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Regular and singular solutions of a quasilinear equation with weights [1]

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Abstract. In this article we study the behavior near 0 of the nonnegative solutions of the equation

$$-\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u) = b(x)|u|^{\delta-1}u, \quad x \in \Omega \setminus \{0\},$$

where Ω is a domain of $\mathbb{R}^N$ containing 0, and $\delta > p-1 > 0$, a, b are nonnegative weight functions. We give a complete classification of the solutions in the radial case, and punctual estimates in the nonradial one. We also consider the Dirichlet problem in Ω.

1. Introduction and main results

Let Ω be a bounded regular domain of $\mathbb{R}^N$ ($N \geqslant 1$) containing 0. In this work we are concerned with the singularity problem of the behavior near 0 of the nonnegative solutions of the problem

$$-\mathrm{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big) = b(x)|u|^{\delta-1}u \quad \text{in } \Omega', \tag{SP}$$

where $\Omega' = \Omega \setminus \{0\}$. Here $\delta > p-1 > 0$, and a, b are nonnegative weight functions in Ω, and a is positive almost everywhere. We can suppose that $\Omega \Supset \mathcal{B} = \mathcal{B}(0,1)$. We also take an interest in the regular Dirichlet problem in Ω,

$$\begin{cases} -\mathrm{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big) = b(x)|u|^{\delta-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{DP}$$

which is closely linked to the singularity problem.

Many authors have dealt with the nonweighted case, i.e., with nonnegative solutions to the equation

$$-\Delta_p u = -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = |u|^{\delta-1}u, \tag{1.1}$$

where $\delta > p-1 > 0$. Two critical values of δ appear: the first one is

$$\frac{N(p-1)+p}{N-p} = P^* - 1, \tag{1.2}$$

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where $P^* = Np/(N-p)$ ($P^* = +\infty$ if $p \leqslant N$), and P^* is the critical value of the Sobolev imbedding $W^{1,p}(\Omega) \hookrightarrow L^{\delta+1}(\Omega)$. It is well known that the Dirichlet problem

$$\begin{cases} -\Delta_p u = |u|^{\delta-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

admits nontrivial nonnegative solutions whenever $\delta + 1 < P^*$, and this condition is necessary when Ω is starshaped. The second value is given by

$$\overline{P} = \frac{N(p-1)}{N-p}, \tag{1.4}$$

which we will call Serrin's exponent, involved in the singularity problem

$$-\Delta_p u = |u|^{\delta-1}u \quad \text{in } \Omega', \tag{1.5}$$

as well as in the Harnack properties of problem (1.3). Notice the relation

$$\overline{P} = \frac{P^*}{p'}, \tag{1.6}$$

where $p' = p/(p-1)$.

The first results about problem (1.5) were obtained for the radial case in [7–9] for $p = 2$, and later in [22] for general p. The behavior of radial solutions of (1.5) as δ crossed the value $\overline{P}$ was described in [15]: if $1 < p < N$, any u positive radial solution to (1.1) defined near 0 is bounded, or

$$\begin{array}{ll} u(r) \approx r^{(p-N)/(p-1)} & \text{when } \delta < \overline{P}, \\ u(r) \approx r^{(p-N)/(p-1)}|\log r|^{(N-p)/(p(p-1))} & \text{when } \delta = \overline{P}, \\ u(r) \approx r^{-p/(\delta+1-p)} & \text{when } \overline{P} < \delta < P^* - 1. \end{array}$$

In [2] one can find a complete classification of local and global radial solutions of any sign, for any δ. In the nonradial case, the behavior near 0 when $p = 2$ was studied in [18] for $\delta < N/(N-2)$, and at the same time in [13] for any $\delta < 2^* - 1$, where local and global results are established; see also [1] for the case $\delta = N/(N-2)$. In the general case $p > 1$, the behavior near 0 or infinity of nonradial positive solutions was obtained when $\delta < \overline{P}$ in [2]. Very recently the results of [13] have been extended to the case $\delta < P^* - 1$.

We are concerned here with the generalization of some of those results to the weighted problems (DP) and (SP). Several studies have been done for $b = 0$, see for example [21,6,19,17,16]. Up to now the only studies for $b \neq 0$ are related to the radial Dirichlet problem, see [12], or to the case $a = 1$, see in particular [4]. We will study both the radial and nonradial situations. The second one is much more complex. In the general case the weights (even when they are radial) can present many types of singularities, and not only at 0. In particular the presence of the weight a increases significantly the difficulty, since it concerns the derivatives up to the order 2, whereas b only concerns the terms of order 0. In the sequel we use suitable weighted spaces $L^s(\Omega, a)$, $L^s(\Omega, b)$ ($s \geqslant 1$), and Sobolev spaces $W^{1,p}(\Omega, a)$, $W_0^{1,p}(\Omega, a)$, see [16]. By solutions of (SP) (resp. (DP)), we mean functions $u \in$

$W^{1,p}_{\mathrm{loc}}(\Omega', a) \cap L^{\delta}_{\mathrm{loc}}(\Omega', b) \cap L^{\infty}_{\mathrm{loc}}(\Omega')$ (resp. $u \in W^{1,p}_0(\Omega, a) \cap L^{\delta}(\Omega, b)$), satisfying the equation in the sense of distributions. The notion of supersolution, namely

$$-\mathrm{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big) \geqslant b(x)|u|^{\delta-1}u \quad \text{in } \Omega' \text{ (resp. in } \Omega),$$

is also taken in that sense.

Section 2 is devoted to the radial case. Here the weigths are supposed to be radial:

$$a(x) = a(r), \quad b(x) = b(r), \quad r = |x|,$$

and, moreover, we look for radial nonnegative solutions of (SP). We are lead to the problem

$$-\big(A(r)\big|u'\big|^{p-2}u'\big)' = B(r)u^{\delta}, \quad r \in (0,1], \tag{SP$_r$}$$

where

$$A(r) := r^{N-1}a(r), \qquad B(r) := r^{N-1}b(r). \tag{1.7}$$

Concerning the weights, we suppose that

$$A, B \in L^1_{\mathrm{loc}}\big((0,1]\big), \qquad A^{1/(1-p)} \in L^1_{\mathrm{loc}}\big((0,1]\big). \tag{H$_1$}$$

This implies that the equation without second member

$$-\big(A(r)\big|w'\big|^{p-2}w'\big)' = 0, \quad r \in (0,1], \tag{1.8}$$

admits (besides the constant solutions) a solution h, given by

$$h(r) := \int_r^1 A^{1/(1-p)}\,\mathrm{d}t, \tag{1.9}$$

which plays a crucial role in the study. We call h a *fundamental solution* to the weighted p-Laplacian. It is easy to prove that if h is bounded near 0, then any solution of (SP$_r$) is bounded near 0. Thus the interesting case is when $\lim_{r\to 0^+} h(r) = +\infty$, which means

$$A^{1/(1-p)} \notin L^1(0,1). \tag{H$_2$}$$

In Section 2.1 we first give necessary conditions for existence of nontrivial solutions. In particular, under (H$_1$), (H$_2$), we have

$$B \in L^1\big((0,1)\big), \quad \text{and} \quad \sup_{0<r<1} h^{p-1}(r)\beta(r) < +\infty, \tag{1.10}$$

where

$$\beta(r) := \int_0^r B\,\mathrm{d}t. \tag{1.11}$$

In Section 2.2 we recall the results of [12] about the existence of nonnegative radial solutions of problem (SP_r), *bounded in* $\mathcal{B}$. They pointed out a critical value p^* for the existence of such solutions, which is the analogous of P^* for the weighted radial case. It is the critical value of the Hardy–Sobolev inequality in dimension 1 with weights A and B. That means p^* is the supremum of the $q \geqslant p$ such that for any φ absolutely continuous in $(0,1]$ with $\varphi(1) = 0$ and $\int_0^1 |\varphi'|^p A\,\mathrm{d}t < +\infty$,

$$\left(\int_0^1 |\varphi|^q B\,\mathrm{d}t\right)^{1/q} \leqslant C\left(\int_0^1 |\varphi'|^p A\,\mathrm{d}t\right)^{1/p}$$

for some $C = C(N,p,q) > 0$. Now setting

$$\overline{p} := \sup\Big\{d \geqslant p-1\colon \sup_{0<r<1} h^d(r)\beta(r) < +\infty\Big\}, \tag{1.12}$$

which is well defined from (1.10), they proved, under an additional assumption, that

$$p^* = p'\overline{p}. \tag{1.13}$$

We will see that $\overline{p}$ plays the role of a radial Serrin's number associated to the weighted problem (SP_r). To this end we first show that $\overline{p}$ can be characterized as

$$\overline{p} = \sup\left\{\delta \geqslant p-1\colon \int_0^1 h^\delta B\,\mathrm{d}t < +\infty\right\} \tag{1.14}$$

when $p-1 < \overline{p}$. Then in Section 2.3 we give a complete classification of the nonnegative solutions of the problem (SP_r):

Theorem 1.1. *Assume* (H_1) *and* (H_2). *Suppose that u is any nonnegative solution to* (SP_r).

(i) *Then there exists $C > 0$ such that the two following estimates hold near* 0:

$$u(r) \leqslant Ch(r), \tag{1.15}$$

$$u(r) \leqslant C\big(h^{(p-1)}(r)\beta(r)\big)^{-1/(\delta+1-p)} \tag{1.16}$$

$$\leqslant Cr^{-p/(\delta+1-p)}\left(\frac{\int_r^{2r} A\,\mathrm{d}t}{\int_r^{2r} B\,\mathrm{d}t}\right)^{-1/(\delta+1-p)}. \tag{1.17}$$

(ii) *If $p-1 \leqslant \overline{p} < \delta$, then, moreover,* $\lim_{r\to 0^+} u(r)/h(r) = 0$.

(iii) *If $p-1 < \delta < \overline{p}$, or if $\delta = \overline{p}$ and $\int_0^1 h^{\overline{p}} B\,\mathrm{d}t < \infty$, then either u is bounded near* 0*, or* $\lim_{r\to 0^+} u(r)/h(r) > 0$.

(iv) *If $\delta = \overline{p}$, and $\int_0^1 h^{\overline{p}} B\,\mathrm{d}t = \infty$, then* $\lim_{r\to 0^+} u(r)/h(r) = 0$. *If $p-1 < \overline{p}$, then*

$$u(r) \leqslant Ch(r)\big(\ell(r)\big)^{-1/(\overline{p}-p+1)} \quad \textit{near } 0, \tag{1.18}$$

for some $C > 0$, where $\ell(r) = \int_r^1 h^{\overline{p}} B\,\mathrm{d}t$.

Among other things, this theorem points out the two aspects of the problem: the estimate (1.15) is due to the fact that the function is a supersolution of the equation without second member (1.8), and the estimate (1.16) is due to the effect of the second member bu^δ. They extend the well known estimates of the nonweighted case, namely

$$u(r) \leqslant C \min\bigl(r^{(p-N)/(p-1)}, r^{-p/(\delta+1-p)}\bigr) \quad \text{near } 0.$$

Our result also shows the role of the integrability of the fundamental solution h^δ (with respect to the weight B) in the behavior of the solutions, see also [11]. We give a few examples in Section 2.4. We note here, that as a first difference with the nonweighted case, the solutions of (SP$_r$) can behave like the fundamental solution at the critical number $\overline{p}$. Also $\overline{p}$ can be equal to $p-1$. Clearly $\overline{p}$ can be infinite, and not only as in the nonweighted case when $N = p$. Finally, in Section 2.5, under some more assumptions on the weights, we complete our classification results in case $\delta > \overline{p}$.

Section 3 concerns the regular Dirichlet problem in Ω. Here we assume that the weights a, b are globally admissible in a sense we will precise. It implies that $a, b \in L^1(\Omega)$, and a Sobolev inequality holds, with weights a and b, i.e., there exists $\kappa > p$ such that for any $\varphi \in \mathcal{D}(\Omega)$,

$$\left(\int_\Omega |\varphi|^\kappa b \,\mathrm{d}x\right)^{1/\kappa} \leqslant C\left(\int_\Omega |\nabla\varphi|^p a \,\mathrm{d}x\right)^{1/p}$$

with $C = C(N, p, q, \Omega, a, b)$. This in turn guarantees the continuity of the imbedding $W_0^{1,p}(\Omega, a) \hookrightarrow L^\kappa(\Omega, b)$. In Section 3.1, we discuss about global Harnack properties with such weights for the nonnegative solutions of equation

$$-\operatorname{div}\bigl(a(x)|\nabla u|^{p-2}\nabla u\bigr) = H(x)b(x)|u|^{p-2}u, \tag{1.19}$$

in Ω, under suitable conditions on the function H. In Section 3.2 we deduce the existence of bounded solutions of (DP) under a compactness assumption. We get the following, which mainly extends the results of [12] to the nonradial case:

Theorem 1.2. *Assume that (a, b) is globally κ-admissible in Ω, and the imbedding $W_0^{1,p}(\Omega, a) \hookrightarrow L^q(\Omega, b)$ is compact for any $p < q < \kappa$. Let $p - 1 < \delta < \kappa - 1$. Then there exists a nontrivial nonnegative bounded solution u of the Dirichlet problem in Ω*:

$$\begin{cases} u \in W_0^{1,p}(\Omega, a) \cap L^\infty(\Omega), \\ -\operatorname{div}\bigl(a(x)|\nabla u|^{p-2}\nabla u\bigr) = b(x)u^\delta \quad \text{in } \Omega. \end{cases} \tag{1.20}$$

We give some applications in Section 3.3.

Section 4 is devoted to the problem of the behavior near 0 of the solutions of (SP) in Ω' in the nonradial case. In [4] the first results have been given in the case $a \equiv 1$ and b is a power of $|x|$:

Theorem 1.3 ([4]). *Assume $N > p > 1$, and $u \in C^0(\Omega')$ is a nonnegative solution of*

$$-\Delta_p u = -\operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr) = |x|^\sigma u^\delta \quad \text{in } \Omega'.$$

Then if $p-1<\delta<\overline{P}$*, then* u *satisfies the Harnack inequality, and*

$$u(x)\leqslant C\min\bigl(|x|^{(p-N)/(p-1)},|x|^{-(p+\sigma)/(\delta-p+1)}\bigr)\quad \textit{near } 0.$$

With general weights we will need new assumptions of admissibility for the weights, following the guideline of [6,16], in order to obtain punctual estimates.

