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EXISTENCE OF SINGULAR SOLUTIONS OF SEMILINEAR ELLIPTIC SYSTEMS WITH DIRICHLET CONDITIONS

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Abstract. We give existence results and a priori estimates for a semilinear elliptic problem of the form

$$\begin{cases} -\Delta w = w^Q + \mu, & \text{in } \Omega, \\ w = \lambda, & \text{on } \partial\Omega, \end{cases}$$

where $Q > 0$, and μ and λ are nonnegative Radon measures in Ω and $\partial\Omega$, with $\int_\Omega \rho \, d\mu < +\infty$, where ρ is the distance to $\partial\Omega$. We extend the results to the case of systems

$$\begin{cases} -\Delta u = v^p + \mu, \qquad -\Delta v = u^q + \eta, & \text{in } \Omega, \\ u = \lambda, \qquad v = \kappa, & \text{on } \partial\Omega, \end{cases}$$

with $p, q > 0$, with the same assumptions on η and κ.

1. Introduction

Let Ω be a regular bounded domain of $\mathbb{R}^N$ $(N \geq 3)$ with boundary $\partial\Omega$. In this article we look for nonnegative solutions of equation

$$-\Delta w = w^Q, \quad \text{in } \Omega, \tag{1.1}$$

where $Q > 0$, or nonnegative solutions of system

$$\begin{cases} -\Delta u = v^p, \\ -\Delta v = u^q, \quad \text{in } \Omega, \end{cases} \tag{1.2}$$

where $p, q > 0$, singular at one point a of Ω, and more generally on some measurable subset of Ω, with Dirichlet conditions. We also consider the case of singularities on the boundary. Our aim is to prove the existence of solutions,

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generally nonradial, for given measure data at the singularities, and give a priori estimates, under some conditions of subcriticality or admissibility.

Let us recall some known results concerning equation (1.1). It admits a particular radial solution $w(r) = Cr^{-2/(Q-1)}$, with $C = C(N,Q)$ if and only if $Q > N/(N-2)$. In the so-called subcritical case

$$Q < N/(N-2), \tag{$\mathbf{H}_0$}$$

such a solution does not exist. P.L. Lions in [15] has shown that all the solutions, singular at one point a of Ω, behave like a multiple of the fundamental solution E_a associated to the linear problem, i.e.,

$$\begin{cases} -\Delta E_a = \delta_a, & \text{in } \Omega, \\ E_a = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where δ_a is the Dirac mass at a. Moreover he constructed solutions of (1.2) for $Q \neq 1$ which are singular at point $a \in \Omega$, satisfying

$$\begin{cases} -\Delta w = w^Q + \alpha\delta_a, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where the equation holds in $\mathcal{D}'(\Omega)$. Such solutions exist for any $\alpha > 0$ small enough if $Q > 1$, for any $\alpha > 0$ if $Q < 1$. The value $N/(N-2)$ is sharp: if $Q \geq N/(N-2)$ such solutions do not exist. There also exist singular solutions, which behave like the particular solution. Their singularity is weaker, so that is not seen in the equation in $\mathcal{D}'(\Omega)$.

Consider now more generally the problem

$$\begin{cases} -\Delta w = w^Q + \alpha\mu, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where μ is a nonnegative Radon measure in Ω, and $\alpha > 0$. Baras and Pierre [4] have extended the existence results to the case where $\mu > 0$ is bounded, for $1 < Q < N/(N-2)$, and recently Amann and Quittner [2] showed the possible existence of two solutions. To our knowledge, no existence result was given when μ is unbounded, and no a priori estimates, even in the case where μ is bounded. In the supercritical case $Q \geq N/(N-2)$, as shown in [4], there can exist some measures μ, which will be called *Q-admissible in Ω*, for which problem (1.5) admits a solution for $\alpha > 0$ small enough, for example $\mu \in L^r(\Omega)$ for r large enough. Such measures cannot be too concentrated.

Finally in the case of the boundary problem

$$\begin{cases} -\Delta w = w^Q, & \text{in } \Omega, \\ w = \widetilde{\alpha}\lambda, & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where λ is a bounded measure on $\partial\Omega$, Bidaut-Véron and Vivier [6] showed the existence for any $1 < Q < (N+1)/(N-1)$, with the same conditions on $\widetilde{\alpha}$. And they gave an a priori estimate with respect to the measure λ. Their existence result is sharp when λ is a Dirac mass at some point of $\partial\Omega$.

Now consider the system (1.2). It admits particular radial solutions in $\mathbb{R}^N \setminus \{0\}$ of the form $u(r) = Ar^{-2(p+1)/(pq-1)}$, $v(r) = Br^{-2(q+1)/(pq-1)}$, where $A = A(N,p,q)$ and $B = B(N,p,q)$ if and only if

$$\min\left(\frac{pq-1}{2(p+1)}, \frac{pq-1}{2(q+1)}\right) > 1/(N-2). \tag{1.7}$$

This leads to defining a subcritical case, *adapted to a point singularity*, by

$$\min\left(\frac{pq-1}{2(p+1)}, \frac{pq-1}{2(q+1)}\right) < 1/(N-2),$$

which is equivalent to

$$\min(\mathbf{P}, \mathbf{Q}) < \frac{N}{N-2}, \tag{S$_0$}$$

with

$$\mathbf{P} = q\frac{p+1}{q+1}, \quad \mathbf{Q} = p\frac{q+1}{p+1}. \tag{1.8}$$

Notice that $\min(\mathbf{P}, \mathbf{Q}) > 1$ if and only if $pq > 1$. Without loss of generality it will be supposed in all the sequel that

$$p \leq q. \tag{1.9}$$

It implies that $p \leq \mathbf{P} \leq \mathbf{Q} \leq q$ if $pq > 1$. And $(\mathbf{S}_0)$ reduces to $\mathbf{P} < N/(N-2)$. In the case $(\mathbf{S}_0)$ the local behaviour was described in [11] in the radial case, and in [5] in the general case. Up to now, the existence of such singular solutions was studied only in the symmetric case of a ball $B(0,1)$ with a singularity at $\{0\}$, via the Schauder fixed-point theorem; see [11].

2. Statement of main results

In Section 3, we consider the scalar case of equation (1.1). We first prove some preliminary results relative to the Green's function. They are the key tools for the construction of solutions, which we obtain by means of supersolutions. They complete results of [13], [7] and [6].

First we need some notation. Let $\rho(x)$ be the distance from any point $x \in \Omega$ to $\partial\Omega$. We denote by $\mathcal{M}(\Omega)$ and $\mathcal{M}(\partial\Omega)$ the spaces of Radon measures on Ω and $\partial\Omega$, and by $\mathcal{M}^+(\Omega)$ and $\mathcal{M}^+(\partial\Omega)$ the subsets of nonnegative measures. For any $\mu \in \mathcal{M}(\Omega)$ such that $\int_\Omega \rho\, |d\mu| < +\infty$, we can define $\Phi = G(\mu)$ as the solution of the linear problem

$$\begin{cases} -\Delta\Phi = \mu, & \text{in } \Omega, \\ \Phi = 0, & \text{on } \partial\Omega, \end{cases}$$

in the integral or the weak sense (see Section 3 and [6]). Let us denote for any $k \in [1, +\infty)$ and $\gamma \in [0, 1]$

$$L^k(\Omega, \rho^\gamma dx) = \left\{ f \text{ measurable on } \Omega : \int_\Omega \rho^\gamma f^k dx < +\infty \right\}.$$

The main result of this section concerns nonnegative measures μ such that $\int_\Omega \rho^\gamma d\mu < +\infty$ for some $\gamma \in [0, 1]$:

Theorem 2.1. *Let $\mu \in \mathcal{M}^+(\Omega)$. Let $\gamma \in [0, 1]$ and $Q > 0$. Assume that $\int_\Omega \rho^\gamma d\mu = 1$ and*

$$Q < (N + \gamma)/(N - 2 + \gamma). \tag{Hγ}$$

Then $G(\mu) \in L^Q(\Omega, \rho^\gamma dx)$, and

$$G(G^Q(\mu)) \leq CG(\mu) \qquad a.e. \text{ in } \Omega, \tag{2.1}$$

where $C = C(N, Q, \gamma, \Omega, \mu) > 0$ (independent of μ if $Q \geq 1$).

Inequality (2.1) is the key tool for later existence proofs, since it allows us to construct supersolutions of the equation. As a direct consequence, we get the following result, which, to our knowledge, is completely new when $\gamma \neq 0$.

Theorem 2.2. *Let $\mu \in \mathcal{M}^+(\Omega)$ with $\int_\Omega \rho^\gamma d\mu < +\infty$ for some $\gamma \in [0, 1]$. Assume (**H**γ), and $Q \neq 1$. Then problem (1.5) has at least one solution, for any $\alpha > 0$ small enough if $Q > 1$ (respectively for any $\alpha > 0$ if $Q < 1$), such that*

$$w \leq CG(\mu) \qquad a.e. \text{ in } \Omega,$$

with $C = C(N, Q, \gamma, \Omega, \alpha, \mu)$. In particular $w \in L^Q(\Omega, \rho^\gamma dx)$.

This shows that the notion of subcriticality *depends on the behaviour of the measure near the boundary*, and more precisely on $\gamma \in [0, 1]$. The critical value of Q is $N/(N - 2)$ only for bounded measures. If the measure μ only satisfies $\int_\Omega \rho\, d\mu < +\infty$, then it becomes $(N + 1)/(N - 1)$.

Then we prove a priori estimates for problem (1.5).

Theorem 2.3. *Let $\mu \in \mathcal{M}^+(\Omega)$ with $\int_\Omega \rho^\gamma d\mu < +\infty$ for some $\gamma \in [0, 1]$. If Q satisfies (**H**γ), any solution $w \geq 0$ of problem (1.5) such that $w \in L^Q(\Omega, \rho^\gamma dx)$ satisfies an estimate*

$$G(\alpha\mu) \leq w \leq C(G(\alpha\mu) + \rho) \tag{2.2}$$

almost everywhere in Ω*, where* $C = C(N, Q, \Omega, \alpha\mu, w)$*, independent of* w *when* $Q > 1$*.*

As a consequence we get local estimates when μ has a compact support. The result covers in particular the local estimates of [15] for $\mu = \delta_a$. In the general case it is new, even when $\gamma = 0$. We use a bootstrap technique as in [6] for problem (1.6). The main difficulty occurs in the case $\gamma \neq 0$, and the proof relies on fine regularity properties of the Green's operator in suitable weighted spaces.

Then we improve the estimates of Theorem 2.1: in particular when $Q > 2/(N-2+\gamma)$ we show that

$$G(G^Q(\mu)) \leq C_\varepsilon G^{Q-2/(N-2+\gamma)+\varepsilon}(\mu) \tag{2.3}$$

in Ω, for any $\varepsilon > 0$ small enough and C_ε depends on ε. This result allows us to make precise the behaviour of the solutions of (2.1), and it is crucial for the study of system (1.2). Using the same ideas, we can show that in the supercritical case $Q \geq (N+\gamma)/(N-2+\gamma)$, any function $h \in L^r(\Omega, \rho^\gamma dx)$ is Q-admissible in Ω if $r > 1$ is large enough. This was first shown in [4] in the case $\gamma = 0$ of bounded measures, but their proof was not extendable to the general case, and our result gives a larger class of admissible measures.

Finally, combining the results of this section with those of [6], we deduce existence results and estimates for the general problem

$$\begin{cases} -\Delta w = w^Q + \alpha\mu, & \text{in } \Omega, \\ w = \tilde{\alpha}\lambda, & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

with $\mu \in \mathcal{M}^+(\Omega)$ with $\int_\Omega \rho \, d\mu < +\infty$, and $\lambda \in \mathcal{M}^+(\partial\Omega)$ and $\alpha, \tilde{\alpha} \geq 0$, which cover in particular the results of [6].

In Section 4 we first study system (1.2) with measure data in Ω and Dirichlet conditions on $\partial\Omega$:

$$\begin{cases} -\Delta u = v^p + \alpha\mu, & \text{in } \Omega, \\ -\Delta v = u^q + \beta\eta, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

where $\alpha, \beta \geq 0$ and $\mu, \eta \in \mathcal{M}^+(\Omega)$, possibly unbounded, with $\int_\Omega \rho \, d\mu + \int_\Omega \rho \, d\eta < +\infty$. Our first result consists of proving existence in the subcritical cases:

Theorem 2.4. *Let* $\eta, \mu \in \mathcal{M}^+(\Omega)$ *with* $\int_\Omega \rho^\gamma d\eta + \int_\Omega \rho^\gamma d\mu < +\infty$ *for some* $\gamma \in [0,1]$*. Assume that* $pq \neq 1$ *and*

$$\mathbf{P} = \min(\mathbf{P}, \mathbf{Q}) < \frac{N+\gamma}{N-2+\gamma}, \tag{$\mathbf{S}\gamma$}$$

$$G(\mu) \in L^q(\Omega, \rho^\gamma dx). \tag{$\mathbf{C}\gamma$}$$

Then system (2.5) *admits at least a solution for* $\alpha, \beta \geq 0$ *small enough of* $pq > 1$, *for any* $\alpha, \beta \geq 0$ *if* $pq < 1$, *such that*

$$G(G^q(\alpha\mu) + \beta\eta) \leq v \leq CG(G^q(\mu) + \eta), \tag{2.6}$$

$$G(\alpha\mu + G^p(G^q(\alpha\mu) + \beta\eta)) \leq u \leq CG(\mu + G^p(G^q(\mu) + G^p(\eta))), \tag{2.7}$$

where $C = C(N, Q, \Omega, \alpha, \mu, \beta, \eta)$. *In particular* $u^q, v^p \in L^1(\Omega, \rho^\gamma dx)$.

In the case of an isolated singularity at a point $a \in \Omega$, where $\mu = \eta = \delta_a$, Theorem 2.4 applies with $\gamma = 0$. Condition $(\mathbf{C}_0)$ reduces to

$$q < N/(N-2) \quad \text{if } \alpha > 0. \tag{2.8}$$

It is well known that it is a necessary condition of existence, since $u \in L^q_{loc}(\Omega)$. Also the existence result is sharp, which means that $(\mathbf{S}_0)$ is also necessary. In the general case, the question is more complex. We prove in particular that existence of a solution of the scalar problem

$$\begin{cases} -\Delta\Phi = \Phi^{\mathbf{P}} + \varepsilon(G^q(\mu) + \eta), & \text{in } \Omega, \\ \Phi = 0, & \text{on } \partial\Omega, \end{cases}$$

for small $\varepsilon > 0$ implies existence for system (2.5) for small $\alpha, \beta > 0$. Conversely, existence for the system for small $\alpha, \beta > 0$ implies existence of the scalar problem

$$\begin{cases} -\Delta\Psi = \Psi^{\mathbf{Q}} + \varepsilon(\mu + G^p(\eta)), & \text{in } \Omega, \\ \Phi = 0, & \text{on } \partial\Omega. \end{cases}$$

Next we prove a priori estimates for the solutions:

Theorem 2.5. *Let* $\eta, \mu \in \mathcal{M}^+(\Omega)$ *with* $\int_\Omega \rho^\gamma d\eta + \int_\Omega \rho^\gamma d\mu < +\infty$ *for some* $\gamma \in [0, 1]$. *Assume that* ($\boldsymbol{S}\gamma$) *holds. Then any solutions* u *and* v *of problem* (2.5), *such that* $u^q \in L^1(\Omega, \rho^\gamma dx)$, *satisfy estimates* (2.6) *and* (2.7).

