2$

In this case we prove that any solution of (1.1) admits an initial trace reduced to a Radon measure. Moreover, we can treat the case of (2.1).

THEOREM 3.3. *Let $0 < q \leq 1$ and $p < 2$. Let u be a nonnegative weak solution of (2.1) in Q_T , with $a \in L^\infty_{\text{loc}}(\Omega)$. Then there exists a Radon measure $\mu \in \mathcal{M}^+(\Omega)$ such that*

$$\lim_{t \rightarrow 0} \int_{\Omega} u(x, t) \zeta(x) dx = \int_{\Omega} \zeta(x) d\mu(x), \quad \forall \zeta \in C_c(\Omega).$$

Then (2.24) holds, (2.21) and estimates (2.22) are satisfied in any open set $U \subset \subset \Omega$.

Proof. We claim that for any open subset U with compact closure $\bar{U} \subset \Omega$,

$$t \mapsto \int_U u(x, t) dx$$

remains bounded near $t = 0$. Let α, t, θ, ζ and τ be as in Proposition 2.1 with $\alpha > -1$ and $\tau > p/(2 - p)$. We have

$$\begin{aligned} & \int_t^\theta \int_{\Omega} (1 + u)^{\alpha + p - 1} \zeta^{\tau - p} |\nabla \zeta|^p dx dt \\ & \leq \int_t^\theta \int_{\Omega} (1 + u)^{\alpha + 1} \zeta^\tau dx dt \\ & \quad + \int_t^\theta \int_{\Omega} \zeta^{\tau - (1 + \alpha)p/(2 - p)} |\nabla \zeta|^{(1 + \alpha)p/(2 - p)} dx dt, \end{aligned} \quad (3.17)$$

since $p < 2$, and

$$\int_t^\theta \int_{\Omega} (1 + u)^{\alpha + q} \zeta^\tau dx dt \leq \int_t^\theta \int_{\Omega} (1 + u)^{\alpha + 1} \zeta^\tau dx dt,$$

since $q \leq 1$. Owing to (2.7) it follows that

$$\begin{aligned} & \int_{\Omega} (1 + u(x, t))^{\alpha + 1} \zeta^\tau(x) dx \\ & \leq \int_{\Omega} (1 + u(x, \theta))^{\alpha + 1} \zeta^\tau(x) dx + C \int_t^\theta \int_{\Omega} \Omega (1 + u)^{1 + \alpha} \zeta^\tau dx dt + C', \end{aligned} \quad (3.18)$$

where C, C' depend on $\alpha, \zeta, \|a\|_{L^\infty(\Omega)}$ and the exponents. If we set

$$X(t) = \int_t^\theta \int_{\Omega} (1 + u)^{1 + \alpha} \zeta^\tau dx dt,$$

then with another $C > 0$,

$$X' + CX + C + \int_{\Omega} u(x, \theta) \zeta^{\tau}(x) dx \geq 0. \quad (3.19)$$

Integrating (3.19) between t and θ implies that $X(t)$ is bounded on $(0, \theta)$. Now we use (2.8). Since $p < 2$ we can choose α such that $(1 - \alpha)(p - 1) \leq 1$ and we derive

$$\begin{aligned} \int_{\Omega} (1 + u(x, t)) \zeta^{\tau}(x) dx &\leq \int_{\Omega} (1 + u(x, \theta)) \zeta^{\tau}(x) dx \\ &+ C \int_t^{\theta} \int_{\Omega} (1 + u) \zeta^{\tau} dx dt. \end{aligned} \quad (3.20)$$

Putting

$$Y(t) = \int_t^{\theta} \int_{\Omega} (1 + u) \zeta^{\tau} dx dt$$

and integrating differential inequality (2.11) yields to

$$\int_{\Omega} (1 + u(x, t)) \zeta^{\tau}(x) dx \leq (e^{C(\theta-t)} + 1) \int_{\Omega} (1 + u(x, \theta)) \zeta^{\tau}(x) dx,$$

for $0 < t \leq \theta$, which implies the claim. Then (2.18) holds. It implies (2.19), since $q \leq 1$ and $p < 2$, hence we can apply Proposition 2.2 in Ω . ■

Remark 3.3. Theorem 3.3 shows, in particular, that in the case $p - 1 < q \leq 1$, the first alternative of Theorem 3.1 cannot happen, which means $\mathcal{S} = \emptyset$ and $\mathcal{R} = \Omega$.

3.3. The Case $q \leq p - 1$, $p > 2$

In this range of exponents we can deal with the more general equation (2.1). The proof of Theorem 3.1 is a consequence of the following lemma.

LEMMA 3.4. *Assume $0 < q \leq p - 1$ and $p > 2$. Let u be any nonnegative weak solution of (2.1) in Q_T , with $a \in L_{\text{loc}}^{\infty}(\Omega)$. Assume that for any open set $U \subset \subset \Omega$,*

$$t \mapsto \int_U u(x, t) dx$$

remains bounded near $t = 0$. Then for any $0 < \theta < T$,

$$\int_0^{\theta} \int_U u^{p-1}(x, t) dx + \int_0^{\theta} \int_U |\nabla u(x, t)|^{p-1} dx < \infty.$$

Proof. Let $\alpha \in (1 - p, 0)$ be fixed, $\alpha \neq -1$, and $\zeta \in C_c^\infty(\Omega)$ as above. We start from (2.25) where possibly $\alpha + 1 \leq 0$. Therefore, by (2.18), since $q \leq p - 1$,

$$\frac{|\alpha|}{2} \int_t^\theta \int_\Omega (1+u)^{\alpha-1} |\nabla u|^p \zeta^\tau dx dt \leq C + C \int_t^\theta \int_\Omega (1+u)^{p-1+\alpha} \zeta^\tau dx dt. \quad (3.21)$$

If U, U^* are any open sets with $U \subset \subset U^* \subset \subset \Omega$, taking $\zeta \in C_c^\infty(\Omega)$ with $0 \leq \zeta \leq 1$, with value 1 on U , and 0 outside of U^* , we get

$$\begin{aligned} \int_t^\theta \int_U (1+u)^{\alpha+p-1+p/N} dx dt &\leq C + C \int_t^\theta \int_U (1+u)^{\alpha-1} |\nabla u|^p dx dt \\ &\leq C + C \int_t^\theta \int_{U^*} (1+u)^{\alpha+p-1} dx dt. \end{aligned} \quad (3.22)$$

Hence any estimate of $(1+u)^{\alpha+p-1}$ in $L^1((0, \theta), L_{\text{loc}}^1(\Omega))$ implies the same estimate for $(1+u)^{\alpha+p-1+p/N}$. We first take $\alpha_0 = 2 - p$. Since $p + \alpha_0 - 1 = 1$,

$$u^{\sigma_1} \in L^1((0, \theta), L_{\text{loc}}^1(\Omega)),$$

with

$$\sigma_1 = \alpha_0 + p - 1 + p/N = \sigma_1 = 1 + p/N.$$

Defining by induction,

$$\alpha_{n+1} = \alpha_n + p/N, \quad \sigma_n = \alpha_n + p - 1, \quad \forall n \in \mathbb{N},$$

we get

$$(1+u)^{\sigma_{n+1}} \in L^1((0, \theta) \times L_{\text{loc}}^1(\Omega))$$

as long as $\alpha_n = np/N + 2 - p < 0$. Let n_0 be the largest integer such that $\alpha_n < 0$. Then $(1+u)^{\sigma_{n_0+1}} \in L^1((0, \theta) \times L_{\text{loc}}^1(\Omega))$. And $\sigma_{n_0+1} \geq p - 1$. Then in (3.21) one can now take $\alpha < 0$ arbitrarily close to 0 and deduce that

$$\frac{|\alpha|}{2} \int_t^\theta \int_\Omega (1+u)^{\alpha-1} |\nabla u|^p \zeta^\tau dx dt \leq C.$$

Then from (3.22) we get $(1+u)^{\alpha+p-1+p/N} \in L^1((0, \theta) \times L_{\text{loc}}^1(\Omega))$. In particular $u^{p-1} \in L^1(0, \theta), L_{\text{loc}}^1(\Omega)$. From (2.32), we deduce that $|\nabla u| \in L^r((0, \theta) \times L_{\text{loc}}^1(\Omega))$ for any r such that

$$(1 - \alpha)r/(p - r) \leq \alpha + p - 1 + p/N.$$

Since α is arbitrary, this happens for any $r < q_c$, in particular $|\nabla u| \in L^{p-1}((0, \theta) \times L^1_{\text{loc}}(\Omega))$. ■

Proof of Theorem 3.1. Case $q \leq p - 1$, $p > 2$ Let $y \in \Omega$. Then either statement (i) of Theorem 3.1 holds, or there exists an open subset $U^* \subset \Omega$ containing y such that $\int_{U^*} u(x, t) dx$ is bounded near $t = 0$. By Lemma 3.4 and Proposition 2.2 in U^* statement (ii) holds. ■

4. COMPLEMENTARY PROPERTIES FOR $p > 2$

4.1. The Case $q \leq 1 < p - 1$, $\Omega = \mathbb{R}^N$

In this section, we prove that the absorption term is negligible, therefore the initial trace of a solution is reduced to a Radon measure, equivalently $\mathcal{S} = \emptyset$.

THEOREM 4.1. Assume $q \leq 1 < p - 1$ and $u \in C(\mathbb{R}^N \times (0, T))$ is a non-negative weak solution of (1.1) in $\mathbb{R}^N \times (0, T)$. Then the initial trace of u is a Radon measure $\mu \in \mathcal{M}^+(\mathbb{R}^N)$.

Proof. We show that for any $b \in \mathbb{R}^N$ there exists $\rho > 0$ such that

$$\limsup_{t \rightarrow 0} \int_{B_\rho(b)} u(x, t) dx < \infty. \quad (4.1)$$

By contradiction we assume that it does not hold. Then there exists some $b \in \mathbb{R}^N$ such that for any $\rho > 0$ there exists a sequence $\{t_{n,\rho}\}$ converging to 0 with the property that

$$\lim_{t_{n,\rho} \rightarrow 0} \int_{B_\rho(b)} u(x, t_{n,\rho}) dx = \infty. \quad (4.2)$$

Let $k > 0$ be fixed. For any $\rho > 0$ there exists n_ρ such that for any $n \geq n_\rho$,

$$\int_{B_\rho(b)} u(x, t_{n,\rho}) dx \geq k. \quad (4.3)$$

By continuity of the integral with respect to the domain, there exists some $0 < \tilde{\rho} \leq \rho$ such that

$$\int_{B_{\tilde{\rho}}(b)} u(x, t_{n,\rho}) dx = k. \quad (4.4)$$

Moreover, $\tilde{\rho}$ is uniquely determined if we impose it to be the largest as possible (in order to avoid the axiom of the choice in what follows). When $\rho \rightarrow 0$, $t_{n,\rho} \rightarrow 0$ since $t \mapsto u(\cdot, t)$ is continuous from $(0, T)$ into $L^1_{\text{loc}}(\mathbb{R}^N)$. Let

$w = w_\rho$ be the solution of

$$\begin{cases} \partial_t w - \nabla \cdot (|\nabla w|^{p-2} \nabla w) + w^q = 0 & \text{in } \mathbb{R}^N \times (0, \infty), \\ w(\cdot, 0) = u(\cdot, t_{n_\rho, \rho}) \chi_{B_{\tilde{\rho}}(b)}, & \text{in } \mathbb{R}^N. \end{cases} \quad (4.5)$$

Since u is nonnegative it follows by the comparison principle that

$$u(x, t + t_{n_\rho, \rho}) \geq w_\rho(x, t) \quad \text{in } \mathbb{R}^N \times (0, T - t_{n_\rho, \rho}).$$

When $\rho \rightarrow 0$, w_ρ converges [29–32] to \tilde{w}_k solution of

$$\begin{cases} \partial_t \tilde{w}_k - \nabla \cdot (|\nabla \tilde{w}_k|^{p-2} \nabla \tilde{w}_k) + \tilde{w}_k^q = 0 & \text{in } \mathbb{R}^N \times (0, \infty), \\ \tilde{w}_k(\cdot, 0) = k \delta_b & \text{in } \mathbb{R}^N. \end{cases} \quad (4.6)$$

By construction and uniqueness, the function \tilde{w}_k is radial with respect to b , and radially decreasing for $t > 0$. Moreover,

$$u(x, t) \geq \tilde{w}_k(x, t) \quad \text{in } \mathbb{R}^N \times (0, T).$$

By the comparison principle, the mapping $k \mapsto \tilde{w}_k$ is increasing. The key point is to notice that, since $0 < q \leq 1$, $\lim_{k \rightarrow \infty} \tilde{w}_k = w_\infty$ a.e. cannot exist.

When $0 < q < 1$ there exists a scaling invariance of the equation, namely, if w is a solution of (1.1), $N_\ell(w)$ defined by

$$N_\ell(w)(x, t) = \ell^{1/(q-1)} w(b + \ell^\gamma(x - b), \ell t), \quad (4.7)$$

with $\gamma = (q + 1 - p)/(p(q - 1))$ and $\ell > 0$, is also a solution, and consequently

$$N_\ell(w_k) = w_k \ell^{1/(q-1)-\gamma N}.$$

Letting $k \rightarrow \infty$ leads to the invariance property

$$N_\ell(w_\infty) = w_\infty, \quad \forall \ell > 0,$$

from which follows the self-similar form of w_∞ ,

$$w_\infty(x, t) = t^{1/(1-q)} f(t^{-\gamma}(x - b)), \quad \forall (x, t) \in \mathbb{R}^N \times (0, \infty). \quad (4.8)$$

This estimate implies, in particular, that $f(0)$ is finite and

$$\tilde{w}_k(b, t) \leq t^{1/(1-q)} f(0) \leq u(b, t), \quad \forall t \in (0, T),$$

which contradicts the fact that $\tilde{w}_k(b, t) \rightarrow \infty$ when $t \rightarrow 0$, since $q < 1$.

When $q = 1$ (and $p \neq 2$ otherwise the result is well known), Eq. (1.1) is invariant with respect to the transformation $M_\ell(w)$ defined (for $\ell > 0$) by

$$M_\ell(w)(x, t) = \ell w(b + \ell^{(2-p)/p}(x - b), t),$$

which yields to

$$M_\ell(w_k) = w_k \ell^{1+N(p-2)/p}.$$

As in the previous case w_∞ satisfies the invariant property

$$M_\ell(w_\infty) = w_\infty, \quad \forall \ell > 0.$$

This estimate implies

$$0 < w_k(b, T/2) \leq w_\infty(b, T/2) = \ell w_\infty(b, T/2) \leq u(b, T/2), \quad \forall \ell > 0,$$

which is again a contradiction. If U is any bounded open of \mathbb{R}^N it follows by compactness that

$$t \mapsto \int_U u(x, t) dx$$

remains bounded near $t = 0$, and the singular set is empty. ■

Remark 4.1. Actually this proof works for any $p > p_0$. But the case $p < 2$ is still covered by Theorem 3.3.