In Section 4.1 we first prove *two weak estimates*: an estimate of the minimum of u over any sphere of center 0 contained in $\mathcal{B}'=\mathcal{B}\setminus\{0\}$, and an integral estimate of u over any ball $B(x_0,2R)\subset\mathcal{B}'$, based on the multiplication of the equation by negative powers of u. We make the assumption

$$a,b\in L^1_{\mathrm{loc}}(\Omega'),\qquad a^{1/(1-p)}\in L^1_{\mathrm{loc}}(\Omega'),\tag{K$_1$}$$

which extend (H_1). Notice that it does not imply the existence of a fundamental solution.

Theorem 1.4. *Assume* (K_1)*. Let* u *be any nonnegative supersolution of* (SP) *in* Ω'*. Then there exists* $C>0$ *such that for any* $x_0\in\mathcal{B}'$*,*

$$\inf_{|x|=|x_0|}u\leqslant C\biggl(\int_{|x_0|\leqslant|x|\leqslant 1}|x|^{(1-N)p'}a^{1/(1-p)}\,\mathrm{d}x+1\biggr),\tag{1.21}$$

and for any $R>0$ *such that* $B(x_0,4R)\subset\mathcal{B}'$*,*

$$\left(\frac{\int_{\mathcal{B}(x_0,R)}bu^\delta\,\mathrm{d}x}{\int_{\mathcal{B}(x_0,R)}b\,\mathrm{d}x}\right)^{1/\delta}\leqslant CR^{-p/(\delta+1-p)}\left(\frac{\int_{\mathcal{B}(x_0,2R)}a\,\mathrm{d}x}{\int_{\mathcal{B}(x_0,R)}b\,\mathrm{d}x}\right)^{1/(\delta-p+1)}.\tag{1.22}$$

In Section 4.2 we discuss about *local* Harnack properties for the solutions of Eq. (1.19) in any domain D of $\mathbb{R}^N$, under local conditions on H. They suppose that a and the pair (a,b) satisfy local admissibility conditions. In particular we assume that two Sobolev–Hardy inequalities hold in any ball $B(x_0,R)\subset D$, namely that there exists some Q and $K>p$ such that for any $\varphi\in\mathcal{D}(B(x_0,R))$,

$$\left(\frac{\int_{B(x_0,R)}|\varphi|^Q a\,\mathrm{d}x}{\int_{B(x_0,R)}a\,\mathrm{d}x}\right)^{1/Q}\leqslant CR\left(\frac{\int_{B(x_0,R)}|\nabla\varphi|^p a\,\mathrm{d}x}{\int_{B(x_0,R)}a\,\mathrm{d}x}\right)^{1/p}\tag{1.23}$$

and

$$\left(\frac{\int_{B(x_0,R)}|\varphi|^K b\,\mathrm{d}x}{\int_{B(x_0,R)}b\,\mathrm{d}x}\right)^{1/K}\leqslant CR\left(\frac{\int_{B(x_0,R)}|\nabla\varphi|^p a\,\mathrm{d}x}{\int_{B(x_0,R)}a\,\mathrm{d}x}\right)^{1/p}\tag{1.24}$$

for some $C=C(N,p,Q,K,D)>0$. We will say that a is locally Q-admissible and (a,b) is locally K-admissible in D. The first condition, (1.23), ensures that the full Harnack inequality holds for the equation without second member

$$-\mathrm{div}\bigl(a(x)|\nabla u|^{p-2}\nabla u\bigr)=0\quad\text{in } D.\tag{1.25}$$

The second one, (1.24), takes into account the second member. This extends the classical results of [24,27,14] in the nonweighted case, and those of [6,5] in the weighted one.

In Section 4.3 we deduce punctual estimates with general weights, by taking $D = \Omega'$. Our main result is the following:

Theorem 1.5. *Assume that a is locally Q-admissible and (a, b) is locally K-admissible in Ω' for some $Q, K > p$. Let*

$$\overline{Q} = \frac{Q}{p'}, \qquad \overline{K} = \frac{K}{p'},$$

be the Serrin's numbers associated to $Q,\ K$. If $\delta < \min(\overline{Q}, \overline{K})$, then any solution u of (SP) *satisfies the following estimate: for any $x_0 \in \mathcal{B}'$,*

$$u(x_0) \leqslant C\left(|x_0|^{-p} \frac{\int_{\mathcal{B}(x_0,|x_0|/4)} a\,\mathrm{d}x}{\int_{\mathcal{B}(x_0,|x_0|/4)} b\,\mathrm{d}x}\right)^{1/(\delta-p+1)}. \tag{1.26}$$

In particular, when $a = b$, we find the exact estimate of the nonweighted case

$$u(x_0) \leqslant C|x_0|^{-p/(\delta+1-p)} \quad \text{near } 0.$$

It is remarkable to see that it does not depend on the weight a. More generally, we deduce the following:

Theorem 1.6. *Assume that a is locally Q-admissible in Ω', and*

$$\frac{b(x)}{\int_{\mathcal{B}(|x_0|,R)} b\,\mathrm{d}x} \leqslant \frac{a(x)}{\int_{\mathcal{B}(|x_0|,R)} a\,\mathrm{d}x}, \tag{1.27}$$

for any $x_0 \in \mathcal{B}'$ and any $R \leqslant |x_0|/2$. If $\delta < \overline{Q}$, then any solution u of (SP) *in Ω' satisfies the Harnack inequality in $\mathcal{B}'$, and*

$$u(x_0) \leqslant C \min\left(\int_{|x_0|\leqslant|x|\leqslant 1} |x|^{(1-N)p'} a^{1/(1-p)}\,\mathrm{d}x, \left(|x_0|^{-p}\frac{a(x_0)}{b(x_0)}\right)^{1/(\delta-p+1)}\right). \tag{1.28}$$

In Section 4.4 we study the special case of radial weights. Then h is still a solution of problem (SP). If it is bounded, then, under the assumptions of Theorem 1.6, any solution u is bounded near 0. Now, assuming that h is unbounded, we show the following estimates, which extend precisely those of the radial case:

Theorem 1.7. *Assume that the weights $a,\ b$ are radial and satisfy* (H_1) *and* (H_2). *Let u be any nontrivial solution of* (SP).

(i) *If a is locally Q-admissible in Ω', and (a, b) is locally K-admissible in Ω', and $p - 1 < \delta < \min(\overline{Q}, \overline{K})$, then*

$$u(x) \leqslant C\left(h^{p-1}(|x|)\beta(|x|)\right)^{-1/(\delta-p+1)} \quad \textit{near } 0. \tag{1.29}$$

Moreover, $\liminf_{x\to 0} u(x) > 0$ *and* (1.10) *holds.*

(ii) *In particular, if a is locally Q-admissible in Ω', and for any $r \leqslant 1$ and any $r < s < 2r$,*

$$\frac{B(s)}{\int_r^{2r} B\,\mathrm{d}t} \leqslant C\frac{A(s)}{\int_r^{2r} A\,\mathrm{d}t}, \tag{1.30}$$

for some $C > 0$, and $\delta < \overline{Q}$, then u satisfies the Harnack inequality in $\mathcal{B}'$, and

$$u(x) \leqslant C\min\bigl(h(|x|), \bigl(h^{p-1}(|x|)\beta(|x|)\bigr)^{-1/(\delta-p+1)}\bigr). \tag{1.31}$$

In Section 4.5, we give some applications of those theorems. They show that the admissibility assumptions are not very constraining when the weights are singular only at 0, since they only concern Ω' and not Ω. Notice that the condition (1.30) is automatically satisfied in the case of two powers

$$a(x) = |x|^\theta, \qquad b(x) = |x|^\sigma, \tag{1.32}$$

where θ, σ are *any* reals. Moreover, $a(x)$ is Q-admissible, for any $Q < P^*$, where P^* is defined in (1.2), independent on γ, σ. Thus the results on the behavior near 0 require only that $\delta < \overline{P}$. In particular we cover the results of Theorem 1.3. On the contrary, for the existence of a bounded solution of the Dirichlet problem in Ω, we need that the pair (a, b) is κ-admissible, with $\kappa < \inf(P^*, p^*)$. This shows the difference between the problem near 0, which only requires Harnack properties in Ω' and the Dirichlet problem, which needs them in the whole Ω.

2. The radial case

We start this section by proving some basic facts concerning positive solutions to (SP_r). For the sake of completness, recall that, for any $s \geqslant 1$, $L^s((0,1), A)$ is the space of A-measurable functions u in $(0,1)$ such that

$$\|u\|_{L^s((0,1),A)} = \left(\int_0^1 u^s A\,\mathrm{d}t\right)^{1/s} < +\infty.$$

And $W^{1,p}((0,1), A)$ is the completion of

$$\{\varphi \in C^\infty((0,1)):\ \|\varphi\|_{1,p,A} = \|\varphi\|_{L^s((0,1),A)} + \|\varphi'\|_{L^s((0,1),A)} < +\infty\}$$

with respect to the norm $\|\cdot\|_{1,p,a}$. We define in the same way $L^s_{\mathrm{loc}}((0,1), A)$ and $W^{1,p}_{\mathrm{loc}}((0,1), a)$. Under the assumption ($\mathrm{H}_1$), the space $W^{1,p}_{\mathrm{loc}}((0,1), A)$ is contained in $W^{1,1}_{\mathrm{loc}}((0,1))$, hence in $C^0((0,1))$, see [16, Lemma 1.13].

2.1. Existence and upper estimates

First we study the case where h is bounded:

Proposition 2.1. *Assume* (H_1) *holds, with $A^{1/(1-p)} \in L^1(0,1)$. Then any nonnegative solution u of* (SP_r) *is bounded near* 0.

Proof. The function $A|u'|^{p-2}u'$ is nonincreasing, thus it has a limit $\lambda \in (-\infty, +\infty]$ as $r \to 0$. If $\lambda > 0$, then $u'(r) > 0$ for small r, hence u has a finite limit as $r \to 0$. If $\lambda \leqslant 0$, then $A^{1/(p-1)}|u'|$ is bounded, hence from the assumption, u' is integrable near 0 and the conclusion follows.

Now we suppose that h is unbounded.

Proposition 2.2. *Assume* (H_1), (H_2). *If there exists a nontrivial nonnegative solution u to problem* (P_r), *then necessarily $u \in L^\delta((0,1), B)$, $\liminf_{r\to 0} u(r) > 0$, and* (1.10) *holds. Moreover, any solution satisfies the estimates* (1.15), (1.16) *and* (1.17) *near* 0, *and $u(r)/h(r)$ has a finite limit L as $r \to 0$.*

Proof. Defining λ as above, we cannot have $\lambda > 0$: it implies that $u'(r) \geqslant CA(r)^{-1/(p-1)}$ near 0 for some $C > 0$, hence $u + Ch$ is nondecreasing near 0, which is impossible since its limit is $+\infty$. Then $\lambda \leqslant 0$, u is nonincreasing near 0, hence $u \geqslant C > 0$ near 0. And $A^{1/(p-1)}|u'|$ is bounded, hence $u \in L^\delta((0,1), B)$ and $B \in L^1(0,1)$ and β is well defined. Also

$$A|u'|^{p-2}u' = -\left(\frac{u'}{h'}\right)^{p-1}$$

is decreasing to a finite limit. Hence u/h has a finite limit, from l'Hospital's rule, and (1.15) holds. Next we make the change of variables

$$h = h(r), \qquad y(h) = u(r). \tag{2.1}$$

Since u is nonincreasing, y is nondecreasing for $h > 0$ sufficiently large, and the equation in (SP_r) transforms into

$$\frac{\mathrm{d}}{\mathrm{d}h}\left(\left(\frac{\mathrm{d}y}{\mathrm{d}h}\right)^{p-1}\right) + A^{1/(p-1)}By^\delta = \frac{\mathrm{d}}{\mathrm{d}h}\left(\left(\frac{\mathrm{d}y}{\mathrm{d}h}\right)^{p-1}\right) - \frac{\mathrm{d}\beta}{\mathrm{d}h}y^\delta = 0. \tag{2.2}$$

By concavity, we have that for large h

$$y(h) \geqslant Ch\frac{\mathrm{d}y}{\mathrm{d}h} \quad \text{for some } C > 0. \tag{2.3}$$

Integrating between h and $k > h$, we get with a new $C > 0$

$$y^\delta(h)(\beta(h) - \beta(k)) \leqslant \int_h^k \left|\frac{\mathrm{d}\beta}{\mathrm{d}h}\right| y^\delta \,\mathrm{d}\tau \leqslant \left(\frac{\mathrm{d}y}{\mathrm{d}h}\right)^{p-1}(h) \leqslant C\left(\frac{y(h)}{h}\right)^{p-1}.$$

Letting $k \to +\infty$, since $\beta(k) \to 0$, and returning to u, we get

$$u^\delta(r)h^{p-1}(r)\beta(r) \leqslant Cu^{p-1}(r),$$

and hence (1.16) holds. Now taking $h = h(r)$ and $k = h(2r)$, we get

$$u^\delta(r)h^{p-1}(r)\int_r^{2r} B \,\mathrm{d}t \leqslant Cu^{p-1}(r),$$

and from the Hölder inequality

$$r^p \leqslant \left(\int_r^{2r} A\,\mathrm{d}t\right)\left(\int_r^{2r} A^{-1/(p-1)}\,\mathrm{d}t\right)^{(p-1)} \leqslant \left(\int_r^{2r} A\,\mathrm{d}t\right)h^{p-1}(r),$$

hence (1.17) follows. □

Remark 2.1. In the case that h is bounded, it is possible that $B \notin L^1(0,1)$ and there can exist a solution $u > 0$ such that $\lim_{r\to 0} u(r) = 0$. Consider for example the equation $-u'' = r^{-\alpha}u^\delta$, where $A = 1$, $p = 2$, $B = r^{-\alpha}$, with $1 < \alpha < 2$. It admits solutions u such that $\lim_{r\to 0}(u(r)/r) > 0$, from the fixed point theorem.

2.2. The Dirichlet radial problem and the Serrin's radial number

Let us first recall the results of [12] concerning the radial Dirichlet problem in the ball $\mathcal{B}$: assuming that

$$\sup_{0<r<1} h^k(r)\beta(r) < \infty \quad \text{for some } k > p-1, \tag{2.4}$$

and defining

$$p^* := \sup\left\{q \geqslant p\colon \sup_{0<r<1} h^{q/p'}(r)\beta(r) < +\infty\right\} = p'\overline{p},$$

they obtained the following:

Theorem 2.1 ([12]). *Assume* (H$_1$), (H$_2$), *and* $B \in L^1((0,1))$. *Let* $p-1 < \delta < p^*-1$. *Then there exist at least one nontrivial solution* u *of the problem*

$$\begin{cases} u \in W^{1,p}\big((0,1)\big), & u(1) = 0, \\ -\big(A(r)|u'|^{p-2}u'\big)' = B(r)u^\delta & \text{a.e. in } (0,1), \end{cases}$$

and bounded in $[0,1]$.