This result applies in particular to any solutions of the system under condition $(\mathbf{S}1)$. The idea is to prove that under the assumption $(\mathbf{S}\gamma)$, the solutions satisfy a pointwise comparison property, namely

$$u \leq G(\mu) + C_\varepsilon v^{p-2/(N-2-\gamma)+\varepsilon},$$

almost everywhere in Ω, for any small $\varepsilon > 0$ and for some $C_\varepsilon > 0$, whenever $p > 2/(N-2-\gamma)$. Then we are reduced to a scalar inequality for the function v of the form

$$-\Delta v \leq C(v^Q + G^q(\mu) + \eta),$$

where Q is subcritical, so that we can use the results of Section 3. This type of result is much more general. In fact we prove a comparison property, available for *any* solutions u and v of the system, without assuming $(\mathbf{S}\gamma)$:

Theorem 2.6. *Let $\mu, \eta \in \mathcal{M}^+(\Omega)$ such that $\int_\Omega \rho\, d\mu + \int_\Omega \rho\, d\eta < +\infty$. Let u and v be any solutions of system* (2.5). *Then*

$$u \leq G(\alpha\mu) + \ell v^{(p+1)/(q+1)} \tag{2.9}$$

almost everywhere in Ω, with $\ell = ((q+1)/(p+1))^{1/(q+1)}$.

This result extends the preceding one of [5, Theorem 1.2] for that system, and its proof is delicate, because of the lack of regularity of the solutions.

Finally, we consider the general system

$$\begin{cases} -\Delta u = v^p + \alpha\mu, \qquad -\Delta v = u^q + \beta\eta, \qquad \text{in } \Omega, \\ u = \tilde{\alpha}\lambda, \qquad v = \tilde{\beta}\kappa \qquad \text{on } \partial\Omega, \end{cases} \tag{2.10}$$

where $\lambda, \kappa \in \mathcal{M}^+(\partial\Omega)$ and $\tilde{\alpha}, \tilde{\beta} \geq 0$. We extend the previous results to system (2.10), in particular when (**S**1) holds.

All our results apply in particular to the biharmonic problem with unknown $u \geq 0$ superharmonic in Ω:

$$\begin{cases} \Delta^2 u = u^q + \beta\eta, \qquad \text{in } \Omega, \\ u = \tilde{\alpha}\lambda, \qquad \text{on } \partial\Omega, \end{cases}$$

for any $q > 0$, by taking $p = 1$, $\mu = 0$ and $\kappa = 0$.

3. The scalar case

3.1. **Weak solutions of the Laplace equation.** Let $\mathcal{G}$ be the Green's function of the Laplacian in Ω, defined on the set $\left\{(x,y) \in \overline{\Omega} \times \overline{\Omega} : x \neq y\right\}$. Let $\mathcal{P}$ be the Poisson kernel defined on $\Omega \times \partial\Omega$ by

$$\mathcal{P}(x,z) = -\partial\mathcal{G}(x,z)/\partial n.$$

Let $B(x,r)$ the open ball of center x and radius $r > 0$.

Recall that any superharmonic function $U \geq 0$ in Ω satisfies $U \in L^1_{loc}(\Omega)$. From the Herglotz theorem, there exist some unique $\mu \in \mathcal{M}^+(\Omega)$ and $\lambda \in \mathcal{M}^+(\partial\Omega)$ such that U admits an integral representation

$$U = G(\mu) + P(\lambda), \tag{3.1}$$

where, for almost any $x \in \Omega$,

$$G(\mu)(x) = \int_\Omega \mathcal{G}(x,y) d\mu(y), \qquad P(\mu)(x) = \int_{\partial\Omega} \mathcal{P}(x,z) d\mu(z). \tag{3.2}$$

Moreover, $\int_\Omega \rho\, d\mu < +\infty$. Conversely, for any $\mu \in \mathcal{M}(\Omega)$ such that $\int_\Omega \rho d\left|\mu\right| < +\infty$ and $\lambda \in \mathcal{M}(\partial\Omega)$, the function U defined by (3.1) lies in $L^1_{loc}(\Omega)$, and satisfies the equation

$$-\Delta U = \mu \qquad \text{in} \quad \mathcal{D}'(\Omega).$$

We shall say that U is the *integral solution* of the problem

$$\begin{cases} -\Delta U = \mu, & \text{in } \Omega, \\ U = \lambda, & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

It is also characterized as the *weak solution* of the equation, in the sense that $U \in L^1(\Omega)$ and

$$\int_\Omega U(-\Delta\xi)dx = \int_\Omega \xi\, d\mu - \int_{\partial\Omega} \frac{\partial\xi}{\partial n} d\lambda, \tag{3.4}$$

for any $\xi \in C_0^{1,1}(\overline{\Omega})$; see [6]. We call λ the trace of U on $\partial\Omega$. Now for any superharmonic nonnegative function $U \in L^1(\Omega)$, and any $\mu \in \mathcal{M}^+(\Omega)$ such that $\int_\Omega \rho\, d\mu < +\infty$ we will say that $-\Delta U \geq \mu$ in the weak sense if

$$\int_\Omega U(-\Delta\xi)\; dx \geq \int_\Omega \xi\, d\mu \tag{3.5}$$

for any nonnegative $\xi \in C_0^{1,1}(\overline{\Omega})$. It implies that $U \geq G(\mu)$.

Now we set for any $k \in [1, +\infty)$ and $\gamma \in [0,1]$,

$$W^{1,k}(\Omega, \rho^\gamma dx) = \left\{ f \;\in L^k(\Omega, \rho^\gamma dx) : |\nabla f| \in L^k(\Omega, \rho^\gamma dx) \right\},$$

and $W_0^{1,k}(\Omega, \rho^\gamma dx)$ is the completion of $\mathcal{D}(\Omega)$ in $W^{1,k}(\Omega, \rho^\gamma dx)$ with its usual norm. Recall that $W_0^{1,s}(\Omega, \rho^\sigma dx) \subset L^k(\Omega, \rho^\tau dx)$ whenever $1 \leq s \leq k$ with $N/k - N/s + 1 \geq 0$ and $(N+\tau)/k - (N+\sigma)/s + 1 \geq 0$, and the injection is compact when $N/k - N/s + 1 > 0$ and $(N+\tau)/k - (N+\sigma)/s + 1 > 0$; see [12, Theorems 19.10, 19.11].

Let us recall some continuity properties of the Green's and Poisson operator G and P, proved in [6] in weighted spaces. For simplification we consider only nonnegative powers of ρ. We refer to [6] for more precise results and estimates in Marcinkiewicz spaces.

(**P**1) G is bounded from the set $B_\gamma = \left\{ \mu \in \mathcal{M}(\Omega) : \int_\Omega \rho^\gamma d|\mu| < +\infty \right\}$, into $L^k(\Omega, \rho^\tau dx)$ for any $\tau \in [0, \gamma N/(N-2))$ if $\gamma \neq 0$, for $\tau = 0$ if $\gamma = 0$, and any $k \in [1, (N+\tau)/(N-2+\gamma))$.

(**P**2) G is bounded from B_γ into $W_0^{1,s}(\Omega, \rho^\sigma dx)$ for any $\sigma \in [0, N\gamma/(N-1))$ if $\gamma \in (0,1)$, for any $\sigma \in (0, N/(N-1))$ if $\gamma = 1$, for $\sigma = 0$ if $\gamma = 0$, and any $s \in [1, (N+\sigma)/(N-1+\gamma))$. As a consequence, if (μ_n) is bounded in B_γ and converges weakly to μ, then $G((\mu_n))$ converges strongly to $G(\mu)$ in $L^k(\Omega, \rho^\tau dx)$ for any $\tau \in [0, \gamma N/(N-2))$ if $\gamma \neq 0$, $\tau = 0$ if $\gamma = 0$, and any $k \in [1, (N+\tau)/(N-2+\gamma))$.

(**P**3) P is bounded from $\mathcal{M}(\partial\Omega)$ into $W^{1,s}(\Omega, \rho^\sigma dx)$, for any $\sigma > 0$ and $s \in [1, (N+\sigma)/N)$. Hence if λ_n converges weakly to λ, then $P(\lambda_n)$ converges strongly to $P(\lambda)$ in $L^k(\Omega, \rho^\tau dx)$ for any $\tau \geq 0$ and $k \in [1, (N+\tau)/(N-1))$.

3.2. **Notion of solution of the semilinear problem.** Let $\mu \in \mathcal{M}^+(\Omega)$ such that $\int_\Omega \rho\, d\mu < +\infty$, and $\alpha \geq 0$. We will say that $w \geq 0$ is a solution of problem (1.5) if w is superharmonic, with trace 0 on $\partial\Omega$, such that $w^Q \in L^1_{loc}(\Omega)$ and

$$-\Delta w = w^Q + \alpha\mu \qquad \text{in} \quad \mathcal{D}'(\Omega).$$

That means equivalently that $w^Q \in L^1(\Omega, \rho\, dx)$ and w is a weak (or integral) solution of the problem.

Remark 3.1. When μ satisfies $\int_\Omega \rho^\gamma d\mu < +\infty$ for some $\gamma \in [0,1)$, the notion of a solution of problem (1.5) given above appears to be too weak for obtaining a priori estimates. In that case we are led to assume that moreover $w^Q \in L^1(\Omega, \rho^\gamma dx)$. If (**H**$\gamma$) holds, this is equivalent to

$$w \in W_0^{1,s^*}(\Omega, \rho^\gamma dx) \qquad \text{with} \qquad 1/s^* = 1/Q + 1/(N+\gamma).$$

This concept of solution was also introduced in [2] in the case $\gamma = 0$ of bounded measures.

3.3. **Some concavity properties.** Here we give some concavity lemmas that we shall use several times in the sequel.

Lemma 3.1. *Let $w = G(h+\sigma) + P(\lambda)$, with $h \in L^1(\Omega, \rho\, dx)$, $h \geq 0$, and $\sigma \in \mathcal{M}^+(\Omega)$, $\int_\Omega \rho\, d\sigma < +\infty$, and $\lambda \in \mathcal{M}^+(\partial\Omega)$. Then for any $\theta \in (0,1)$, one has $\rho w^{\theta-1} h \in L^1(\Omega)$ and*

$$-\Delta(w^\theta) \geq \theta w^{\theta-1} h, \tag{3.6}$$

in the weak sense.

Proof. Let $h_n, \sigma_n \in \mathcal{D}(\Omega)$, $\lambda_n \in C^\infty(\Omega)$, $h_n, \sigma_n, \lambda_n \geq 0$ be such that h_n converges strongly to h in $L^1(\Omega, \rho\, dx)$, σ_n converges weakly to σ, λ_n converges weakly to λ and $\|\rho\sigma_n\|_{L^1(\Omega)} \leq \int_\Omega \rho\, d\sigma$, $\|\lambda_n\|_{L^1(\partial\Omega)} \leq \int_{\partial\Omega} d\lambda$. Let $w_n = G(h_n + \sigma_n) + P(\lambda_n)$. Let $\varepsilon > 0$. Then

$$-\Delta((w_n+\varepsilon)^\theta) \geq \theta(w_n+\varepsilon)^{\theta-1} h_n,$$

in the classical sense. Hence

$$\int_\Omega (w_n+\varepsilon)^\theta(-\Delta\xi)dx \geq \theta \int_\Omega (w_n+\varepsilon)^{\theta-1} h_n \xi\, dx,$$

for any nonnegative $\xi \in C_0^{1,1}(\overline{\Omega})$, since $\lambda_n \geq 0$. Now w_n converges to w weakly in $L^k(\Omega)$ for any $k \in [1, N/(N-1))$, and after an extraction almost everywhere in Ω, from (**P**1), (**P**2) and (**P**3). Then $(w_n+\varepsilon)^\theta$ converges weakly in $L^{1/\theta}(\Omega)$. And $(w_n+\varepsilon)^{\theta-1}h_n \leq \varepsilon^{\theta-1}h_n\xi$; hence we can pass to the limit and obtain

$$\int_\Omega (w+\varepsilon)^\theta(-\Delta\xi)dx \geq \theta\int_\Omega (w+\varepsilon)^{\theta-1}h\xi\,dx.$$

Now we go to the limit as $\varepsilon \to 0$ from the Fatou Lemma. Then $\xi w^{\theta-1}h \in L^1(\Omega)$; hence $\rho w^{\theta-1}h \in L^1(\Omega)$, and we get (3.6). □

The next lemma is a variant for measures of the result of [7, Lemma 5.4]. We give the proof for better comprehension.

Lemma 3.2. *Let $h \in L^1(\Omega,\rho\,dx)$, with $h \geq 0$. Let*

$$z = G(\mu), \qquad w = G(\eta),$$

with $\mu,\eta \in \mathcal{M}^+(\Omega)$, $\mu \neq 0$, $\int_\Omega \rho\,d\mu+\int_\Omega \rho\,d\eta < +\infty$, such that $-\Delta(w-z) \geq h$ in the weak sense. Let φ be a concave nondecreasing C^2 function on $[0,+\infty)$, such that $\varphi(1) \geq 0$. Then $\varphi'(w/z)h \in L^1(\Omega,\rho\,dx)$ and

$$-\Delta(z\varphi(\frac{w}{z})) \geq \varphi'(\frac{w}{z})h, \tag{3.7}$$

in the weak sense.

Proof. We can write $\eta = h+\mu+\sigma$, with $\sigma \in \mathcal{M}^+(\Omega)$. Let $h_n,\mu_n,\sigma_n \in \mathcal{D}^+(\Omega)$ such that h_n converges strongly to h in $L^1(\Omega,\rho\,dx)$, and μ_n,σ_n converges weakly to μ,σ. Let

$$z_n = G(\mu_n), \qquad w_n = G(h_n+\mu_n+\sigma_n).$$

Then z_n converges strongly to $G(\mu)$ in $L^1(\Omega)$, w_n converges to w in $L^1(\Omega)$, and after an extraction almost everywhere in Ω. Hence $z_n > 0$ in Ω for large n. From concavity,

$$-\Delta(z_n\varphi(\frac{w_n}{z_n})) \geq \varphi'(\frac{w_n}{z_n})(h_n+\sigma_n) \geq \varphi'(\frac{w_n}{z_n})h_n,$$

in the classical sense (see [7]) since $\varphi(1) \geq 0$ and $\varphi' \geq 0$. Also

$$0 \leq z_n\varphi(\frac{w_n}{z_n}) \leq z_n(\varphi(0)+\varphi'(0)\frac{w_n}{z_n}) \leq C(z_n+w_n),$$

for some $C > 0$. Then $z_n\varphi(w_n/z_n)$ converges in $L^1(\Omega)$. For any nonnegative $\xi \in C_0^{1,1}(\overline{\Omega})$ we have

$$\int_\Omega z_n\varphi(\frac{w_n}{z_n})(-\Delta\xi)dx \geq \int_\Omega \varphi'(\frac{w_n}{z_n})h_n\xi\,dx.$$

Thus we can pass to the limit with Lebesgue's theorem and Fatou's lemma, which gives (3.7). □

3.4. **Green's properties.** We begin by a simple result, with an elementary proof, which guides the whole study:

Lemma 3.3. *Let* $0 \leq Q < N/(N-2)$. *Then* $E_a^Q \in L^1(\Omega)$, *and there exists* $C_a = C_a(N, Q, \Omega, a)$ *such that*

$$G(E_a^Q) \leq C_a E_a \qquad \text{in } \Omega \backslash \{a\} . \tag{3.8}$$

Proof. We have $|x-a|^{-(N-2)Q} \in L^1(\Omega)$; hence $E_a^Q \in L^1(\Omega)$, and $E_a(x) \leq C |x-a|^{2-N}$ with $C = C(N)$. Denoting by D the diameter of Ω, the function

$$x \longmapsto h(x) = \begin{cases} |x-a|^{2-(N-2)Q}, & \text{if } Q > 2/(N-2), \\ D - |x-a|^{2-(N-2)Q}, & \text{if } Q < 2/(N-2), \\ \ln(D/|x-a|) & \text{if } Q = 2/(N-2), \end{cases} \tag{3.9}$$

satisfies $-\Delta h = C |x-a|^{-(N-2)Q}$ in $\mathcal{D}'(\Omega)$, with $C = C(N, Q, \Omega) > 0$; hence,

$$G(E_a^Q) \leq Ch \leq C |x-a|^{2-N} , \tag{3.10}$$

with $C = C(N, Q, \Omega) > 0$. Let $r > 0$ be small enough, such that $X = \overline{B}(a, r) \subset \Omega$. Then (3.8) holds in $X \backslash \{a\}$, since $|x-a|^{(N-2)} E_a$ is minorated on X. And

$$G(E_a^Q)(x) \leq C_a \rho(x) \leq C_a E_a, \tag{3.11}$$

in $\Omega \backslash X$, with another $C_a > 0$, since $G(E_a^Q) \in C^1(\overline{\Omega \backslash X})$, and from the Höpf lemma. Hence (3.8) holds. □

Now we prove Theorem 2.1, which gives a much stronger result than (3.8), but needs more precise estimates of the Green's function. It shows in particular that in Lemma 3.3 in fact C_a does not depend on a.