Remark 4.2. The argument used below implies that if u is any continuous, nonnegative solution of (1.12) in $\mathbb{R}^N \times (0, T)$, then for any relatively compact subset $U \subset \mathbb{R}^N$, $\int_U u(x, t) dx$ remains bounded when t remains bounded.

4.2. The Case $1 < q \leq p - 1$, $\Omega = \mathbb{R}^N$

Since $q > 1$, the main difference with the previous section comes from the existence of the flat singular solution W defined in (1.15).

THEOREM 4.2. *Let $1 < q \leq p - 1$. Assume $u \in C(\mathbb{R}^N \times (0, T))$ is a non-negative weak solution of (1.1) in $\mathbb{R}^N \times (0, T)$. Then*

- (i) *either $\mathcal{S} = \mathbb{R}^N$ and in fact $u \geq W$,*
- (ii) *$\mathcal{S} = \emptyset$ and the initial trace of u is a Radon measure $\mu \in \mathcal{M}^+(\mathbb{R}^N)$.*

Proof. The scheme of the proof is very similar to the one of Theorem 4.1. Either for any $b \in \mathbb{R}^N$ there exists $\rho > 0$ such that (4.1) holds, hence (2.18) holds. And Lemma 3.4 applies as above, since $q < p - 1$ and $p > 2$, thus (2.20) holds, and statement (ii) follows by Proposition 2.2. Or there exists some $b \in \mathbb{R}^N$ such that for any $\rho > 0$ there exists a sequence $\{t_{n,\rho}\}$ converging to 0 satisfying (4.2). Let $k > 0$ be fixed. For any ρ there exists $t_{n\rho,\rho} > 0$ such that (4.3) holds, and there exists some $0 < \tilde{\rho} \leq \rho$ such that (4.4) holds. Letting $\rho \rightarrow 0$, implies that u is bounded from below by the solution

of (4.6), which does exist since $1 < q < q_c$, and the convergence result quoted in Theorem 4.1 applies. Since this estimate from below holds for any $k > 0$, it follows that

$$u \geq \tilde{w}_\infty = \lim_{k \rightarrow \infty} \tilde{w}_k.$$

Such a \tilde{w}_∞ exists and is a solution of (1.1) by the classical theory. Moreover,

$$\tilde{w}_k \leq \tilde{w}_\infty \leq W. \quad (4.9)$$

Using the same scaling operator N_ℓ defined in (4.7) infers that \tilde{w}_∞ is a self-similar solution of (1.1) which means

$$\tilde{w}_\infty(x, t) = t^{-1/(q-1)} f(t^{-\gamma}(x - b)), \quad \forall (x, t) \in \mathbb{R}^N \times (0, \infty) \quad (4.10)$$

with $\gamma = (q + 1 - p)/(p(q - 1))$. The function f is nonnegative and radially symmetric. Because of (4.9) f is smaller than $(1/(q - 1))^{1/(q-1)}$. Moreover, it satisfies the following equation:

$$\begin{cases} r^{1-N}(r^{N-1}|f'|^{p-2}f')' + \gamma r f' + \frac{1}{q-1}f - f^q = 0 & \text{in } (0, \infty), \\ f'(0) = 0. \end{cases} \quad (4.11)$$

Clearly $f \equiv (1/(q - 1))^{1/(q-1)}$ is a solution and any solution satisfies in particular $f(0) \leq (1/(q - 1))^{1/(q-1)}$. It remains to show that in any case, f is constant. Suppose it were not true. Since $\gamma \leq 0$ f is a nonincreasing neighborhood of $r = 0$, and it remains decreasing on $(0, \infty)$ (by contradiction). If $\gamma < 0$, then, writing

$$(r^{N-1}|f'|^{p-2}f')' \leq -\gamma r^N f',$$

a straightforward but lengthy computation implies that

$$f'(r) \leq -|\gamma|^{1/(p-2)} C_{N,p} r^{2/(p-2)},$$

for some $C_{N,p} > 0$, contradicting the nonnegativity of f . If $\gamma = 0$ (or equivalently $q = p - 1$), f admits a nonnegative limit l at ∞ . If this limit is not zero, there holds

$$(r^{N-1}|f'|^{p-2}f')' \leq r^{N-1} \left(f^{p-1} - \frac{1}{p-2}f \right) \leq m r^{N-1},$$

for some $m < 0$ and $r \geq 1$. This inequality implies

$$|f'|^{p-2}f'(r) \leq m \frac{r}{N} + C_1,$$

for $r > 1$ and some real C_1 , contradicting the nonnegativity of f . If $l = 0$ there exists $R > 0$ such that

$$-\nabla \cdot (|\nabla f|^{p-2} \nabla f) \geq \frac{1}{2(p-2)} f \quad \text{in } \{y \in \mathbb{R}^N : |y| > R\}.$$

Such an inequality admits no nonnegative, nontrivial solution [12, 28]. Actually, this is a particular case of a more general result stating that any nonnegative solution of

$$-\nabla \cdot (|\nabla f|^{p-2} \nabla f) \geq f^s$$

in an exterior domain of \mathbb{R}^N is identically zero whenever $0 < s \leq N \times (p-1)/(N-p)$ (no condition on s if $p \geq N$). Thus f is constant, hence $u \geq W$. In particular, for any bounded open set $U \subset \mathbb{R}^N$, $\lim_{t \rightarrow 0} \int_U u(x, t) dx = \infty$, which means that $\mathcal{S} = \mathbb{R}^N$. ■

Remark 4.3. If we assume that the solution u of (1.1) is such that $u(\cdot, t)$ is bounded in \mathbb{R}^N for any $t > 0$, then $u \leq W$, thus the statement of Theorem 4.2 becomes

- (i) either $u \equiv W$,
- (ii) or the initial trace of u is a measure $\mu \in \mathcal{M}^+(\mathbb{R}^N)$.

4.3. The Case $q \leq 1 < p - 1$, Ω Bounded

Although the absorption term is dominated by the diffusion one, the fact that Ω is bounded, infers that the singular set of the initial trace of a solution may be nonempty, and, in such a case, it is whole Ω .

THEOREM 4.3. *Assume $q \leq 1 < p - 1$, Ω is bounded with a smooth boundary and u is a continuous and nonnegative solution of (1.1) in $(0, T) \times \Omega$. Then*

- (i) either $\mathcal{S} = \Omega$ and, more precisely,

$$\liminf_{t \rightarrow 0} t^{1/(p-2)} u(x, t) \geq v_\Omega(x), \quad \forall x \in \Omega,$$

- (ii) or $\mathcal{S} = \emptyset$ and the initial trace of u is a Radon measure $\mu \in \mathcal{M}^+(\Omega)$.

For the proof, we start from the following classical result.

LEMMA 4.4. *Assume $p > 2$ and Ω is bounded, then there exists a unique nonzero function $v = v_\Omega \in W_0^{1,p}(\Omega) \cap C^1(\bar{\Omega})$ satisfying*

$$\begin{cases} -\nabla \cdot (|\nabla v|^{p-2} \nabla v) = \frac{1}{p-2} v & \text{in } \Omega, \\ v \geq 0 & \text{in } \Omega. \end{cases} \quad (4.12)$$

Moreover, $v > 0$ in Ω .

The existence of a nonnegative nontrivial solution is obtained by minimization. The positivity follows by the Hopf boundary lemma as in [41] and the regularity from [24]. This function v_Ω plays a fundamental role in describing the set of solutions of (1.12) in $\Omega \times (0, \infty)$ since the function

$$(x, t) \mapsto V_\Omega(x, t) = t^{-1/(p-2)} v_\Omega(x) \quad (4.13)$$

is a solution with full blow-up at $t = 0$ and vanishes on $\partial\Omega \times (0, \infty)$.

LEMMA 4.5. *Assume $0 < q \leq p - 1$, $p > 2$ and Ω is bounded. For $k > 0$ and $b \in \Omega$ let $\tilde{w} = \tilde{w}_{k,b}$ be the solution of*

$$\begin{cases} \partial_t \tilde{w} - \nabla \cdot (|\nabla \tilde{w}|^{p-2} \nabla \tilde{w}) + \tilde{w}^q = 0 & \text{in } \Omega \times (0, \infty), \\ \tilde{w} = 0 & \text{in } \partial\Omega \times (0, \infty), \\ \tilde{w}(\cdot, 0) = k\delta_b(\cdot) & \text{in } \Omega. \end{cases} \quad (4.14)$$

Then $\lim_{k \rightarrow \infty} \tilde{w}_{k,b} = \tilde{w}_{\infty,b}$ exists; $\tilde{w}_{\infty,b}$ is a solution of (1.1) dominated by V_Ω , and

$$\lim_{t \rightarrow 0} t^{1/(p-2)} \tilde{w}_{\infty,b}(x, t) = v_\Omega(x)$$

uniformly on Ω .

Proof. The function \tilde{w} is a limit of solutions of (1.1) with smooth initial data and zero lateral boundary data. Therefore $\tilde{w}_{k,b} \leq V_\Omega$ in $\Omega \times (0, \infty)$. The sequence $\{\tilde{w}_{k,b}\}$ is increasing and its limit $\tilde{w}_{\infty,b}$ is a solution of (1.1) in $\Omega \times (0, \infty)$. Moreover $\tilde{w}_{\infty,b}$ is dominated by V_Ω . For $\ell > 0$ we set

$$S_\ell(\tilde{w})(x, t) = \ell^{1/(p-2)} \tilde{w}(x, \ell t).$$

Then $S_\ell(\tilde{w}) = \tilde{w}^\ell$ satisfies

$$\begin{cases} \partial_t \tilde{w}^\ell - \nabla \cdot (|\nabla \tilde{w}^\ell|^{p-2} \nabla \tilde{w}^\ell) + \ell^{(p-1-q)/(p-2)} (\tilde{w}^\ell)^q = 0 & \text{in } \Omega \times (0, \infty), \\ \tilde{w}^\ell = 0 & \text{in } \partial\Omega \times (0, \infty), \\ \tilde{w}^\ell(\cdot, 0) = \ell^{1/(p-2)} k \delta_b(\cdot) & \text{in } \Omega. \end{cases} \quad (4.15)$$

Since

$$\partial_t \tilde{w}^\ell - \nabla \cdot (|\nabla \tilde{w}^\ell|^{p-2} \nabla \tilde{w}^\ell) + (\tilde{w}^\ell)^q = (1 - \ell^{(p-1-q)/(p-2)}) (\tilde{w}^\ell)^q,$$

$S_\ell(\tilde{w})$ is a subsolution for $\ell \geq 1$, and therefore

$$S_\ell(\tilde{w}_{k,b}) \leq \tilde{w}_{\ell^{1/(p-2)}k,b}.$$

Letting $k \rightarrow \infty$ yields

$$S_\ell(\tilde{w}_{\infty,b}) \leq \tilde{w}_{\ell^{1/(p-2)}\infty,b}.$$

Taking $t = \tau/\ell$ (with $\tau > 0$ arbitrary) and replacing $1/\ell$ by $t/\tau \in (0, 1]$ yields to

$$\tilde{w}_{\infty,b}(x, t) \geq \left(\frac{t}{\tau}\right)^{-1/(p-2)} \tilde{w}_{\infty,b}(x, \tau) \quad \text{in } \Omega \times (0, \tau].$$

This implies that $t \mapsto t^{1/(p-2)} \tilde{w}_{\infty,b}(x, t)$ is nonincreasing. Since $\tilde{w}_{\infty,b}(b, \tau) > 0$ for τ small enough, there holds

$$\Phi(x) \leq t^{1/(p-2)} \tilde{w}_{\infty,b}(x, t) \leq v_\Omega(x),$$

for some function $x \mapsto \Phi(x)$ which is positive at $x = b$ (if $0 < q < 1$ all the functions $\tilde{w}_{k,a}$ vanishes for t large enough). Put

$$\psi(x, s) = t^{1/(p-2)} \tilde{w}_{\infty,b}(x, t) \quad \text{where } s = \ln t,$$

then

$$\begin{cases} \partial_s \psi - \frac{1}{p-2} \psi - \nabla \cdot (|\nabla \psi|^{p-2} \nabla \psi) + e^{s(p-1-q)/(p-2)} \psi^q = 0 & \text{in } \Omega \times \mathbb{R}, \\ \psi = 0 & \text{on } \partial\Omega \times \mathbb{R}. \end{cases} \quad (4.16)$$

Moreover, $s \mapsto \psi(x, s)$ is nonincreasing and $\Phi(x) \leq \psi(x, s) \leq v_\Omega(x)$ for $s \leq \ln \tau$. Therefore $\lim_{s \rightarrow -\infty} \psi(x, s) = \omega(x)$ exists for any $x \in \bar{\Omega}$. Since $\psi(\cdot, s)$ is bounded in $L^\infty(\Omega)$ uniformly with respect to s , it is bounded in $C^1(\bar{\Omega})$, which infers $\lim_{s \rightarrow -\infty} \psi(\cdot, s) = \omega(\cdot)$ in $C^1(\bar{\Omega})$. Multiplying (4.16) by

$\xi \in W_0^{1,p}(\Omega)$ and integrating on $\Omega \times (n-1, n)$ ($n < 0$) yields to

$$\begin{aligned} & \int_{\Omega} (\psi(n, x) - \psi(n-1, x)) \xi(x) dx \\ & - \int_{n-1}^n \int_{\Omega} \left(\frac{1}{p-2} \psi \xi + |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla \xi \right) dx ds \\ & = \int_{n-1}^n \int_{\Omega} e^{s(p-1-q)/(p-2)} \psi^q \xi dx ds. \end{aligned}$$

Since $\lim_{s \rightarrow -\infty} \psi(\cdot, s)$ is independent of s , we obtain

$$\int_{\Omega} \left(\frac{1}{p-2} \omega \xi + |\nabla \omega|^{p-2} \nabla \omega \cdot \nabla \xi \right) dx ds = 0.$$

Since $\omega \geq \Phi$, it infers $\omega = v_{\Omega}$ by Lemma 4.4, which ends the proof. ■

Proof of Theorem 4.3. As in the proof of Theorem 4.2, either (ii) holds, or there exists $b \in \Omega$ such that $u \geq \tilde{w}_{k,b}$ in Q_T for any $k > 0$. In that case, Lemma 4.4 implies that (i) holds. ■

Remark 4.4. The above technique also applies to Eq. (1.12). Actually, Lemma 4.5 is valid under the following form: Assume $p > 2$, Ω is bounded, and let $\tilde{w} = \tilde{w}_{k,b}$ be the solution of

$$\begin{cases} \partial_t \tilde{w} - \nabla \cdot (|\nabla \tilde{w}|^{p-2} \nabla \tilde{w}) = 0 & \text{in } \Omega \times (0, \infty), \\ \tilde{w} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ \tilde{w}(\cdot, 0) = k\delta_b(\cdot) & \text{in } \Omega. \end{cases} \quad (4.17)$$

Then $\lim_{k \rightarrow \infty} \tilde{w}_{k,b} = V_{\Omega}$ (see the proof of Lemma 4.5). Actually, the proof is simpler since Eq. (1.12) is invariant under the similarity transformation S_{ℓ} . As a consequence the following result holds.