Notice that the condition (2.4) is not necessary to define $\overline{p}$ and p^* from Proposition 2.2, it only guarantees that $p < p^*$, that means $p-1 < \overline{p}$.

Let us give an equivalent definition of $\overline{p}$:

Proposition 2.3. *Assume* (H$_1$), (H$_2$). *If* $p-1 < \overline{p}$, *then* $\overline{p}$ *is also characterized by* (1.14).

Proof. Let

$$\mathcal{W} := \left\{d \geqslant p-1\colon \int_0^1 h^d B\,\mathrm{d}t < +\infty\right\}, \qquad \mathcal{U} := \left\{d \geqslant p-1\colon \sup_{t\in(0,1)} h^d(t)\beta(t) < +\infty\right\}. \tag{2.5}$$

The set $\mathcal{U}$ is not empty, since it contains $p-1$, from Proposition 2.2. First observe that $\mathcal{W} \subset \mathcal{U}$, since for any $0 < r \leqslant 1$ and $d \in \mathcal{W}$, we have $\beta(r)h^d(r) \leqslant \int_0^r h^d B\,\mathrm{d}t$, and thus $\widetilde{p} \leqslant \overline{p}$. Now for any $d,\ d_1$ such that $p-1 < d < d_1 < \overline{p}$ and $0 < s < r \leqslant r_0 < 1$, we have by integrating by parts

$$\begin{aligned}\int_s^r h^d B\,\mathrm{d}t &= \int_s^r h^d \beta'\,\mathrm{d}t \leqslant h^d(r)\beta(r) - d\int_s^r h^{d-1}h'\beta\,\mathrm{d}t \\ &\leqslant h^{d_1}(r)\beta(r)h^{d-d_1}(r_0) - d\Big(\sup_{r\in(0,1]} h^{d_1}(r)\beta(r)\Big)\int_s^r h^{d-d_1-1}h'\,\mathrm{d}t \\ &\leqslant h^{d-d_1}(r_0)\Big(\sup_{r\in(0,1]} h^{d_1}(r)\beta(r)\Big)\frac{d_1}{d_1-d},\end{aligned}$$

hence $d \in \mathcal{W}$ and thus $(p-1, \overline{p}) \subset \mathcal{W}$ and $\overline{p} = \sup \mathcal{W}$. $\square$

Remark 2.2. When $p-1 < \overline{p}$, we have $p-1 \in \mathcal{W}$ from the Hölder inequality, hence

$$\text{either} \quad \mathcal{W} = \mathcal{U} = [p-1, \overline{p}), \quad \text{or} \quad \mathcal{W} = [p-1, \overline{p}) \subsetneqq \mathcal{U} = [p-1, \overline{p}].$$

The two cases can happen, see Section 2.4. If $p-1 = \overline{p}$, then $\mathcal{U} = \{p-1\}$ and $\mathcal{W} = \mathcal{U}$ or $\emptyset$.

Remark 2.3. From [12], the value p^* can be computed as

$$p^* = p' \liminf_{r\to 0^+} \frac{|\log \beta(r)|}{\log(h(r))},$$

hence

$$\overline{p} = \liminf_{r\to 0^+} \frac{|\log \beta(r)|}{\log(h(r))}. \tag{2.6}$$

2.3. Description of the behavior

Here we prove Theorem 1.1 and give some remarks.

Proof of Theorem 1.1. (i) The estimates follow from Proposition 2.2. Let $L = \lim_{r\to 0}(u(r)/h(r))$.

(ii) Assume $\delta > \overline{p}$, then $\sup_{0<r<1} h^\delta(r)\beta(r) = +\infty$. Then there exists a sequence (r_n) tending to 0, such that $\lim_{n\to\infty} h^\delta(r_n)\beta(r_n) = \infty$. From (1.16), we have

$$\frac{u(r_n)}{h(r_n)} \leqslant C\big(h^\delta(r_n)\beta(r_n)\big)^{-1/(\delta-p+1)},$$

hence $\lim_{n\to\infty} u(r_n)/h(r_n) = 0$. Then necessarily $L = 0$.

(iii) Assume $p-1 < \delta < \overline{p}$. Suppose that $L = 0$. Then, from concavity, the function y/h is nonincreasing and tends to 0 at infinity. Then for any $\varepsilon > 0$ there exists $h_1 > 0$ such that $\sup_{h\geqslant h_1} y(h)/h \leqslant \varepsilon$; and we have $\lim_{h\to+\infty} \mathrm{d}y/\mathrm{d}h = 0$, hence

$$\left(\frac{\mathrm{d}y}{\mathrm{d}h}\right)^{p-1}(h) = \int_h^\infty A^{1/(p-1)} B y^\delta\,\mathrm{d}\tau, \tag{2.7}$$

and for any $h > h_1$,

$$\left(\frac{\mathrm{d}y}{\mathrm{d}h}\right)^{p-1}(h) \leqslant \left(\frac{y}{h}\right)^{\delta} \int_h^{\infty} A^{1/(p-1)} b h^{\delta}\,\mathrm{d}\tau = \left(\frac{y}{h}\right)^{\delta} \int_0^r h^{\delta} B\,\mathrm{d}t$$

and $\lim_{r\to 0} \int_0^r h^{\delta} B\,\mathrm{d}t = 0$ from (1.14); hence we can choose h_1 large enough such that

$$\left(\frac{\mathrm{d}y}{\mathrm{d}h}\right)^{p-1}(h) \leqslant \varepsilon^{(p-1-\delta)} \left(\frac{y}{h}\right)^{\delta}.$$

Thus integrating on (h_1, h), we deduce that

$$y^{1-\delta/(p-1)}(h) \geqslant y^{1-\delta/(p-1)}(h_1) - (\varepsilon h_1)^{1-\delta/(p-1)} > 0.$$

Then $y = u$ is bounded. If $\int_0^1 h^{\overline{p}} B\,\mathrm{d}t < \infty$, the proof given above is still valid for $\delta = \overline{p}$.

(iv) Suppose $\delta = \overline{p}$ and $\int_0^1 h^{\overline{p}} B\,\mathrm{d}t = +\infty$. Then $L = 0$. Indeed if $L > 0$, then by using (2.1) we find that

$$\left(\frac{\mathrm{d}y}{\mathrm{d}h}\right)^{p-1}(h) \geqslant \frac{L}{2} \int_{h_1}^{h} A^{1/(p-1)} B h^{\delta}\,\mathrm{d}\tau = \frac{L}{2} \int_r^{r_1} h^{\delta} B\,\mathrm{d}t$$

for $h > h_1 = h(r_1)$ large enough. Hence $\lim_{h\to+\infty} \mathrm{d}y/\mathrm{d}h = +\infty$, which contradicts (2.3). Hence (2.7) holds again. From (2.3) we also deduce

$$-\frac{\mathrm{d}}{\mathrm{d}h}\left(\left(\frac{\mathrm{d}y}{\mathrm{d}h}\right)^{p-1}\right) = A^{1/(p-1)} B y^{\overline{p}} \geqslant C A^{1/(p-1)} B h^{\overline{p}} \left(\frac{\mathrm{d}y}{\mathrm{d}h}\right)^{\overline{p}},$$

that is

$$-\frac{\mathrm{d}}{\mathrm{d}h}\left(\left(\frac{\mathrm{d}y}{\mathrm{d}h}\right)^{p-1}\right) = -\frac{\mathrm{d}\beta}{\mathrm{d}h} y^{\overline{p}} \geqslant -C \frac{\mathrm{d}\beta}{\mathrm{d}h} h^{\overline{p}} \left(\frac{\mathrm{d}y}{\mathrm{d}h}\right)^{\overline{p}}.$$

Thus setting $w = (\mathrm{d}y/\mathrm{d}h)^{p-1}$,

$$-w^{-\overline{p}/(p-1)} \frac{\mathrm{d}w}{\mathrm{d}h} \geqslant -C \frac{\mathrm{d}\beta}{\mathrm{d}h} h^{\overline{p}} = C \frac{\mathrm{d}\ell}{\mathrm{d}h}.$$

Integrating this relation, we find, since $\lim_{r\to 0} \ell(r) = +\infty$,

$$w^{1-\overline{p}/(p-1)} \geqslant C\ell,$$

with another $C > 0$, hence

$$\frac{\mathrm{d}y}{\mathrm{d}h} \leqslant C \ell^{-1/(\overline{p}-p+1)}.$$

And thus from (2.7),

$$\int_h^\infty A^{1/(p-1)} B y^{\overline{p}} \, \mathrm{d}\tau \leqslant C \ell^{-(p-1)/(\overline{p}-p+1)}. \tag{2.8}$$

On the other hand, y/h is nonincreasing, which yields

$$\int_h^\infty A^{1/(p-1)} B y^{\overline{p}} \, \mathrm{d}\tau \geqslant \left(\frac{y(h)}{h}\right)^{\overline{p}} \int_h^\infty A^{1/(p-1)} B h^{\overline{p}} \, \mathrm{d}\tau = \left(\frac{y(h)}{h}\right)^{\overline{p}} \ell(h),$$

and thus, combining with (2.8) we obtain

$$\left(\frac{y(h)}{h}\right)^{\overline{p}} \leqslant C \ell^{-\overline{p}/(\overline{p}-p+1)},$$

and (1.18) follows. □

Remark 2.4. Among other things, when $p-1 < \overline{p}$, this theorem shows that, under the assumption (1.10), the condition

$$\int_0^1 h^\delta B \, \mathrm{d}t < +\infty$$

is a necessary and sufficient condition for any unbounded solution u to (SP_r) to behave like the fundamental solution h. Arguing as in [10], one can prove the existence of such solutions, such that

$$u(1) = 0, \qquad A(1)\left|u'(1)\right|^{p-2} u'(1) = -\gamma,$$

where $\gamma > 0$ is chosen so that $\int_0^1 h^\delta B \, \mathrm{d}t < \gamma^{(p-1-\delta)/(p-1)}$.

2.4. Some examples

Here we give some applications extending the results of [15,3]. Also we show the links between the sets $\mathcal{U}$ and $\mathcal{W}$ defined in (2.5).

Example 1. We consider the problem (SP_r) with the weights equal to powers,

$$a(x) = |x|^\theta, \quad b(x) = |x|^\sigma, \quad \theta, \sigma \in \mathbb{R}.$$

Here $A(r) = r^{N-1+\theta}$, $B(r) = r^{N-1+\sigma}$. Then (H_1) is obviously satisfied, (H_2) means that $\theta > p - N$, (1.10) means $\sigma > -N$, and $\sigma + p \geqslant \theta$. Clearly, $\beta(r) = r^{N+\sigma}/(N+\sigma)$ and $h(r) \approx r^{(p-N-\theta)/(p-1)}$ near 0. Also, we find

$$\overline{p} = \frac{(N+\sigma)(p-1)}{N+\theta-p}, \qquad p^* = \frac{(N+\sigma)p}{N+\theta-p},$$

and the estimates (1.15), (1.16), reduce to

$$u(r) \leqslant C \min\bigl(r^{(p-N-\theta)/(p-1)}, r^{-(\sigma+p-\theta)/(\delta+1-p)}\bigr) \quad \text{near } 0.$$

Notice that here $\lim_{r\to 0} h^{\overline{p}}(r)\beta(r) = \lim_{r\to 0}\int_r^1 h^{\overline{p}} B\,\mathrm{d}t = c > 0$, thus $\mathcal{U} = \mathcal{W} = [p-1, \overline{p}]$. In the case $\delta = \overline{p}$, we find the estimate

$$u(r) \leqslant C r^{(p-N-\theta)/(p-1)} |\log r|^{-(N+\theta-p)/(p-1)(p+\sigma-\theta)} \quad \text{near } 0.$$

Example 2. Here we assume that

$$a(x) = 1, \quad b(x) = |x|^{\sigma}\left|\log\left|\frac{x}{2}\right|\right|^{k}, \quad \sigma, k \in \mathbb{R},$$

hence $A(r) = r^{N-1}$, $B(r) = r^{N-1+\sigma}|\log(r/2)|^{k}$. The assumption ($\mathrm{H}_2$) means that $N > p$, and we have $h(r) \approx r^{(p-N)/(p-1)}$, and $\beta(r) \approx r^{N-1+\sigma}|\log(r/2)|^{k}$ near 0. Then $h^{p-1}(r)\beta(r) \approx r^{p+\sigma}|\log r|^{k}$ near 0. Hence (1.10) means $\sigma > -N$, and $\sigma > -p$ or $\sigma = -p$ and $k \leqslant 0$. Also

$$\overline{p} = \frac{(N+\sigma)(p-1)}{N-p},$$

and the estimates (1.15), (1.16), reduce to

$$u(r) \leqslant C \min\bigl(r^{(p-N)/(p-1)}, \bigl(r^{(\sigma+p)}|\log r|^{k}\bigr)^{-1/(\delta+1-p)}\bigr) \quad \text{near } 0.$$

Notice that if $\sigma = -p$, then $\overline{p} = p - 1$. Now

$$h^{\overline{p}}(r)\beta(r) \approx |\log r|^{k} \quad \text{and} \quad \int_r^1 h^{\overline{p}} B\,\mathrm{d}t \approx \int_r^1 t^{-1}|\log t|^{k}\,\mathrm{d}t.$$

Hence $\overline{p} \in \mathcal{U}$ if and only if $k \leqslant 0$, and $\overline{p} \in \mathcal{W}$ if and only if $k < -1$.