Proof of Theorem 2.1. We have $G^Q(\mu) \in L^1(\Omega, \rho^\gamma dx)$ from [6], since $Q < (N+\gamma)/(N-2+\gamma)$. First assume that $Q \geq 1$. We have

$$G(\mu)(x) = \int_\Omega \mathcal{G}(x,y) d\mu(y) = \int_\Omega \frac{E_y(x)}{\rho^\gamma(y)} \rho^\gamma(y) d\mu(y);$$

hence from the Jensen inequality,

$$G^Q(\mu)(x) \leq \int_\Omega \left(\frac{E_y(x)}{\rho^\gamma(y)} \right)^Q \rho^\gamma(y) d\mu(y),$$

$$G(G^Q(\mu))(x) \leq \int_\Omega G((\frac{E_y}{\rho^\gamma(y)})^Q)(x) \rho^\gamma(y) d\mu(y) = \int_\Omega G(E_y^Q)(x) \rho^{\gamma(1-Q)}(y) d\mu(y).$$

Now

$$G(E_y^Q)(x) \rho^{\gamma(1-Q)}(y) = \int_\Omega \mathcal{G}(x,z) \mathcal{G}(y,z) \left(\frac{\mathcal{G}^Q(y,z)}{\rho^\gamma(y)} \right)^{Q-1} dz.$$

Also we have the estimate

$$\mathcal{G}(y,z) \leq C\min(|y-z|^{2-N}, \rho(y)\,|y-z|^{1-N}), \tag{3.12}$$

with $C = C(N,\Omega)$ (see [6]), which implies

$$\mathcal{G}(y,z) \leq C\rho^{\gamma}(y)\,|y-z|^{2-N-\gamma},$$

with $C = C(N,\gamma,\Omega)$. Now recall the 3-G inequality (see [8]):

$$\frac{\mathcal{G}(x,z)\mathcal{G}(y,z)}{\mathcal{G}(x,y)} \leq C\left(|x-z|^{2-N} + |y-z|^{2-N}\right) \tag{3.13}$$

where $C = C(N,\Omega)$. It implies

$$G(E_y^Q)(x)\rho^{\gamma(1-Q)}(y) \leq C\mathcal{G}(x,y)I(x,y),$$

with $C = C(N,Q,\gamma,\Omega)$, and

$$\begin{aligned} I(x,y) &= \int_\Omega |y-z|^{(2-N-\gamma)(Q-1)}\left(|x-z|^{2-N} + |y-z|^{2-N}\right)dz \\ &\leq \int_\Omega (|x-z| + |y-z|)^{2-N+(2-N-\gamma)(Q-1)}\,dz \leq C, \end{aligned}$$

and $C = C(N,Q,\gamma,\Omega)$, since $Q < (N+\gamma)(N-2+\gamma) \leq N/(N-2)$. Thus

$$G(G^Q(\mu))(x) \leq C\int_\Omega \mathcal{G}(x,y)d\mu(y) \leq CG(\mu)(x).$$

Notice that $C = C(N,\Omega)$ when $Q = 1$. Now assume $Q < 1$. Then

$$G(G^Q(\mu)) \leq G(1) + G(G(\mu)). \tag{3.14}$$

Now $G(1) \leq C\rho$ in Ω, with $C = C(N,\Omega)$. Now considering a compact $K \subset \Omega$ contained in the support of μ, the restriction μ_K of μ to K satisfies $G(\mu_K) \in C^1(\overline{\Omega}\backslash K)$ from [9, p. 578]. Then $G(\mu) \geq G(\mu_K) \geq C\rho$ in $\overline{\Omega}\backslash K$ from the Höpf lemma, with $C = C(N,\Omega,\mu)$. Then in turn $G(\mu) \geq C\rho$ in Ω, with another $C > 0$. Hence (2.1) follows. □

Remark 3.2. Theorem 2.1 is true for more general second-order operators, namely those which satisfy the 3-G inequality.

3.5. **First existence results.** Here we prove Theorem 2.2, which means existence in the subcritical cases where $(\mathbf{H}\gamma)$ holds for some $\gamma \in [0,1]$. It is the direct consequence of Theorem 2.1 and the following general existence result, proved for example in [13], at least for $Q > 1$. We give the detailed proof for a better comprehension of the sequel.

Theorem 3.4. *Assume that for some $C_0 > 0$,*

$$G(G^Q(\mu)) \leq C_0 G(\mu), \tag{3.15}$$

almost everywhere in Ω. Then problem (1.5) *admits a solution, for any $\alpha \geq 0$ small enough if $Q > 1$, for any $\alpha \geq 0$ if $Q < 1$.*

Proof. One can assume $\alpha \neq 0$. Let $W = A_0 G(\alpha\mu)$, where $A_0 > 0$ is a parameter. Then under the condition (3.15),

$$G(W^Q + \alpha\mu) \leq (C_0 A_0^Q \alpha^Q + \alpha) G(\mu),$$

hence

$$W \geq G(W^Q + \alpha\mu), \tag{3.16}$$

as long as

$$C_0 A_0^Q \alpha^{Q-1} + 1 \leq A_0. \tag{3.17}$$

If $Q > 1$, then (3.17) is satisfied for any $A_0 > 1$ and for small α. If $Q < 1$, the relation is satisfied for any $\alpha > 0$, after choosing A_0 large enough. Now the existence of a supersolution in the sense of (3.16) implies the existence of a solution. Indeed by induction we can construct a nondecreasing sequence (w_n) such that $w_0 = \alpha G(\mu)$,

$$w_n = G(w_{n-1}^Q) + \alpha G(\mu), \qquad \forall n \geq 1,$$

and $w_n \leq W$. Then (w_n) is bounded in $L^1(\Omega)$ from (**P**1); hence, from the Beppo–Levy theorem, $w_n \to w$ and $w_n^Q \to w^Q$ in $L^1(\Omega)$ and almost everywhere in Ω. Then $G(w_{n-1}^Q) \to G(w^Q)$ in $L^1(\Omega)$ from (**P**1), so that w is a solution of (1.5). And this solution satisfies

$$w \leq A_0 G(\alpha\mu), \qquad \text{if } Q > 1, \tag{3.18}$$

$$w \leq \max(2\alpha, (2C_0)^{1/(1-Q)}) G(\mu), \qquad \text{if } Q < 1, \tag{3.19}$$

where (3.19) follows by taking $A_0 = \max(2, (2C_0)^{1/(1-Q)}/\alpha)$. □

3.6. **Necessary and sufficient conditions of existence.**

Definition 3.1. For given $Q > 0$, $Q \neq 1$, and $\mu \in \mathcal{M}^+(\Omega)$ such that $\int_\Omega \rho \, d\mu < +\infty$, $\mu \neq 0$, we shall say that μ *is Q-admissible in* Ω if the problem (1.5) admits a solution for $\alpha \geq 0$ small enough.

When $Q < 1$, and even any $Q < (N+1)/(N-1)$, any measure $\mu \in \mathcal{M}^+(\Omega)$ such that $\int_\Omega \rho \, d\mu < +\infty$ is Q-admissible in Ω from Theorem 2.2. When $Q > 1$, the condition (3.15) is a necessary condition of existence of a solution of problem (1.5) for $\alpha > 0$ small enough, from [13]. In fact as in [7] we can obtain more precise results. We denote $Q' = Q/(Q-1)$.

Proposition 3.5. *If* $Q > 1$ *and the problem* (1.5) *admits a solution, then*

$$G(G^Q(\alpha\mu)) \leq \frac{1}{Q-1} G(\alpha\mu). \tag{3.20}$$

Proof. The proof of [7] in the case $\mu \in L^1(\Omega, \rho\, dx)$ extends to the general case: we can suppose that $\alpha = 1$ and $\mu \neq 0$. Assume that (1.5) admits a solution w. We can apply Lemma 3.2 to functions w and $z = G(\mu)$, with φ given by

$$\varphi(s) = \begin{cases} (1 - s^{1-Q})/(Q-1), & \text{for } s \geq 1, \\ s - 1, & \text{for } s < 1. \end{cases}$$

Then

$$-\Delta(G(\mu)\varphi(\frac{w}{G(\mu)})) \geq \varphi'(\frac{w}{G(\mu)})w^Q = G^Q(\mu) \tag{3.21}$$

in the weak sense; hence

$$\frac{1}{Q-1} G(\mu) \geq G(\mu)\varphi(\frac{w}{G(\mu)}) \geq G(G^Q(\mu)).$$

□

Remark 3.3. Condition (3.15), equivalent to existence, implies that

$$G^Q(\mu) \in L^1(\Omega, \rho\, dx). \tag{3.22}$$

Indeed we have $w^Q \in L^1(\Omega, \rho\, dx)$, and $w^Q \geq G^Q(\alpha\mu)$. Condition (3.22) is strictly weaker than (3.15). Indeed if $Q > N/(N-2)$, from [4], there exists a function $f \in L^r(\Omega)$ with $1 \leq r < N/2Q'$, such that the corresponding equation has no solution. Hence f does not satisfy (3.15), but $G(f) \in W^{2,r}(\Omega)$; hence $G(f) \in L^s(\Omega)$ for any $1 \leq s < rN/(N-2r)$. In particular we can choose r such that $Q < rN/(N-2r)$, since $Q > N/(N-2)$; hence $G^Q(f) \in L^1(\Omega)$.

Remark 3.4. Assume here that $Q > 1$. Let us define

$$K(\mu) = \sup_{x \in \Omega} \left| \frac{G(G^Q(\mu))(x)}{G(\mu)(x)} \right|$$

as in [7]. If problem (1.5) has a solution, then from (3.20),

$$\alpha \leq ((Q-1)K(\mu))^{1/(1-Q)}. \tag{3.23}$$

And reciprocally, if μ is Q-admissible in Ω, taking $A_0 = Q/(Q-1)$ in the proof of Theorem 2.2, we deduce that, for any α such that

$$\alpha < (QC_0)^{1/(1-Q)}/Q', \tag{3.24}$$

in particular for any α such that

$$\alpha < (QK(\mu))^{1/(1-Q)}/Q', \tag{3.25}$$

problem (1.5) admits at least a solution w such that, from (3.18),

$$G(\alpha\mu) \leq w \leq Q' G(\alpha\mu). \tag{3.26}$$

In particular, under the assumptions of Theorem 2.1, using (2.1) we can take $C_0 = C\left(\int_\Omega \rho^\gamma d\mu\right)^{Q-1}$, where $C = C(N, Q, \gamma, \Omega)$. Then condition (3.24) means that

$$\alpha \int_\Omega \rho^\gamma d\mu \leq C^*, \tag{3.27}$$

for some $C^* = C^*(N, Q, \gamma, \Omega))$ independent of μ. We shall use this result in Section 3.8.

Remark 3.5. Other necessary and sufficient conditions for existence have been given in [4] and [13]: for given $Q > 1$, a measure $\mu \in \mathcal{M}^+(\Omega)$ such that $\int_\Omega \rho\, d\mu < +\infty$, $\mu \neq 0$, the following conditions are equivalent:

(i) μ is Q-admissible in Ω,

(ii) (3.15) holds for some $C_0 > 0$,

(iii) $G^Q(\mu)$ is Q-admissible in Ω,

(iv) there exists $C > 0$ such that, for any $g \in L^\infty(\Omega)$ with compact support,

$$\int_\Omega G(\mu) g\, dx \leq C \int_\Omega \frac{g^{Q'}}{(G(g))^{Q'-1}} dx,$$

(v) there exists $C > 0$ such that, for any Borel set $A \subset \Omega$,

$$\int_A \rho\, d\mu \leq C cap(A),$$

where $cap(A)$ is the weighted capacity of the set A defined by

$$cap(A) = \inf\left\{ \int_\Omega g^{Q'} dx : g \in L^{Q'}(\Omega),\ g \geq 0,\ G(g) \geq \rho \text{ on } A \right\}.$$

Notice that the conditions (iv) or (v) are generally hard to verify. Obviously (ii) implies (iii). For proving that (iii) implies (i), one writes problem (1.5) in the form

$$w = \alpha G(\mu) + h, \qquad -\Delta h = (h + G(\alpha\mu))^Q, \tag{3.28}$$

and observes that the problem

$$-\Delta h = M_Q(h^Q + G^Q(\alpha\mu))$$

admits a solution for $\alpha > 0$ small enough, which is a supersolution of (3.28). We shall give a new example of Q-admissible measures in Section 3.8.

Remark 3.6. In Theorem 2.2, we have excluded the linear case $Q = 1$. In that case, (3.15) is satisfied for some $C_0 = C_0(N, \Omega)$ independent of μ, from Theorem 2.1. Then a sufficient condition for existence (for any $\alpha \geq 0$)

is that $C_0 < 1$. Denoting by λ_1 the first eigenvalue of $-\Delta$ with Dirichlet conditions, we have also a necessary condition: $\lambda_1 > 1$. Indeed let $\varphi_1 > 0$ be an eigenfunction for λ_1. If there exists a solution w, then

$$\int_\Omega w(-\Delta\varphi_1)dx = \lambda_1 \int_\Omega w\varphi_1\, dx = \int_\Omega w\varphi_1\, dx + \alpha \int_\Omega \varphi_1\, d\mu.$$

3.7. **A priori estimates.** Now we prove a priori estimates. Clearly, estimate (3.18) is not sufficient, since we have no uniqueness of the solutions; see [2]. Our result concerns more generally *subsolutions* of the equation. The proof lies in a bootstrap technique. It is an adaptation to the interior problem of the one of [6, Theorem 1.2] for the boundary problem (1.6). The main difficulties come in the case $\gamma \neq 0$.

Theorem 3.6. *Let $\mu \in \mathcal{M}^+(\Omega)$ with $\int_\Omega \rho^\gamma d\mu < +\infty$ for some $\gamma \in [0,1]$. Assume that ($\boldsymbol{H}\gamma$) holds. Let $w \geq 0$ be any function in Ω, such that*

$$w \leq G(w^Q + \alpha\mu), \tag{3.29}$$

almost everywhere in Ω, and $w \in L^Q(\Omega, \rho^\gamma dx)$. Then

$$w \leq C(G(\alpha\mu) + \rho), \tag{3.30}$$

almost everywhere in Ω, where $C = C(N, Q, \Omega, \alpha\mu, \|w\|_{L^Q(\Omega,\rho^\gamma dx)})$.