THEOREM 4.6. Assume $2 < p$, Ω is bounded with a smooth boundary and u is a continuous and nonnegative solution of (1.12) in $(0, T) \times \Omega$. Then

- (i) either $u(x, t) \geq V_{\Omega}(x, t) = t^{-1/(p-2)} v_{\Omega}(x, t)$,
- (ii) or the initial trace of u is a Radon measure $\mu \in \mathcal{M}^+(\Omega)$.

4.4. The Case $1 < q \leq p-1$, Ω Bounded

In this range of exponents the function W defined by (1.15) is a singular solution of (1.1). Therefore, if $1 < q < p-1$ we have to compare u with the two functions V_{Ω} and W . In the case $q = p-1$, V_{Ω} has to be replaced by W_{Ω}

defined by

$$W_\Omega(x, t) = t^{-1/(p-2)} w_\Omega(x), \quad (4.18)$$

where w_Ω is a solution of

$$\begin{cases} -\nabla \cdot (|\nabla w|^{p-2} \nabla w) + w^{p-1} = \frac{1}{p-2} w & \text{in } \Omega, \\ w \geq 0 & \text{on } \partial\Omega. \end{cases} \quad (4.19)$$

Notice that $t^{-1/(p-2)} = t^{-1/(q-1)}$ and that the constant solution $(1/(p-2))^{1/(p-2)}$ is a particular solution of (4.19).

THEOREM 4.7. *Assume $q = p - 1 > 2$, Ω is bounded with a smooth boundary and u is a continuous and nonnegative solution of (1.1) in $(0, T) \times \Omega$. Then*

- (i) *either $u(x, t) \geq W_\Omega(x, t) = t^{-1/(p-2)} w_\Omega(x, t)$,*
- (ii) *or the initial trace of u is a Radon measure $\mu \in \mathcal{M}^+(\Omega)$.*

The proof is similar to the one of Theorem 3.1, by using the next lemmas.

LEMMA 4.8. *Assume $p > 2$ and Ω is bounded, then there exists a unique nonzero function $w = w_\Omega \in W_0^{1,p}(\Omega) \cap C^1(\bar{\Omega})$ satisfying (4.19).*

The proof is similar to the one of Lemma 4.4.

LEMMA 4.9. *Assume $p > 2$ and Ω is bounded, and let $\tilde{w} = \tilde{w}_{k,a}$ (for $k > 0$ and some $a \in \Omega$) be the solution of*

$$\begin{cases} \partial_t \tilde{w} - \nabla \cdot (|\nabla \tilde{w}|^{p-2} \nabla \tilde{w}) + \tilde{w}^{p-1} = 0 & \text{in } \Omega \times (0, \infty), \\ \tilde{w} = 0 & \text{in } \partial\Omega \times (0, \infty), \\ \tilde{w}(\cdot, 0) = k\delta_a(\cdot) & \text{in } \Omega. \end{cases} \quad (4.20)$$

Then $\lim_{k \rightarrow \infty} \tilde{w}_{k,a} = W_\Omega$ uniformly on Ω .

The proof is simpler than the one of Lemma 4.5 since Eq. (1.1) and domain Ω are invariant under the similarity transformation S_ℓ defined by $S_\ell(\tilde{w})(x, t) = \ell^{1/(p-2)} \tilde{w}(x, \ell t)$. This implies

$$S_\ell(\tilde{w}_{k,a}) = \tilde{w}_{\ell^{1/(p-2)}k,a}, \quad \forall k, \ell > 0.$$

Therefore $\lim_{k \rightarrow \infty} \tilde{w}_{k,a} = \tilde{w}_{\infty,a}$ is self-similar since $S_\ell(\tilde{w}_{\infty,a}) = \tilde{w}_{\infty,a}$. Then

$$\tilde{w}_{\infty,a}(x; t) = t^{-1/(p-2)} \tilde{w}_{\infty,a}(x, 1).$$

Because $w_{\infty,a}(x, 1)$ is not identically zero and satisfies (4.19), $w_{\infty,a}(x, 1) = w_\Omega$.

The case $1 < q < p - 1$ appears more difficult to deal with, and up to now we do not have a full answer (notice that here $t^{-1/(p-2)} < t^{-1/(q-1)}$ near $t = 0$).

THEOREM 4.10. *Assume $1 < q < p - 1$, Ω is bounded with a smooth boundary and u is a continuous and nonnegative solution of (1.1) in $(0, T) \times \Omega$. Then*

(i) *either*

$$\liminf_{t \rightarrow 0} t^{1/(p-2)} u(x, t) \geq v_\Omega(x), \quad \forall x \in \Omega,$$

(ii) *or the initial trace of u is a Radon measure $\mu \in \mathcal{M}^+(\Omega)$. Moreover, if $\limsup_{t \rightarrow 0} t^{1/(p-2)} u_{\partial\Omega}(x, t) = c$, for some $c > 0$, then*

$$\limsup_{t \rightarrow 0} t^{1/(p-2)} u(x, t) \leq v_\Omega(x) + c. \quad (4.21)$$

Finally, if $\liminf_{t \rightarrow 0} t^{1/(q-1)} u_{\partial\Omega}(x, t) = \rho > 0$ and $\lim_{t \rightarrow 0} u(x, t) = \infty$, uniformly in Ω , then

$$\liminf_{t \rightarrow 0} t^{1/(q-1)} u(x, t) \geq \min(\rho, (1/(q-1))^{1/(q-1)}), \quad (4.22)$$

uniformly on Ω .

The next result is an extension of Lemma 4.4.

LEMMA 4.11. *Assume $p > 2$ and Ω is bounded. Then for any $m \geq 0$ there exists a unique nonzero function $v = v_{\Omega,a} \in W_0^{1,p}(\Omega) \cap C^1(\bar{\Omega})$ satisfying*

$$\begin{cases} -\nabla \cdot (|\nabla v|^{p-2} \nabla v) = \frac{1}{p-2} (v + m) & \text{in } \Omega, \\ v \geq 0 & \text{on } \partial\Omega. \end{cases} \quad (4.23)$$

Proof. Uniqueness comes again from the argument of [24]. For existence we consider the functional J on $W_0^{1,p}(\Omega)$ defined by

$$J(\psi) = \int_{\Omega} \left(\frac{1}{p} |\nabla \psi|^p - \frac{1}{2(p-2)} (\psi^+)^2 - \frac{m}{p-2} \psi^+ \right) dx.$$

Clearly,

$$J(\psi) \rightarrow \infty \quad \text{as } \|\psi\|_{W_0^{1,p}} \rightarrow \infty,$$

and since for $t > 0$,

$$J(t\psi) = \frac{t^p}{p} \int_{\Omega} |\nabla \psi|^p dx - \frac{t^2}{2(p-2)} \int_{\Omega} (\psi^+)^2 dx - \frac{mt}{p-2} \int_{\Omega} \psi^+ dx,$$

there exists $t_1 > 0$ such that $J(t_1\psi) > 0$. Therefore, the infimum of J in $W_0^{1,p}(\Omega)$ is negative and achieved at some v for which there holds

$$-\nabla \cdot (|\nabla v|^{p-2} \nabla v) = \frac{1}{p-2} (v^+ + m\chi_{u>0}) \quad \text{in } \Omega.$$

From the maximum principle $u \geq 0$ and consequently (4.11) holds. ■

Remark 4.5. An equivalent formulation states that for any $d \geq 0$ there exists a unique \tilde{v} in $W^{1,p}(\Omega)$ solution of

$$\begin{cases} -\nabla \cdot (|\nabla \tilde{v}|^{p-2} \nabla \tilde{v}) = \frac{1}{p-2} \tilde{v} & \text{in } \Omega, \\ \tilde{v} = d & \text{on } \partial\Omega, \\ \tilde{v} \geq d & \text{in } \Omega. \end{cases} \quad (4.24)$$

Actually $\tilde{v} = v_{\Omega} + m$ with $m = d(p-2)$.

Proof of Theorem 4.10. The dichotomy between cases (i) and (ii) is clear from the proof of the previous theorems, therefore let us assume

$$\limsup_{t \rightarrow 0} t^{1/(p-2)} u_{\partial\Omega}(x, t) = m.$$

For any $\varepsilon > 0$ there exists $t_{\varepsilon} > 0$ such that

$$u_{\partial\Omega}(x, t) \leq (m + \varepsilon) t^{-1/(p-2)} \quad (4.25)$$

on $\Omega \times (0, t_{\varepsilon}]$. Let Ω' be a relatively compact smooth open domain containing $\bar{\Omega}$ and denote by $v_{\Omega'}$ the solution of problem (4.12) relative to Ω' . From uniqueness and positivity $v_{\Omega'} > v_{\Omega}$ on $\bar{\Omega}$. Consequently, for any $0 < \delta < t_{\varepsilon}$, there holds

$$u(x, t) \leq (v_{\Omega'}(x) + m + \varepsilon)(t - \delta)^{-1/(p-2)}$$

on $(\delta, t_{\varepsilon}]$. We can take Ω' to be such that $\text{dist}(\Omega, \partial\Omega') < 1/n$ and let $n \rightarrow \infty$. Then the sequence $\{v_{\Omega'}\} = \{v_{\Omega',n}\}$ decreases and converges to v_{Ω} . Letting

successively $\delta \rightarrow 0$, $n \rightarrow \infty$, $t \rightarrow 0$ and $\varepsilon \rightarrow 0$ yields to

$$\limsup_{t \rightarrow 0} t^{1/(p-2)} u(x, t) \leq v_{\Omega}(x) + m. \quad (4.26)$$

For the last assertion we just have to notice that for $\sigma > 0$, and $0 < \tau < \min(\rho, (q-1)^{-1/(q-1)})$, the function $t \mapsto \min(\rho - \tau, (q-1)^{-1/(q-1)} - \tau)(t + \sigma)^{-1/(q-1)}$ is a subsolution of (1.1) which is smaller than u at $t = 0$. Therefore there exists $t_{\tau} > 0$, independent of σ such that

$$u(x, t) \geq \min(\rho - \tau, (q-1)^{-1/(q-1)} - \tau)(t + \sigma)^{-1/(q-1)}$$

on $\Omega \times (t_{\tau}]$. Letting $\sigma \rightarrow 0$ yields to

$$\liminf_{t \rightarrow 0} t^{1/(q-1)} u(x, t) \geq \min(\rho - \tau, (1/(q-1))^{1/(q-1)} - \tau),$$

which implies the claim by letting $\rho \rightarrow 0$. ■

Remark 4.6. If the function u has zero boundary data, then the conclusion of Theorem 4.10 becomes more precise:

(i) either

$$\liminf_{t \rightarrow 0} t^{1/(p-2)} u(x, t) = v_{\Omega}(x), \quad \forall x \in \Omega,$$

(ii) or the initial trace of u is a Radon measure $\mu \in \mathcal{M}^+(\Omega)$.

This conclusion also holds for Theorems 4.3, 4.6, and 4.7.

5. SOLUTION WITH A GIVEN INITIAL TRACE

In this section, we consider, the problem of the existence of a solution of (1.1) in $Q_T = \Omega \times (0, T)$ with a given initial trace

$$\begin{cases} \partial_t u - \nabla \cdot (|\nabla u|^{p-2} \nabla u) + u^q = 0 & \text{in } Q_T, \\ Tr_{\Omega}(u) = v \in \mathcal{B}_{\text{reg}}^+(\Omega) \approx (\mathcal{S}, \mu), \end{cases} \quad (5.1)$$

for a given couple $(\mathcal{S}, \mu) \in CM^+(\Omega)$. Let $\mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}$.

DEFINITION 5.1. A nonnegative function u is a solution of (5.1) if u is a weak solution of (1.1), and

- (i) for any open subset U such that $U \cap \mathcal{S} \neq \emptyset$,

$$\lim_{t \rightarrow 0} \int_U u(x, t) dx = \infty,$$

- (ii) for any $0 < \theta < T$, $u \in L^\infty((0, \theta); L^1_{\text{loc}}(\mathcal{R}))$, $|\nabla u| \in L^{p-1}_{\text{loc}}(\mathcal{R} \times [0, T))$, and u satisfies (3.4) and (3.5).

There are many possibilities according to $p - 1$ is smaller or larger than q , and Ω is bounded or equal to \mathbb{R}^N . Even in the case where the initial data is a Radon measure μ , i.e. $\mathcal{S} = \emptyset$, some difficulties occur. The first one may come from the concentration of the measure whenever $1 < p \leq p_0$. A detailed analysis of this problem will be made in a forthcoming paper. When $\Omega = \mathbb{R}^N$ and $p > 2$, another difficulty comes from the growth of the measure at infinity as it was pointed out in [7, 8] for Eq. (1.12). In view of all the difficulties mentioned above, we will concentrate essentially on the case $\Omega = \mathbb{R}^N$, and consider the general initial value problem associated to (1.1) in

$$\begin{cases} \partial_t u - \nabla \cdot (|\nabla u|^{p-2} \nabla u) + u^q = 0 & \text{in } Q_\infty, \\ \text{Tr}_{\mathbb{R}^N}(u) = v \in \mathcal{B}^+_{\text{reg}}(\mathbb{R}^N) \approx (\mathcal{S}, \mu), \end{cases} \quad (5.2)$$

for a given couple $(\mathcal{S}, \mu) \in CM^+(\mathbb{R}^N)$. We recall that $\mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}$.

If $q \leq 1$, if the problem has a solution, then $\mathcal{S} = \emptyset$ and the initial trace is reduced to a Radon measure, from Theorems 3.3 and 4.1, at least if u is continuous. If $1 < q \leq p - 1$, then $\mathcal{S} = \mathbb{R}^N$ or $\mathcal{S} = \emptyset$.

Thus the most interesting case occurs when

$$q > \max(1, p - 1), \quad (5.3)$$

where we can solve the problem for any Borel measure v . In that case, we have a pointwise estimate, which plays an important part: any nonnegative weak solution $u \in C(Q_\infty)$ of (1.1) in Q_∞ satisfies

$$u(x, t) \leq W(t) = \left(\frac{1}{t(q-1)} \right)^{1/(q-1)}, \quad \forall (x, t) \in Q_\infty. \quad (5.4)$$

In all the cases we have to solve first the initial value problem with a Radon measure as initial data. In the sequel, we shall distinguish according to the cases $q < q_c$, with (5.3) called *the subcritical case*, and $q \geq q_c$ with $q > 1$ that we call *the supercritical case*. Notice that if $q \geq q_c$ and $p > p_0$, then $q > \max(1, p - 1)$.

5.1. The Subcritical Case with a Radon Measure

In the case of Eq. (1.12), with no absorption term, it was shown in [8] that a solution exists with any Radon measure as initial data, when $p_0 < p < 2$. But there are restrictions on the behavior of μ at infinity when $p > 2$. Concerning (1.1), we can construct a solution when q is subcritical and $p_0 < p < 2$, or $q \geq p - 1$. In the case $p > p_1$ our proof relies on some arguments of [3], which were also used in [1, 17] for elliptic problems. We combine this with the regularity estimates for a power of u . This allows to reach the range $p > p_0$. Contrary to the proof of [8], we do not use the local boundedness of the solution, and actually, our result is more general, see Remark 5.1.