Now consider the critical case $\delta = \overline{p}$ when $\sigma > -p$. If $k < -1$, Theorem 1.1(iii) applies, so that any unbounded solution behaves like h. Moreover there do exist such solutions. Indeed, it suffices to consider the problem (SP_r) with

$$u(r_0) = 0, \qquad r_0^{N-1}\bigl|u'(r_0)\bigr|^{p-2}u'(r_0) = -1,$$

and $r_0 > 0$ small to satisfy $|\log r_0|^{\kappa+1}/|k+1| < ((N-p)/(p-1))^{p-1}$. If $-1 \leqslant k \leqslant 0$, Theorem 1.1(iv) applies, and gives the estimate

$$u(r) \leqslant C r^{(p-N)/(p-1)} |\log r|^{-(k+1)/(\delta+1-p)} \quad \text{near } 0.$$

2.5. More precise asymptotics

Here we give more precise information on the behavior of the solutions in the case $\delta > \overline{p} > p-1$, in terms of the function h. We will do it under some additional assumptions on the weights. First observe that if

$$\liminf_{r\to 0^+} h^{\overline{p}}(r)\beta(r) > 0,$$

then from (1.16) we have an estimate of u in terms of h, namely

$$u(r) \leqslant Ch(r)^{(\overline{p}-p+1)/(\delta+1-p)} \quad \text{near } 0. \tag{2.9}$$

Let us define M by the relation

$$\frac{M(p-1)}{M-p} = \overline{p},$$

so that M plays the role of a dimension associated to $\overline{p}$. Now performing the change of variables

$$u(r) = w(s), \qquad s = \frac{M-p}{p-1} h^{-(p-1)/(M-p)}(r),$$

we are lead to the equation

$$-\frac{\mathrm{d}}{\mathrm{d}s}\left(s^{M-1}\left|\frac{\mathrm{d}w}{\mathrm{d}s}\right|^{p-2}\frac{\mathrm{d}w}{\mathrm{d}s}\right) = Q(r)s^{M-1}w^{\delta},$$

where

$$Q(r) = \frac{h^{\overline{p}+1}(r)B(r)}{|h'(r)|}.$$

If Q were constant, the problem would reduce to a nonweighted one, in the variable s and in dimension M, for which we know the complete classification of the solutions. We can hope to obtain a similar result when Q has a positive limit as r tends to 0. In the preceding works of [15,2], the usual changes of variable reduce the study to an autonomous equation of the second order in a cylinder. Here the equation is generally *nonautonomous*, and the problem of the convergence offers a particular interest. We obtain the following:

Theorem 2.2. *Assume* (H$_1$), (H$_2$), *with* $A,\ B$ *continuous in* $(0,1)$. *Let* $\delta > \overline{p} > p-1$, *with* $\delta+1 \neq p^*$. *Let* u *be any solution to* (SP$_r$).

Assume that the mapping

$$Q = \frac{h^{\overline{p}+1}B}{|h'|} = -h^{\overline{p}+1}\frac{\mathrm{d}\beta}{\mathrm{d}h}$$

has a derivative in $L^1((0,1))$, *and*

$$\ell_Q := \lim_{r \to 0} Q(r) > 0.$$

Then either $(u/h^{\overline{p}-p+1)/(\delta-p+1)})(r)$ *has a finite limit* $\ell > 0$ *as* r *goes to* 0, *or* u *is bounded.*

Proof. The assumption $\ell_Q > 0$ implies that $h^{\overline{p}}(r)\beta(r)$ has also a positive limit from l'Hospital's rule, since $Q = p\beta'/(h^{-\overline{p}})'$, hence (2.9) holds. We make the change of variable

$$u(r) = h^\tau(r)v(t), \quad t = \log\bigl(h(r)\bigr), \quad \tau = \frac{\overline{p} - p + 1}{\delta + 1 - p}.$$

This is the same as making the change of variable $y(h) = h^\tau v(t),\ t = \log h$, in (2.2). Since we have

$$\frac{\mathrm{d}y}{\mathrm{d}h} = h^{\tau-1}\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \tau v\right) \geqslant 0, \tag{2.10}$$

we obtain the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \tau v\right)^{p-1}\right) - (1-\tau)(p-1)\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \tau v\right)^{p-1} + Qv^\delta = 0. \tag{2.11}$$

It is not autonomous in general because of the coefficient Q. By (2.9), v is bounded, and $\mathrm{d}v/\mathrm{d}t$ is bounded from (2.10) and (2.3). As in [2], we can write (2.11) as a system:

$$\begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}t} = -\tau v + z^{1/(p-1)}, \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = (1-\tau)(p-1)z - Qv^\delta. \end{cases}$$

Related to this system we consider an energy function given by

$$V(t) = \frac{z^{p'}}{p'} - \tau vz + Q\frac{v^{\delta+1}}{\delta+1} - \Lambda\theta^{p-1}\frac{v^p}{p},$$

where the constant Λ is linked to the Sobolev exponent p^* by

$$\Lambda = p - 1 - p\tau = \frac{(p-1)(\delta+1-p^*)}{\delta+1-p} \neq 0.$$

After some computations we get

$$\frac{\mathrm{d}V}{\mathrm{d}t}(t) = \Lambda X(t) + \frac{\mathrm{d}Q}{\mathrm{d}t}\frac{v^{\delta+1}}{\delta+1}, \tag{2.12}$$

where

$$X = \left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)\left(z - \tau^{p-1}v^{p-1}\right) = \left(z^{1/(p-1)} - \tau v\right)\left(z - \tau^{p-1}v^{p-1}\right).$$

Since v, z, Q, and V are bounded, and $(\mathrm{d}Q/\mathrm{d}t)(t) = Q'(r)h(r)/h'(r)$ belongs to $L^1(1,\infty)$, we deduce that $X \in L^1(1,\infty)$ from (2.12). Now for all $x,\ y \geqslant 0$ it holds that

$$(x-y)\left(x^{p-1} - y^{p-1}\right) \geqslant \begin{cases} c_p(x-y)^2\left(|x|+|y|\right)^{p-2} & \text{if } p < 2, \\ |x-y|^p & \text{if } p \geqslant 2, \end{cases}$$

where $c_p > 0$ depends only on p. Hence, by setting $x = z^{1/(p-1)}$, $y = \tau v$, we find that $\mathrm{d}v/\mathrm{d}t \in L^2((1,+\infty))$ if $p < 2$, since v and z are bounded, and $\mathrm{d}v/\mathrm{d}t \in L^p((1,+\infty))$ if $p \geqslant 2$. Since $\mathrm{d}v/\mathrm{d}t$ is uniformly continuous, it follows that $\lim_{t\to\infty} \mathrm{d}v/\mathrm{d}t = 0$. Since $X(t) \geqslant 0$, the function

$$t \mapsto E(t) = V(t) - \int_1^t \frac{\mathrm{d}Q}{\mathrm{d}t}\frac{v^{\delta+1}}{\delta+1}\,\mathrm{d}s$$

is bounded and monotone, hence it has a finite limit. Then also V has a finite limit, since v is bounded and $\mathrm{d}Q/\mathrm{d}t \in L^1((1,+\infty))$. From the expression of V, we deduce that v has a finite limit $\ell \geqslant 0$ as $t \to +\infty$, hence $\lim_{r\to 0}(u/h^\tau)(r) = \ell$. Then we find

$$\lim_{t\to\infty} z(t) = (\tau\ell)^{p-1}, \qquad \lim_{t\to\infty} \frac{\mathrm{d}z}{\mathrm{d}t} = (1-\tau)(p-1)(\tau\ell)^{p-1} - \ell_Q\ell^\delta,$$

which yields

$$\ell = \left((1-\tau)(p-1)\tau^{p-1}\ell_Q^{-1}\right)^{1/(\delta-p+1)}, \quad \text{or} \quad \ell = 0.$$

Suppose that $\ell = 0$. Following the procedure in [15], we write our equation in the form

$$z(t) = \mathrm{e}^{(1-\tau)(p-1)t}\int_t^\infty \mathrm{e}^{-(1-\tau)(p-1)s} Q v^\delta\,\mathrm{d}s,$$

hence $z(t) \leqslant Cv^\delta(s)$, since v is nonincreasing for large t. Indeed at each point where $\mathrm{d}v/\mathrm{d}t = 0$, we have $\mathrm{d}^2v/\mathrm{d}t^2 \geqslant 0$. Then

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \tau v \leqslant Cv^{\delta/(p-1)},$$

hence

$$v(t) \leqslant \left(K\mathrm{e}^{(\tau(\delta-p+1)/(p-1))t} + \frac{C}{\tau}\right)^{-(p-1)/(\delta-p+1)},$$

which implies that $v(t)\mathrm{e}^{\tau t}$ is bounded, so that u is bounded. $\square$

Remark 2.5. The theorem applies in particular if Q is monotone and bounded near 0. This is the case in Example 1, and moreover, if we replace the function h in Theorem 2.2 by the function $r \mapsto \widetilde{h}(r) = \int_r^{+\infty} A^{-1/(p-1)}\,\mathrm{d}t$, then the function Q is constant.

Remark 2.6. One can also precise the behavior of the solutions in the critical case $\delta = \overline{p}$. We will not mention here the results, because of their technicality.

3. The nonradial regular Dirichlet problem

We consider here the general problem (DP) in the nonradial case, with eventually nonradial weights. For any $s \geqslant 1$, we denote by $L^s(\Omega, a)$ the space of a-measurable functions u in Ω such that

$$\|u\|_{L^s(\Omega,a)} = \left(\int_\Omega u^s a\,\mathrm{d}x\right)^{1/s} < +\infty.$$

And $W^{1,p}(\Omega, a)$ is the completion of

$$\left\{\varphi \in C^\infty(\Omega)\colon\ \|\varphi\|_{1,p,a} = \|\varphi\|_{L^s(\Omega,a)} + \big\|\,|\nabla\varphi|\big\|_{L^s(\Omega,a)} < +\infty\right\}$$

with respect to the norm $\|\cdot\|_{1,p,a}$, and $W_0^{1,p}(\Omega, a)$ the completion of $\mathcal{D}(\Omega)$. We define $L^s_{\mathrm{loc}}(\Omega, a)$ and $W^{1,p}_{\mathrm{loc}}(\Omega, a)$ in the same way, see [16,17].

3.1. Global Harnack properties

We will say that (a, b) is globally admissible in Ω if it satisfies the conditions:

(i) *Integrability*:

$$a \in L^1_{\mathrm{loc}}(\Omega), \quad b \in L^1(\Omega), \quad a^{1/(1-p)} \in L^1_{\mathrm{loc}}(\Omega). \tag{G$_1$}$$

(ii) *Sobolev–Hardy inequality*: there exists $\kappa > p$ such that, for any $\varphi \in \mathcal{D}(\Omega)$,

$$\left(\int_\Omega |\varphi|^\kappa b\,\mathrm{d}x\right)^{1/\kappa} \leqslant C\left(\int_\Omega |\nabla\varphi|^p a\,\mathrm{d}x\right)^{1/p}, \tag{G$_2$}$$

with $C = C(N, p, q, \Omega, a, b)$.

(iii) *Poincaré inequality*: for any $\varphi \in \mathcal{D}(\Omega)$,

$$\int_\Omega |\varphi|^p a\,\mathrm{d}x \leqslant C\int_\Omega |\nabla\varphi|^p a\,\mathrm{d}x, \tag{G$_3$}$$

with $C = C(N, p, \Omega, a) > 0$.

We will associate to κ two numbers: $\eta > p$ and $\overline{\kappa}$ defined by

$$\frac{\eta p}{\eta - p} := \kappa, \qquad \overline{\kappa} := \frac{\kappa}{p'} = \frac{\eta(p-1)}{\eta - p}. \tag{3.1}$$

Remark 3.1. The assumption (G_1) ensures that any $u \in W^{1,p}(\Omega, a)$ is in $L^1_{\mathrm{loc}}(\Omega)$, so that the gradient is well defined in $\mathcal{D}'(\Omega)$, and that $W^{1,p}(\Omega, a)$ and $W_0^{1,p}(\Omega, a)$ are Banach spaces, see [16,17].

First we give a global regularity result in Ω, which is an extension of well-known results in the non-weighted case, in case $p = 2$, see [14, Theorem 8.15]. We give the proof for a better comprehension of the results in Section 4.

Theorem 3.1. *Assume that* (G_1) *and* (G_2) *hold. Let* $u \in W_0^{1,p}(\Omega, a)$ *be any nonnegative solution of equation*

$$-\mathrm{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big) = H(x)b(x)u^{p-1} \quad \text{in } \Omega, \tag{3.2}$$

where $H \in L^s(\Omega, b)$ *for some* $s > \eta/p$, *with* η *given by* (3.1), $H \geqslant 0$. *Then for any* $\lambda \geqslant p$,

$$\sup_{\Omega} u \leqslant C\|u\|_{L^\lambda(\Omega,b)}, \tag{3.3}$$

where $C = C(N, a, b, s, \Omega, \|H\|_{L^s(\Omega,b)})$.

Proof. For any $\varphi \in \mathcal{D}(\Omega)$, we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi a \,\mathrm{d}x = \int_\Omega H u^{p-1}\varphi b \,\mathrm{d}x.$$

From (G_1), (G_2) and the assumption on H, it also holds by density for any $\varphi \in W_0^{1,p}(\Omega, a)$. Indeed $\varphi \in L^\kappa(\Omega, b)$ and defining $1 < t < \kappa$ by the relation $1/t = 1 - 1/s - (p-1)/\kappa$, we have

$$\int_\Omega H u^{p-1}|\varphi| b \,\mathrm{d}x \leqslant \left(\int_\Omega H^s b \,\mathrm{d}x\right)^{1/s}\left(\int_\Omega u^\kappa b \,\mathrm{d}x\right)^{(p-1)/\kappa}\left(\int_\Omega \varphi^t b \,\mathrm{d}x\right)^{1/t} < +\infty,$$

since $b \in L^1(\Omega)$. Let $\beta \geqslant 1$ and $\gamma = \beta + p - 1$. For any $n > \varepsilon > 0$, we set $u_\varepsilon = u + \varepsilon$, and consider a function $F \in C^1([\varepsilon, +\infty))$ defined by

$$F(z) = z^{\gamma/p} - \varepsilon^{\gamma/p} \quad \text{on } [\varepsilon, n], \quad F \text{ linear on } [n, +\infty).$$

Let us set $\varphi = G(u_\varepsilon)$, where

$$G(y) = \int_\varepsilon^y |F'(s)|^p \,\mathrm{d}s.$$

Since $\varphi \leqslant C_\varepsilon u$, we have $\varphi \in L^p(\Omega, a)$ and thus $\varphi \in W_0^{1,p}(\Omega, a)$ from the chain rule, see [16, Theorem 1.18 and Lemma 1.25]. Hence φ is an admissible test function, and we get

$$\int_\Omega |\nabla u|^p G'\big((u_\varepsilon)\big) a \,\mathrm{d}x \leqslant \int_\Omega H u_\varepsilon^{p-1} G(u_\varepsilon) b \,\mathrm{d}x \leqslant \int_\Omega H u_\varepsilon^p G'(u_\varepsilon) b \,\mathrm{d}x.$$

Thus we obtain

$$\int_\Omega \left|\nabla\left(F(u_\varepsilon)\right)\right|^p a\,\mathrm{d}x \leqslant \int_\Omega H u_\varepsilon^p \left|F'(u_\varepsilon)\right|^p b\,\mathrm{d}x.$$