Proof. One can assume $Q \geq 1$. Indeed if $Q < 1$, then

$$w \leq G(w + (1 + \alpha\mu))$$

so that we are reduced to the case $Q = 1$ with measure $1 + \alpha\mu$.

i) The case $\gamma = 0$. That means μ is bounded. We can assume $\alpha = 1$. Here we follow the technique of bootstrap of [15]. Let us set

$$w_1 = (w - G(\mu))^+ \leq G(w^Q);$$

hence $w \leq G(\mu) + w_1$. Now $w \in L^Q(\Omega)$ by hypothesis; hence $\mu + w^Q$ is a bounded measure. Then $w \in L^s(\Omega)$ for any $s \in [1, N/(N-2))$ from (**P**1). Since $Q < N/(N-2)$, $w^Q \in L^{k_0}(\Omega)$ for some $k_0 > 1$. We can choose k_0 such that

$$N/2Q' < k_0 < \min(N/2, N/(N-2)Q).$$

Hence $G(w^Q) \in W_0^{2,k_0}(\Omega)$, and from the Sobolev injection, $w_1^Q \in L^{k_1}(\Omega)$, with

$$k_1 = Nk_0/(Q(N-2k_0)).$$

Now $w^Q \leq M_Q(G^Q(\mu) + w_1^Q)$, with $M_Q = \max(1, 2^{Q-1})$; hence from (2.1),

$$w_1 \leq C(G(G^Q(\mu)) + w_2) \leq C_1(G(\mu) + w_2),$$

where $w_2 = G(w_1^Q)$; hence

$$w \le C_1'(G(\mu) + w_2),$$

and $w_2 \in W_0^{2,k_1}(\Omega)$. By induction for any $n \ge 2$, we can define $w_n = G(w_{n-1}^Q)$ such that

$$w_n \le C_n(G(\mu) + w_{n+1}), \qquad w \le C_n'(G(\mu) + w_{n+1}),$$

where C_n and C_n' depend only on N, Q, Ω and μ. And $w_n^Q \in L^{k_n}(\Omega)$, till $k_{n-1} < N/2$, with k_n given by

$$k_n = Nk_{n-1}/(Q(N - 2k_{n-1})). \tag{3.31}$$

But the whole sequence (k_n) is increasing to infinity: otherwise $k_n \to \ell = N(Q-1)/2Q$, which is a contradiction, since $\ell < k_0$. Hence there exists some $n_0 = n_0(N, Q)$ such that $w_{n_0} \in C^0(\overline{\Omega})$. Then

$$w \le C_{n_0}'(G(\mu) + G(1)) \le C_{n_0}''(G(\mu) + \rho) \tag{3.32}$$

in Ω, and C_{n_0}'' depends on N, Q, Ω, μ, and $\left\| w^Q \right\|_{L^1(\Omega,\rho dx)}$, from (**P**1). Then (3.30) follows.

ii) The case $0<\gamma \le 1$. Let $m \ge 2$ be some fixed integer such that

$$\gamma/m < N + \gamma - (N - 2 + \gamma)Q.$$

Now $w^Q \in L^1(\Omega, \rho^\gamma dx)$ by hypothesis; hence $\int_\Omega \rho^\gamma (d\mu + w^Q dx) < +\infty$. Then from (**P**1), $w \in L^k(\Omega, \rho^\tau dx)$, for any $k \in [1, (N + \tau)/(N - 2 + \gamma))$, and any $\tau \in [0, \gamma]$. For any $n \in [0, m]$, let

$$\tau_n = \gamma(1 - n/m) \in [0, 1].$$

Let $w_0 = w$; hence

$$w_0^Q \in L^{r_0}(\Omega, \rho^{\tau_0} dx), \quad \text{with} \quad 1 < r_0 < (N + \tau_0)/(N - 2 + \gamma)Q.$$

Here again we define w_1 as above, and $w_1 \le w$, so that we can define $w_2 = G(w_1^Q)$ in $L^1(\Omega)$. Now $w_1 \in L^k(\Omega, \rho^\tau dx)$ for any $k \in [1, (N+\tau)/(N-2+\tau_0))$ and any $\tau \in [0, \tau_0]$. Taking $\tau = \tau_1$, we get

$$w_1^Q \in L^{r_1}(\Omega, \rho^{\tau_1} dx), \quad \text{with} \quad 1 < r_1 < (N + \tau_1)/(N - 2 + \tau_0)Q,$$

since $N + \tau_1 - (N - 2 + \tau_0)Q = N + \gamma - (N - 2 - \gamma)Q - \gamma/m > 0$. For any $n \le m$, assume by induction that $w_{n-1} = G(w_{n-2}^Q)$ in $L^1(\Omega)$, and that

$$w_{n-1}^Q \in L^{r_{n-1}}(\Omega, \rho^{\tau_{n-1}} dx), \quad \text{with} \quad 1 < r_{n-1} < (N+\tau_{n-1})/(N-2+\tau_{n-2})Q.$$

Then we can define $w_n = G(w_{n-1}^Q)$ in $L^1(\Omega)$, and, for any $k \in [1, (N+\tau_n)/(N-2+\tau_{n-1}))$, we have $w_n \in L^k(\Omega, \rho^{\tau_n}dx)$. Now

$$(N+\tau_n) - (N-2+\tau_{n-1})Q > (n-1)(Q-1)\gamma/m \geq 0;$$

hence

$$w_n^Q \in L^{r_n}(\Omega, \rho^{\tau_n}dx), \quad \text{with} \quad 1 < r_n < (N+\tau_n)/(N-2+\tau_{n-1})Q.$$

Now in case $n = m$, we have $\tau_m = 0$. This proves that $w_m^Q \in L^{r_m}(\Omega)$,with $r_m > 1$, and we are reduced to the first case: there exists an integer $n_0 = n_0(N, Q)$ such that $w_{n_0+m} \in C^0(\overline{\Omega})$. Then (3.30) follows again. □

Now Theorem 2.3 follows.

Proof of Theorem 2.3. From Theorem 3.6, it remains to prove, as in [10], that for any solution w of (1.5), $\left\|w^Q\right\|_{L^1(\Omega,\rho^\gamma dx)}$ is bounded independently of w when $Q > 1$. Let again $\varphi_1 > 0$ be an eigenfunction for λ_1. Then

$$\int_\Omega w(-\Delta\varphi_1)dx = \lambda_1 \int_\Omega w\varphi_1\, dx = \int_\Omega w^Q\varphi_1\, dx + \alpha \int \varphi_1\, d\mu,$$

and the result follows from the Hölder inequality. □

Notice that the assumption $w \in L^Q(\Omega, \rho^\gamma dx)$ in Theorem 2.3 is always satisfied when $\gamma = 1$. In the case of measures with compact support, with $\gamma = 0$, we can deduce local estimates for the solutions:

Theorem 3.7. *Let $\mu \in \mathcal{M}^+(\Omega)$ with compact support K. Assume ($\boldsymbol{H}_0$). Let $w \geq 0$ be any solution of the problem* (1.5). *Then for any regular domain Ω' such that $K \subset \Omega' \subset\subset \Omega$, there exists $C' > 0$ such that*

$$w(x) \leq C'(G(\alpha\mu) + 1) \qquad \text{a.e. in } \Omega'. \tag{3.33}$$

Proof. By hypothesis, we have

$$-\Delta w = w^Q \qquad \text{in } \mathcal{D}'(\Omega \backslash K),$$

and $w \in L^k(\Omega)$ for any $k \in [1, N/(N-2)$, from (**P**1), since μ is bounded. Then by the same bootstrap, we have $w \in C^\infty(\Omega\backslash K)$; hence w is bounded on $\partial\Omega'$. Let y be harmonic in Ω', such that $y = w$ on $\partial\Omega'$, and let $z = w - y \geq 0$ in Ω'. Since y is bounded, there exists $C' > 0$ such that

$$\begin{cases} -\Delta z = (z+y)^Q + \alpha\mu \leq C'z^Q + \alpha\mu + C' & \text{in } \mathcal{D}'(\Omega') \\ \qquad z = 0 \qquad \text{on } \partial\Omega'. \end{cases}$$

Hence from Theorem 3.6, denoting by G' the Greeen operator in Ω'

$$z(x) \leq C'(G'(\alpha\mu) + 1) \qquad \text{in } \Omega',$$

with another $C' > 0$, which implies (3.33). □

3.8. **New Green's properties and existence results.** In view of the study of systems with measures we need more precise properties of the functions G and P. First let us improve Lemma 3.3.

Lemma 3.8. *Under the assumptions of Lemma* 3.3, *we have*

$$G(E_a^Q) \leq C_a E_a^s, \tag{3.34}$$

for any $s \in (0,1]$ *such that* $s \geq Q - 2/(N-2)$, *with* $C_a = C_a(N, Q, s, \Omega, a)$.

Proof. From (3.10) and (3.9), the inequality holds in $X = B(a,r) \subset \Omega$, since either $s \geq Q - 2/(N-2) > 0$, or $s > 0 \geq Q - 2/(N-2)$. And it holds from (3.11) in $\Omega \backslash X$, since $E_a^s \geq C_a \rho^s \geq C_a \rho$ in $\Omega \backslash X$, hence in Ω. If $Q > 2/(N-2)$, we have more precisely

$$G(E_a^Q)(x) \leq C_a \left|x-a\right|^{N-N-2)Q} E_a \quad \text{in } \Omega.$$

$\square$

Now we will extend property (3.34) to measures, which is also an essential point for the study of the system.

Theorem 3.9. *Let* $\mu \in \mathcal{M}^+(\Omega)$. *Let* $\gamma \in [0,1]$ *and* $Q > 0$. *Assume that* $\int_\Omega \rho^\gamma d\mu = 1$ *and (*$\boldsymbol{H}\gamma$*) holds. Then for any* s *such that*

$$\max(0, Q - \frac{2}{N-2+\gamma}) < s \leq 1, \tag{3.35}$$

there exists $C = C(N, Q, \gamma, s, \Omega, \mu) > 0$ (*independent of* μ *if* $Q \geq 1$), *such that*

$$G(G^Q(\mu)) \leq C\, G^s(\mu) \qquad \text{a.e. in } \Omega. \tag{3.36}$$

Proof. Returning to the proof of Theorem 2.1 with $Q \geq 1$, we have seen that

$$G(G^Q(\mu))(x) \leq \int_\Omega G(E_y^Q)(x) \rho^{\gamma(1-Q)}(y) d\mu(y).$$

Now let $s \in (0,1]$ be fixed. We now write

$$\begin{aligned} G(E_y^Q)(x)\rho^{\gamma(1-Q)}(y) &= \rho^{\gamma(1-Q)}(y) \int_\Omega \mathcal{G}(x,z)\mathcal{G}^Q(y,z)dz \\ &= \rho^{\gamma(1-s)}(y) \int_\Omega \mathcal{G}^s(x,z)\mathcal{G}^s(y,z)\mathcal{G}^{1-s}(x,z) \left(\frac{\mathcal{G}(y,z)}{\rho^\gamma(y)}\right)^{Q-s} dz. \end{aligned}$$

Hence from (3.13) and (3.12),

$$G(E_y^Q)(x)\rho^{\gamma(1-Q)}(y) \leq C\mathcal{G}^s(x,y)\rho^{\gamma(1-s)}(y) I_s(x,y),$$

with $C = C(N, Q, \gamma, \Omega)$, and

$$
\begin{aligned}
&I_s(x,y) \\
&= \int_\Omega \left(|x-z|^{(2-N)s} + |y-z|^{(2-N)s} \right) |x-z|^{(2-N)(1-s)} |y-z|^{(2-N-\gamma)(Q-s)} dz \\
&\leq \int_\Omega \left(|x-z| + |y-z| \right)^{2-N+(2-N-\gamma)(Q-s)} dz \leq C,
\end{aligned}
$$

with $C = C(N, Q, \gamma, s, \Omega)$, since $s > Q - 2/(N-2+\gamma)$. Thus

$$G(G^Q(\mu))(x) \leq C \int_\Omega \mathcal{G}^s(x,y) \rho^{\gamma(1-s)}(y) d\mu(y).$$

Hence from the Jensen inequality, since $s \leq 1$, and $\int_\Omega \rho^\gamma d\mu = 1$,

$$
\begin{aligned}
G(G^Q(\mu))(x) &\leq C \int_\Omega \left(\frac{\mathcal{G}(x,y)}{\rho^\gamma(y)} \right)^s \rho^\gamma(y) d\mu(y) \\
&\leq C \left(\int_\Omega \frac{\mathcal{G}(x,y)}{\rho^\gamma(y)} \rho^\gamma(y) d\mu(y) \right)^s \leq C G^s(\mu)(x).
\end{aligned}
$$

Now assume $Q < 1$. Then

$$G(G^Q(\mu)) \leq G(1 + G(\mu)) \leq G(1) + CG^s(\mu) \leq C(\rho^s + G^s(\mu)) \leq CG^s(\mu)$$

with $C = C(N, \Omega, \mu)$, from the Höpf lemma. Hence (3.36) follows. □

Remark 3.7. Assume that $2/(N-2) < Q < N/(N-2)$ and $\mu \in \mathcal{M}^+(\Omega)$ has a compact support K. Then we also get (3.36) with $s = Q - 2/(N-2)$. Indeed we can suppose $\int_\Omega d\mu = 1$. Then from (3.10) and the Hölder inequality,

$$
\begin{aligned}
&G(G^Q(\mu))(x) \leq \int_K G(E_y^Q)(x) d\mu(y) \leq \int_K G(|x-y|^{(2-N)Q})(x) d\mu(y) \\
&\leq C \int_K |x-y|^{(2-N)Q+2} d\mu(y) \leq C \left(\int_K |x-y|^{(2-N)} d\mu(y) \right)^{Q-2/(N-2)},
\end{aligned}
$$

with $C = C(N, \Omega)$. Now if we consider a domain Ω_K such that $K \subset \Omega_K \subset\subset \Omega$, we have $|x-y|^{(2-N)} \leq C_K \mathcal{G}(x,y)$, for any $x, y \in \Omega_K$, with $C_K = C_K(N, K, \Omega)$; hence with another C_K,

$$G(G^Q(\mu))(x) \leq C_K G^{Q-2/(N-2)}(\mu)(x), \tag{3.37}$$

in Ω_K. Since μ has a compact support, we have $G(\mu) \geq C_K \rho$ in $\overline{\Omega} \backslash K$. Moreover we can write $G^Q(\mu) = w_1 + w_2$ with $w_1 \in L^1(\Omega)$ with support in K and $w_2 \in L^\infty(\Omega)$. Hence $G(G^Q(\mu)) \in C^1(\overline{\Omega} \backslash K)$, so that $G(G^Q(\mu)) \leq C_K \rho$

in $\overline{\Omega}\backslash K$. Since $\rho \leq C\rho^{Q-2/(N-2)}$, relation (3.37) is also true in $\overline{\Omega}\backslash K$, hence finally in Ω.

Remark 3.8. Using Theorem 3.9 one can make precise the a priori estimates of Theorem 2.3 by an asymptotic expansion: under the assumptions of Theorem 2.3, any solution satisfies

$$w - G(\alpha\mu) \leq CG^s(\alpha\mu),$$

for some $C > 0$, for any s satisfying (3.35) since

$$w = G(\alpha\mu) + G(w^Q) \leq G(\alpha\mu) + CG(G^Q(\alpha\mu)).$$

By iteration one can get a complete expansion of the solution.