THEOREM 5.1. *Let $\mu \in \mathcal{M}^+(\mathbb{R}^N)$. Assume that*

$$p > p_0 \text{ (or } p > 1, \text{ if } \mu \in L^1_{\text{loc}}(\mathbb{R}^N)), \quad (5.5)$$

$$p - 1 < q, \text{ or } p < 2, \quad (5.6)$$

$$q < q_c. \quad (5.7)$$

Then there exists a solution $u \in C((0, \infty), L^1_{\text{loc}}(\mathbb{R}^N))$ to

$$\begin{cases} \partial_t u - \nabla \cdot (|\nabla u|^{p-2} \nabla u) + u^q = 0 & \text{in } Q_\infty, \\ u(\cdot, 0) = \mu & \text{in } \mathbb{R}^N. \end{cases} \quad (5.8)$$

Proof. Let $\mu_n \in \mathcal{D}(\mathbb{R}^N)$ be nonnegative and converging to μ in the weak sense, for example $\mu_n = (\mu \chi_{B_{n-1}}) * \rho_n$, where (ρ_n) is a regularizing sequence with support in $B_{1/n}$; hence $\text{supp } \mu_n \subset B_n$. Then

$$\int_{B_\rho} \mu_n(x) dx \leq \int_{B_{\rho+1}} d\mu(x), \quad \forall \rho > 0. \quad (5.9)$$

We consider the approximate problem

$$\begin{cases} \partial_t u_n - \nabla \cdot (|\nabla u_n|^{p-2} \nabla u_n) + u_n^q = 0 & \text{in } Q_\infty, \\ u_n(0) = \mu_n & \text{in } \mathbb{R}^N. \end{cases} \quad (5.10)$$

It admits a (unique) weak solution, such that $u_n \in L^p_{\text{loc}}([0, \infty), W^{1,p}_{\text{loc}}(\mathbb{R}^N))$. And u_n is a strong solution, namely $u_n \in C([0, \infty), L^2_{\text{loc}}(\mathbb{R}^N))$, $u_n^q \in C([0, \infty), L^1_{\text{loc}}(\mathbb{R}^N))$, and $|\nabla u_n| \in C([0, \infty), L^p_{\text{loc}}(\mathbb{R}^N))$ and $\partial_t u_n \in L^2(Q_\infty)$.

Step 1 (A Priori Estimates). Let $\rho, T > 0$ be fixed, and

$$Q_{\rho,T} = (0, T) \times B_\rho.$$

Let $\zeta \in \mathcal{D}(B_{2\rho})$ with values in $[0, 1]$ such that $\zeta = 1$ on B_ρ , and $\tau > 0$ large enough. Set $\theta > 0$. Applying (2.9) to u_n , and letting $t \rightarrow 0$ infers

$$\begin{aligned} & \frac{1}{4} \int_{\Omega} u_n(x, \theta) \zeta^\tau(x) dx + \frac{1}{2} \int_0^\theta \int_{\Omega} u_n^q \zeta^\tau dx dt \\ & \leq C + \int_{\Omega} \mu_n(x) \zeta^\tau(x) dx + \tau \int_0^\theta \int_{\Omega} \zeta^{\tau-1} |\nabla u_n|^{p-1} |\nabla \zeta| dx dt. \end{aligned}$$

First, we suppose that $q > p - 1$. Then it implies estimate (3.11) and we get

$$\frac{1}{8} \int_{\Omega} u_n(x, \theta) \zeta^\tau(x) dx + \int_0^\theta \int_{\Omega} u_n^q \zeta^\tau dx dt \leq \int_{\Omega} \mu_n(x) \zeta^\tau(x) dx + C, \quad (5.11)$$

where $C = C(p, q, N, T, \zeta, \tau)$ is independent on u_n . Next we suppose that $q \leq p - 1 < 1$. By (2.7), (2.8) and Young's inequality, for any $\varepsilon > 0$,

$$\begin{aligned} & \tau \int_0^\theta \int_{\Omega} \zeta^{\tau-1} |\nabla u_n|^{p-1} |\nabla \zeta| dx dt \\ & \leq \varepsilon \frac{|\alpha|}{2} \int_0^\theta \int_{\Omega} (1 + u_n)^{\alpha-1} |\nabla u_n|^p \zeta^\tau dx dt + \varepsilon \int_t^\theta \int_{\Omega} u_n \zeta^\tau dx dt + C_\varepsilon, \\ & \leq 2\varepsilon \int_{\Omega} (u_n(x, \theta)) \zeta^\tau(x) dx + \varepsilon \int_t^\theta \int_{\Omega} u_n^q \zeta^\tau dx dt + \varepsilon \int_t^\theta \int_{\Omega} u_n \zeta^\tau dx dt + C_\varepsilon. \end{aligned}$$

Hence

$$\begin{aligned} & \frac{1}{8} \int_{\Omega} u_n(x, \theta) \zeta^\tau(x) dx + \frac{1}{4} \int_0^\theta \int_{\Omega} u_n^q \zeta^\tau dx dt \\ & \leq C + \int_{\Omega} \mu_n(x) \zeta^\tau(x) dx + \varepsilon \int_0^\theta \int_{\Omega} u_n \zeta^\tau dx dt + C_\varepsilon. \end{aligned} \quad (5.12)$$

Integrating on $(0, T)$, and choosing $\varepsilon = \frac{1}{8}T$, we get

$$\int_0^\theta \int_{\Omega} u_n \zeta^\tau dx dt \leq C + C \int_{\Omega} \mu_n(x) \zeta^\tau(x) dx, \quad (5.13)$$

and then again (5.11) holds from (5.12) and (5.13). In both cases,

$(u_n)_{n>\rho}$ is bounded in $L^\infty((0, T), L^1(B_\rho))$, $(u_n^q)_{n>\rho}$ is bounded in $L^1(Q_{\rho,T})$.

Besides this,

$$(u_n^{p-1})_{n>\rho} \text{ is bounded in } L^1(Q_{\rho,T})$$

by the Hölder inequality, whenever $q > p - 1$, or because $p - 1 < 1$. Then, by Proposition 2.2,

$$(u_n^\sigma)_{n>\rho} \text{ is bounded in } L^1(Q_{\rho,T}), \quad \forall \sigma \in (0, q_c), \quad (5.14)$$

$$(|\nabla u_n|^r)_{n>\rho} \text{ is bounded in } L^1(Q_{\rho,T}), \quad \forall r \in (0, Nq_c/(N+1)). \quad (5.15)$$

Let $\alpha \in (\max(1 - p, -1), 0)$ and

$$v_n(x, t) = (1 + u_n(x, t))^\beta, \quad \beta = (\alpha + p - 1)/p. \quad (5.16)$$

Clearly,

$$(v_n^{p+ p/N\beta})_{n>\rho} \text{ is bounded in } L^1(Q_{\rho,T}),$$

and it follows from (2.26) that

$$(|\nabla v_n|^p)_{n>\rho} \text{ is bounded in } L^1(Q_{\rho,T}).$$

Consequently, v_n is bounded in $L^p_{\text{loc}}([0, \infty), W^{1,p}(B_\rho))$. Next, we estimate the t -derivative of v_n :

$$\begin{aligned} \partial_t v_n &= \beta(1 + u_n)^{\beta-1} \partial_t u_n = \beta(1 + u_n)^{\beta-1} (-u_n^q + \nabla \cdot (|\nabla u_n|^{p-2} \nabla u_n)) \\ &= \beta(H_n + \nabla \cdot K_n), \end{aligned}$$

where

$$\begin{aligned} H_n &= -(1 + u_n)^{\beta-1} u_n^q - (\beta - 1)(1 + u_n)^{\beta-2} |\nabla u_n|^p, \\ K_n &= (1 + u_n)^{\beta-1} |\nabla u_n|^{p-2} \nabla u_n. \end{aligned}$$

Because $\beta < 1$, H_n is bounded in $L^1(B_\rho)$, by (2.26). And $|K_n| \leq |\nabla u_n|^{p-1}$. Then from the estimate of $|\nabla u_n|^r$, K_n is bounded in $L^s(Q_{\rho,T})$ for any $s \in [1, s_c)$, where

$$s_c = 1 + 1/(N+1)(p-1). \quad (5.17)$$

In particular $\nabla \cdot K_n$ is bounded in $L^s((0, T), W^{-1,s}(\Omega))$. It implies that $(\partial_t v_n)_{n>\rho}$ is bounded in $L^1(B_\rho) + L^s((0, T), W^{-1,s}(\Omega))$ for any $s \in [1, s_c)$.

Step 2 (Convergence). For any $\rho > 0$, and any $T > 0$, the sequence $(v_n)_{n>\rho}$ is relatively compact in $L^1(Q_{\rho,T})$. We choosing ρ, T as integers. There exists a subsequence, still denoted (v_n) , such that (v_n) converges a.e. in Q_∞ toward

some locally integrable function v . Then (u_n) converges a.e. in \mathcal{Q}_∞ to $u = v^{1/\beta} - 1$. In particular,

$$(u_n) \text{ converges locally in measure to } u. \quad (5.18)$$

Following [3], we deduce that

$$(\nabla u_n) \text{ is locally a Cauchy sequence in measure.} \quad (5.19)$$

Indeed let $\lambda, \varepsilon > 0$ and set

$$E_{n,m,\lambda} = \{|\nabla(u_n - u_m)| > \lambda\} = \{(x, t) \in \mathcal{Q}_{\rho,T} : |\nabla u_n - \nabla u_m| > \lambda\}.$$

We have to prove that $|E_{n,m,\lambda}| \leq \varepsilon$ for m, n large enough. We write $E_{n,m,\lambda} \subset E_{n,m,\lambda}^1 \cup E_{n,m,\lambda}^2 \cup E_{n,m,\lambda}^3$ where

$$E_{n,m,\lambda}^1 = \{(x, t) \in \mathcal{Q}_{\rho,T} : |\nabla u_n| > b\} \cup \{|\nabla u_m| > b\} \cup \{u_n > b\} \cup \{u_m > b\},$$

$$E_{n,m,\lambda}^2 = \{(x, t) \in \mathcal{Q}_{\rho,T} : |u_n - u_m| > k\},$$

$$E_{n,m,\lambda}^3 = \{(x, t) \in \mathcal{Q}_{\rho,T} : |\nabla u_n| \leq b, |\nabla u_m| \leq b, u_n \leq b, u_m \leq b,$$

$$|u_n - u_m| \leq k, |\nabla(u_n - u_m)| > \lambda\},$$

and $k, b > 0$ are parameters. From the estimates on u_n and ∇u_n , we can choose $b = b_\varepsilon$ such that $|E_{n,m,\lambda}^1| \leq \varepsilon/3$ for any $m, n \in \mathbb{N}$. Next, we have

$$\partial_t(u_n - u_m) - \nabla \cdot (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_m|^{p-2} \nabla u_m) + u_n^q - u_m^q = 0. \quad (5.20)$$

Set $\tau > 1$ and $\Theta_k(s) = \int_0^s T_k(\theta) d\theta$, then $|\Theta_k(s)| \leq k|s|$. Multiplying (5.20) by $T_k(u_n - u_m)\zeta^\tau$, we get

$$\begin{aligned} & \int_{B_{2\rho}} \Theta_k(u_n - u_m)(T)\zeta^\tau dx + \int_0^T \int_{B_{2\rho}} (u_n^q - u_m^q) T_k(u_n - u_m) \zeta^\tau dx dt \\ & + \int_0^T \int_{B_{2\rho} \cap \{|u_n - u_m| \leq k\}} ((|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_m|^{p-2} \nabla u_m)(\nabla u_n - \nabla u_m)) \zeta^\tau dx dt \\ & = \int_{B_{2\rho}} \Theta_k(\mu_n - \mu_m) \zeta^\tau dx \\ & + \int_0^T \int_{B_{2\rho}} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_m|^{p-2} \nabla u_m) T_k(u_n - u_m) \zeta^{\tau-1} \nabla \zeta dx dt. \end{aligned}$$

Next from (5.9),

$$\int_{B_{2\rho}} \Theta_k(\mu_n - \mu_m) \zeta^\theta dx \leq k \left(\int_{B_{2\rho}} (\mu_n + \mu_m) dx \right) \leq 2kC \int_{B_{2\rho+1}} \mu dx = kC_\rho \quad (5.21)$$

with $C_\rho = 2C \int_{B_{2\rho}} \mu dx$, and from (5.15),

$$\begin{aligned} & \int_0^T \int_{B_{2\rho}} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_m|^{p-2} \nabla u_m) T_k(u_n - u_m) \zeta^{\tau-1} \nabla \zeta dx dt \\ & \leq k \int_0^T \int_{B_{2\rho}} (|\nabla u_n|^{p-1} + |\nabla u_m|^{p-1}) \zeta^{\tau-1} |\nabla \zeta| dx dt \leq kC_\rho, \end{aligned}$$

with another $C_\rho > 0$. Hence

$$\int_0^T \int_{B_\rho \cap \{|u_n - u_m| \leq k\}} ((|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_m|^{p-2} \nabla u_m) (\nabla u_n - \nabla u_m)) dx dt \leq kC_\rho.$$

When $p < 2$, this implies

$$\int_0^T \int_{B_\rho \cap \{|u_n - u_m| \leq k\}} \frac{|\nabla u_n - \nabla u_m|^2}{|\nabla u_n|^{2-p} + |\nabla u_m|^{2-p}} dx dt \leq kC_\rho,$$

hence $|E_{n,m,\lambda}^3| \leq 2b_\varepsilon^{2-p} kC_\rho / \lambda^2 \leq \varepsilon/3$ as soon as $k \leq k_\varepsilon$ small enough. When $p \geq 2$, we have

$$\int_0^T \int_{B_\rho \cap \{|u_n - u_m| \leq k\}} |\nabla u_n - \nabla u_m|^p dx dt \leq kC_\rho,$$

hence $|E_{n,m,\lambda}^3| \leq kC_\rho / \lambda^p \leq \varepsilon/3$ as soon as $k \leq k_\varepsilon$ small enough. After having chosen such a k , we deduce that there exists $n_\varepsilon \in \mathbb{N}$ such that $|E_{n,m,\lambda}^2| \leq \varepsilon/3$ for any $m, n \geq n_\varepsilon$, from (5.18). Hence $|E_{n,m,\lambda}| \leq \varepsilon$ for any $m, n \geq n_\varepsilon$, and (5.19) follows. This also implies that

$$(|\nabla u_n|^{p-2} \nabla u_n) \text{ is locally a Cauchy sequence in measure.} \quad (5.22)$$

But from (5.15),

$$(|\nabla u_n|^{p-2} \nabla u_n)_{n \geq \rho} \text{ is bounded in } (L^s(Q_{\rho,T}))^N, \quad \forall s \in [1, s_c).$$