And the function $F(u_\varepsilon) \in W_0^{1,p}(D,a)$, hence from ($G_2$),

$$\left(\int_\Omega \left(F(u_\varepsilon)\right)^\kappa b\,\mathrm{d}x\right)^{1/\kappa} \leqslant C\left(\int_\Omega \left|\nabla\left(F(u_\varepsilon)\right)\right|^p a\,\mathrm{d}x\right)^{1/p} \leqslant C\left(\int_\Omega H u_\varepsilon^p \left|F'(u_\varepsilon)\right|^p b\,\mathrm{d}x\right)^{1/p}$$
$$\leqslant \frac{\gamma}{p}\left(\int_\Omega H u_\varepsilon^{\gamma/p} b\,\mathrm{d}x\right)^{1/p}.$$

Making $n \to +\infty$, we get

$$\left(\int_\Omega \left(u_\varepsilon^{\gamma/p} - \varepsilon^{\gamma/p}\right)^\kappa b\,\mathrm{d}x\right)^{1/\kappa} \leqslant \left(\frac{\gamma}{p}\right)^p \int_\Omega H u_\varepsilon^{\gamma/p} b\,\mathrm{d}x.$$

Now $b \in L^1(\Omega)$ and thus $H \in L^1(\Omega, b)$. Then we can let $\varepsilon \to 0$, and setting $v = u^{\gamma/p}$, we get

$$\|v\|_{L^\kappa(\Omega,b)} \leqslant C\gamma\left(\int_\Omega H v^p b\,\mathrm{d}x\right)^{1/p} \leqslant \gamma\|H\|_{L^s(\Omega,b)}^{1/p}\|v\|_{L^{ps'}(\Omega,b)}$$
$$\leqslant \gamma\|H\|_{L^s(\Omega,b)}^{1/p}\left(\varepsilon\|v\|_{L^\kappa(\Omega,b)} + \varepsilon^{-\tau}\|v\|_{L^p(D,b)}\right),$$

where $\tau = \eta/(ps-\eta) > 0$, for any $\varepsilon > 0$, by interpolation. Taking $\varepsilon = (1/2\gamma)\|H\|_{L^s(\Omega,b)}^{1/p}$, we deduce

$$\|v\|_{L^\kappa(\Omega,b)} \leqslant C\left(\gamma\|H\|_{L^s(\Omega,b)}^{1/p}\right)^{1+\tau}\|v\|_{L^p(\Omega,b)}.$$

Then returning to u, it comes

$$\left(\int_\Omega u^{\gamma\kappa/p} b\,\mathrm{d}x\right)^{p/\gamma\kappa} \leqslant C\gamma^{(1+\tau)p/\gamma}\left(\int_\Omega u^\gamma b\,\mathrm{d}x\right)^{1/\gamma},$$

with another $C > 0$ depending on $\|H\|_{L^s(\Omega,b)}^{1/p}$. Taking a sequence $\gamma_n = \lambda(Q/p)^n$, with $\lambda \geqslant p$, we get (3.3) by iteration as in [27,14]. □

3.2. *Existence of a bounded solution*

Here we prove the existence result of Theorem 1.2. First we prove the existence of a weak solution of the problem:

Proposition 3.1. *Assume that (a, b) is globally κ-admissible in Ω, and the imbedding $W_0^{1,p}(\Omega, a) \hookrightarrow L^\kappa(\Omega, b)$ is compact for any $1 < p < k < \kappa$. Suppose $\delta < \kappa - 1$. Then there exists a nontrivial nonnegative solution u of the Dirichlet problem in Ω:*

$$\begin{cases} u \in W_0^{1,p}(\Omega, a), \\ -\mathrm{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big) = b(x)u^\delta \quad \text{in } \Omega. \end{cases} \tag{3.4}$$

Proof. The idea of the proof is classical and still used in the radial case in [12]. We consider the minimization problem

$$\inf\{J(w)\colon\ w \in S\},$$

where

$$J(w) = \int_\Omega |\nabla w|^p a\,\mathrm{d}x \quad \text{and} \quad S = \left\{ w \in W_0^{1,p}(\Omega, a),\ \int_\Omega |w|^{\delta+1} b\,\mathrm{d}x = 1 \right\}.$$

Under the assumption (G_1), the space $W_0^{1,p}(\Omega, a)$ is reflexive since $p > 1$, see [16]. The set S is nonempty: it contains some elements of $\mathcal{D}(\Omega)$, since $a, b \in L^1_{\mathrm{loc}}(\Omega)$. From ($G_2$) and the compactness assumption, S is weakly closed. From (G_3), $w \mapsto (J(w)^{1/p})$ is an equivalent norm on $W_0^{1,p}(\Omega, a)$. Then J achieves its minimum at some point $u_1 \geqslant 0$, since $J(w) = J(|w|)$ for any $w \in W_0^{1,p}(\Omega, a)$. Hence, from the Lagrange multiplier rule, there exists a real λ such that

$$\int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \nabla v a\,\mathrm{d}x = \lambda \int_\Omega u_1^\delta v b\,\mathrm{d}x,$$

for any $v \in W_0^{1,p}(\Omega, a)$. Taking $v = u_1$, one gets $\lambda > 0$. Then $u = \lambda^{1/(\delta-p+1)}u_1$ is a solution of the problem (3.4). □

From now on we are in a position to prove Theorem 1.2:

Proof of Theorem 1.2. From Proposition 3.1 we have constructed a nontrivial solution of the problem, such that

$$\int_\Omega u^{\delta+1} b\,\mathrm{d}x < +\infty.$$

We can apply Theorem 3.1 with $H := u^{\delta+1-p}$. We have $H \in L^s(\Omega, b)$ for

$$s = \frac{\delta+1}{\delta+1-p} > \frac{\eta}{p},$$

since $\delta < \kappa - 1$. Then we get $u \in L^\infty(\Omega)$ from (3.3) with $\lambda = \delta + 1$. □

Remark 3.2. Our process of proof is rather different of the one of [12]. They prove the existence of some weak solutions of problem (SP_r) in a suitable Sobolev space, and then they show that such solutions are bounded near 0 by using monotonicity and descent methods. We consider directly the Dirichlet problem (DP) instead of (SP) and use Harnack inequality. This supposes that (G_1) holds, which was not supposed in the radial case.

3.3. Applications

Let us begin with the case of powers of $|x|$. We have the following.

Theorem 3.2. *Let θ, σ be any reals, such that $\theta, \sigma > -N$, and $\theta < N(p-1)$. Assume that*

$$\delta + 1 < \inf(P^*, p^*) = \inf\left(\frac{pN}{N-p}, \frac{p(N+\sigma)}{N+\theta-p}\right)$$

(*hence $\sigma + p > \theta$*). *Then there exists a nontrivial nonnegative bounded solution u of the Dirichlet problem in Ω*:

$$\begin{cases} u \in W_0^{1,p}(\Omega, a) \cap L^\infty(\Omega), \\ -\operatorname{div}\big(|x|^\theta |\nabla u|^{p-2}\nabla u\big) = |x|^\sigma u^\delta \quad \textit{in } \Omega. \end{cases} \tag{3.5}$$

Proof. We can apply Theorem 1.2. Indeed the conditions $\theta, \sigma > -N$, imply that $a, b \in L^1(\Omega)$, and $\theta < N(p-1)$ ensures that $a^{-1/(p-1)} \in L^1(\Omega)$, hence ($G_1$) holds. And ($G_2$), ($G_3$) and the compactness property hold from [17], with $\kappa = \inf(P^*, p^*)$. □

Now for any $x \in \Omega$, let $d(x)$ be the distance from x to the boundary $\partial\Omega$. Here we consider some weights which are powers of d and deduce similarly the following:

Theorem 3.3. *Let θ, σ be any reals. Assume that $-N < \theta < p-1$ and $-1 < \sigma$ and*

$$\delta + 1 < \inf(P^*, p^*) = \inf\left(\frac{pN}{N-p}, \frac{p(N+\sigma)}{(N+\theta-p)}\right). \tag{3.6}$$

Then there exists a nontrivial nonnegative bounded solution u of the Dirichlet problem in Ω:

$$\begin{cases} u \in W_0^{1,p}(\Omega, d^\theta) \cap L^\infty(\Omega), \\ -\operatorname{div}\big(d^\theta(x) |\nabla u|^{p-2}\nabla u\big) = d^\sigma(x) u^\delta, \quad x \in \Omega. \end{cases} \tag{3.7}$$

Proof. We can again apply Theorem 1.2. Indeed the conditions $-N < \theta < p-1$ and $-1 < \sigma$ imply (G_1). Also (3.6) implies (G_2) and (G_3), and the compactness property follows from [17, Theorem 19.2]. □

The problem of characterizing admissible weights has been studied rather extensively during the last twenty years and continue to be under active research in the case of two different weights a, b. Necessary and sufficient conditions have be given in [19] in terms of capacity, but they are hard to check. Hence many authors tried to give only sufficient conditions. A well-known class of weights was introduced by Muckenhoupt [21]: a function a in a domain D of $\mathbb{R}^N$, positive a.e. in D, such that $a \in L^1_{\mathrm{loc}}(D)$, lies in *the Muckenhoupt-class* $\mathcal{A}_r(D)$ ($r \geqslant 1$) if there exists a constant $C > 0$, such that, for any ball $\mathcal{B}(x, R) \subset \mathbb{R}^N$,

$$\left(\int_{\mathcal{B}(x,R)\cap D} a \,\mathrm{d}x\right)^{1/r} \left(\int_{\mathcal{B}(x,R)\cap D} a^{-1/(r-1)} \,\mathrm{d}x\right)^{1/r'} \leqslant C R^N, \tag{3.8}$$

if $r > 1$, or

$$\int_{\mathcal{B}(x,R)\cap D} a\,\mathrm{d}x \leqslant CR^N \inf_{\mathcal{B}(x,R)\cap D} a,$$

if $r = 1$. Let us recall some properties of this class. We have $\mathcal{A}_1(D) \subset \mathcal{A}_r(D)$ for any $r > 1$. Hence any superharmonic function, positive a.e. in $\mathbb{R}^N$, is in $\mathcal{A}_1(\mathbb{R}^N)$. On the other hand, any function $a \in \mathcal{A}_p(\mathbb{R}^N)$ with $p > 1$ satisfies a reverse Hölder inequality: there exists $p_a > 1$, and $C > 0$, such that

$$\left(\fint_{\mathcal{B}(x,R)} a^{p_a}\,\mathrm{d}x\right)^{1/p_a} \leqslant C \fint_{\mathcal{B}(x,R)} a\,\mathrm{d}x,$$

for any ball $\mathcal{B}(x,R) \subset \mathbb{R}^N$. As a consequence, there exists q_a such that $1 < q_a < p$ and $a \in \mathcal{A}_{q_a}(\mathbb{R}^N)$. Concerning the Dirichlet problem, the main result is that if $a \in \mathcal{A}_p(\mathbb{R}^N)$, then the pair (a,a) is globally admissible in any *bounded* domain Ω, and it satisfies (G_2) in Ω for any $\kappa > p$ such that

$$\kappa(Nq_a - p) < Npq_a,$$

see [16, Theorem 15.23]. As a consequence, if $a \in \mathcal{A}_1(\mathbb{R}^N)$ and $1 < p \leqslant N$, then (G_2) holds in $\mathcal{B}$ for any $p < \kappa < P^*$. Up to our knowledge, the question of compactness was not solved in the general case of a pair (a,b). Notice that any power $a(x) = |x|^\theta$ lies in $\mathcal{A}_p(\mathbb{R}^N)$ if and only if $-N < \theta < N(p-1)$. Also the weight $a(x) = d^\theta(x)$ lies in $\mathcal{A}_p(\Omega)$ if and only if $-1 < \theta < p-1$.

Remark 3.3. These classes have been extended to two different weights, see [16]: a pair (a,b) of non-negative functions in $L^1_{\mathrm{loc}}(D)$, such that a is positive a.e. in D, lies in the class $\widetilde{\mathcal{A}_r}(D)$ $(r > 1)$, whenever there exists a constant $C > 0$, such that, for any ball $\mathcal{B}(x,R) \subset \mathbb{R}^N$,

$$\left(\int_{\mathcal{B}(x,R)\cap\Omega} b\,\mathrm{d}x\right)^{1/r}\left(\int_{\mathcal{B}(x,R)\cap\Omega} a^{-1/(r-1)}\,\mathrm{d}x\right)^{1/r'} \leqslant CR^N.$$

It can be shown that for any r such that $1 < r < p < Nr$, the Sobolev inequality (G_2) holds with $\kappa = Npr/(Nr-p)$. This class appears to be quite restrictive, since it limits the growth of b with respect to a: in the case of the problem with the weights given by (1.32), the pair $(a,b) = (|x|^\theta, |x|^\sigma)$ lies in $\widetilde{\mathcal{A}_r}(\mathcal{B})$ if and only if $\theta,\sigma > -N$, and $\sigma \geqslant \theta$, and this last condition is not required at Theorem 3.2. In the same way in the case of problem (3.7), the pair $(a,b) = (d^\theta(x), d^\sigma(x))$ lies in $\widetilde{\mathcal{A}_r}(\mathcal{B})$ if and only if $\theta,\sigma > -N$, and $\theta < r-1$, $\sigma > -1$ and $\sigma \geqslant \theta$, which is more than what is required in Theorem 3.3, see [17, Remark 15.19].

4. The nonradial singularity problem

We consider here the general problem (SP) in the nonradial case, with eventually nonradial weights.