Now we prove more general results also adapted to supercritical cases, with a quite different method.

Theorem 3.10. *Let $\widetilde{Q} > 0$ and $R \in [\widetilde{Q}, \widetilde{Q}+1)$, $R \neq 1$.*
(i) Let $\eta \in \mathcal{M}^+(\Omega)$ such that $\int_\Omega \rho\, d\eta < +\infty$ and η is R-admissible in Ω. Then there exists $C > 0$ such that

$$G(G^{\widetilde{Q}}(\eta)) \leq CG^s(\eta) \tag{3.38}$$

with $s = \widetilde{Q} + 1 - R$.
(ii) Let $\mu \in \mathcal{M}^+(\Omega)$ such that $\int_\Omega \rho\, d\mu < +\infty$ and $G^{R/s}(\mu)$ is R-admissible in Ω. Then μ is $(\widetilde{Q}/s)$-admissible in Ω.

Proof. (i) When $R = \widetilde{Q}$, (3.38) follows from Theorem 2.1. Now suppose that $R \neq \widetilde{Q}$. Then $s \in (0,1)$. By hypothesis, the problem

$$\begin{cases} -\Delta\Phi = \Phi^R + \varepsilon\eta, & \text{in } \Omega, \\ \Phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.39}$$

has a solution for $\varepsilon > 0$ small enough, such that

$$\Phi \leq CG(\varepsilon\eta),$$

from Theorem 3.4. Let $\Psi = \Phi^s$. Then from Lemma 3.1, we have

$$-\Delta\Psi \geq s\Phi^{s-1}(-\Delta\Phi) \geq s\Phi^{R+s-1} = s\Phi^{\widetilde{Q}},$$

in the weak sense. Then $\Psi \geq sG(\Phi^{\widetilde{Q}})$. And $\Phi \geq G(\varepsilon\eta)$; hence

$$sG(G^{\widetilde{Q}}(\varepsilon\eta)) \leq \Phi^s \leq C^sG^s(\varepsilon\eta),$$

which means

$$G(G^{\widetilde{Q}}(\eta)) \leq CG^s(\eta), \tag{3.40}$$

with $C = C(N, \widetilde{Q}, R, \eta)$.

(ii) Take $\eta = G^{R/s}(\mu)$ in the preceding proof, and let Φ be the solution of (3.39). We set $F = \Psi + G(\varepsilon^{s/R}\mu) = \Phi^s + G(\varepsilon^{s/R}\mu)$. Then

$$\begin{aligned} -\Delta F &\geq \varepsilon^{s/R}\mu + s\Phi^{s-1}(-\Delta\Phi) = \mu + s\Phi^{s-1}(\Phi^R + \varepsilon G^{R/s}(\mu)) \\ &\geq \varepsilon^{s/R}\mu + CF^{(s-1)/s}F^{R/s}, \end{aligned}$$

with $C = sM_{R/s}^{-1}$. Then F is a supersolution of the equation

$$-\Delta w = Cw^{\widetilde{Q}/s} + \varepsilon^{s/R}\mu.$$

Then μ is $\widetilde{Q}/s$-admissible in Ω. $\square$

Thus we find again Theorem 3.9 from Theorem 2.1 and the first part of Theorem 3.10. Indeed let us take $\widetilde{Q} = Q$, and s satisfying (3.35), and $R = Q + 1 - s$. If $s \neq Q$, then $R \neq 1$. Then $R < (N+\gamma)/(N-2+\gamma)$, hence η is R-admissible in Ω from Theorem 2.1, and $R \neq 1$; hence (3.40) holds. If $s = Q \leq 1$, we get the result as in Theorem 3.9.

Now the second part of Theorem 3.10 gives us an interesting result of existence in an uppercritical case:

Corollary 3.11. *Let $Q \geq (N+\gamma)/(N-2+\gamma)$, with $\gamma \in [0,1]$. Let*

$$r > (N+\gamma)/2Q'. \tag{3.41}$$

Then
(i) any measure $\mu \in \mathcal{M}^+(\Omega)$ such that $\int_\Omega \rho d\mu < +\infty$ and $G^Q(\mu) \in L^r(\Omega, \rho^\gamma dx)$ is Q-admissible in Ω,
(ii) any function $h \in L^r(\Omega, \rho^\gamma dx)$ is Q-admissible in Ω.

Proof. (i) We have $Q > 1$ and $r > 1$. We apply the second part of Theorem 3.10 to μ with now $\widetilde{Q} = Qs$, and

$$s = \frac{1}{1 + Q(r-1)} \quad \text{and} \quad R = Qsr = \widetilde{Q}r. \tag{3.42}$$

From (3.41), R is subcritical: $R < (N+\gamma)/(N-2+\gamma)$. Then $G^{R/s}(\mu) = G^{Qr}(\mu) \in L^1(\Omega, \rho^\gamma dx)$. Hence $G^{R/s}(\mu)$ is R-admissible in Ω, from Theorem 2.2. Then μ is Q-admissible in Ω from Theorem 3.10, since $Q = \widetilde{Q}/s$.

(ii) From Remark 3.5, $G^Q(\mu)$ is also Q-admissible in Ω. Now the set H of functions $h \in L^r(\Omega, \rho^\gamma dx)$ of the form $G^Q(\mu)$ is dense in $L^r(\Omega, \rho^\gamma dx)$, since H contains the functions φ^Q for any $\varphi \in \mathcal{D}^+(\Omega)$, and $\mathcal{D}^+(\Omega)$ is dense into $L^{Qr,+}(\Omega, \rho^\gamma dx)$. Then for any $h \in L^r(\Omega, \rho^\gamma dx)$ there exists $h_n \in H$ such that $h_n \to h$ in $L^r(\Omega, \rho^\gamma dx)$. Then for a given $\alpha > 0$, there exists a function w_n a solution of

$$\begin{cases} -\Delta w_n = w_n^Q + \alpha h_n, & \text{in } \Omega, \\ w_n = 0 & \text{on } \partial\Omega, \end{cases}$$

as long as $\alpha\int_\Omega \rho^\gamma h_n dx \leq C^*$, from (3.24), where $C^* = C^*(N, Q, \gamma, \Omega)$, in particular for $\alpha = C^*/2\int_\Omega \rho^\gamma h\, dx$ and n large enough. Then

$$G(G^Q(\alpha h_n)) \leq \frac{1}{Q-1} G(\alpha h_n),$$

from Proposition 3.5, and $G(h_n) \to G(h)$ in $L^1(\Omega)$ from (**P**1), and after an extraction almost everywhere. Hence we can go to the limit and get

$$G(G^Q(\alpha h)) \leq \frac{1}{Q-1} G(\alpha h),$$

so that h is Q-admissible in Ω from Theorem 2.2. □

Remark 3.9. In the case $\gamma = 0$ of bounded measures, we get existence for any $\mu \in W^{-2,r}(\Omega)$. Thus we extend the existence result of [4, Corollary 3.2] for any $h \in L^r(\Omega)$, except for the critical case $r = N/2Q'$ when $Q > N(N-2)$. Here one can get the implication (i)$\Longrightarrow$(ii) directly, since for any $h \in L^r(\Omega)$, we have $G(h) \in W^{2,r}(\Omega)$; hence $G(h) \in L^r(\Omega)$ from the Sobolev injection, since $r > N/2Q'$.

3.9. **Equation with interior and boundary measures.** First we recall and extend the results of [6], which are the equivalent of Theorems 2.1, 2.2 and 2.3 at the boundary.

Theorem 3.12. *(i) Let $\lambda \in \mathcal{M}^+(\partial\Omega)$. Assume that (**H**1) holds. Then $P(\lambda) \in L^Q(\Omega, \rho\, dx)$, and*

$$G(P^Q(\lambda)) \leq C_0 P(\lambda) \qquad a.e.\ in\ \Omega, \tag{3.43}$$

for some $C_0 > 0$.

(ii) Let $\lambda \in \mathcal{M}^+(\partial\Omega)$ satisfying (3.43), and $Q \neq 1$. Then there exists a solution w of problem (1.6), for any $\tilde{\alpha} \geq 0$, small enough if $Q > 1$, such that

$$P(\tilde{\alpha}\lambda) \leq w \leq CP(\tilde{\alpha}\lambda) \qquad in\ \Omega, \tag{3.44}$$

and (3.43) is also a necessary condition for existence if $Q > 1$.

*(iii) When (**H**1) holds, any solution of (1.6) satisfies the estimate*

$$P(\tilde{\alpha}\lambda) \leq w \leq C(P(\tilde{\alpha}\lambda) + \rho) \qquad in\ \Omega. \tag{3.45}$$

Proof. These results follow from [6, Theorems 1.1 to 1.3] for $Q > 1$, and in fact (3.43) holds also for $Q = 1$. It remains true for $0 < Q < 1$, since

$$G(P^Q(\lambda)) \leq G(1 + P(\lambda)) \leq G(1) + CP(\lambda).$$

Now $G(1) \leq C\rho$ in Ω, with $C = C(N, \Omega)$. And $P(\lambda) \geq C\rho$ in Ω, with $C = C(N, \lambda, \Omega)$. The proofs of existence and a priori estimates can be extended as in Theorems 2.2 and 2.3. □

Remark 3.11. For any $\lambda \in \mathcal{M}^+(\partial\Omega)$, and any $Q > 0$, the existence implies that $P(\lambda) \in L^Q(\Omega, \rho\, dx)$, as in Remark 3.3.

Definition 3.2. For given $Q > 0$, $Q \neq 1$, and $\lambda \in \mathcal{M}^+(\partial\Omega)$, $\lambda \neq 0$, we shall say that λ *is Q-admissible on $\partial\Omega$* if the problem (1.6) admits a solution for $\tilde{\alpha} \geq 0$ small enough.

Now with the preceding result for the problem with an interior measure, we can consider more generally a problem with interior and boundary measures:

$$\begin{cases} -\Delta w = w^Q + \alpha\mu, & \text{in } \Omega, \\ w = \tilde{\alpha}\lambda & \text{on } \partial\Omega. \end{cases} \tag{3.46}$$

Theorem 3.13. *Let $\mu \in \mathcal{M}^+(\Omega)$ such that $\int_\Omega \rho d\mu < +\infty$, and $\lambda \in \mathcal{M}^+(\partial\Omega)$, $\lambda, \mu \neq 0$, and $\alpha, \tilde{\alpha} \geq 0$. If μ is Q-admissible in Ω and λ is Q-admissible on $\partial\Omega$, in particular if ($\boldsymbol{H}1$) holds, then problem (3.46) has a solution for α and $\tilde{\alpha}$ small enough, such that with another C*

$$w \leq C(G(\mu) + P(\lambda)). \tag{3.47}$$

Moreover if ($\boldsymbol{H}1$) holds, then any solution w of problem (3.46) satisfies the estimate

$$G(\alpha\mu) + P(\tilde{\alpha}\lambda) \leq w \leq C(G(\mu) + P(\lambda) + \rho) \tag{3.48}$$

where $C = C(N, Q, \Omega, \tilde{\alpha}\lambda, \alpha\mu)$.

Proof. In fact this problem can be reduced to an interior one. Let us set

$$w = P(\tilde{\alpha}\lambda) + y.$$

Then (3.46) is equivalent to

$$\begin{cases} -\Delta y = (P(\tilde{\alpha}\lambda) + y)^Q + \alpha\mu, & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.49}$$

Now (3.43) implies $P^Q(\lambda) \in L^1(\Omega, \rho\, dx)$ and

$$G(G^Q(P^Q(\lambda))) \leq CG(P^Q(\lambda)); \tag{3.50}$$

hence $P^Q(\lambda)$ is Q-admissible in Ω, so that the problem

$$\begin{cases} -\Delta z = M_Q z^Q + M_Q P^Q(\tilde{\alpha}\lambda) + \alpha\mu, & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.51}$$

with $M_Q = \max(1, 2^{Q-1})$, admits a solution z, for $\alpha, \tilde{\alpha} \geq 0$ small enough if $Q > 1$, for any $\alpha, \tilde{\alpha} \geq 0$ if $Q < 1$. Moreover,

$$z \leq CG(P^Q(\lambda) + \mu) \leq C(P(\lambda) + G(\mu)), \tag{3.52}$$

for some $C > 0$, from Theorem 3.4. Then z is a supersolution of problem (3.49). Then this problem admits a solution $y \leq z$. Hence problem (3.46) has a solution satisfying (3.47). If (H1) holds, for any solution w of (3.46), y is a subsolution of (3.51); hence from Theorem (2.3) it satisfies

$$y \leq CG(P^Q(\lambda) + \mu + \rho) \leq C(P(\lambda) + G(\mu) + \rho),$$

and (3.48) follows. □

Remark 3.12. Reciprocally, if problem (3.46) admits a solution with $\alpha, \beta > 0$, then (3.15) and (3.43) hold, since existence for $\alpha, \tilde{\alpha} > 0$ implies existence for $\alpha > 0, \tilde{\alpha} = 0$ and $\alpha = 0, \tilde{\alpha} > 0$. Also (3.43) is equivalent to (3.50).

In other words, λ is Q-admissible on $\partial\Omega$ if and only if $P^Q(\lambda)$ is Q-admissible in Ω.

As in Section 3.8, we can improve (3.43):

Theorem 3.14. *Let $\widetilde{Q} > 0$ and $R \in [\widetilde{Q}, \widetilde{Q} + 1)$, $R \neq 1$.*
i) Let $\lambda \in \mathcal{M}^+(\partial\Omega)$ such that λ is R-admissible on $\partial\Omega$. Then there exists $C > 0$ such that

$$G(P^{\widetilde{Q}}(\lambda)) \leq C\, P^s(\lambda) \qquad \text{in } \Omega, \tag{3.53}$$

*with $s = \widetilde{Q} + 1 - R$. In particular if (**H**1) holds, then (3.53) holds for any $s \in (\max(0, Q - 2/(N-2)), 1]$.*
ii) Let $\kappa \in L^{1/s}(\partial\Omega)$ such that $\kappa^{1/s}$ is R-admissible on $\partial\Omega$. Then κ is $(\widetilde{Q}/s)$-admissible on $\partial\Omega$.

Proof. i) This estimate cannot be reduced to an interior one. We can assume $\lambda \neq 0$. From Theorem 3.12, for $\varepsilon > 0$ small enough, the problem

$$\begin{cases} -\Delta\Phi = \Phi^R & \text{in } \Omega, \\ \Phi = \varepsilon\lambda & \text{on } \partial\Omega, \end{cases} \tag{3.54}$$

admits at least a solution such that $P(\varepsilon\lambda) \leq \Phi \leq C(P(\varepsilon\lambda) + \rho)$. Let again $\Psi = \Phi^s$. From Lemma 3.1, we still have $-\Delta\Psi \geq s\Phi^{\widetilde{Q}}$ in the weak sense. And Ψ is superharmonic; hence from the Herglotz theorem, $\Psi \geq G(-\Delta\Psi)$. Then

$$\varepsilon^p sG(P^{\widetilde{Q}}(\lambda)) \leq G(-\Delta\Psi) \leq \Phi^s \leq C^s P^s(\varepsilon\lambda);$$

hence (3.53) follows as above.

ii) Let us set $\lambda = \kappa^{1/s}$. Then problem (3.54) still admits a solution Φ for $\varepsilon > 0$ small enough, and $\Psi = \Phi^s \geq sG(\Phi^{\widetilde{Q}}) = sG(\Psi^{\widetilde{Q}/s})$ as above. Moreover $\Phi \geq P(\kappa^{1/s})$; hence from Jensen's inequality, $\Psi \geq P^s(\kappa^{1/s}) \geq (P(1))^{1-s}P(\kappa)$. Then

$$\Psi \geq C_\varepsilon(G(\Psi^{\widetilde{Q}/s}) + P(\kappa))$$

for some $C_\varepsilon > 0$. That means that Ψ is a supersolution of the problem

$$\begin{cases} -\Delta h = C_\varepsilon h^{\widetilde{Q}/s} & \text{in } \Omega, \\ \Phi = C_\varepsilon \kappa & \text{on } \partial\Omega, \end{cases} \tag{3.55}$$

in the integral sense. Then this problem admits a solution; hence κ is $(\widetilde{Q}/s)$-admissible on $\partial\Omega$. □

Remark 3.13. Here also one can give an expansion of the solutions by using Theorems 3.14 and 3.13.