After extraction of another subsequence, there exists some $y = (y_1, \dots, y_N)$, $w = (w_1, \dots, w_N)$ such that

$$(\nabla u_n) \rightarrow y \text{ a.e. in } Q_\infty, \quad (|\nabla u_n|^{p-2} \nabla u_n) \rightarrow w = |y|^{p-2} y \text{ a.e. in } Q_\infty$$

and

$$(|\nabla u_n|^{p-2} \nabla u_n) \rightarrow w \text{ strongly in } (L^s(Q_{\rho,T}))^N, \quad \forall s \in [1, s_c).$$

Let us denote

$$u_n^k = T_k(u_n) \quad \text{and} \quad u^k = T_k(u).$$

Now for any fixed $k > 0$, u_n^k converges a.e. to u^k . Moreover, it converges weakly in the space $L^p((0, T), W^{1,p}(B_\rho))$, and the limit function is u^k . In particular,

$$\nabla u_n^k \rightarrow \nabla u^k \quad \text{weakly in } (L^p(Q_{\rho,T}))^N.$$

But $\nabla u_n^k = \nabla u_n \times 1_{\{|u_n| \leq k\}}$, hence ∇u_n^k converges a.e. to $y \times 1_{\{|u| \leq k\}}$. Then $\nabla u^k = y \times 1_{\{|u| \leq k\}}$ a.e. in Q_∞ . Since we have defined the gradient of u by $\nabla u = y$, it follows that

$$(|\nabla u_n|^{p-2} \nabla u_n) \rightarrow |\nabla u|^{p-2} \nabla u \quad \text{strongly in } (L^s(Q_{\rho,T}))^N, \quad \forall s \in [1, s_c). \quad (5.23)$$

Either $p > p_0$, hence $1 < q_c$, so that (5.14) implies that

$$(u_n) \rightarrow u \quad \text{strongly in } L^\sigma(Q_{\rho,T}), \quad \forall \sigma \in [1, q_c). \quad (5.24)$$

Or $\mu \in L^1_{\text{loc}}(\mathbb{R}^N)$. Moreover u_n converges strongly to μ in $L^1(B_\rho)$ for any $\rho > 0$. Hence for any $\varepsilon > 0$, and if $m, n \geq n(\varepsilon)$,

$$\frac{1}{k} \int_{B_{2\rho}} \Theta_k(\mu_n - \mu_m) \zeta^0 dx \leq \int_{B_{2\rho}} |\mu_n - \mu_m| dx \leq \varepsilon.$$

Since $|\Theta_k(s)| \geq k|s|/2$ on the set $\{|s| \geq k\}$, we have for any $T > 0$,

$$\begin{aligned} & \frac{1}{2} \int_{B_\rho \cap \{|u_n(\cdot, T) - u_m(\cdot, T)| \geq k\}} |u_n(x, T) - u_m(x, T)| dx \\ & \leq \frac{1}{k} \int_{B_{2\rho}} \Theta_k(u_n - u_m)(T) \zeta^\tau dx. \end{aligned}$$

Hence

$$\begin{aligned}
& \frac{1}{2} \int_{B_\rho \cap \{|u_n(\cdot, T) - u_m(\cdot, T)| \geq k\}} |u_n(x, T) - u_m(x, T)| dx \\
& + \frac{1}{k} \int_0^T \int_{B_\rho \cap \{|u_n - u_m| \leq k\}} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_m|^{p-2} \nabla u_m) (\nabla u_n - \nabla u_m) dx dt \\
& \leq \varepsilon + \frac{1}{k} \int_0^T \int_{B_{2\rho}} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_m|^{p-2} \nabla u_m) T_k(u_n - u_m) \zeta^{\tau-1} \nabla \zeta dx dt. \\
& \leq \varepsilon + C_\rho \int_0^T \int_{B_{2\rho}} ||\nabla u_n|^{p-2} \nabla u_n - |\nabla u_m|^{p-2} \nabla u_m| dx dt \leq 2\varepsilon,
\end{aligned} \tag{5.25}$$

for $m, n \geq n'(\varepsilon)$ large enough, independent on k . Letting $k \rightarrow 0$, this proves that

$$(u_n) \rightarrow u \text{ strongly in } C([0, T], L^1(B_\rho)). \tag{5.26}$$

In any case, from (5.24) or (5.26),

$$(u_n) \rightarrow u \text{ strongly in } L^1(Q_{\rho, T}). \tag{5.27}$$

Next, we show that the limit function admits μ as an initial trace, since $q < q_c$. We have the property

$$(u_n^q) \rightarrow u^q \text{ strongly in } L^\tau(Q_{\rho, T}), \quad \forall \tau \in [1, q_c/q]. \tag{5.28}$$

For any $\zeta \in C_c^\infty(\mathbb{R}^N)$ and $t > 0$, we have

$$\begin{aligned}
& \int_{\mathbb{R}^N} u_n(x, t) \zeta(x) dx - \int_{\mathbb{R}^N} \mu_n(x) \zeta(x) dx \\
& = \int_0^t \int_\Omega (|\nabla u_n|^{p-2} \nabla u_n \nabla \zeta + u_n^q \zeta) dx dt.
\end{aligned} \tag{5.29}$$

Up to the extraction of a subsequence, we can pass to the limit in each term, for almost any $t > 0$, and get

$$\int_{\mathbb{R}^N} u(x, t) \zeta(x) dx - \int_{\mathbb{R}^N} \zeta(x) d\mu(x) = \int_0^t \int_\Omega (|\nabla u|^{p-2} \nabla u \nabla \zeta + u^q \zeta) dx dt,$$

from (5.23), (5.28). Since the right-hand side tends to 0 as t goes to 0, hence

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(x, t) \zeta(x) dx = \int_{\mathbb{R}^N} \zeta(x) d\mu(x). \tag{5.30}$$

For any $\varphi \in C_c^\infty(\mathbb{R}^N \times [0, \infty))$ and $\theta > 0$, we have

$$\begin{aligned} & \int_0^\theta \int_{\mathbb{R}^N} (-u_n \varphi_t + |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi + u_n^q \varphi) dx dt \\ &= \int_{\mathbb{R}^N} \varphi(0, x) \mu_n(x) dx - \int_{\mathbb{R}^N} u_n(x, \theta) \varphi(x, \theta) dx, \end{aligned}$$

for any $n > \rho$, by using also (5.27). Then u satisfies (3.5) in $\mathcal{R} = \mathbb{R}^N$.

It remains to prove that u is a weak solution. First suppose that $\mu \in L_{\text{loc}}^1(\mathbb{R}^N)$. Going to the limit in (5.25) when $m \rightarrow \infty$ from the Fatou Lemma and Lebesgue theorem, we get for any $T > 0$

$$\begin{aligned} & \frac{1}{k} \int_0^T \int_{B_\rho \cap \{|u_n - u| \leq k\}} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) (\nabla u_n - \nabla u) dx dt \\ & \leq \varepsilon + C_\rho \int_0^T \int_{B_{2\rho}} ||\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u| dx dt \leq 2\varepsilon. \end{aligned}$$

Moreover,

$$\begin{aligned} X_{k,n} &= \int_0^T \int_{B_\rho} (|\nabla u_n^k|^{p-2} \nabla(u_n^k) - |\nabla u^k|^{p-2} \nabla u^k) (\nabla u_n^k - \nabla u^k) dx dt \\ &\leq \int_0^T \int_{B_\rho \cap \{|u_n - u| \leq 2k\}} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) (\nabla u_n - \nabla u) dx dt. \end{aligned}$$

Hence for fixed k , $X_{k,n} \rightarrow 0$. Therefore,

$$\int_0^T \int_{B_\rho} |\nabla u_n^k|^p dx dt \rightarrow \int_0^T \int_{B_\rho} |\nabla u^k|^p dx dt$$

which implies

$$\nabla u_n^k \rightarrow \nabla u^k \text{ strongly in } L^p(Q_{\rho,T}).$$

Next, we consider the general case where μ is a measure. Then for almost all $\tau > 0$ and $\rho > 0$ $u_n(\cdot, \tau)$ converges strongly to $u(\cdot, \tau)$ in $L^1(B_\rho)$. Let $0 < \tau < T$; if we replace the interval $[0, T]$ by $[\tau, T]$, we deduce as above that $u \in C^1((0, \infty), L^1(B_\rho))$, $(u_n) \rightarrow u$ strongly in $C([\tau, T], L^1(B_\rho))$, and

$$\nabla u_n^k \rightarrow \nabla u^k \text{ strongly in } L^p((\tau, T) \times B_\rho). \quad (5.31)$$

In fact this happens for any $0 < \tau < T$. Let $\varphi \in C_c^\infty(Q_\infty)$. Consider $0 < \tau < T$ and $\rho > 0$ such that the support of φ is contained in $(\tau, T) \times B_\rho$. Let $h \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ where $H'(r) = h(r)$, and $h(r)$ constant outside $[-k, k]$

for some $k > 0$. Multiplying Eq. (5.10) by $h(u_n)\varphi$, we get

$$\begin{aligned} 0 &= \int_0^\infty \int_{\mathbb{R}^N} (-H(u_n)\partial_t \varphi + |\nabla u_n|^{p-2} \nabla(h(u_n)\varphi) + h(u_n)u_n^q \varphi) dx dt \\ &= \int_0^\infty \int_{\mathbb{R}^N} (-H(u_n)\partial_t \varphi + |\nabla u_n^k|^p h'(u_n)\varphi + h(u_n)|\nabla u_n|^{p-2} \nabla u_n \cdot \nabla u \varphi \\ &\quad + h(u_n)u_n^q \varphi) dx dt. \end{aligned} \quad (5.32)$$

Owing to (5.28), (5.27) and (5.31), we can pass to the limit in each term. Thus the proof is complete. ■

Remark 5.1. In the case $q > \max(1, p-1)$, the proof is much shorter, since (5.4) implies that $\{u_n\}$ is locally bounded, uniformly with respect to n ; therefore the use of the T_k -truncature of u_n is useless. Moreover, because

$$\partial_t u_n - \nabla \cdot (|\nabla u_n|^{p-2} \nabla u_n) + u_n^q = 0$$

holds in Q_∞ and $\{u_n^q\}$ is locally bounded uniformly with respect to n , the sequence $\{u_n\}$ is equicontinuous in the local uniform topology of Q_∞ , hence, up to a subsequence, it converges uniformly on any compact subset of Q_∞ . Thus in particular u is continuous on Q_∞ . Actually, for a given open bounded domain Ω in \mathbb{R}^N , the regularity results of [5, 9], assert that, any $v \in C((0, T), L^2(\Omega)) \cap L^2((0, T), W^{1,p}(\Omega)) \cap L^\infty(Q_T)$ solution of

$$\partial_t v - \nabla \cdot (|\nabla v|^{p-2} \nabla v) = h \quad (5.33)$$

in $Q_T = \Omega \times (0, T)$, with $h \in L^\infty(Q_T)$, is Hölder continuous, and the same is true with ∇v : there exists $\alpha \in (0, 1)$ such that for any compact set $K \subset \Omega$, and any $(x_i, t_i) \in K \times [T/2, 2T/3]$.

$$|v(x_1, t_1) - v(x_2, t_2)| \leq \gamma(|x_1 - x_2| + |t_1 - t_2|^{1/p})^\alpha,$$

where $\gamma = \gamma(N, p, \|h\|_{L^\infty(\Omega \times (0, T))}, \|v\|_{L^\infty(G \times (0, T))}, \text{dist}(K, \partial\Omega))$.

Remark 5.2. If we replace u^q by $|u|^{q-1}u$ in (5.8) the existence result of changing sign solution with a given signed Radon measure μ as initial data holds under some minor modifications in the proof. Also, since the proof does not use local boundedness of the solutions, we can treat where there is a forcing term f in the equation. Consider the problem

$$\begin{cases} \partial_t u - \nabla \cdot (|\nabla u|^{p-2} \nabla u) + |u|^{q-1}u = f & \text{in } \mathbb{R}^N \times (0, T), \\ u_n(0) = \mu & \text{in } \mathbb{R}^N, \end{cases} \quad (5.34)$$

where f and μ are given Radon measures in $\mathbb{R}^N \times (0, T)$ and \mathbb{R}^N , respectively. Under the assumptions of Theorem 5.1 on p and q , there exists a solution of problem (5.34) in the sense of distributions in $\mathbb{R}^N \times (0, T)$, and such that (5.30) holds for any $\xi \in C_c^\infty(\mathbb{R}^N)$. If moreover $f \in L_{\text{loc}}^1(\mathbb{R}^N \times (0, T))$, then u is a weak solution as before, that means for any $\varphi \in C_c^\infty(\mathbb{R}^N \times (0, T))$,

$$\int_0^T \int_{\mathbb{R}^N} (-H(u) \partial_t \varphi + |\nabla u|^{p-2} \nabla u \cdot \nabla (h(u) \varphi) + h(u)(|u|^{q-1} u - f) \varphi) dx dt = 0.$$

In the same way, we can prove the existence of a solution of the Cauchy–Dirichlet problem

$$\begin{cases} \partial_t u - \nabla \cdot (|\nabla u|^{p-2} \nabla u) + |u|^{q-1} u = f & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = \mu & \text{in } \Omega, \end{cases} \quad (5.35)$$

in any bounded regular domain Ω , for any bounded Radon measures f and μ in $\Omega \times (0, T)$ and Ω , respectively. The Dirichlet condition on $\partial\Omega \times (0, \infty)$ is given in the sense $T_k(u) \in L_{\text{loc}}^p((0, \infty), W_0^{1,p}(\Omega))$. In case $f \in L^1(\Omega \times (0, T))$, $\mu \in L^1(\Omega)$, one finds again the result of [15].

5.2. Constructive Solutions

Up to now, no uniqueness result is known for a measure, except when it is a (sum of) Dirac measure(s). That is the reason why we define a notion of solutions corresponding to the previous construction which will permit some operations and comparison between them.

DEFINITION 5.2. Let $p > 1$, $q > 0$, and $\mu \in \mathcal{M}^+(\mathbb{R}^N)$. Suppose that there exists a weak nonnegative solution u of (1.1) in Q_∞ with initial trace μ . Then u is called a *constructive solution* if there exists a sequence $\{\mu_n\}$ of continuous functions with compact support converging weakly to μ , such that the corresponding sequence $\{u_n\}$ of solutions to (5.8) with initial data μ_n converges a.e. to u . We shall denote by $\mathcal{M}^{*+}(\mathbb{R}^N)$ the set of initial traces of all the constructive solutions.