4.1. *Weak estimates with general weights*

Here we give the first estimates on any solution of (SP) in the nonradial case and prove Theorem 1.4. For any domain $D \subset \mathbb{R}^N$, and any $f \in L^1(D,a)$, we set

$$\fint_{D,a} f = \frac{\int_D f a \,\mathrm{d}x}{\int_D a \,\mathrm{d}x}.$$

The first estimate, (1.21), extends to the supersolutions of (1.18):

Theorem 4.1. *Assume (K_1). Let $u \in L^\infty_{\mathrm{loc}}(\Omega') \cap W^{1,p}_{\mathrm{loc}}(\Omega', a)$ be any nonnegative function in Ω' such that the distribution in Ω'*

$$g := -\mathrm{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big) \tag{4.1}$$

lies in $L^1_{\mathrm{loc}}(\Omega')$, and $g \geqslant 0$. Then there exists $C > 0$ such that for any $x_0 \in \mathcal{B}'$,

$$\inf_{|x|=|x_0|} u \leqslant C\bigg(\int_{|x_0|\leqslant|x|\leqslant 1} |x|^{(1-N)p'} a^{1/(1-p)} \,\mathrm{d}x + 1\bigg). \tag{4.2}$$

Proof. We follow the method of [2] relative to the nonweighted case. We use a test function introduced by Serrin in [24]. Let $C_1 = 2\sup_{|x|=1} u(x)$ and $u_1 = u - C_1$. For any $r < 1$, let $m_1(r) = \inf_{|x|=r} u_1(x)$. We define

$$v_r(x) = \begin{cases} 0 & \text{if } |x| > r \text{ and } u_1(x) \leqslant 0, \text{ or if } |x| \geqslant 1, \\ u_1(x) & \text{if } r < |x| < 1 \text{ and } 0 \leqslant u_1(x) \leqslant m_1(r), \\ m_1(r) & \text{if } r < |x| < 1 \text{ and } u_1(x) > m_1(r), \text{or if } |x| \leqslant r. \end{cases}$$

For any $\varphi \in \mathcal{D}(\Omega')$, we have

$$\int_{\Omega'} |\nabla u|^{p-2}\nabla u \nabla\varphi a \,\mathrm{d}x = \int_{\Omega'} g\varphi \,\mathrm{d}x, \tag{4.3}$$

and by density, (4.3) also holds for any $\varphi \in W^{1,p}(\Omega, a) \cap L^\infty(\Omega)$ with compact support in Ω'. We take

$$\varphi = v_r(x) - m_1(r)\eta,$$

where η is radial, with values in $[0,1]$, such that $\eta = 0$ for $|x| \geqslant 1$, and $\eta = 1$ near 0. Then

$$\int_{\Omega'} |\nabla u|^{p-2}\nabla u \nabla v_r a \,\mathrm{d}x + \int_{\Omega'} g\big(m_1(r) - v_r\big) \,\mathrm{d}x = m_1(r)C_2,$$

where C_2 does not depend on r:

$$C_2 = \int_{\Omega'} |\nabla u|^{p-2}\nabla u \nabla\eta a \,\mathrm{d}x + \int_{\Omega'} g(1-\eta) \,\mathrm{d}x.$$

Then

$$m_1(r)C_2 \geqslant \int_{\mathcal{B}'} |\nabla u|^{p-2}\nabla u \nabla v_r a \,\mathrm{d}x = \int_{\mathcal{B}'} |\nabla v_r|^p a \,\mathrm{d}x.$$

Defining the capacity of any compact set $X \subset \mathcal{B}$ with the weight a by

$$c_{p,a}(X,\mathcal{B}) = \inf\left\{\int_{\mathcal{B}} |\nabla \varphi|^p a \,\mathrm{d}x\colon\ \varphi \in \mathcal{D}(\mathcal{B}),\ \varphi \geqslant 1 \text{ on } X\right\},$$

we get, since $v_r(x) \geqslant m_1(r)$ on the ball $\mathcal{B}_r = \mathcal{B}(0,r)$,

$$m_1(r)C_2 \geqslant m_1^p(r)c_{p,a}(\mathcal{B}_r,\mathcal{B}).$$

From [16] this capacity can be estimated by

$$c_{p,a}(\mathcal{B}_r,\mathcal{B}) \geqslant C\left(\int_{r\leqslant|x|\leqslant 1} |x|^{(1-N)p'} a^{1/(1-p)} \,\mathrm{d}x\right)^{1-p},$$

for some $C = C(N,p) > 0$, hence with a new $C > 0$,

$$\inf_{|x|=r} u(x) \leqslant m_1(r) + C_1 \leqslant C\int_{r\leqslant|x|\leqslant 1} |x|^{(1-N)p'} a^{1/(1-p)} \,\mathrm{d}x + C_1,$$

and (4.2) follows. □

Now we prove the second estimate, (1.22), which will end the proof of Theorem 1.4. It relies on the ideas of [4,20].

Theorem 4.2. *Assume* (K$_1$). *Let u be any nonnegative supersolution of* (SP) *in Ω'. Then there exists $C > 0$ such that for any $x_0 \in \mathcal{B}'$, and any ball $\mathcal{B}(x_0,2R) \subset \Omega'$,*

$$\left(\fint_{B(x_0,R),b} u^\delta\right)^{1/\delta} \leqslant CR^{-p/(\delta+1-p)}\left(\frac{\int_{\mathcal{B}(|x_0|,2R)} a \,\mathrm{d}x}{\int_{\mathcal{B}(|x_0|,R)} b \,\mathrm{d}x}\right)^{1/(\delta-p+1)}. \tag{4.4}$$

Proof. For any $\varphi \in \mathcal{D}(\Omega')$, we have

$$\int_{\Omega'} |\nabla u|^{p-2}\nabla u \nabla \varphi a \,\mathrm{d}x = \int_{\Omega'} u^\delta \varphi b \,\mathrm{d}x,$$

and by density, it also holds for any $\varphi \in W^{1,p}(\Omega',a) \cap L^\infty(\Omega')$ with compact support in Ω. Let $\varepsilon > 0$, and $u_\varepsilon = u + \varepsilon > 0$. Let $\zeta \in \mathcal{D}(\Omega')$, $\zeta \geqslant 0$. Since $u \in L^\infty(\Omega')$, we can take

$$\varphi = \zeta^\lambda u_\varepsilon^\alpha,$$

with $1-p<\alpha<0$, and $\lambda>0$ large enough. We obtain

$$\begin{aligned}\int_{\Omega'} u_\varepsilon^\alpha u^\delta \zeta^\lambda b\,\mathrm{d}x + |\alpha|\int_{\Omega'} u_\varepsilon^{\alpha-1}|\nabla u|^p\zeta^\lambda a\,\mathrm{d}x &\leqslant \lambda\int_{\Omega'} u_\varepsilon^\alpha \zeta^{\lambda-1}|\nabla u|^{p-1}|\nabla\zeta| a\,\mathrm{d}x\\ &\leqslant \frac{|\alpha|}{2}\int_{\Omega'}\zeta^\lambda u_\varepsilon^{\alpha-1}|\nabla u|^p a\,\mathrm{d}x + C\int_{\Omega'} u_\varepsilon^{\alpha+p-1}\zeta^{\lambda-p}|\nabla\zeta|^p a\,\mathrm{d}x,\end{aligned}$$

where $C>0$ depends on α. Hence

$$\int_{\Omega'} u_\varepsilon^\alpha u^\delta \zeta^\lambda b\,\mathrm{d}x + \int_{\Omega'} u_\varepsilon^{\alpha-1}|\nabla u|^p\zeta^\lambda a\,\mathrm{d}x \leqslant C\int_{\Omega'} u_\varepsilon^{\alpha+p-1}\zeta^{\lambda-p}|\nabla\zeta|^p a\,\mathrm{d}x.$$

Then from the Hölder inequality, setting $\theta=\delta/(p-1+\alpha)>1$, and letting ε tend to 0, we get

$$\begin{aligned}&\int_{\Omega'} u^{\delta+\alpha}\zeta^\lambda b\,\mathrm{d}x + \int_{\Omega'} u^{\alpha-1}|\nabla u|^p\zeta^\lambda a\,\mathrm{d}x\\ &\quad\leqslant C\biggl(\int_{\Omega'} u^\delta\zeta^\lambda b\,\mathrm{d}x\biggr)^{1/\theta}\biggl(\int_{\Omega'}\zeta^{\lambda-p\theta'}|\nabla\zeta|^{p\theta'}a^{\theta'}b^{1-\theta'}\,\mathrm{d}x\biggr)^{1/\theta'},\end{aligned}$$

with a new constant $C>0$. Now we choose

$$\varphi=\zeta^\lambda$$

as a test function. We get

$$\int_{\Omega'} u^\delta\zeta^\lambda b\,\mathrm{d}x \leqslant \lambda\int_{\Omega'}\zeta^{\lambda-1}|\nabla u|^{p-1}|\nabla\zeta| a\,\mathrm{d}x,$$

hence for any $\alpha\in(1-p,0)$,

$$\int_{\Omega'} u^\delta\zeta^\lambda b\,\mathrm{d}x \leqslant \lambda\biggl(\int_{\Omega'} u^{\alpha-1}\zeta^\lambda|\nabla u|^p a\,\mathrm{d}x\biggr)^{1/p'}\biggl(\int_{\Omega'} u^{(1-\alpha)(p-1)}\zeta^{\lambda-p}|\nabla\zeta|^p a\,\mathrm{d}x\biggr)^{1/p}.$$

Since $\delta>p-1$, we can fix an $\alpha\in(1-p,0)$ such that $\tau=\delta/(1-\alpha)(p-1)>1$. Then we get

$$\begin{aligned}\int_{\Omega'} u^\delta\zeta^\lambda b\,\mathrm{d}x &\leqslant C\biggl(\int_{\Omega'} u^\delta\zeta^\lambda b\,\mathrm{d}x\biggr)^{1/\theta p'+1/\tau p}\biggl(\int_{\Omega'}\zeta^{\lambda-\theta' p}|\nabla\zeta|^{\theta' p}a^{\theta'}b^{1-\theta'}\,\mathrm{d}x\biggr)^{1/\theta' p'}\\ &\quad\times\biggl(\int_{\Omega'}\zeta^{\lambda-\tau' p}|\nabla\zeta|^{\tau' p}a^{\tau'}b^{1-\tau'}\,\mathrm{d}x\biggr)^{1/\tau' p}.\end{aligned}\tag{4.5}$$

Since $1/\theta p'+1/\tau p=(p-1)/\delta=1-(1/\theta' p'+1/\tau' p)$, we find

$$\begin{aligned}&\biggl(\int_{\Omega'} u^\delta\zeta^\lambda b\,\mathrm{d}x\biggr)^{(\delta-p+1)/\delta}\\ &\quad\leqslant C\biggl(\int_{\Omega'}\Bigl(\frac{a}{b}\Bigr)^{\theta'}\zeta^{-p\theta'}|\nabla\zeta|^{p\theta'}\zeta^\lambda b\,\mathrm{d}x\biggr)^{1/\theta' p'}\biggl(\int_{\Omega'}\Bigl(\frac{a}{b}\Bigr)^{\tau'}\zeta^{-p\tau'}|\nabla\zeta|^{p\tau'}\zeta^\lambda b\,\mathrm{d}x\biggr)^{1/p\tau'}.\end{aligned}$$

Then from the Hölder inequality with coefficients $1/\theta'$ and $1/\tau'$,

$$\left(\int_{\Omega'} u^\delta \zeta^\lambda b\,\mathrm{d}x\right)^{(\delta-p+1)/\delta} \leqslant C\left(\int_{\Omega'} \zeta^{\lambda-p}|\nabla\zeta|^p a\,\mathrm{d}x\right)^{1/p'}\left(\int_{\Omega'} \zeta^\lambda b\,\mathrm{d}x\right)^{(1-\theta')/\theta' p'} \times\left(\int_{\Omega'} \zeta^{\lambda-p}|\nabla\zeta|^p a\,\mathrm{d}x\right)^{1/p}\left(\int_{\Omega'} \zeta^\lambda b\,\mathrm{d}x\right)^{(1-\tau')/p\tau'},$$

hence

$$\left(\int_{\Omega'} u^\delta \zeta^\lambda b\,\mathrm{d}x\right)^{(\delta-p+1)/\delta} \leqslant C\left(\int_{\Omega'} \zeta^{\lambda-p}|\nabla\zeta|^p a\,\mathrm{d}x\right)\left(\int_{\Omega'} \zeta^\lambda b\,\mathrm{d}x\right)^{-(p-1)/\delta},$$

that is

$$\left(\int_{\Omega'} u^\delta \zeta^\lambda b\,\mathrm{d}x\right)^{(\delta-p+1)/\delta}\left(\int_{\Omega'} \zeta^\lambda b\,\mathrm{d}x\right)^{(p-1)/\delta} \leqslant C\int_{\Omega'} \zeta^{\lambda-p}|\nabla\zeta|^p a\,\mathrm{d}x. \tag{4.6}$$

Consider any ball $\mathcal{B}(x_0,2R)\subset\Omega'$. We take $\zeta(x)=\xi(x-x_0)$ where ξ has its support in $\mathcal{B}(0,2R)$, with values in $[0,1]$, such that $\xi\equiv 1$ in $\mathcal{B}(0,R)$ and $|\nabla\xi|\leqslant C/R$, and deduce

$$\left(\int_{\mathcal{B}(x_0,R)} u^\delta b\,\mathrm{d}x\right)^{(\delta-p+1)/\delta}\left(\int_{\mathcal{B}(x_0,R)} b\,\mathrm{d}x\right)^{(p-1)/\delta} \leqslant CR^{-p}\int_{\mathcal{B}(x_0,2R)} a\,\mathrm{d}x,$$

and (4.4) follows. □

4.2. Local Harnack properties

Here we extend some local Harnack properties in any domain D of $\mathbb{R}^N$, which are known in the nonweighted case, to the weighted one, in order to apply it to our problem with $D=\Omega'$. We deal with the equation

$$-\mathrm{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big)=H(x)b(x)u^{p-1}\quad\text{in } D, \tag{4.7}$$

where now H satisfies local estimates in a suitable L^s space, with $H\geqslant 0$. Much work has been done for the equation without second member (1.25) but we did not find the precise result we needed for the equation with coefficients (4.7), even if a part of the result is mentioned in [5] for $p=2$. Thus we give here a complete proof , which is an extension of the results of [24,27] using the Moser technique.

According to [16], we will say that a is *locally admissible in D* if it satisfies the following conditions, for some $C_a=C_a(N,p,D,a)>0$:

(i) *Integrability*:

$$a\in L^1_{\mathrm{loc}}(D),\qquad a^{1/(1-p)}\in L^1_{\mathrm{loc}}(D); \tag{C$_1$}$$

(ii) *Doubling condition*: for any ball $\mathcal{B}(x_0, 4R) \subset D$,

$$\int_{\mathcal{B}(x_0,2R)} a \,\mathrm{d}x \leqslant C_a \int_{\mathcal{B}(x_0,R)} a \,\mathrm{d}x. \tag{C$_2$}$$

(iii) *Sobolev inequality*: there exists some $Q > p$ such that for any ball $B(x_0, 2R) \subset D$ and any $\varphi \in \mathcal{D}(B(x_0, R))$,

$$\left(\oint_{B(x_0,R),a} |\varphi|^Q \right)^{1/Q} \leqslant C_a R \left(\oint_{B(x_0,R),a} |\nabla\varphi|^p \right)^{1/p}. \tag{C$_3$}$$

(iv) *Poincaré inequality*: for any ball $B(x_0, 2R) \subset D$ and any bounded $\varphi \in C^\infty(B(x_0, R))$,

$$\int_{B(x_0,R)} |\varphi - \overline{\varphi}_a|^p a \,\mathrm{d}x \leqslant C_a R^p \int_{B(x_0,R)} |\nabla\varphi|^p a \,\mathrm{d}x, \tag{C$_4$}$$

where $\overline{\varphi}_a = \oint_{B(x_0,R),a} \varphi$.