Remark 3.14. If (**H**1) holds, we can make (3.53) precise for a pointwise singularity for any $a \in \partial\Omega$,

$$\begin{aligned} G(P^Q(\delta_a)) &\leq C \begin{cases} |x-a|^{N+1-(N-1)Q}\, P(\delta_a) & \text{if } Q \geq 1/(N-1), \\ \rho & \text{if } Q < 1/(N-1), \\ \rho(1+\ln(|x-a|) & \text{if } Q = 1/(N-1), \end{cases} \\ &\leq P^s(\delta_a) \end{aligned} \tag{3.56}$$

for any $s > 0$ such that $s \in [Q-2/(N-1), 1]$, with $C = C(N,\Omega,Q)$, independent of a. This was proved in [6] when $Q > 1$, and the proof is similar when $Q \leq 1$.

At last Corollary 3.11 gives us an existence result for the boundary problem (1.6) in the supercritical case:

Corollary 3.15. *Let $Q \geq (N+1)/(N-1)$ and $\lambda \in L^\tau(\partial\Omega)$ with*

$$\tau > (N-1)(Q-1)/2. \tag{3.57}$$

Then problem (1.6) *admits a solution for $\tilde{\alpha} \geq 0$ small enough.*

Proof. Let us define s, r and R by

$$s = \frac{1}{\tau} = \frac{1}{1+Q(r-1)}, \qquad R = Qsr.$$

We apply the second part of Theorem 3.14 to the measure λ, with $\widetilde{Q} = sQ$, and $R = Qsr = \widetilde{Q}r$. Assumption (3.57) reduces to $r > (N+1)/2Q'$. Thus R is subcritical: $R < (N+1)/(N-1)$. And $\lambda^\tau \in L^1(\partial\Omega)$; hence λ^τ is R-admissible on $\partial\Omega$. Then λ is Q-admissible on $\partial\Omega$. □

Remark 3.15. Notice that in the critical case $Q = (N+1)/(N-1)$, condition (3.57) reduces to $\tau > 1$, just as in Corollary 3.11, condition (3.41) reduces to $r > 1$ in the critical case $Q = (N+\gamma)/(N-2+\gamma)$.

4. The case of a system

4.1. **Setting of the problem.** We first consider the case of a system with interior measure data and Dirichlet conditions on $\partial\Omega$. First consider the case of an isolated singularity at the point $a \in \Omega$. We study the solutions of the system

$$\begin{cases} -\Delta u = v^p + \alpha\delta_a, & \text{in } \Omega, \\ -\Delta v = u^q + \beta\delta_a, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

with $\alpha, \beta \geq 0$, in the weak (or integral) sense, which means

$$u = G(v^p) + \alpha E_a, \qquad v = G(u^q) + \beta E_a.$$

The existence has been proved in [11] in the radial case. In the nonradial one, some partial results are given in [16] for the case of the biharmonic operator by constructing supersolutions radial with respect to a. The precise behaviour near a of the solutions has been obtained in [5, Theorems 4.3 and 5.1], for regular functions in $\overline{\Omega}\backslash\{a\}$. In particular one has the following estimates:

Theorem 4.1. *Assume ($\boldsymbol{S}_0$). Let $u, v \in C^2(\Omega\backslash\{a\})$ be any solutions of* (4.1)*. Then there exists a constant $C > 0$ such that*

$$u + v \leq C\,|x-a|^{2-N} \tag{4.2}$$

near a.

This result comes from a pointwise comparison property between u and v in Ω : Any $C^2(\Omega\backslash\{a\})$ solutions of system (4.1) satisfy for any domain $\Omega' \subset\subset \Omega$

$$u \leq \alpha E_a + \ell v^{(p+1)/(q+1)} + \max_{\partial\Omega'} u \qquad \text{in } \Omega'\backslash\{a\}, \tag{4.3}$$

with $\ell = ((q+1)/(p+1))^{1/(q+1)}$. As a consequence, v satisfies an inequality

$$-\Delta v \leq C(v^{\mathbf{P}} + E_a^q), \qquad \text{in } \Omega'\backslash\{a\}$$

where $\mathbf{P} < N/(N-2)$. Then the estimate on v followed from the Harnack inequality. We shall give another proof in the sequel, and will also prove that any solutions are in $C^2(\Omega\backslash\{a\})$ in case ($\mathbf{S}_0$) holds. Notice that if $\alpha > 0$, then from (2.8) one has $q = \max(p,q) < N/(N-2)$, and the local study of the system is easy: from the Hölder inequality

$$-\Delta(u+v) \leq u^q + v^p \leq (u+v)^q + 1 \qquad \text{in } \Omega\backslash\{a\},$$

and q is subcritical, so that we are reduced to the scalar case; see [5, Theorem 4.1]. The delicate case corresponds to $\alpha = 0$.

Now consider more generally the system (2.5) in Ω :

$$\begin{cases} -\Delta u = v^p + \alpha\mu & \text{in } \Omega, \\ -\Delta v = u^q + \beta\eta & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mu, \eta \in \mathcal{M}^+(\Omega)$, $\mu, \eta \neq 0$, and $\alpha, \beta \geq 0$. The equations hold in the integral sense as above. In particular, $v^p, u^q \in L^1(\Omega, \rho\, dx)$, and $u, v \in L^1(\Omega)$. Here also we look for existence results, estimates, and comparison results. The case of measures is much more difficult, for several reasons:

1°) *The existence with* $\alpha \neq 0$ *does not necessarily imply* $q < N/(N-2)$, contrary to the case of a Dirac mass. It *does not even imply that* μ *is* q-admissible in Ω. It only implies that $G^q(\mu) \in L^1(\Omega, \rho\, dx)$, since $u^q \geq G^q(\alpha\mu)$. 2°) *The functions* u, v *are no more regular*, and we have to solve technical difficulties due to the lack of regularity.

4.2. **Green's estimates.** Let us define for any $\gamma \in [0,1]$,

$$\mathbf{m}_\gamma = q(p - 2/(N - 2 + \gamma)). \tag{4.4}$$

Notice that condition $(\mathbf{S}\gamma)$ implies

$$\mathbf{m}_\gamma < \mathbf{P} < (N + \gamma)/(N - 2 + \gamma).$$

Then using Theorem 3.9 we find the following.

Corollary 4.2. *Let* $\eta \in \mathcal{M}^+(\Omega)$ *such that* $\int_\Omega \rho^\gamma d\eta < +\infty$ *with* $\gamma \in [0,1]$. *If* $(\boldsymbol{S}\gamma)$ *holds, then* $G^q(G^p(\eta)) \in L^1(\Omega)$, *and*

$$G^q(G^p(\eta)) \leq CG^{\mathbf{m}}(\eta), \tag{4.5}$$

for any $\mathbf{m} \in (\max(0, \mathbf{m}_\gamma), q]$, *with* $C = C(N, p, q, \mathbf{m}, \eta)$ (*independent of* η *if* $p > 1$). *In particular*

$$G^q(G^p(\eta)) \leq CG^{\mathbf{P}}(\eta), \tag{4.6}$$

and

$$G(G^q(G^p(\eta))) \leq CG(\eta) \tag{4.7}$$

with $C = C(N, p, q, \gamma, \eta)$.

Remark 4.1. For any $p, q > 0$, the condition (4.6) is equivalent to

$$G(G^p(\eta)) \leq CG^{(p+1)/(q+1)}(\eta). \tag{4.8}$$

Notice that condition (4.7) *is not symmetric in* p *and* q, since we have supposed $p \leq q$, and p can be different from q.

4.3. **First existence results.** First let us give a general existence result based on supersolutions.

Lemma 4.3. *Let $\mu, \eta \in \mathcal{M}^+(\Omega)$ such that $\int_\Omega \rho\, d\mu + \int_\Omega \rho\, d\eta < +\infty$. Assume that there exist $U \in L^q(\Omega, \rho\, dx)$ and $V \in L^p(\Omega, \rho\, dx)$ such that*

$$U \geq G(V^p + \alpha\mu), \qquad V \geq G(U^q + \beta\eta)$$

almost everywhere in Ω. Then there exists at least a solution of problem (2.5), *such that $G(\alpha\mu) \leq u \leq U$ and $G(\beta\eta) \leq v \leq V$.*

Proof. We can construct nondecreasing sequences $(u_n)_{n\geq 0}, (v_n)_{n\geq 1} \in L^1(\Omega)$ such that $u_0 = 0$, and

$$\begin{cases} v_{n+1} = G(u_n^q + \beta\eta), & \forall n \geq 0, \\ u_n = G(v_n^p + \alpha\mu), & \forall n \geq 1, \end{cases}$$

$$G(\beta\eta) \leq v_n \leq V, \qquad G(\alpha\mu) \leq u_n \leq U, \qquad \forall n \geq 1.$$

The function $v_1 = G(\beta\eta)$ is well defined, and $v_1 \leq V$. Then $v_1^p \in L^1(\Omega, \rho\, dx)$, so that u_1 is well defined in $L^1(\Omega)$, and

$$u_1 \leq G(V^p + \alpha\mu) \leq U.$$

Then $u_1^q \in L^1(\Omega, \rho dx)$, so that v_2 is well defined in $L^1(\Omega)$, and

$$v_1 \leq v_2 \leq G(U^q + \beta\eta) \leq V.$$

Then the construction follows by induction. Thus (v_n^p) and (u_n^q) are bounded in $L^1(\Omega, \rho\, dx)$; hence (u_n) and (v_n) are bounded in $L^1(\Omega)$, from (**P**1). Then $u_n \to u$, $v_n \to v$, $u_n^q \to \chi_1$ and $v_n^p \to \chi_2$ in $L^1(\Omega)$ and almost everywhere in Ω from the Beppo–Levy theorem. Then $\chi_1 = u^q$, $\chi_2 = v^p$, and

$$u = G(v^p + \alpha\mu), \qquad v = G(u^q + \beta\eta),$$

since G is continuous from $L^1(\Omega, \rho\, dx)$ to $L^1(\Omega)$, from (**P**1). Hence (u, v) is a solution of (2.5), such that $u \leq U$ and $v \leq V$. □

Remark 4.2. The set Σ of $(\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+$ for which there exists a solution of (2.5), is a neighborhood of $(0,0)$. If $\left(\overline{\alpha}, \overline{\beta}\right) \in \Sigma$, then $(\alpha, \beta) \in \Sigma$ for any $\alpha \in [0, \overline{\alpha}]$ and $\beta \in \left[0, \overline{\beta}\right]$, from Lemma 4.3. In particular, Σ is star-shaped.

Now we prove Theorem 2.4. Notice that we suppose only that $G^q(\mu) \in L^1(\Omega, \rho^\gamma dx)$, and *not that μ is q*-admissible in Ω.

Proof of Theorem 2.4. We will construct a supersolution (U, V) of system (2.5). Let $\left(\overline{\alpha}, \overline{\beta}\right) \neq (0,0)$, and $\theta = G^q(\overline{\alpha}\mu) + \overline{\beta}\eta$. Since (**C**$\gamma$) holds, we have $\int_\Omega \rho^\gamma d\theta < +\infty$. Since (**S**$\gamma$) holds, from Corollary 4.2, there exists $c_0 > 0$ such that

$$G(G^q(G^p(\theta))) \leq c_0 G(\theta).$$

Now let $\alpha = t^{1/q}\overline{\alpha}$ and $\beta = t\overline{\beta}$, with $t \in [0,1]$, and $V = a_0 G(t\theta)$ and $U = G(V^p + \alpha\mu)$, where $c_1 > 0$ is a parameter. Then

$$U^q + \beta\eta \leq M_q(a_0^{pq} G^q(G^p(t\theta)) + t\theta)$$

with $M_q = \max(1, 2^{q-1})$. Hence

$$G(U^q + \beta\eta) \leq M_q(a_0^{pq} t^{pq} c_0 + t) G(\theta) \leq V$$

as long as

$$M_q(a_0^{pq} t^{pq-1} c_0 + 1) \leq a_0. \tag{4.9}$$

If $pq > 1$, the relation is satisfied for any $a_0 > 0$ for small t. If $pq < 1$, it is satisfied for any $t > 0$, after choosing a_0 large enough. Then system (2.5) admits a solution (u, v), for any $\alpha, \beta > 0$ small enough if $pq > 1$, for any $\alpha, \beta > 0$ if $pq < 1$. More precisely, we construct a solution for $\alpha = t^{1/q}\overline{\alpha}$ and $\beta = t\overline{\beta}$ such that

$$v \leq a_0 G(G^q(\alpha\mu) + \beta\eta).$$

In any case, (2.6) and (2.7) follow. □

Remark 4.3. In this theorem the measure $G^q(\alpha\mu) + \beta\eta$ is involved. Notice that the question of existence (for small parameters α and β) for problem (2.5) relative to measures $(\alpha\mu, \beta\eta)$ is equivalent to the existence for the same problem with measures $(0, G^q(\alpha\mu) + \beta\eta)$, and also with measures $(G^p(\beta\eta) + \alpha\mu, 0)$. Indeed if for example (u, v) is a solution of problem (2.5) relative to $(\alpha\mu, \beta\eta)$, defining $(\widetilde{u}, \widetilde{v})$ by $\widetilde{u} = u - G(\alpha\mu)$ and $\widetilde{v} = v$, then

$$\widetilde{u} = G(\widetilde{v}^p), \quad \widetilde{v} \geq M'_q G(\widetilde{u}^q + G^q(\alpha\mu)) + \beta\eta$$

with $M'_q = \min(1, 2^{1-1/q})$; hence the problem with $(0, G^q(\alpha\mu) + \beta\eta)$ has a solution from Lemma 4.3. Reciprocally, if (U, V) is a corresponding solution of the problem with $(0, G^q(\alpha\mu) + \beta\eta)$, then $\widetilde{U} = U + G(\alpha\mu)$ and $\widetilde{V} = V$ satisfy

$$\widetilde{U} = G(\widetilde{V}^p) + G(\alpha\mu), \quad \widetilde{V} \geq M_q^{-1} G(\widetilde{U}^q) + G(\beta\eta);$$

hence the problem with $(\alpha\mu, \beta\eta)$ has a solution. The particular role of $G^q(\alpha\mu) + \beta\eta$ and not $G^p(\beta\eta) + \alpha\mu$ is due to the dissymmetry caused by the fact that we can have $p < q$.