Remark 5.3. Notice that the solutions of (5.8) constructed in Theorem 5.1 are constructive, hence in that case $\mathcal{M}^{*+}(\mathbb{R}^N) = \mathcal{M}^+(\mathbb{R}^N)$. Now assume (5.5) and $q \geq q_c$ (hence $q > p - 1$), then all the estimates of Theorem 5.1 are still valid, and all the convergences up to (5.27), with the noticeable exception of (5.28). When $q > 1$, the limit function u is a solution of (1.1), by (5.4) and Remark 5.1. The function u does admits an initial trace μ' , which satisfies $\mu' \leq \mu$ instead of equality. Actually, u is a weak solution of (1.1) with

initial trace μ if and only if for any $\varphi \in C_c(\mathbb{R}^N \times [0, \infty))$,

$$\int_0^T \int_{\mathbb{R}^N} u_n^q \varphi \, dx \, dt \rightarrow \int_0^T \int_{\mathbb{R}^N} u^q \varphi \, dx \, dt, \quad (5.36)$$

or equivalently, if and only if

$$u_n^q \rightarrow u^q \text{ strongly in } L^1_{\text{loc}}(\mathbb{R}^N \times [0, \infty)), \quad (5.37)$$

since $u_n^q \rightarrow u^q$ a.e. in Q_∞ . Indeed if (5.36) holds, then for any $\varphi \in C_c^\infty(\mathbb{R}^N \times [0, \infty))$ we can pass to the limit in (5.29) and get (5.30), and in (5.32). Then u is a weak solution of (1.1) with initial trace μ . Conversely, if u is such a solution, then (5.29) and (5.30) hold. Therefore (5.36) holds by difference for any $\varphi \in C_c^\infty(\mathbb{R}^N \times [0, \infty))$, and then, by density, for any $\varphi \in C_c(\mathbb{R}^N \times [0, \infty))$. As a consequence a function u is constructive if and only if there exists a sequence $\{\mu_n\}$ of continuous functions with compact support converging weakly to μ , such that the corresponding sequence $\{u_n\}$ of solutions to (5.8) with initial data μ_n converges a.e. to u and (5.36) holds.

Next we give some useful properties of the constructive solutions. They are settled upon the following measure theory result the proof of which, due to E. Lesigne, is given in the appendix.

LEMMA 5.2. *Let $\tilde{\mu}, \mu \in \mathcal{M}^+(\mathbb{R}^N)$ with $\tilde{\mu} \leq \mu$. Let $\mu_n \in C_c(\mathbb{R}^N)$ be a sequence of nonnegative functions converging weakly to μ . Then there exists a sequence of nonnegative functions $\tilde{\mu}_n \in C_c(\mathbb{R}^N)$ such that*

$$\tilde{\mu}_n(x) \leq \mu_n(x), \quad \forall x \in \mathbb{R}^N \quad \text{and} \quad \forall n \in \mathbb{N}^*$$

converging weakly to $\tilde{\mu}$.

PROPOSITION 5.3. *Let $\tilde{\mu}, \mu \in \mathcal{M}^+(\mathbb{R}^N)$ with $\tilde{\mu} \leq \mu$.*

(i) *Assume (5.5)–(5.7). Then there exist constructive solutions \tilde{u}, u with respective initial traces $\tilde{\mu}, \mu$, such that $\tilde{u} \leq u$ a.e. in Q_∞ .*

(ii) *Assume (5.5) and $q \geq q_c$, that is $q \geq q_c > 1$. If there exists a constructive solution u with initial trace μ , then there exists a constructive solution \tilde{u} with initial trace $\tilde{\mu}$ such that $\tilde{u} \leq u$ a.e. in Q_∞ . Moreover $\mathcal{M}^{*+}(\mathbb{R}^N)$ is a positive cone in $\mathcal{M}^+(\mathbb{R}^N)$.*

Proof. (i) Let $\tilde{\mu}_n, \mu_n$, be defined in Lemma 5.2, and $\tilde{u}_n \leq u_n$ the corresponding solutions. Then $\tilde{u}_n \leq u_n$ in Q_∞ from the maximum principle, hence $\tilde{u} \leq u$ a.e. in Q_∞ .

(ii) Let u be a constructive solution with initial trace μ , and $\tilde{\mu}_n, \mu_n$, and $\tilde{u}_n \leq u_n$ be defined as above. Then u_n converges to u a.e., in Q_∞ and \tilde{u}_n converges to some \tilde{u} a.e. in Q_∞ . And $u_n^q \rightarrow u^q$, strongly in $L^1_{\text{loc}}(\mathbb{R}^N \times [0, \infty))$,

hence also $\tilde{u}_n^q \rightarrow \tilde{u}^q$ from a variant of the Lebesgue theorem, see for example [26], so that \tilde{u} is constructive.

Set $\mu \in \mathcal{M}^{*+}(\mathbb{R}^N)$, $\lambda > 0$ and let u be a constructive solution with initial trace μ . If $\lambda < 1$, then $\lambda\mu \leq \mu$, hence from above, there exists a constructive solution \tilde{u} with initial trace $\lambda\mu$ such that $\tilde{u} \leq u$ a.e. in Q_∞ . Next, we assume $\lambda > 1$ and let $\mu_n \in C_c(\mathbb{R}^N)$ converging weakly to μ and u_n be the solution with initial trace μ_n . We perform the change of scale

$$S_\lambda(u_n)(x, t) = \lambda u_n(x, \lambda^{p-2}t). \quad (5.38)$$

Then $v_n = S_\lambda(u_n)$ satisfies

$$\partial_t v_n - \nabla \cdot (|\nabla v_n|^{p-2} \nabla v_n) + v_n^q = (1 - \lambda^{p-1-q})v_n^q,$$

in Q_∞ , hence v_n is a supersolution, with initial trace $\lambda\mu$. If w_n denotes the solution with initial trace $\lambda\mu_n$, then $w_n \leq v_n$ from the comparison principle, and w_n converges to some w a.e. in Q_∞ . Moreover, $u_n^q \rightarrow u^q$, strongly in $L^1_{\text{loc}}(\mathbb{R}^N \times [0, \infty))$, hence also $v_n^q = (S_\lambda(u_n))^q \rightarrow (S_\lambda(u))^q$, thus also $w_n^q \rightarrow w^q$. This proves that w is a constructive solution with initial trace $\lambda\mu$. ■

Remark 5.5. Although we believe that $\mathcal{M}^{*+}(\mathbb{R}^N)$ is stable by addition, we have not been able to give a proof to this property.

Remark 5.6. Under the assumptions of Theorem 5.1, suppose that $\mu \in L^1_{\text{loc}}(\Omega)$. Let u be a constructive solution. Then $u \in C([0, T], L^1_{\text{loc}}(\Omega))$, and u is in the class of existence and uniqueness introduced in [15]. Indeed for any open sets $U \subset\subset U^* \subset\subset \Omega$, and any $k > 0$, taking $\xi \in C^\infty_c(\Omega)$, with values in the interval $[0, 1]$, 1 in U , and 0 outside of U^* , and $h(u_n) = T_{k+1}(u_n) - T_k(u_n)$ in (2.4), we get for any $0 < t < \theta < T$,

$$\begin{aligned} & \int_t^\theta \int_{U \cap \{k \leq u \leq k+1\}} |\nabla u_n|^p dx dt + \int_t^\theta \int_{U \cap \{u_n \geq k+1\}} u_n^q dx dt \\ & \leq C \int_t^\theta \int_{U^* \cap \{u \geq k\}} |\nabla u|^{p-1} dx dt + \int_{U^*} H(u(x, t)) dx \\ & \leq C \int_0^\theta \int_{U^* \cap \{u \geq k\}} |\nabla u|^{p-1} dx dt + \int_{U^*} (u(x, t) - k)^+ dx. \end{aligned}$$

Letting t go to 0,

$$\begin{aligned} \int_0^\theta \int_{U \cap \{k \leq u \leq k+1\}} |\nabla u_n|^p dx dt & \leq C \int_0^\theta \int_{U^* \cap \{u_n \geq k\}} |\nabla u_n|^{p-1} dx dt \\ & \quad + \int_{U^*} (\mu - k)^+ dx. \end{aligned}$$

Letting n go to ∞ ,

$$\begin{aligned} & \int_0^\theta \int_{U \cap \{k \leq u \leq k+1\}} |\nabla u|^p dx dt \\ & \leq C \int_0^\theta \int_{U^* \cap \{u \geq k\}} |\nabla u|^{p-1} dx dt + \int_{U^*} (\mu - k)^+ dx. \end{aligned}$$

Finally, letting k go to ∞ ,

$$\lim_{k \rightarrow \infty} \int_0^\theta \int_{U \cap \{k \leq u \leq k+1\}} |\nabla u|^p dx dt = 0.$$

5.3. The Subcritical Case with a Generalized Borel Measure

The main result of this section is the following.

THEOREM 5.4. *Let $\Omega = \mathbb{R}^N$ and*

$$\max(1, p-1) < q < q_c. \quad (5.39)$$

Then for any $v \in \mathcal{B}_{\text{reg}}^+(\mathbb{R}^N)$, there exists at least one solution to (5.2), and $u \in C(Q_\infty)$.

Under (5.39) we recall that there exist singular solutions \tilde{w}_k of (1.16) with initial data $k\delta_0$ for any $k > 0$. When $k \rightarrow \infty$, $\{\tilde{w}_k\}$ increases and converges to \tilde{w}_∞ , which is a singular solution of (1.1) invariant under the similarity transformations N_ℓ defined in (4.7). Therefore, it takes the form

$$\tilde{w}_\infty(x, t) = t^{-1/(q-1)} f(t^{-\gamma} x), \quad (5.40)$$

where $\gamma = (q + p - 1)/(p(q - 1)) > 0$, and f is radial and the unique nontrivial nonnegative solution of the problem

$$\begin{cases} r^{1-N} (r^{N-1} |f'|^{p-2} f')' + \gamma r f' + \frac{1}{q-1} f - f^q = 0 & \text{in } (0, \infty), \\ f'(0) = 0, \\ \lim_{r \rightarrow \infty} r^{p/(q+1-p)} f(r) = 0, \end{cases} \quad (5.41)$$

see [24, 38] for the case $p > 2$, where f has a compact support, and [22] for the case $p < 2$. The following result points out the pointwise blow-up over a singular point in the subcritical case.

LEMMA 5.5. *Assume (5.39) and let $u \in C(\mathbb{R}^N \times (0, T))$ be a non-negative weak solution of (1.1) in $\mathbb{R}^N \times (0, T)$ with initial trace*

$tr_{\mathbb{R}^N}(u) = (\mathcal{S}, \mu) \in CM^+(\mathbb{R}^N)$. If $y \in \mathcal{S}$, then

$$u(x, t) \geq \tilde{w}_\infty(x - y, t), \quad \forall (x, t) \in \mathbb{R}^N \times (0, T). \quad (5.42)$$

Proof. The proof is a variant of the one of Theorem 4.1 and is based on a construction due to Marcus and Véron [33, 34]. Assuming that $y \in \mathcal{S}$ infers that for any open neighborhood U of y ,

$$\lim_{t \rightarrow 0} \int_U u(x, t) dx = \infty.$$

Therefore, for any $k > 0$ there exist two sequences $\{t_n\}$ and $\{r_n\}$ decreasing to 0 such that

$$\int_{B_{r_n}(y)} u(x, t_n) dx = k.$$

Then $u \geq \tilde{w}_{k,y}$ where $\tilde{w}_{k,y}(x, t) = \tilde{w}_k(x - y, t)$ is the fundamental solution of (1.1) with initial data $k\delta_y(\cdot)$. Letting $k \rightarrow \infty$ implies the claim. ■

Remark 5.7. If \mathbb{R}^N is replaced by a general open subset Ω , the lower estimate on u takes the following form:

$$u(x, t) \geq \tilde{w}_{\infty,R}(x - y, t),$$

where $\tilde{w}_{\infty,R}$ is the increasing limit as $k \rightarrow \infty$ of the solution of $\tilde{w} = \tilde{w}_{k,R}$

$$\begin{cases} \partial_t \tilde{w} - \nabla \cdot (|\nabla \tilde{w}|^{p-2} \nabla \tilde{w}) + \tilde{w}^q = 0 & \text{in } B_R \times (0, \infty), \\ \tilde{w} = 0 & \text{in } \partial B_R \times (0, \infty), \\ \tilde{w}(\cdot, 0) = k\delta_0(\cdot) & \text{in } B_R, \end{cases} \quad (5.43)$$

and $R > 0$ is chosen in such a way that $\bar{B}_R(y) \subset \Omega$.

Proof of Theorem 5.4. Suppose $v = (\mathcal{S}, \mu)$, and let $\{a_k\}_{k \in \mathbb{N}^*}$ be a countable dense subset of \mathcal{S} . If $n \in \mathbb{N}^*$, we define $\mu_k \in \mathcal{M}^+(\mathbb{R}^N)$ by

$$\mu_k = \mu + k \sum_{j=1}^k \delta_{a_j}.$$

By Proposition 5.3, there exists a sequence $\{u_k\}$ of constructive continuous solutions of problem (5.8) with initial data μ_k , such that

$$0 \leq \tilde{w}_{a_j,k} \leq u_k \leq u_{k+1} \leq W, \quad \forall k > 0 \quad \text{and } j = 1, \dots, k,$$

where $\tilde{w}_{a_j,k}$ is the solution of (5.8) with initial data $k\delta_{a_j}$. If $k \rightarrow \infty$, $\{u_k\}$ converges to some function u which satisfies

$$0 \leq \tilde{w}_{a_j,\infty} \leq u \leq W$$

in Q_∞ , hence u is locally bounded in Q_∞ . The techniques developed in Theorem 5.1 on $(0, T)$ apply here on (τ, T) for any $0 < \tau < T$, and infer that u is a weak solution of (1.1) in Q_∞ . Moreover, as in Remark 5.1, the sequence $\{u_k\}$ is equicontinuous in the local uniform topology of Q_∞ . Up to a subsequence it converges uniformly on any compact subset of Q_∞ . Thus u is continuous on Q_∞ . An easy calculation shows that for any $\rho > 0$ and $j \geq 1$,

$$\lim_{t \rightarrow 0} \int_{B_\rho(a_j)} w_{a_j, \infty} dx = \infty.$$

Because $\{a_k\}$ is dense in \mathcal{S} , the singular set of the initial trace of u contains \mathcal{S} . But on the other hand, for any open subsets $V \subset \subset V^* \subset \subset \mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}$, if we take a test function ζ with support in V^* in the proof of Theorem 5.1, we obtain estimates (5.14) and (2.22) of Lemma 2.2 in V for u_k . They also hold for u , since μ_k and μ have the same restriction to \mathcal{R} . Therefore the regular set of the initial trace of u contains \bar{V} . Finally, for any $\theta > 0$, letting $k \rightarrow \infty$ in the equality

$$\begin{aligned} & \int_0^\theta \int_{V^*} (-u_k \partial_t \varphi + |\nabla u|^{p-2} \nabla u_k \cdot \nabla u_k \varphi) + u_k^q \varphi dx dt \\ &= \int_{V^*} \varphi(x, 0) d\mu_k - \int_{V^*} u_k(x, \theta) \varphi(x, \theta) dx, \end{aligned}$$

where $\varphi \in C_c^\infty(V^* \times [0, \infty))$, implies (3.5) in V^* , since $q < q_c$. This proves that the regular part of the initial trace of u is μ and consequently $Tr_{\mathbb{R}^n}(u) = \nu \approx (\mathcal{S}, \mu)$. ■

Remark 5.8. If we endow the set $\mathcal{B}_+^{\text{reg}}(\mathbb{R}^N)$ of the following order relation:

$$\nu_1 \lesssim \nu_2 \Leftrightarrow \begin{cases} \mathcal{S}_1 \subseteq \mathcal{S}_2 \\ \mu_{1, \mathcal{R}_2} \leq \mu_2 \end{cases} \quad \text{if } \nu_i \approx (\mathcal{S}_i, \mu_i) \quad \text{with } \mathcal{R}_i = \mathbb{R}^N \setminus \mathcal{S}_i,$$

the solutions u_i of (5.2) with respective initial trace ν_i satisfy $u_1 \leq u_2$ in Q_∞ .