We will associate to Q the numbers: $\nu > p$ and $\overline{Q}$ defined by

$$\frac{\nu p}{\nu - p} := Q, \qquad \overline{Q} := \frac{Q}{p'} = \frac{\nu(p-1)}{\nu - p}. \tag{4.8}$$

Also, we will say that *the pair* (a, b) *is locally admissible in* D if the following conditions are satisfied:

(v) *Integrability*:

$$a, b \in L^1_{\mathrm{loc}}(D); \tag{C$_5$}$$

(vi) *Hardy–Sobolev inequality*: there exists $K > p$ such that for any ball $B(x_0, 2R) \subset D$ and any $\varphi \in \mathcal{D}(B(x_0, R))$,

$$\left(\oint_{B(x_0,R),b} |\varphi|^Q \right)^{1/Q} \leqslant C_{a,b} R \left(\oint_{B(x_0,R),a} |\nabla\varphi|^p \right)^{1/p}, \tag{C$_6$}$$

where $C_{a,b} = C_{a,b}(N, p, D, a, b) > 0$.

We define similarly two numbers $\eta > p$ and $\overline{K}$ by

$$\frac{\eta p}{\eta - p} := K, \qquad \overline{K} := \frac{K}{p'} = \frac{\eta(p-1)}{\eta - p}. \tag{4.9}$$

Theorem 4.3. *Assume that a is locally Q-admissible and (a, b) is locally K-admissible in D, for some $Q, K > p$. Let η be defined by* (4.8), (4.9). *Let $H \in L^s_{\mathrm{loc}}(D, b)$ for some $s > \eta/p$, with $H \geqslant 0$. Let u be a nonnegative solution of* (4.7). *Then*

(i) *for any ball* $B(x_0, 4R) \subset D$ *and any* $m > p - 1$,

$$\sup_{B(x_0,R)} u \leqslant C\left(\left(\oint_{B(x_0,2R),a} u^m\right)^{1/m} + \left(\oint_{B(x_0,2R),b} u^m\right)^{1/m}\right), \tag{4.10}$$

where $C = C(N, p, a, b, s, m, C_R)$, *and*

$$C_R = R^p\left(\oint_{B(x_0,2R),b} H^s\right)^{1/s} \frac{\int_{B(x_0,2R)} b\,\mathrm{d}x}{\int_{B(x_0,R)} a\,\mathrm{d}x}; \tag{4.11}$$

(ii) *for any ball* $B(x_0, 4R) \subset D$ *and any* $0 < m < \overline{Q}$,

$$\left(\oint_{B(x_0,2R),a} u^m\right)^{1/m} \leqslant C' \inf_{B(x_0,R)} u, \tag{4.12}$$

with $C' = C'(N, p, m, a) > 0$. *As a consequence for any* $0 < m < \overline{Q}$,

$$\sup_{B(x_0,R)} u \leqslant C\left(\oint_{B(x_0,2R),b} u^m\right)^{1/m}, \tag{4.13}$$

with $C = C(N, p, a, b, s, m, C_R)$.

Proof. (i) We have supposed from the begining that $u \in L^\infty_{\mathrm{loc}}(D)$. By replacing u by $u + \varepsilon$ and making $\varepsilon \to 0$, we can suppose that $u > 0$ in D and $u^{-1} \in L^\infty_{\mathrm{loc}}(D)$. For any $\varphi \in \mathcal{D}(D)$, we have

$$\int_D |\nabla u|^{p-2}\nabla u \nabla\varphi a\,\mathrm{d}x = \int_D H u^{p-1}\varphi b\,\mathrm{d}x.$$

From (C_5), (C_6) and the assumption on H, this also holds for any $\varphi \in W^{1,p}(D, a)$ with compact support in D. Let $\zeta \in \mathcal{D}(D)$, $\zeta \geqslant 0$, with compact support in a ball $B(x_0, 8R_0) \subset D$. We take

$$\varphi = u^\beta\zeta^p, \quad \beta > 0,$$

and get

$$\beta\int_D u^{\beta-1}|\nabla u|^p\zeta^p a\,\mathrm{d}x \leqslant p\int_D u^\beta\zeta^{p-1}|\nabla u|^{p-2}\nabla u\nabla\zeta a\,\mathrm{d}x + \int_D H u^{\beta+p-1}\zeta^p b\,\mathrm{d}x.$$

Hence from the Hölder inequality,

$$\frac{\beta}{2}\int_D u^{\beta-1}|\nabla u|^p\zeta^p a\,\mathrm{d}x \leqslant C\beta^{1-p}\int_D u^{\beta+p-1}|\nabla\zeta|^p a\,\mathrm{d}x + \int_D H u^{\beta+p-1}\zeta^p b\,\mathrm{d}x.$$

Let $v = u^{\gamma/p}$, with $\gamma = \beta + p - 1$. Then

$$\int_D |\nabla v|^p\zeta^p a\,\mathrm{d}x \leqslant C\frac{\gamma^p}{\beta}\left(\beta^{1-p}\int_D v^p|\nabla\zeta|^p a\,\mathrm{d}x + \int_D H v^p\zeta^p b\,\mathrm{d}x\right).$$

Let us consider any $R < 2R_0$ and take supp $\zeta \subset B(x_0, R) \subset D$. We have from ($C_3$) and ($C_6$)

$$\begin{aligned}
&\left(\int_{B(x_0,R)} a\,\mathrm{d}x\right)^{1/p}\left(\left(\oint_{D,a}(v\zeta)^Q\right)^{1/Q}+\left(\oint_{D,b}(v\zeta)^K\right)^{1/K}\right)\\
&\quad\leqslant CR\left(\int_D |\nabla(v\zeta)|^p a\,\mathrm{d}x\right)^{1/p}\leqslant CR\left(\left(\int_D v^p|\nabla\zeta|^p a\,\mathrm{d}x\right)^{1/p}+\left(\int_D |\nabla v|^p\zeta^p a\,\mathrm{d}x\right)^{1/p}\right)\\
&\quad\leqslant CR\left(\left(\left(1+\frac{\gamma}{\beta}\right)\int_D v^p|\nabla\zeta|^p a\,\mathrm{d}x\right)^{1/p}+\frac{\gamma}{\beta^{1/p}}\left(\int_D Hv^p\zeta^p b\,\mathrm{d}x\right)^{1/p}\right).
\end{aligned}$$

Now from Hölder inequality, and setting $\|H\| = (\int_D H^s b\,\mathrm{d}x)^{1/s}$ and $\tau = \eta/(ps-\eta) > 0$,

$$\left(\int_D Hv^p\zeta^p b\,\mathrm{d}x\right)^{1/p}\leqslant \|H\|^{1/p}\|v\zeta\|_{L^{ps'}(D,b)}\leqslant \|H\|^{1/p}\left(\varepsilon\|v\zeta\|_{L^K(D,b)}+\varepsilon^{-\tau}\|v\zeta\|_{L^p(D,b)}\right),$$

for any $\varepsilon > 0$, by interpolation. Hence

$$\begin{aligned}
&\left(\oint_{D,a}(v\zeta)^Q\right)^{1/Q}+\left(\oint_{D,b}(v\zeta)^K\right)^{1/K}\leqslant CR\left(\left(1+\frac{\gamma}{\beta}\right)\left(\oint_{D,a} v^p|\nabla\zeta|^p\right)^{1/p}\right.\\
&\quad+\varepsilon\frac{\gamma}{\beta^{1/p}}\|H\|^{1/p}\left(\oint_{D,b}(v\zeta)^K\right)^{1/K}\left(\int_{B(x_0,R)} a\,\mathrm{d}x\right)^{-1/p}\left(\int_{B(x_0,R)} b\,\mathrm{d}x\right)^{1/K}\\
&\quad\left.+\varepsilon^{-\tau}\frac{\gamma}{\beta^{1/p}}\|H\|^{1/p}\left(\oint_{D,b}(v\zeta)^p\right)^{1/p}\left(\int_{B(x_0,R)} a\,\mathrm{d}x\right)^{-1/p}\left(\int_{B(x_0,R)} b\,\mathrm{d}x\right)^{1/p}\right).
\end{aligned}$$

Let us take

$$\varepsilon = \beta^{1/p}(2CR\gamma)^{-1}\|H\|^{-1/p}\left(\int_{B(x_0,R)} a\,\mathrm{d}x\right)^{1/p}\left(\int_{B(x_0,R)} b\,\mathrm{d}x\right)^{-1/K}.$$

After some computations, and with a new constant $C > 0$, we get

$$\begin{aligned}
&\left(\oint_{D,a}(v\zeta)^Q\right)^{1/Q}+\left(\oint_{D,b}(v\zeta)^K\right)^{1/K}\\
&\quad\leqslant CR\left(1+\frac{\gamma}{\beta}\right)\left(\oint_{D,a} v^p|\nabla\zeta|^p\right)^{1/p}+C\left(\frac{\gamma L_R^{1/p}}{\beta^{1/p}}\right)^{1+\tau}\left(\oint_{D,b}(v\zeta)^p\right)^{1/p},
\end{aligned}$$

where

$$L_R = R^p\left(\oint_{B(x_0,R),b} H^s\right)^{1/s}\frac{\int_{B(x_0,R)} b\,\mathrm{d}x}{\int_{B(x_0,R)} a\,\mathrm{d}x}. \tag{4.14}$$

Then as soon as $\beta \geqslant \beta_0 > 0$, with another $C > 0$ depending on β_0, we have

$$\left(\oint_{D,a} (v\zeta)^Q\right)^{1/Q} + \left(\oint_{D,b} (v\zeta)^K\right)^{1/K}$$
$$\leqslant C\gamma^{1+\tau}\left(R\left(\oint_{D,a} v^p|\nabla\zeta|^p\right)^{1/p} + L_R^{(1+\tau)/p}\left(\oint_{D,b} (v\zeta)^p\right)^{1/p}\right).$$

Let us define sequences $R_n = R_0(1+2^{-n})$ and ζ_n with support in $\mathcal{B}_n = \mathcal{B}(x_0, R_n)$, with values in $[0,1]$, such that $\zeta_n = 1$ on $\mathcal{B}_{n+1}$, and $|\nabla\zeta_n| \leqslant C2^n/R_0$. Then we find

$$\left(\oint_{\mathcal{B}_{n+1},a} v^Q\right)^{1/Q} + \left(\oint_{\mathcal{B}_{n+1},b} v^K\right)^{1/K} \leqslant C\gamma^{1+\tau}\left(2^n\left(\oint_{\mathcal{B}_n,a} v^p\right)^{1/p} + L_{R_n}^{(1+\tau)/p}\left(\oint_{\mathcal{B}_n,b} v^p\right)^{1/p}\right).$$

It can be verified that $L_{R_n} \leqslant 2^p C_{R_0}$. Hence with a new constant C depending on C_{R_0},

$$\left(\oint_{\mathcal{B}_{n+1},a} v^Q\right)^{1/Q} + \left(\oint_{\mathcal{B}_{n+1},b} v^K\right)^{1/K} \leqslant C\gamma^{1+\tau}2^n\left(\left(\oint_{\mathcal{B}_n,a} v^p\right)^{1/p} + \left(\oint_{\mathcal{B}_n,b} v^p\right)^{1/p}\right).$$

Let us set $\rho = \min(Q, K)$. Then in particular

$$\left(\oint_{\mathcal{B}_{n+1},a} v^\rho + \oint_{\mathcal{B}_{n+1},b} v^\rho\right)^{1/\rho} \leqslant C\gamma^{1+\tau}2^n\left(\oint_{\mathcal{B}_n,a} v^p + \oint_{\mathcal{B}_n,b} v^p\right)^{1/p}.$$

Recovering u, we find

$$\left(\oint_{\mathcal{B}_{n+1},a} u^{\gamma\rho/p} + \oint_{\mathcal{B}_{n+1},b} u^{\gamma\rho/p}\right)^{p/\gamma\rho} \leqslant C^{1/\gamma}\gamma^{(1+\tau)p/\gamma}2^{np/\gamma}\left(\oint_{\mathcal{B}_n,a} u^\gamma + \oint_{\mathcal{B}_n,b} u^\gamma\right)^{1/\gamma},$$

where $C > 0$ is a new constant. Taking any $m > p-1$, and a sequence $\gamma_n = m(Q/p)^n$, we get by summation as in [27,14],

$$\left(\oint_{\mathcal{B}_n,a} u^{\gamma_n} + \oint_{\mathcal{B}_n,b} u^{\gamma_n}\right)^{1/\gamma_n} \leqslant C\left(\oint_{\mathcal{B}_0,a} u^m + \oint_{\mathcal{B}_0,b} u^m\right)^{1/m}$$

hence

$$\sup_{B(x_0,R_0)} u = \lim\left(\oint_{\mathcal{B}_n,a} u^{\gamma_n}\right)^{1/\gamma_n} \leqslant C\left(\oint_{B(x_0,2R_0),a} u^m + \oint_{B(x_0,2R_0),b} u^m\right)^{1/m}.$$

(ii) The function u is a supersolution of the equation without second member, hence (4.12) follows from the assumptions (C_3) and (C_4), for any $0 < m < \overline{Q}$, see [16, Theorem 3.59]. Recall that (C_4) allows to prove a John Niremberg lemma adapted to the weight a. Also (4.13) follows from (4.10) and (4.12). □

Remark 4.1. In fact the estimate (4.10) remains true without sign condition on H, and for any subsolution of (4.7).

4.3. *Punctual estimates with general weights*

Here we prove our main result concerning the singularity problem (SP), namely the punctual estimate (1.26).