Now we apply Theorem 2.4 to the case of a pointwise singularity and show that the result is sharp:

Corollary 4.4. *Assume $pq \neq 1$. Let $a \in \Omega$. Then system* (4.1) *has a solution for any $\alpha, \beta \geq 0$ (small enough if $pq > 1$) with $(\alpha, \beta) \neq (0,0)$ if and only if*

$$\min(\mathbf{P}, \mathbf{Q}) < \frac{N}{N-2}, \qquad \textit{and} \quad q < N/(N-2) \quad \textit{if}\, \alpha > 0.$$

Proof. The existence follows from Theorem 2.4. Reciprocally if system (4.1) admits a solution with $\alpha > 0$, then we have seen that $q < N/(N-2)$; hence ($\mathbf{S}_0$) holds. Now suppose that there exists a solution with $\beta > 0$ and $\alpha = 0$. Then $v \geq \beta E_a$, so that $u \geq \beta^p G(E_a^p)$. Let $B_r = B(a,r) \subset\subset \Omega$. Then there exists a constant $C_r > 0$ such that

$$\mathcal{G}(x,y) \geq C_r \left|x-y\right|^{2-N}$$

in $B(a,r)$ and $E_a(y) \geq C_r \left|y-a\right|^{2-N}$ in B_r. Now for any $x \in B(a,r/2)$

$$G(E_a^p)(x) = \int_\Omega \mathcal{G}(x,y)\mathcal{G}^p(y,a)dy \geq C_r^{p+1} \int_{B(x,|x-a|)} \left|x-y\right|^{2-N} \left|y-a\right|^{(2-N)p} dy$$

and $B(x,|x-a|) \subset B(a,2\,|x-a|)$; hence, with new constants C_r,

$$G(E_a^p)(x) \geq C_r \left|x-a\right|^{(2-N)(p+1)} \left|B(x,|x-a|)\right| \geq C_r \left|x-a\right|^{(2-N)p+2}.$$

Then $u^q \geq C\left|x-a\right|^{((2-N)p+2)q}$ near a. But $u^q \in L^1_{loc}(\Omega)$; hence $((2-N)p+2)q > N$, which means ($\mathbf{S}_0$) holds. □

4.4. **Sufficient conditions of existence.** The proof of Theorem 2.4 gives a sufficient condition for existence.

Theorem 4.5. *Let $pq > 1$. Assume that the measure $\theta = G^q(\mu) + \eta$ satisfies*

$$G(G^q(G^p(\theta))) \leq c_0 G(\theta) \tag{4.10}$$

for some $c_0 > 0$. Then the problem (2.5) *admits a solution, for any $\alpha, \beta \geq 0$, $(\alpha,\beta) \neq (0,0)$ small enough, satisfying* (2.6) *and* (2.7).

Remark 4.4. In the simple case $p = q$, the condition (4.10) is necessary and sufficient, and it is also equivalent to the scalar condition

$$\mu, \eta \text{ are } p\text{-admissible in } \Omega,$$

as can be shown easily by addition of the two equations. In the general case, the question is more complex, because of the dissymmetry. In the case of a pointwise singularity, condition (4.10) is also necessary, from Corollaries 4.2 and 4.4. For general measures, the problem is open.

Now we can give new sufficient conditions for existence, by using Theorems 3.10 and 4.5:

Corollary 4.6. *Suppose that $pq > 1$ and*

$$\theta = G^q(\mu) + \eta \quad \text{ is } \mathbf{P}\text{-admissible in } \Omega. \tag{4.11}$$

Then problem (2.5) *admits a solution for (α, β) small enough.*

Proof. Let us apply the first part of Theorem 3.10 to the measure θ, with $R = \mathbf{P}$ and $\widetilde{Q} = p$. Then $s = (p+1)/(q+1)$; hence there exists $C > 0$ such that

$$G(G^p(\theta)) \leq CG^{(p+1)/(q+1}(\theta).$$

Then with another $C > 0$,

$$G(G^q(G^p(\theta)) \leq CG(G^{\mathbf{P}}(\theta)) \leq CG(\theta). \qquad \square$$

This result reduces a part of the study of the existence to the scalar case: it shows that the existence of a solution of the equation

$$\begin{cases} -\Delta\Phi = \Phi^{\mathbf{P}} + \varepsilon\theta, & \text{in } \Omega, \\ \Phi = 0 \quad \text{on } \partial\Omega, \end{cases}$$

for small $\varepsilon > 0$ implies the existence of a solution of system (2.5) for small $\alpha, \beta > 0$.

Remark 4.5. The condition

$$\omega = \mu + G^p(\eta) \quad \text{ is } q\text{-admissible in } \Omega, \tag{4.12}$$

is also sufficient. Indeed it is stronger than (4.11): it implies

$$G(G^{\mathbf{P}}(G^q(\mu)) \leq CG(G^{\mathbf{P}}(\mu)) \leq CG(\mu)$$

since μ is $\mathbf{P}$-admissible in Ω, because $\mathbf{P} \leq q$. Also if $\eta \neq 0$, from the Hölder inequality, since $p \leq q$,

$$G(G^{\mathbf{P}}(\eta)) \leq CG^{(p+1)/(q+1)}(G^q(\eta)) \leq CG^{(p+1)/(q+1)}(\eta) \leq C(G(\eta)+1) \leq CG(\eta).$$

4.5. **Necessary conditions of existence.** Now let us give necessary conditions of existence. Suppose that system (2.5) has a nontrivial solution. Since by definition $u^q, v^p \in L^1(\Omega, \rho\, dx)$, and $u \geq G(\alpha\mu)$ and $v \geq G(\beta\eta)$, necessarily

$$G^q(\alpha\mu),\ G^p(\beta\eta) \in L^1(\Omega, \rho\, dx). \tag{4.13}$$

Now we prove a condition, which reduces the system to the scalar case, and has to be compared to Corollary 4.6.

Theorem 4.7. *Assume that the system* (2.5) *has at least a solution with* $\alpha \neq 0$ (*respectively* $\beta \neq 0$). *Then*

$$\mu \ (\textit{respectively } G^p(\eta)) \quad \textit{is } \mathbf{Q}\textit{-admissible in } \Omega. \tag{4.14}$$

Proof. We can assume μ nonidentically 0. Let

$$f = v^{(p+1)/(q+1)} \qquad \text{and} \quad g = u + f \in L^1(\Omega). \tag{4.15}$$

Now $v = G(u^q+\eta)$, and $p \leq q$; hence from Lemma 3.1 with $\theta = (p+1)/(q+1)$ if $p < q$, we have $\rho v^{(p-q)/(q+1)} u^q \in L^1(\Omega)$ and

$$-\Delta(v^{(p+1)/(q+1)}) \geq \frac{p+1}{q+1} v^{(p-q)/(q+1)} u^q$$

in the weak sense. Then

$$-\Delta g \geq \frac{p+1}{q+1} f^{(p-q)/(p+1)}(u^q + f^q) + \alpha\mu;$$

hence

$$-\Delta g \geq C g^{\mathbf{Q}} + \alpha\mu \tag{4.16}$$

in the weak sense, with $C = C(p,q) > 0$. That means g *is a supersolution of a scalar equation with the new parameter* $\mathbf{Q}$.

Then there exists a solution of

$$\begin{cases} -\Delta h = h^{\mathbf{Q}} + \alpha\mu, & \text{in } \Omega, \\ h = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.17}$$

and the conclusion follows. □

Remark 4.6. The existence also implies that

$$G^q(\mu) \text{ (respectively } \eta) \quad \text{is } p\text{-admissible in } \Omega. \tag{4.18}$$

Indeed if $\beta \neq 0$, $G^p(\eta)$ is $\mathbf{Q}$-admissible in Ω from (4.14), then it is p-admissible in Ω since $p \leq \mathbf{Q}$; hence η is p-admissible in Ω from Remark 3.5. But the existence for the pair (μ, η) also implies the existence for $(0, G^q(\mu) + \eta)$; hence if $\alpha \neq 0$, then $G^q(\mu)$ is p-admissible in Ω.

As a consequence, we also get properties relative to the size of α, β:

Proposition 4.8. *Assume $pq > 1$ and $\mu, \eta \neq 0$. Then the set Σ of $(\alpha,\beta) \in \mathbb{R}^+ \times \mathbb{R}^+$ for which there exists a solution of* (2.5) *is bounded.*

Proof. The problem (4.17) has a solution; hence α is bounded, since $\mathbf{Q} > 1$. Let $\varphi_1 > 0$ be the eigenfunction for λ_1 such that $\int_\Omega \varphi_1\, dx = 1$. Now from (4.16),

$$\lambda_1 \int_\Omega g\varphi_1\, dx = \int_\Omega g(-\Delta\varphi_1) dx \geq \int_\Omega g^{\mathbf{Q}} \varphi_1\, dx \tag{4.19}$$

since $\partial\varphi_1/\partial n \leq 0$. Then $\int_\Omega g\varphi_1\, dx \leq \lambda_1^{1/(\mathbf{Q}-1)}$, since $\mathbf{Q} > 1$, from the Hölder inequality. Now from (4.15) and

$$\int_\Omega g\varphi_1\, dx \geq \int_\Omega u\varphi_1\, dx \geq \int_\Omega G(v^p)\varphi_1\, dx \geq \beta^p \int_\Omega G^p(\eta)\varphi_1\, dx;$$

hence β is bounded. □

Remark 4.7. Consider the limit case $pq = 1$. As in Remark 3.6, a sufficient condition for existence (for any $\alpha, \beta \geq 0$), is that $M_q c_0 < 1$, from (4.9). We have also $\lambda_1 > 1$ as a necessary condition, from (4.19).

Remark 4.8. Another consequence of this section is that we can find again the result of Corollary 3.11 by using a system. Let $Q \geq (N+\gamma)/(N-2+\gamma)$ with $\gamma \in [0,1]$ and r satisfying (3.41). Let $\mu \in \mathcal{M}^+(\Omega)$ such that $\int_\Omega \rho \, d\mu < +\infty$ and $G^Q(\mu) \in L^r(\Omega, \rho^\gamma dx)$. Let us define p and q by

$$q = Qr \quad \text{and } p = Q/(Q(r-1)+1). \tag{4.20}$$

Then $p \leq q$, and

$$\mathbf{Q} = p(q+1)/(p+1) = Q, \quad \text{and} \quad \mathbf{P} = q(p+1)/(q+1) = pr;$$

hence, $\mathbf{P} <(N+\gamma)/(N+\gamma-2)$. Then system (2.5) with coefficients p and q and measure data (μ, η) with $\eta = 0$ admits a solution for small α, since $G^q(\mu) \in L^1(\Omega, \rho^\gamma dx)$. Then μ is $\mathbf{Q}$-admissible in Ω, from Theorem 4.7, which proves the first part of Corollary 3.11. In fact it was our initial proof of the result.

4.6. **Comparison properties.** Using the results of Section 3.8, we get a first result of comparison of the solutions in the subcritical cases, fundamental for obtaining a priori estimates:

Corollary 4.9. *Let $\mu, \eta \in \mathcal{M}^+(\Omega)$ be such that $\int_\Omega \rho^\gamma d\mu + \int_\Omega \rho^\gamma d\eta < +\infty$. Under the assumption ($\boldsymbol{S}\gamma$), any solution (u,v) of system (2.5) such that $u^q \in L^1(\Omega, \rho^\gamma dx)$ satisfies the inequality*

$$u \leq G(\alpha\mu) + Cv^s \tag{4.21}$$

in Ω, for any $s \in (\max(0, p - 2/(N-2+\gamma)), 1]$, where $C = C(N,p,q,s,\gamma)$. In particular

$$u \leq G(\alpha\mu) + Cv^{(p+1)/(q+1)}, \tag{4.22}$$

with $C = C(N,p,q,\eta)$.

Proof. We apply Theorem 3.9 to measure

$$\chi = -\Delta v = u^q + \beta\eta,$$

with $Q = p$. Since $\int_\Omega \rho^\gamma d\chi < +\infty$, and $p \leq \mathbf{P} < (N+\gamma)/(N-2+\gamma)$, we find

$$G(G^p(\chi)) \leq CG^s(\chi),$$

which means

$$G(v^p) = u - G(\alpha\mu) \leq Cv^s;$$

hence (4.21) follows, and (4.22) with $s = (p+1)/(q+1)$. $\square$

Now we prove that the result of comparison (4.22) between the solutions u and v is in fact completely general, available for any solutions of the system. In the proof we have to solve some technical difficulties, due to the lack of regularity of the solutions. We need a lemma, which is an extension of the Kato inequality.

Lemma 4.10. *Let $U = G(\mu) + P(\lambda)$, with $\mu \in \mathcal{M}(\Omega)$ and $\lambda \in \mathcal{M}^+(\partial\Omega)$. Let $F \in L^1(\Omega, \rho\, dx)$. Suppose that*

$$\mu \geq F \quad \text{in } \mathcal{D}'(\Omega), \quad \text{and} \quad FU^- \geq 0, \quad \text{a.e. in } \Omega.$$

Then $U \geq 0$ almost everywhere in Ω.

Proof. Let $\sigma = -\Delta U - F = \mu - F \geq 0$, and $\sigma_n \in \mathcal{D}(\Omega)$, $f_n \in \mathcal{D}(\Omega)$, $\lambda_n \in C^\infty(\partial\Omega)$, $\sigma_n \geq 0$ and $\lambda_n \geq 0$ be such that σ_n converges weakly to σ, f_n converges strongly to F in $L^1(\Omega, \rho\, dx)$, and λ_n converges weakly to λ. Let $U_n = G(f_n + \sigma_n) + P(\lambda_n)$. For any $\varepsilon > 0$, let

$$j_\varepsilon(t) = (\varepsilon^2 + t^2)^{1/2} - \varepsilon, \quad \text{if } t \leq 0, \qquad j_\varepsilon(t) = 0, \quad \text{if } t \geq 0.$$

Then

$$-\Delta(j_\varepsilon(U_n)) \leq j'_\varepsilon(U_n)(f_n + \sigma_n) \leq j'_\varepsilon(U_n) f_n$$

in the classical sense, since j_ε is convex and nonincreasing. Hence

$$\int_\Omega j_\varepsilon(U_n)(-\Delta\xi) dx \leq \int_\Omega j'_\varepsilon(U_n) f_n \xi\, dx$$

for any nonnegative $\xi \in C_0^{1,1}(\overline{\Omega})$, since $U_n \geq 0$ on $\partial\Omega$. Now U_n converges to U strongly in $L^1(\Omega)$, from (**P**2) and (**P**3), and after an extraction almost everywhere in Ω. Then we can pass to the limit on each side from the Lebesgue theorem, since $j(t) \leq |t|$ and $|j'(t)| \leq 1$. And $j'_\varepsilon(U)F \leq 0$ by hypothesis; hence

$$\int_\Omega j_\varepsilon(U)(-\Delta\xi) dx \leq 0.$$

Then we pass to the limit as $\varepsilon \to 0$ and get

$$\int_\Omega U^-(-\Delta\xi) dx \leq 0.$$

Taking $\xi = G(1)$, we deduce that $U^- = 0$; hence $U \geq 0$ almost everywhere in Ω. □

Remark 4.9. This result is a consequence of [14, Lemma 1.5] when $\mu = F \in L^1(\Omega, \rho\, dx)$. More generally, for any $U = G(\mu) + P(\lambda)$ with $\mu \in \mathcal{M}(\Omega)$ and

$\lambda \in \mathcal{M}(\partial\Omega)$, such that $\mu \geq F$ in $\mathcal{D}'(\Omega)$, where $F \in L^1(\Omega, \rho\, dx)$, we have, for any nonnegative $\xi \in C_0^{1,1}(\overline{\Omega})$,

$$\int_\Omega U^-(-\Delta\xi)dx + \int_{\{U<0\}} F\xi\, dx \leq \int_{\partial\Omega} (-\frac{\partial\xi}{\partial n})d(\lambda^-).$$

We deduce the following, which proves in particular Theorem 2.6:

Theorem 4.11. *Let $\mu, \eta \in \mathcal{M}^+(\Omega)$ be such that $\int_\Omega \rho\, d\mu + \int_\Omega \rho\, d\eta < +\infty$. Let $u, v \geq 0$ be such that $u^q, v^p \in L^1(\Omega, \rho\, dx)$, satisfying*

$$\begin{cases} 0 \leq -\Delta u \leq v^p + \alpha\mu, \\ \qquad -\Delta v \geq u^q, \end{cases}$$

in the weak sense. Then

$$u(x) \leq G(\alpha\mu) + \ell v(x)^{(p+1)/(q+1)} \tag{4.23}$$

almost everywhere in Ω, with $\ell = ((q+1)/(p+1))^{1/(q+1)}$.