5.4. The Super-Critical Case

In this section we assume

$$q \geq q_c > 1. \tag{5.44}$$

In that case it is important to notice that neither every measure in an open subset of \mathbb{R}^N , nor every closed subset of \mathbb{R}^N are eligible for being, respectively, the regular part and the singular part of the initial trace of a positive solution of (1.1) in Q_∞ . The examples of the Dirac measure or the

pointwise singular set show this fact. The conditions for a measure to be eligible should probably be expressed in terms of capacities, but the theory is not known up to now.

Given an open subset \mathcal{R} of \mathbb{R}^N and $\varepsilon > 0$, we denote

$$\mathcal{S} = \mathbb{R}^N \setminus \mathcal{R}, \quad \mathcal{S}^\varepsilon = \{x \in \mathbb{R}^N : \text{dist}(x, \mathcal{S}) \leq \varepsilon\}, \quad \mathcal{R}^\varepsilon = \mathbb{R}^N \setminus \mathcal{S}^\varepsilon.$$

If $\mu \in \mathcal{M}^+(\mathcal{R})$ we define a measure $\mu_\varepsilon \in \mathcal{M}^+(\mathbb{R}^N)$ by

$$\mu_\varepsilon(E) = \mu(E \cap \mathcal{R}^\varepsilon), \quad \forall E \subset \mathbb{R}^N, \quad E \text{ Borel set.}$$

DEFINITION 5.3. Let $p > 1$, $q \geq q_c$ and \mathcal{R} be an open subset of \mathbb{R}^N . A Radon measure $\mu \in \mathcal{M}^+(\mathcal{R})$ is called a (p, q) -trace if for any $\varepsilon > 0$ there exists a nonnegative constructive solution u_{μ_ε} of (1.1) in Q_∞ with initial trace μ_ε . We denote by $\mathcal{M}^{p,q}(\mathcal{R})$ the set of all (p, q) -traces on \mathcal{R} .

Remark 5.8. It follows from Proposition 5.3 that if $\mu \in \mathcal{M}^{p,q}(\mathcal{R})$ and $\tilde{\mu} \in \mathcal{M}^+(\mathcal{R})$ are such that $\tilde{\mu} \leq \mu$, then $\tilde{\mu} \in \mathcal{M}^{p,q}(\mathcal{R})$. Moreover, $\mathcal{M}^{p,q}(\mathcal{R})$ is a positive cone in $\mathcal{M}^+(\mathcal{R})$.

LEMMA 5.6. Assume $q \geq q_c > 1$ and let \mathcal{R} be an open subset of \mathbb{R}^N and $\mu \in \mathcal{M}^{p,q}(\mathcal{R})$. For $0 < \varepsilon' < \varepsilon$ there exist constructive solutions of (1.1) corresponding to μ_ε and $\mu_{\varepsilon'}$ satisfying

$$u_{\mu_\varepsilon} \leq u_{\mu_{\varepsilon'}}. \quad (5.45)$$

Moreover, $u_\mu = \lim_{\varepsilon \rightarrow 0} u_{\mu_\varepsilon}$ is a solution of (1.1) in Q_∞ with initial data μ on \mathcal{R} .

Proof. Since $\varepsilon' < \varepsilon$, we have $\mathcal{R}^\varepsilon \subset \mathcal{R}^{\varepsilon'}$, hence $\mu_\varepsilon \leq \mu_{\varepsilon'}$, hence by Proposition 5.3, there exist nonnegative constructive corresponding solutions such that

$$u_{\mu_\varepsilon} \leq u_{\mu_{\varepsilon'}} \leq W.$$

As in Theorem 5.4, they converge a.e. to $u_\mu = \sup u_{\mu_\varepsilon}$, and u_μ is a weak solution of (1.1) in Q_∞ , and it converges uniformly on any compact subset of Q_∞ , hence u_μ is continuous on Q_∞ . The fact that u_μ has initial data μ on \mathcal{R} is proved in the following way. Let $\varphi \in C_c^\infty(\mathcal{R} \times [0, \infty))$ be nonnegative. For ε small enough, the support of $\zeta(\cdot, t)$ lies in a compact subset of \mathcal{R}^ε independently of t . Since u_{μ_ε} admits μ_ε as initial trace, we have

for almost all $\theta > 0$

$$\begin{aligned} \int_0^\theta \int_{\mathcal{R}} u_{\mu_\varepsilon}^q \varphi \, dx \, dt &= \int_0^\theta \int_{\mathcal{R}} (u_{\mu_\varepsilon} \partial_t \varphi - |\nabla u_{\mu_\varepsilon}|^{p-2} \nabla u_{\mu_\varepsilon} \cdot \nabla \varphi) \, dx \, dt \\ &\quad + \int_{\mathcal{R}} \varphi(x, 0) \, d\mu_\varepsilon - \int_{\mathcal{R}} \varphi(x, \theta) u_{\mu_\varepsilon}(x, \theta) \, dx. \end{aligned}$$

But $\int_{\mathcal{R}} \varphi(x, 0) \, d\mu_\varepsilon = \int_{\mathcal{R}} \varphi(x, 0) \, d\mu$, and we can pass to the limit in the right-hand side. Then by the Beppo-Levi theorem,

$$\int_0^\theta \int_{\mathcal{R}} u_{\mu_\varepsilon}^q \zeta \, dx \, dt \rightarrow \int_0^\theta \int_{\mathcal{R}} u_\mu^q \zeta \, dx \, dt < \infty.$$

Then

$$u_{\mu_\varepsilon}^q \rightarrow u_\mu^q \quad \text{strongly in } L_{\text{loc}}^1(\mathcal{R} \times [0, \infty)).$$

Hence as in Remark 5.3,

$$\begin{aligned} \int_0^\theta \int_{\mathcal{R}} u_\mu^q \varphi \, dx \, dt &= \int_0^\theta \int_{\mathcal{R}} (u_\mu \partial_t \varphi - |\nabla u_\mu|^{p-2} \nabla u_\mu \cdot \nabla \varphi) \, dx \, dt \\ &\quad + \int_{\mathcal{R}} \varphi(x, 0) \, d\mu - \int_{\mathcal{R}} \varphi(x, \theta) u_\mu(x, \theta) \, dx. \end{aligned}$$

Moreover, for almost all $t > 0$, and any $\zeta \in C_c^\infty(\mathcal{R})$,

$$\begin{aligned} \int_{\mathbb{R}^N} u_\mu(x, t) \zeta(x) \, dx - \int_{\mathbb{R}^N} \zeta(x) \, d\mu(x) \\ = \int_0^t \int_{\Omega} (|\nabla u_\mu|^{p-2} \nabla u_\mu \cdot \nabla \zeta + u_\mu^q \zeta) \, dx \, dt \end{aligned}$$

by passing to the limit in the corresponding equality for u_{μ_ε} which implies

$$\lim_{t \rightarrow 0} \int_{\mathcal{R}^c} u_\mu(x, t) \zeta(x) \, dx = 0.$$

By definition, this means that the solution u_μ admits μ as initial trace in \mathcal{R} . ■

LEMMA 5.7. *Assume $q \geq q_c > 1$ and let \mathcal{R} be an open subset of \mathbb{R}^N and $\mathcal{S} = \mathbb{R}^N \setminus \mathcal{R}$. For $\varepsilon, k > 0$, let u_{k, \mathcal{S}^c} be the solution of (1.1) in Q_∞ with initial data $k \chi_{\mathcal{S}^c \cap B_k}$. Then $k \mapsto u_{k, \mathcal{S}^c}$ is increasing and*

$$u_{\mathcal{S}^c} = \lim_{k \rightarrow \infty} u_{k, \mathcal{S}^c}$$

is a solution of (1.1) in Q_∞ with initial trace $v_{\mathcal{S}^e} \approx (\mathcal{S}^e, 0) \in \mathcal{B}_+^{\text{reg}}(\mathbb{R}^N)$. Moreover,

$$\lim_{t \rightarrow 0} t^{1/(q-1)} u_{\mathcal{S}^e}(x, t) = \left(\frac{1}{q-1} \right)^{1/(q-1)}, \quad \forall x \text{ interior to } \mathcal{S}^e, \quad (5.46)$$

and this limit holds uniformly on any compact subset interior to \mathcal{S}^e . Moreover, for $0 < \varepsilon' < \varepsilon$ there holds

$$u_{\mathcal{S}^{e'}} \leq u_{\mathcal{S}^e}. \quad (5.47)$$

Proof. The existence and uniqueness of u_{k, \mathcal{S}^e} follows from the classical theory for the Cauchy problem for Eq. (1.1), and from the maximum principle, $k \mapsto u_{k, \mathcal{S}^e}$ is increasing. Since there always holds

$$u_{k, \mathcal{S}^e}(x, t) \leq W(t) \quad (5.48)$$

in Q_∞ , then $u_{\mathcal{S}^e} = \lim_{k \rightarrow \infty} u_{k, \mathcal{S}^e}$ exists. As above, it is a weak solution of (1.1) in Q_∞ . Moreover, inequality (5.47) holds clearly for $0 < \varepsilon' < \varepsilon$ as a consequence of the approximation process.

In order to prove (5.46), we consider a ball B_r . For $\sigma > 0$ small enough the functional H_σ defined on $W_0^{1,p}(B_r) \cap L^{q+1}(B_r)$ by

$$H_\sigma(\varphi) = \int_{B_r} \left(\frac{\sigma}{p} |\nabla \varphi|^p + \frac{1}{q+1} |\varphi|^{q+1} - \frac{1}{2} \varphi^2 \right) dx$$

achieves a negative minimal value. Let $\Phi_{r,\sigma}$ be a positive minimizer, solution of the problem

$$\begin{cases} -\sigma \nabla \cdot (|\nabla \Phi_{r,\sigma}|^{p-2} \nabla \Phi_{r,\sigma}) + \Phi_{r,\sigma}^q = \Phi_{r,\sigma} & \text{in } B_r, \\ \Phi_{r,\sigma} = 0 & \text{on } \partial B_r. \end{cases} \quad (5.49)$$

From the maximum principle

$$0 < \Phi_{r,\sigma} \leq 1 \quad \text{in } B_r.$$

Put $w(x, t) = W(t) \Phi_{r,\sigma}$. Then

$$\begin{aligned} & \partial_t w - \nabla \cdot (|\nabla w|^{p-2} \nabla w) + w^q \\ &= -W^{p-1} \nabla \cdot (|\nabla \Phi_{r,\sigma}|^{p-2} \nabla \Phi_{r,\sigma}) + U^q (\Phi_{r,\sigma}^q - \Phi_{r,\sigma}) \\ &= \left(W^q - \frac{1}{\sigma} W^{p-1} \right) (\Phi_{r,\sigma}^q - \Phi_{r,\sigma}). \end{aligned}$$

Set $t_\sigma = (q-1)\sigma^{(q-1)/(q+1-p)}$. For $0 < t \leq t_\sigma$, we have

$$W(t)^q - \frac{1}{\sigma} W(t)^{p-1} \geq 0.$$

Since $\Phi_{r,\sigma}^q - \Phi_{r,\sigma} \leq 0$, the function w is a subsolution. Now consider $b \in \mathcal{S}_\varepsilon$ such that the ball $B_r(b)$ be compactly imbedded in \mathcal{S}_ε . Replacing t by $t + \delta$ in the definition of w , and B_r by a ball $B_r(b)$, and letting δ go to 0, yields to the lower estimate

$$u_{\mathcal{S}^c}(x, t) \geq W(t)\Phi_{r,\sigma}(x - b) \quad \text{in } B_r(b) \times (0, t_\sigma]. \quad (5.50)$$

Using the scaling invariance of the equation, the transformed function $N_\ell(u_{\mathcal{S}^c})$ defined by (4.7) satisfies (1.1), and

$$W(t)\Phi_{r,\sigma}(\ell^\beta(x - b)) \leq N_\ell(u_{\mathcal{S}^c})(x, t) \leq W(t), \quad (5.51)$$

in $B_{r\ell^{-\beta}}(b) \times (0, t_\sigma\ell^{-1}]$. By the previous estimates and the local regularity theory, see Remark 5.3, there exists a sequence $\{\ell_n\}$ converging to 0 such that $N_{\ell_n}(u_{\mathcal{S}^c})$ converges to a function U solution of (1.1) in Q_∞ . Since $\Phi_{r,\sigma} \geq \theta > 0$ on $B_{r/2}$ for some $\theta > 0$, the function U satisfies

$$\theta W(t) \leq U(x, t) \leq W(t), \quad (5.52)$$

in Q_∞ . Because $W(t + \lambda) < U(x, t) \leq W(t)$ for any $t > 0$ and $\lambda > 0$ it follows classically $U = W$ by letting λ go to 0. This equality implies also

$$\lim_{\ell \rightarrow 0} N_\ell(u_{\mathcal{S}^c}) = W.$$

Taking $t = 1$ and $x = b$ yields to

$$\lim_{t \rightarrow 0} t^{1/(q-1)} u_{\mathcal{S}^c}(b, t) = \left(\frac{1}{q-1} \right)^{1/(q-1)},$$

and this holds for any b interior to \mathcal{S}^c . Since the equation is invariant by x -translations, the uniformity on any compact interior to \mathcal{S}^c follows easily by contradiction. ■

LEMMA 5.8. *Under the assumptions of Lemmas 5.6 and 5.7, let $\partial_\mu \mathcal{S}$ be the singular set of the initial trace of u_μ , and*

$$u_{\mathcal{S}} = \lim_{\varepsilon \rightarrow 0} u_{\mathcal{S}^c}.$$

Then $u_{\mathcal{S}}$ is a solution of (1.1) in Q_∞ . If we denote by $\mathcal{S}_{p,q}^$ the singular set of its initial trace, there always holds*

$$\mathcal{S}_{p,q}^* \cup \partial_\mu \mathcal{S} \subset \mathcal{S}.$$

Proof. Since the initial trace u_μ in \mathcal{R} is equal to μ , we have $\partial_\mu \mathcal{S} \subset \mathcal{S}$. Since $\varepsilon \mapsto u_{\mathcal{S}^\varepsilon}$ is increasing, $u_{\mathcal{S}} = \lim_{\varepsilon \rightarrow 0} u_{\mathcal{S}^\varepsilon}$ exists and is a solution of (1.1) in Q_∞ . Since \mathcal{R}^ε is the regular set of the initial trace of $u_{\mathcal{S}^\varepsilon}$, we deduce that for any $\varepsilon > 0$,

$$\begin{aligned} & \int_0^\theta \int_{\mathcal{R}^\varepsilon} (-u_{\mathcal{S}} \partial_t \varphi + |\nabla u_{\mathcal{S}}|^{p-2} \nabla u_{\mathcal{S}} \cdot \nabla u_{\mathcal{S}} \varphi) + u_{\mathcal{S}}^q \varphi) dx dt \\ &= - \int_{\mathcal{R}^\varepsilon} u_{\mathcal{S}}(x, \theta) \varphi(x, \theta) dx, \end{aligned}$$

for any $\theta > 0$ and $\varphi \in C_c^\infty(\mathcal{R}^\varepsilon \times [0, \infty))$, by passing to the limit in the corresponding relation for $u_{\mathcal{S}^\varepsilon}$. Similarly we get, for almost all $t > 0$ and any $\xi \in C_c^\infty(\mathcal{R}^\varepsilon)$,

$$\begin{aligned} & \int_{\mathbb{R}^N} u_{\mathcal{S}}(x, t) \xi(x) dx - \int_{\mathbb{R}^N} \xi(x) d\mu(x) \\ &= \int_0^t \int_{\Omega} (|\nabla u_{\mathcal{S}}|^{p-2} \nabla u_{\mathcal{S}} \nabla \xi + u_{\mathcal{S}}^q \xi) dx dt, \end{aligned}$$

which implies

$$\lim_{t \rightarrow 0} \int_{\mathcal{R}^\varepsilon} u_{\mathcal{S}}(x, t) \xi(x) dx = 0.$$

Then $u_{\mathcal{S}}$ admits 0 as an initial trace in \mathcal{R}^ε . Hence $\mathcal{R}^\varepsilon \subset \mathbb{R}^N \setminus \mathcal{S}_{p,q}^*$, for any $\varepsilon > 0$, hence $\mathcal{R} \subset \mathbb{R}^N \setminus \mathcal{S}_{p,q}^*$, i.e. $\mathcal{S}_{p,q}^* \subset \mathcal{S}$. ■

The main result in this section is the following.