Proof of Theorem 1.5. (i) Let u be any solution of (SP). Then it satisfies Eq. (4.7) with

$$H = u^{\delta-p+1}.$$

Now we apply Theorem 4.3 with

$$s = \frac{\delta}{\delta - p + 1} > \frac{\eta}{p},$$

since $\delta < \overline{K}$. From Theorem 4.2, there exists $C > 0$ such that, for any $x_0 \in \mathcal{B}'$, and any ball $\mathcal{B}(x_0, 8R) \subset \Omega'$,

$$\left(\fint_{B(x_0,2R),b} u^\delta \right)^{1/s} \leqslant C R^{-p} \frac{\int_{\mathcal{B}(x_0,4R)} a \,\mathrm{d}x}{\int_{\mathcal{B}(x_0,2R)} b \,\mathrm{d}x}.$$

Hence the function C_R defined in (4.11) satisfies

$$C_R = R^p \left(\fint_{B(x_0,2R),b} u^\delta \right)^{1/s} \frac{\int_{B(x_0,2R)} b \,\mathrm{d}x}{\int_{B(x_0,R)} a \,\mathrm{d}x} \leqslant C \frac{\int_{\mathcal{B}(x_0,4R)} a \,\mathrm{d}x}{\int_{\mathcal{B}(x_0,R)} a \,\mathrm{d}x},$$

so that C_R is bounded independently of R and x_0 from the doubling condition (C_2). Then Theorem 4.3 applies, and we can take $m = \delta < \overline{Q}$ in (4.13), since $\delta < \overline{Q}$,and $R = |x_0|/8$. It gives, with the condition (C_2),

$$\begin{aligned} u(x_0) &\leqslant \sup_{B(x_0,|x_0|/8)} u \leqslant C \left(\fint_{B(x_0,|x_0|/4)} u^\delta \right)^{1/\delta} \leqslant C |x_0|^{-p/(\delta-p+1)} \frac{\int_{\mathcal{B}(x_0,|x_0|/2)} a \,\mathrm{d}x}{\int_{\mathcal{B}(x_0,|x_0|/4)} b \,\mathrm{d}x} \\ &\leqslant C |x_0|^{-p/(\delta-p+1)} \frac{\int_{\mathcal{B}(x_0,|x_0|/4)} a \,\mathrm{d}x}{\int_{\mathcal{B}(x_0,|x_0|/4)} b \,\mathrm{d}x}, \end{aligned}$$

which ends the proof of (1.26). □

Now we can deduce Theorem 1.6.

Proof of Theorem 1.6. If a is locally Q-admissible in Ω', and (1.27) holds, then clearly (a, b) is also locally Q-admissible in Ω', and for any $m > 0$,

$$\fint_{B(x_0,|x_0|/2),b} u^m \leqslant C \fint_{B(x_0,|x_0|/2),a} u^m$$

hence the Harnack inequality follows from (4.10) and (4.12). Then we deduce the estimate (1.28) from Theorem 4.2, (1.26) and (1.27).

4.4. Estimates with radial weights

Here we study the case of radial weights. First we can precise the weak estimate (4.2) of Theorem 4.1 and consequently the estimate (1.21) of Theorem 1.4.

Corollary 4.1. *Assume that a is radial and satisfies* (H_1)*. Let u be any function satisfying the assumptions of Theorem* 4.1*, in particular any supersolution of* (SP) *in Ω'. Then there exists $C > 0$, such that for any $x_0 \in \mathcal{B}'$,*

$$\inf_{|x|=|x_0|} u \leqslant C\big(h(|x_0|) + 1\big). \tag{4.15}$$

Proof. This follows from (4.2) by a staightforward computation,

$$\int_{|x_0|\leqslant|x|\leqslant 1} |x|^{(1-N)p'} a^{1/(1-p)}\,\mathrm{d}x = \int_{|x_0|}^{1} r^{(1-N)p'+(1-N)/(1-p)+N-1} A^{1/(1-p)}\,\mathrm{d}r = h\big(|x_0|\big).$$

Now we consider the case where h is unbounded. Then we also improve the estimate (4.4) of Theorem 4.2. Moreover, we show that the conditions (1.10) are still necessary conditions of existence of a possibly nonradial solution, extending the results of [2]. □

Theorem 4.4. *Assume that a and b are radial, with* (H_1) *and* (H_2)*. Let u be any supersolution of* (SP)*. Then $b \in L^1_{\mathrm{loc}}(\Omega)$, $u \in L^{\delta}_{\mathrm{loc}}(\Omega, b)$, and there exists $C > 0$, such that for any $r < 1$,*

$$\int_{\mathcal{B}(0,r)} u^{\delta} b\,\mathrm{d}x \leqslant C\big(h^{\delta}(r)\beta(r)\big)^{-(p-1)/(\delta-p+1)}, \tag{4.16}$$

and

$$\left(⨍_{\mathcal{B}(0,r),b} u^{\delta}\right)^{1/\delta} \leqslant C\big(h^{(p-1)}(r)\beta(r)\big)^{-1/(\delta+1-p)}. \tag{4.17}$$

Proof. Let us return to (4.6): there exists $C > 0$ such that, for any $\zeta \in \mathcal{D}(\Omega')$,

$$\left(\int_{\Omega'} u^{\delta}\zeta^{\lambda} b\,\mathrm{d}x\right)^{(\delta-p+1)/\delta}\left(\int_{\Omega'} \zeta^{\lambda} b\,\mathrm{d}x\right)^{(p-1)/\delta} \leqslant C\int_{\Omega'} \zeta^{\lambda-p}|\nabla\zeta|^p a\,\mathrm{d}x.$$

Let $r_0 \in (0,1)$ be fixed, and $h_0 = h(r_0)$. Let $n \geqslant 1$ be a fixed integer. From (H_2), the function h maps $(0,1)$ onto $(0,+\infty)$. Thus we can chose the test function under the form

$$\zeta(x) = \xi\big(h(|x|)\big),$$

where $h \mapsto \xi(h) \in \mathcal{D}((0,+\infty))$ with values in $[0,1]$, such that $\xi(\tau) = 1$ for $h_0 \leqslant \tau \leqslant nh_0$, $\xi(\tau) = 0$ for $\tau \leqslant h_0/2$ or $\tau \geqslant (n+1/2)h_0$, and $|\mathrm{d}\xi/\mathrm{d}h| \leqslant C/h_0$, with C independent on h_0. We find

$$\begin{aligned}\int_{\Omega'} a\zeta^{\lambda-p}|\nabla\zeta|^p\,\mathrm{d}x &\leqslant \frac{C}{h_0^p}\left(\int_{r(h)}^{r(h/2)} A(r)\big|h'(r)\big|^p\,\mathrm{d}r + \int_{r((n+1/2)h_0)}^{r(nh_0)} A(r)\big|h'(r)\big|^p\,\mathrm{d}r\right)\\ &= -\frac{C}{h_0^p}\left(\int_{r(h_0)}^{r(h_0/2)} h'(r)\,\mathrm{d}r + \int_{r((n+1/2)h_0)}^{r(nh_0)} h'(r)\,\mathrm{d}r\right) = \frac{C}{h_0^p}h_0 = Ch_0^{1-p}.\end{aligned}$$

Then

$$\left(\int_{\mathcal{C}_{r(nh_0),r(h_0)}} u^\delta b\,\mathrm{d}x\right)^{(\delta-p+1)/\delta}\left(\int_{\mathcal{C}_{r(nh_0),r(h_0)}} b\,\mathrm{d}x\right)^{(p-1)/\delta} \leqslant Ch_0^{1-p}.$$

Making $n \to +\infty$, we find that $b \in L^1_{\mathrm{loc}}(\Omega)$, so that β is well defined, and moreover $u \in L^\delta_{\mathrm{loc}}(\Omega, b)$, and since

$$\int_{\mathcal{C}_{r(nh_0),r(h_0)}} b\,\mathrm{d}x = \int_{r(nh_0)}^{r(h_0)} B(r)\,\mathrm{d}r = \beta\big(r(h_0)\big) - \beta\big(r(nh_0)\big),$$

we get

$$\beta^{(p-1)/\delta}(r_0)\left(\int_{\mathcal{B}(0,r_0)} u^\delta b\,\mathrm{d}x\right)^{(\delta-p+1)/\delta} \leqslant Ch_0^{1-p}$$

which proves (4.16) and (4.17). □

Let us apply it to the case of admissible weights:

Proof of Theorem 1.7. (i) The estimate (1.29) follows from (4.17) in Theorem 4.4 and (4.13) in Theorem 4.3 as for Theorem 1.5. Since a is locally admissible, and u is a supersolution of (1.8), u is lower semicontinuous in Ω' and $u > 0$ in Ω' from the strict maximum principle, see [16, Theorem 7.12]. Following the ideas of [2], we show that u satisfies a stronger form of the maximum principle. Let us set $m = \inf_{|x|=1} u(x) > 0$. For any integer $n \geqslant 2$, the function $x \mapsto w_n(x) = m(1 - h(|x|)/h(1/n))$ is a solution of Eq. (1.8) for $1/n \leqslant |x| \leqslant 1$. From the comparison principle [16, Theorem 7.6], we have $w_n(x) \leqslant u(x)$ for $1/n \leqslant |x| \leqslant 1$. Going to the limit as $n \to +\infty$, we deduce that $u(x) \geqslant m$, a.e. in $\mathcal{B}'$, hence $\liminf_{x\to 0} u(x) > 0$. Then in particular (1.10) holds from (4.17).

(ii) This follows from (1.29) and Theorem 1.6. □

4.5. Applications

Notice that the assumptions of Theorem 1.6 are relatively weak, compared to those of Theorem 1.2. Consider for example the case of two powers of $|x|$.

Theorem 4.5. *Let θ, σ be any reals such that $\sigma + p \geqslant \theta > -N$, $\sigma > -N$. Let u be any nonnegative solution of*

$$-\mathrm{div}\big(|x|^\theta |\nabla u|^{p-2}\nabla u\big) = |x|^\sigma u^\delta \quad \textit{in } \Omega'. \tag{4.18}$$

Then if $p - 1 < \delta < \overline{P}$, then u satisfies the Harnack inequality. Moreover, either $\theta + N \leqslant p$ and u is bounded near 0, or $\theta + N > p$ and

$$u(x) \leqslant C\min\big(|x|^{(p-N-\theta)/(p-1)}, |x|^{-(p+\sigma-\theta)/(\delta-p+1)}\big) \quad \textit{near } 0. \tag{4.19}$$

Proof. We can apply Theorem 1.7. First $a = |x|^\theta$ is locally *admissible in* Ω': obviously $a \in L^1_{\mathrm{loc}}(\Omega')$ and verifies (C_1), since a is positive and continuous in Ω'. Moreover a satisfies the doubling condition: for any ball $\mathcal{B}(x_0, 4R) \subset \Omega'$, one has $R < |x_0|/4$, hence

$$\int_{\mathcal{B}(x_0,2R)} |x|^\theta \,\mathrm{d}x \leqslant 2^N |\mathcal{B}(x_0,R)|\, |x_0|^\theta \max\left(\left(\frac{3}{2}\right)^\theta, \left(\frac{1}{2}\right)^\theta\right),$$

$$\int_{\mathcal{B}(x_0,R)} |x|^\theta \,\mathrm{d}x \geqslant |\mathcal{B}(x_0,R)|\, |x_0|^\theta \min\left(\left(\frac{3}{4}\right)^\theta, \left(\frac{5}{4}\right)^\theta\right).$$

And a satisfies (C_3) with any $Q < P^* \leqslant +\infty$: for any ball $B(x_0, 2R) \subset \Omega'$ and any $\varphi \in \mathcal{D}(B(x_0,R))$, one has similarly $R < |x_0|/2$, hence from the usual Sobolev imbedding, with new constants C depending on Q and θ,

$$\left(\oint_{B(x_0,R),a} |\varphi|^Q\right)^{1/Q} \leqslant C\left(\oint_{B(x_0,R),1} |\varphi|^Q\right)^{1/Q} \leqslant CR\left(\oint_{B(x_0,R),1} |\nabla\varphi|^p\right)^{1/p}$$
$$\leqslant CR\left(\oint_{B(x_0,R),a} |\nabla\varphi|^p\right)^{1/p},$$

and a satisfies (C_4). Indeed for any constant $c > 0$, one has

$$\int_{B(x_0,R)} |\varphi - \overline{\varphi}_a|^p a \,\mathrm{d}x \leqslant 2^p R^p \int_{B(x_0,R)} |\varphi - c|^p a \,\mathrm{d}x,$$

where $\overline{\varphi}_a = \oint_{B(x_0,R),a} \varphi$, and the result follows as above from the Poincaré-inequality without weight, after taking $c = \oint_{B(x_0,R),1} \varphi$. Also ($C_5$) holds, and ($C_6$) follows as ($C_3$), or using (1.30). □

Theorem 4.5 covers in particular Theorem 1.3. It extends immediately to the case

$$a(x) = |x|^\theta \left|\log\left|\frac{x}{C}\right|\right|^\ell, \qquad b(x) = |x|^\sigma \left|\log\left|\frac{x}{C}\right|\right|^k,$$

where $C > \operatorname{diam}(\Omega)$, with the same assumptions on θ, σ, for any real numbers k, ℓ. For any $p - 1 < \delta < \overline{P}$, we get Harnack inequality. Then u is bounded near 0 when $\theta + N < p$ and satisfies (1.31) when $\theta + N \geqslant p$. This covers in particular the examples given at Section 2.4.

More generally, let us define a *local Muckenhoupt class* for any domain $D \subset \mathbb{R}^N$. We will say that a function a, positive a.e. in D, lies in $\mathcal{A}_{r,\mathrm{loc}}(D)$ if $a \in L^1_{\mathrm{loc}}(D)$, and $a^{-1/(r-1)} \in L^1_{\mathrm{loc}}(D)$ if $r > 1$, and there exists a constant $C > 0$, such that, for any ball $\mathcal{B}(x, 2R) \subset D$,

$$\left(\int_{\mathcal{B}(x,R)} a \,\mathrm{d}x\right)^{1/r} \left(\int_{\mathcal{B}(x,R)} a^{-1/(r-1)} \,\mathrm{d}x\right)^{1/r'} \leqslant CR^N$$

if $r > 1$, or

$$\int_{\mathcal{B}(x,R)} a \,\mathrm{d}x \leqslant CR^N \inf_{\mathcal{B}(x,R)} a$$

if $r = 1$. One verifies that $a \in \mathcal{A}_{r,\mathrm{loc}}(D)$ if and only if $a \in \mathcal{A}_r(D')$ for any domain $D' \Subset D$. It is known that any weight $a \in \mathcal{A}_p(\mathbb{R}^N)$ is locally admissible in D. In fact it remains true for any $a \in \mathcal{A}_{p,\mathrm{loc}}(D)$. Hence *Theorem* 1.6 *applies to any weight* $a \in \mathcal{A}_{p,\mathrm{loc}}(\Omega')$, and in particular to any $a \in \mathcal{A}_{1,\mathrm{loc}}(\Omega')$. Observe that the local Muckenhoupt class $\mathcal{A}_{p,\mathrm{loc}}(\Omega')$ is very large, compared to the global one $\mathcal{A}_p(\Omega)$.

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