Proof. Let $\varepsilon > 0$. The function $f = (v+\varepsilon)^{(p+1)/(q+1)}$ is superharmonic, since $p \leq q$. From Lemma 3.1, we have $(v+\varepsilon)^{(p-q)/(q+1)}u^q \in L^1(\Omega, \rho\, dx)$ and

$$-\Delta f \geq \frac{p+1}{q+1}(v+\varepsilon)^{(p-q)/(q+1)}u^q = \frac{p+1}{q+1}f^{(p-q)/(p+1)}u^q$$

in the weak sense, hence in $\mathcal{D}'(\Omega)$. That means we can write

$$-\Delta f = \frac{p+1}{q+1}f^{(p-q)/(p+1)}u^q + \mu_0$$

with $\mu_0 \geq 0$. Now $f^{p(q+1)/(p+1)} = (v+\varepsilon)^p \in L^1(\Omega, \rho\, dx)$ and

$$-\Delta(u - \alpha G(\mu)) \leq v^p \leq f^{p(q+1)/(p+1)} = f^{(p-q)/(p+1)}f^q$$

in the weak sense. Then by difference

$$-\Delta(\ell f + \alpha G(\mu) - u) \geq \ell^{-q}f^{(p-q)/(p+1)}(u^q - \ell^q f^q)$$

in $\mathcal{D}'(\Omega)$, and $u^q - \ell^q f^q \geq 0$ almost everywhere on the set $\{u \geq \ell f + \alpha G(\mu)\}$. Now we can apply Lemma 4.10 to the functions

$$U = \ell f + \alpha G(\mu) - u \quad \text{and} \quad F = \ell^{-q}f^{(p-q)/(p+1)}(u^q - \ell^q f^q).$$

Then $U \geq 0$ almost everywhere in Ω and

$$u(x) \leq \alpha G(\mu) + \ell(v+\varepsilon)^{(p+1)/(q+1)}$$

almost everywhere in Ω . Going to the limit as $\varepsilon \to 0$, we get (2.9). □

Remark 4.10. Under the assumptions of Theorem 2.6, the function $U = u - \alpha G(\mu)$ satisfies the inequality

$$-\Delta U \geq U^{\mathbf{Q}} \tag{4.24}$$

in the weak sense, which is interesting to compare with (4.16) and (4.25). In particular when $\mu = \eta = 0$, we find the system of inequalities in Ω

$$\begin{cases} -\Delta u \geq u^{\mathbf{Q}}, \\ -\Delta v \leq v^{\mathbf{P}}. \end{cases}$$

4.7. **A priori estimates.**

Proof of Theorem 2.5. Let u, v be any solutions of the problem (2.5). Then from Corollary 4.9 we have, for some $C > 0$,

$$-\Delta v \leq (G(\alpha\mu) + Cv^{(p+1)/(q+1)})^q + \beta\eta$$

in the weak sense. Hence with another C,

$$-\Delta v \leq Cv^{\mathbf{P}} + CG^q(\alpha\mu) + \beta\eta. \tag{4.25}$$

That means that v *is a subsolution of a problem of the form* (1.5) *with* $Q = \mathbf{P}$ *and with the measure* $CG^q(\alpha\mu) + \beta\eta$. Now $v = G(u^q + \beta\eta)$, and $\int_\Omega \rho^\gamma(u^q dx + \beta\, d\eta) < +\infty$ by hypothesis. Thus we can apply Theorem 2.3 and Remark 2.6 to deduce that with another C,

$$v \leq CG(G^q(\mu) + \eta).$$

And (2.6) follows, since also $u \geq G(\mu)$; hence

$$v = G(u^q + \beta\eta) \geq G(G^q(\alpha\mu) + \beta\eta). \tag{4.26}$$

Then

$$u = G(v^p + \mu) \leq CG(G^p(G^q(\mu)) + G^p(\eta) + \mu), \tag{4.27}$$

and (2.7) follows, since from (4.26)

$$u \geq G(G^p(G^q(\alpha\mu) + \beta\eta) + \mu).$$

Remark 4.11. If moreover the measure μ is q-admissible in Ω, then

$$G(G^p(G^q(\mu)) \leq CG(G^p(\mu)) \leq CG(\mu)$$

for some $C > 0$, since $p < (N + \gamma)/(N - 2 + \gamma)$. Thus the estimates imply

$$G(\alpha\mu + \beta\eta) \leq u + v \leq CG(\mu + \eta).$$

In particular in the case of an isolated singularity $a \in \Omega$, the measure δ_a is q-admissible in Ω from (2.8). Thus we find again the result of [5].

In the case of measures μ and η with compact support, we get local results as in the scalar case:

Theorem 4.12. *Let $\mu, \eta \in \mathcal{M}^+(\Omega)$ with $\mu + \eta$ with compact support K. Assume ($\boldsymbol{S}_0$). Then for any regular domain Ω' such that $K \subset \Omega' \subset\subset \Omega$, there exists $C' > 0$ such that*

$$v \leq C'(G(G^q(\mu) + \eta) + 1) \qquad a.e.\ in\ \Omega'. \tag{4.28}$$

Proof. We have $v \in L^k_{loc}(\Omega)$ for any $k \in [1, N/(N-2))$, and from (4.25),

$$-\Delta v \leq C v^{\mathbf{P}} + C G^q(\alpha\mu) \qquad \text{in } \mathcal{D}'(\Omega \backslash K).$$

Now $G(\mu) \in C^1(\overline{\Omega} \backslash K)$, and $\mathbf{P} < N/(N-2)$; hence by bootstrap $v \in L^\infty(\Omega \backslash K)$. This implies that $u \in C^0(\Omega \backslash K)$; hence v and $u \in C^2(\Omega \backslash K)$. In particular v is bounded on $\partial\Omega'$. Let y be harmonic in Ω', such that $y = v$ on $\partial\Omega'$, and let $z = v - y \geq 0$ in Ω'. Then there exists $C' > 0$ such that

$$\begin{aligned} -\Delta z &\leq C(z+y)^{\mathbf{P}} + C G^q(\alpha\mu) + \beta\eta \\ &\leq C' z^{\mathbf{P}} + C G^q(\alpha\mu) + \beta\eta + C' \qquad \text{in } \mathcal{D}'(\Omega'), \end{aligned}$$

and $z = 0$ on $\partial\Omega'$. Hence (4.28) holds from Theorem 3.6. □

4.8. **System with interior and boundary measures.** Here we extend the previous results to the general case of systems

$$\begin{cases} -\Delta u = v^p + \alpha\mu, \qquad -\Delta v = u^q + \beta\eta, \qquad \text{in } \Omega, \\ u = \tilde{\alpha}\lambda, \qquad v = \tilde{\beta}\kappa \qquad \text{on } \partial\Omega \end{cases} \tag{4.29}$$

where $\mu, \eta \in \mathcal{M}^+(\Omega)$ such that $\int_\Omega \rho\, d\mu + \int_\Omega \rho\, d\eta < +\infty$, and $\lambda, \kappa \in \mathcal{M}^+(\partial\Omega)$, and $\alpha, \beta, \tilde{\alpha}, \tilde{\beta} \geq 0$. The problem is still taken in the integral sense, so that any solutions satisfy $u^q, v^p \in L^1(\Omega, \rho\, dx)$. And $u \geq G(\alpha\mu) + P(\tilde{\alpha}\lambda)$ and $v \geq G(\beta\eta) + P(\tilde{\beta}\kappa)$, so that in particular the existence implies

$$G^q(\alpha\mu + P^p(\tilde{\beta}\kappa)) + P^q(\tilde{\alpha}\lambda) \in L^1(\Omega, \rho\, dx), \tag{4.30}$$

$$G^p(\beta\eta + P^q(\tilde{\alpha}\lambda)) + P^p(\tilde{\beta}\kappa) \in L^1(\Omega, \rho\, dx). \tag{4.31}$$

Here also we can reduce the problem (4.29) to an interior problem. We get the following:

Theorem 4.13. *Let $\mu, \eta \in \mathcal{M}^+(\Omega)$ be such that $\int_\Omega \rho\, d\mu + \int_\Omega \rho\, d\eta < +\infty$, and $\lambda, \kappa \in \mathcal{M}^+(\partial\Omega)$.*
(i) Assume that ($\boldsymbol{S}1$) holds, and

$$G^q(\mu) \in L^1(\Omega, \rho\, dx), \qquad P^q(\lambda) \in L^1(\Omega, \rho\, dx), \tag{\textbf{D}1}$$

or, more generally, that the measure

$$\Theta = G^q(\mu + P^p(\kappa)) + \eta + P^q(\lambda) \quad is\ \mathbf{P}\text{-}admissible\ in\ \Omega. \tag{4.32}$$

Then system (4.29) *admits a solution for* $\alpha, \beta, \tilde{\alpha}$ *and* $\tilde{\beta}$ *small enough if* $pq > 1$, *for any* $\alpha, \beta, \tilde{\alpha}, \tilde{\beta} \geq 0$ *if* $pq < 1$, *such that for some* $C > 0$,

$$\begin{aligned} v &\leq C(G(G^q(\mu + P^p(\kappa))) + \eta + P^q(\lambda)) + P(\kappa), \\ u &\leq C(\mu + P^p(\kappa) + G^p(G^q(\mu + P^p(\kappa)) + G^p(\eta + P^q(\lambda)) + P(\lambda). \end{aligned}$$

(*ii*) *If* (**S**1) *and* (**D**1) *hold, then any solutions of* (4.29) *such that* $u \in L^q(\Omega, \rho\, dx)$ *satisfy these estimates, and*

$$u \leq G(\alpha\mu) + P(\tilde{\alpha}\lambda) + Cv^s$$

in Ω, *for any* $\max(0, p - 2/(N - 2 + \gamma) < s \leq 1$, *and for some* $C > 0$.
(*iii*) *Any solutions of* (4.29) *satisfy with another* $C > 0$

$$u \leq G(\alpha\mu) + P(\tilde{\alpha}\lambda) + Cv^{(p+1)/(q+1)}.$$

Proof. First (**S**1) and (**D**1) imply (4.32). Indeed $P(\kappa) \in L^k(\Omega, \rho\, dx)$ for any $k \in [1, (N+1)/(N-1))$, from [6], hence for $k = p$, since $p < (N+1)/(N-1)$ from (**S**1). Now let us apply Theorem 3.14 to κ, with $Q = p$ and $R = \mathbf{P}$. We find

$$G^q(P^p(\kappa)) \leq CP^{\mathbf{P}}(\kappa)$$

for some $C > 0$. Now $\mathbf{P} < (N+1)/(N-1)$; hence with a new $C > 0$,

$$G(G^q(P^p(\kappa))) \leq CG(P^{\mathbf{P}}(\kappa)) \leq CG(\kappa);$$

thus $G^q(P^p(\kappa))$ is $\mathbf{P}$-admissible in Ω. Then from (**D**1), (4.32) holds.

Now assume (4.32). Here again we can reduce the problem to an interior one, by setting $u = P(\tilde{\alpha}\lambda) + U$ and $v = P(\tilde{\beta}\kappa) + V$. We get

$$\begin{cases} -\Delta U = (P(\tilde{\beta}\kappa) + V)^p + \alpha\mu, & \text{in } \Omega, \\ -\Delta V = (P(\tilde{\alpha}\lambda) + U)^q + \beta\eta & \text{in } \Omega, \\ U = V = 0 & \text{on } \partial\Omega; \end{cases} \tag{4.33}$$

hence we can apply Theorem 2.4 to (U, V) and get (i). Then (ii) and (iii) follow from Theorems 2.5 and 2.6, and Corollary 4.9. $\square$

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References

[1] D. Adams and M. Pierre, *Capacitary strong type estimates in semilinear problems,* Ann. Inst. Fourier, Grenoble 41 (1991), 117–135.
[2] H. Amann and P. Quittner, *Elliptic boundary value problems involving measures; existence, regularity, and multiplicity,* Adv. Differential Equations, 3 (1998)753–813.
[3] P. Baras and M. Pierre, *Singularités éliminables pour des équations semi-linéaires*, Ann. Inst. Fourier, Grenoble 34 (1984), 185–206.

[4] P. Baras and M. Pierre, *Critères d'existence de solutions positives pour des équations semi-linéaires non monotones,* Anal. non Linéaire, Ann. Inst. Henri Poincaré, 2, 43 (1985), 185–212.
[5] M.F. Bidaut-Véron, *Local behaviour of solutions of a class of nonlinear elliptic systems,* Advances in Diff. Equ., 5 (1-3) (2000), 147–192.
[6] M.F. Bidaut-Véron and L. Vivier, *An elliptic semilinear equation with source term involving boundary measures: the subcritical case,* Revista Matematica Iberoamericana (to appear).
[7] H. Brézis and X. Cabré, *Some simple nonlinear PDE's without solutions*, Boll. Unione Mat. Italiana, 8 (1998), 223–262.
[8] K.L. Chung and Z. Zhao, "From Brownian Motion to Schrödinger's equation," MR 96F:60140, Springer–Verlag, Berlin–Heidelberg–New-York, 1995.
[9] R. Dautrey and J.L. Lions, "Analyse Mathématique et Calcul Numérique," Masson, 2, 1987.
[10] D. De Figueiredo, P.L. Lions, and R. Nussbaum, *A priori estimates and existence of positive solutions of semilinear elliptic equations,* J. Math. Pures et Appl., 61 (1982), 41–63.
[11] M. García-Huidobro, R. Manásevich, E. Mitidieri, and C. Yarur, *Existence and nonexistence of singular positive solutions for a class of semilinear elliptic systems,* Arch. Rat. Mech. Anal. 140 (1997), 253–284.
[12] A. Kufner and B. Opic, "Hardy Type Inequalities," Pitman Research in Math. Series, Longman Scientific Technical, 1990.
[13] N. Kalton and I. Verbitsky, *Nonlinear equations and weighted norm inequalities,* Trans. Amer. Math. Soc., 351, 9 (1999), 3441–3497.
[14] M. Marcus and L. Veron, *The boundary trace of positive solutions of semilinear elliptic equations: the supercritical case,* J. Math. Pures Appl. (1998), 481–524.
[15] P.L. Lions, *Isolated singularities in semilinear problems,* Jl. Diff. Eq., 38 (1980), 441–450.
[16] R. Soranzo, *Isolated singularities of positive solutions of a superlinear biharmonic equation,* Quaderni Matematici, University of Trieste (1994), 1–32.
[17] K.O. Widman, *Inequalities for the Green's function and boundary continuity of the gradient of solutions of elliptic differential equations,* Math. Scand. 21 (1967), 17–37.
[18] Z. Zhao, *Green function for Schrödinger operator and conditioned Feynman–Kac gauge,* J. Math. Anal. Appl. 116 (1986), 309–334.