THEOREM 5.9. *Assume $q \geq q_c > 1$ and let $v \approx (\mathcal{S}, \mu) \in \mathcal{B}_+^{\text{reg}}(\mathbb{R}^N)$. Then a sufficient condition for the existence of a nonnegative solution u of (1.1) in Q_∞ with initial trace v , is the following:*

- (i) $\mu \in \mathcal{M}^{p,q}(\mathcal{R})$.
- (ii) *There holds*

$$\mathcal{S}_{p,q}^* \cup \partial_\mu \mathcal{S} = \mathcal{S}. \quad (5.53)$$

Proof. (i) We first assume that $p > 2$. For $\varepsilon > \eta > 0$ and $k > 0$ we consider the functions u_{μ_ε} and u_{k,\mathcal{S}^η} introduced in Lemmas 5.6 and 5.7. The support of $u_{\mu_\varepsilon}(\cdot, 0)$ and $u_{k,\mathcal{S}^\eta}(\cdot, 0)$ are disjoint. Since $p > 2$ the speed of propagation of the support of u_{μ_ε} and u_{k,\mathcal{S}^η} is finite and depends locally on the amount of mass concentrated near the free boundary [32]. Consequently, there exists $t_{\varepsilon,\eta,k} > 0$ such that, for $0 < t < t_{\varepsilon,\eta,k}$ the support of $u_{\mu_\varepsilon}(\cdot, t)$ and $u_{k,\mathcal{S}^\eta}(\cdot, t)$ are disjoint. It implies that $u_{\mu_\varepsilon} + u_{k,\mathcal{S}^\eta}$ is a solution of (5.8) on $Q_{t_{\varepsilon,\eta,k}} = \mathbb{R}^N \times (0, t_{\varepsilon,\eta,k})$ with initial data $\mu_\varepsilon + k\chi_{\mathcal{S}^\eta \cap B_k}$. Therefore, we can

define a solution $u_{\mu_\varepsilon, k, \mathcal{S}^\eta}$ by

$$u_{\mu_\varepsilon, k, \mathcal{S}^\eta}(x, t) = \begin{cases} u_{\mu_\varepsilon}(x, t) + u_{k, \mathcal{S}^\eta}(x, t) & \text{in } \mathbb{R}^N \times (0, t_{\varepsilon, \eta, k}), \\ v_{k, \varepsilon, \eta}(x, t - t_{\varepsilon, \eta, k}) & \text{in } \mathbb{R}^N \times (t_{\varepsilon, \eta, k}, \infty), \end{cases} \quad (5.54)$$

where $v_{k, \varepsilon, \eta}$ is a solution of (5.8) in Q_∞ with initial data $u_{\mu_\varepsilon}(\cdot, t_{\varepsilon, \eta, k}) + u_{k, \mathcal{S}^\eta}(\cdot, t_{\varepsilon, \eta, k})$. By Theorem 5.1 it exists since the initial data is locally integrable. Moreover, since u_{μ_ε} and u_{k, \mathcal{S}^η} are constructive, we can construct $v_{k, \varepsilon, \eta}$ such that

$$v_{k, \varepsilon, \eta} \geq \sup(u_{\mu_\varepsilon}, u_{k, \mathcal{S}^\eta}) \quad \text{in } \mathbb{R}^N \times (t_{\varepsilon, \eta, k}, \infty),$$

hence

$$u_{\mu_\varepsilon, k, \mathcal{S}^\eta} \geq \sup(u_{\mu_\varepsilon}, u_{k, \mathcal{S}^\eta}) \quad \text{in } Q_\infty.$$

Letting $k \rightarrow \infty$ infers that $u_{\mu_\varepsilon, k, \mathcal{S}^\eta}$ increases and converges to a solution $u_{\mu_\varepsilon, \mathcal{S}^\eta}$ of (1.1). Therefore,

$$u_{\mu_\varepsilon, \mathcal{S}^\eta} \geq \sup(u_{\mu_\varepsilon}, u_{\mathcal{S}^\eta}) \quad \text{in } Q_\infty.$$

Because of Lemmas 5.6 and 5.7 the function $u_{\mu_\varepsilon, k, \mathcal{S}^\eta}$ is monotone decreasing with respect to ε and increasing with respect to k and η (such are the initial data). Therefore $u_{\mu_\varepsilon, \mathcal{S}^\eta}$ is monotone decreasing with respect to ε and increasing with respect to η . We let successively η and ε go to 0. Then there exists

$$u_{\mu, \mathcal{S}} = \lim_{\varepsilon \rightarrow 0} \lim_{\eta \rightarrow 0} u_{\mu_\varepsilon, \mathcal{S}^\eta}.$$

As in the previous limit process, $u_{\mu, \mathcal{S}}$ is solution of (1.1) in Q_∞ , and

$$\sup(u_\mu, u_{\mathcal{S}}) \leq u_{\mu, \mathcal{S}} \leq W.$$

If we call $\tilde{\mathcal{S}}$ the singular part of the initial trace of $u_{\mu, \mathcal{S}}$, then by definition of a singular set,

$$\mathcal{S}_{p, q}^* \cup \partial_\mu \mathcal{S} = \mathcal{S} \subset \tilde{\mathcal{S}}.$$

But, for any $b \in \mathcal{R}$, if $0 < \varepsilon_0 < \text{dist}(b, \mathcal{S})/3$, then for any $T > 0$, $\int_{B_{\varepsilon_0}(b)} u_{\mu_\varepsilon, \mathcal{S}^\eta} \times (x, \theta) dx$ remains bounded for $\theta \in (0, T]$ independently of $0 < \eta < \varepsilon < \varepsilon_0$. Let $\zeta \in C_c^\infty(\mathbb{R}^N)$ with support in \mathcal{R}^{e_0} such that $\zeta = 1$ in $B_{\varepsilon_0}(b)$, and $\tau > 0$ large enough. By estimate (5.11), we derive

$$\frac{1}{8} \int_\Omega u_{\mu_\varepsilon, \mathcal{S}^\eta}(x, \theta) \zeta^\tau(x) dx + \int_0^\theta \int_\Omega u_{\mu_\varepsilon, \mathcal{S}^\eta}^q \zeta^\tau dx dt \leq \int_\Omega \mu(x) \zeta^\tau(x) dx + C,$$

with $C = C(p, q, N, T, \zeta, \tau)$. Consequently, the same holds for $\int_{B_{v_0}(b)} u_{\mu, \mathcal{S}}(x, \theta) dx$, and this implies $B_{v_0}(b) \subset \tilde{\mathcal{R}} = \mathbb{R}^N \setminus \tilde{\mathcal{S}}$. By complement $\tilde{\mathcal{S}} \subset \mathcal{S}$ and finally

$$\mathcal{S} = \tilde{\mathcal{S}}.$$

For the regular part of the initial trace of $u_{\mu, \mathcal{S}}$, we prove that it coincides with μ in \mathcal{R} , as in Lemma 5.6. It follows that the initial trace of $u_{\mu, \mathcal{S}}$ is the couple (\mathcal{R}, μ) .

(ii) Now assume that $1 < p < 2$. Then the speed of propagation of the support of the solutions to (1.1) is infinite. However, for any $\kappa < p - 1$ and $\gamma > 0$, the following equation:

$$\partial v - \nabla \cdot (|\nabla v|^{p-2} \nabla v) + v^q + \gamma v^\kappa = 0 \quad (5.55)$$

has the finite speed of propagation property, [25]. Moreover the positive classical solutions depends monotonically of γ . Therefore, we follow the construction of the case $p > 2$ in constructing first the functions $u_{\mu_e}^\gamma$ and $u_{k, \mathcal{S}^\eta}^\gamma$ solutions of (5.55) with respective initial data μ_e and $k\chi_{\mathcal{S}^\eta B_k}$. Those solutions exist and are dominated by $u_{\mu_e}^\gamma$ and $u_{k, \mathcal{S}^\eta}^\gamma$, respectively. We define $u_{\mu_e, k, \mathcal{S}^\eta}^\gamma$ by additivity in a similar way as in (5.54). Now $u_{\mu, \mathcal{S}}$ is well defined by the expression

$$u_{\mu, \mathcal{S}} = \lim_{e \rightarrow 0} \lim_{\eta \rightarrow 0} \lim_{k \rightarrow \infty} \lim_{\gamma \rightarrow 0} u_{\mu_e, k, \mathcal{S}^\eta}^\gamma.$$

The remaining of the proof is as in the first case. \blacksquare

Remark 5.6. In the case $p = 2$, it is proved in [33] that the two assertions (i) and (ii) in Theorem 5.9 are necessary and sufficient conditions for the existence of a solution with initial trace $v \approx (\mathcal{S}, \mu)$. Moreover, the solution which is constructed is a maximal solution. Finally, conditions (i) and (ii) are expressed in terms of N -dimensional Bessel capacities $C_{2/q, q'}$:

$$\mu \in \mathcal{M}^{2, q}(\mathcal{R}) \Leftrightarrow \mu(E) = 0, \quad \forall \text{ Borel set } E \text{ with } C_{2/q, q'}(E) = 0.$$

And the set $\mathcal{S}_{p, q}^* = \mathcal{S}_q^*$ is the “nonremovable part” of \mathcal{S} , and more precisely,

$$\mathcal{S}_q^* = \{x \in \mathbb{R}^N : C_{2/q, q'}(\mathcal{S} \cap U) > 0, \forall U \text{ open neighborhood of } x\},$$

and $\partial_\mu \mathcal{S}$ is the set of blow-up points of μ , that is

$$\partial_\mu \mathcal{S} = \{x \in \mathcal{S} : \mu(U \cap \mathcal{S}) = \infty, \forall U \text{ open neighborhood of } x\}.$$

APPENDIX

We prove below Lemma 5.2. We assume that $\{\mu_n\} \subset C_c(\mathbb{R}^N)$ are nonnegative and converges to μ in the sense of measures. Since $\tilde{\mu} \leq \mu$, the Radon–Nikodym theorem applies and there exists a function $g \in L^1_\mu(\mathbb{R}^N)$ such that

$$d\tilde{\mu} = g d\mu.$$

Moreover,

$$0 \leq g \leq 1, \quad \mu\text{-a.e. in } \mathbb{R}^N.$$

Now there exists a sequence $\{g_n\} \subset C_c(\mathbb{R}^N)$ of nonnegative functions, such that $g_n \rightarrow g$ in $L^1_\mu(\mathbb{R}^N)$. By truncation it can also be assumed that $0 \leq g_n \leq 1$, $\forall n \in \mathbb{N}$. From the assumption, for any $\varphi \in C_c(\mathbb{R}^N)$ and any $k \in \mathbb{N}^*$ there exists $n_k \in \mathbb{N}^*$ such that $\forall n \geq n_k$

$$\left| \int_{\mathbb{R}^N} \varphi g_k \mu_n dx - \int_{\mathbb{R}^N} \varphi g_k d\mu \right| \leq \frac{1}{k}.$$

We define a new sequence $\{g'_n\}$ by setting

$$g'_n = g_k \quad \text{if } n_k \leq n < n_{k+1}.$$

Then

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} \varphi g'_n \mu_n dx - \int_{\mathbb{R}^N} \varphi d\tilde{\mu} \right| \\ & \leq \left| \int_{\mathbb{R}^N} \varphi g'_n \mu_n dx - \int_{\mathbb{R}^N} \varphi g'_n d\mu \right| + \left| \int_{\mathbb{R}^N} \varphi g'_n d\mu - \int_{\mathbb{R}^N} \varphi g d\mu \right|, \end{aligned}$$

and, for $n \geq n_k$,

$$\left| \int_{\mathbb{R}^N} \varphi g'_n \mu_n dx - \int_{\mathbb{R}^N} \varphi d\tilde{\mu} \right| \leq \frac{1}{k} + \left| \int_{\mathbb{R}^N} \varphi g'_n d\mu - \int_{\mathbb{R}^N} \varphi g d\mu \right|.$$

Since, by Lebesgue's theorem,

$$\lim_{n' \rightarrow \infty} \int_{\mathbb{R}^N} \varphi g'_n d\mu - \int_{\mathbb{R}^N} \varphi g d\mu = 0,$$

we conclude that for any continuous function φ with compact support, there exists a subsequence $\{g'_n\}$ of $\{g_n\}$ such that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \varphi g'_n \mu_n dx - \int_{\mathbb{R}^N} \varphi d\tilde{\mu} = 0.$$

Since the set $C_c(\mathbb{R}^N)$ admits a countable dense subset $\{\varphi_j\}_{j \in \mathbb{N}}$ for the topology of uniform convergence on compact subsets of \mathbb{R}^N (which is defined by the countable set of semi-norms $\|\eta\|_m = \max_{|x| \leq m} |\eta(x)|$, for $m \in \mathbb{N}_*$ and $\eta \in C_c(\mathbb{R}^N)$), there exists a diagonal sequence $\{\tilde{\mu}_n\}$ extracted from $\{g'_n \mu_n\}$ such that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \varphi_j \tilde{\mu}_n dx = \int_{\mathbb{R}^N} \varphi_j d\tilde{\mu}, \quad \forall j \in \mathbb{N}.$$

The conclusion follows by density.

ACKNOWLEDGMENT

The authors are grateful to Emmanuel Lesigne for providing them with the proof of Lemma 5.2.

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