Local behaviour of the solutions of a class of nonlinear elliptic systems

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Abstract

Here we study the behaviour near a punctual singularity of the positive solutions of semilinear elliptic systems in $\mathbb{R}^N (N \geq 3)$ given by

$$\left\{ \begin{array}{l} \Delta u + |x|^a u^s v^p = 0, \\ \Delta v + |x|^b u^q v^t = 0, \end{array} \right.$$

(where $a, b, p, q, s, t \in \mathbb{R}$, $p, q > 0, s, t \geq 0$). We describe the first undercritical case, and the sublinear and linear cases. The proofs do not use any variational methods, but lie essentially upon comparison properties between the two solutions u and v, and the properties of the subsolutions and supersolutions of the scalar equation

$$\Delta f + |x|^\sigma f^\eta = 0$$

($\sigma, \eta \in \mathbb{R}$, $\eta > 0$). This extends the classical study of the scalar equation when $0 < \eta <$ max $(N, (N+\sigma))/(N-2)$.

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1 Introduction

In this article we study some semilinear elliptic systems of reaction-diffusion equations of the following type

$$\begin{cases} \Delta u+|x|^{a}u^{s}v^{p}=0, \\ \Delta v+|x|^{b}u^{q}v^{t}=0, \end{cases} \tag{1.1}$$

in a domain Ω of $\mathbb{R}^N$ $(N\geq 3)$, with boundary $\partial\Omega$, where a,b,p,q,s,t are given reals, and u,v are supposed to be nonnegative. We assume that $s,t\geq 0$, and $p,q>0$. It means that the system (1.1) is completely coupled. With no restriction it will be supposed that

$$p+s\leq q+t. \tag{1.2}$$

Our main purpose is to study the local existence and the local behaviour of the solutions near an isolated singularity. Let us denote $B_\rho(x)=\left\{y\in\mathbb{R}^N\,||y-x|<\rho\right\}$, and $B_\rho=B_\rho(0)$ for any $\rho>0$. We can always assume that the possible singularity is located at the origin and that $\Omega\supset B_{1,}$ and u,v lie in $C^2(\overline{\Omega}\backslash\{0\})$. The asymptotical behaviour at infinity will follow from Kelvin transform. We are also concerned with the regular Dirichlet problem in Ω.

The particular case of system

$$\begin{cases} \Delta u+|x|^{a}\,u^{s}v^{t+1}=0, \\ \Delta v+|x|^{a}\,u^{s+1}v^{t}=0, \end{cases} \tag{1.3}$$

where $s,t>0$, $p=t+1$, $q=s+1$, $a=b$, was studied in [2] whenever $s+t+1<(N+2)/(N-2)$; see also [24], [25]. This system was introduced in [11], where De Thélin and Vélin consider the Dirichlet regular problem in Ω. Notice that it is gradient-type: it is the Euler system of the functional

$$L(u,v)=\int_\Omega\left(\frac{s+1}{2}\left|\nabla u\right|^2+\frac{t+1}{2}\left|\nabla v\right|^2-|x|^{a}\,u^{s+1}v^{t+1}\right).$$

When $a=0$, one can distinguish a first undercritical case appears where $s+t+1<N/(N-2)$, a second one beyond this value where $s+t+1<(N+2)/(N-2)$, and a so-called supercritical case beyond $(N+2)/(N-2)$.

Another interesting example is the case of the system

$$\begin{cases} \Delta u+|x|^{a}\,v^{p}=0, \\ \Delta v+|x|^{b}\,u^{q}=0, \end{cases} \tag{1.4}$$

where $s=t=0$. This system is Hamiltonian-type: it is the Euler system of the functional

$$\mathcal{L}(u,v)=\int_\Omega\left(\nabla u.\nabla v-|x|^{b}\,\frac{u^{q+1}}{q+1}-|x|^{a}\,\frac{v^{p+1}}{p+1}\right).$$

It has been considered by many authors when $a=b=0$ and $pq>1$. Here we find a first undercritical case where

$$\max(2(p+1)/(pq-1),2(q+1)/(pq-1)>N-2, \tag{1.5}$$

then a second one beyond this region, where

$$1/(p+1) + 1/(q+1) > (N-2)/N, \tag{1.6}$$

that is

$$2(p+1)/(pq-1) + 2(q+1)/(pq-1) > N-2; \tag{1.7}$$

and finally a supercritical case beyond that new region. In the first undercritical case the radial local behaviour was described by Garcia, Manasevitch, Mitidieri and Yarur in [15]. Soranzo [30] gave the nonradial local behaviour in the special case of the biharmonic problem, where $p = 1 < q$, $a = 0$. Its proof is valid only if $q < \min(N, N+b)/(N-2)$. This does not cover the first undercritical case: when $b = 0$, (1.5) means $q < N/(N-4)$. In the second undercritical case, Clément, de Figueiredo, Felmer, Mitidieri [8], [10] and Van der Vorst [33] obtained existence and uniqueness results for the regular Dirichlet problem in Ω. In the supercritical case, ground states appear. The question of existence or nonexistence of ground states, still partially open, has been discussed by Serrin and Zou [28], [29], whether (1.6) is satisfied or not; see also [10], [20], [21].

In the general case of system (1.1), which has no variational structure, very few results are known. Qi [23] gave some properties of radial ground states, and nonexistence results when $s, t \geq 1$, which are nonoptimal in case of system (1.3). Our approach is not variational, it is based upon comparison results between the functions u and v.

In **Section 2** we give necessary existence conditions and elementary properties for system (1.1). Any solution of the form $(u, 0)$ or $(0, v)$ of (1.1) with u or v harmonic will be called *trivial solution*. Let us set $r = |x|$ and denote by

$$\bar{v}(r) = |\partial B_r|^{-1} \int\limits_{\partial B_r} v$$

the mean value of v. The first natural step of the study is to look for particular radial solutions of that system, which play a fundamental part. We find such solutions under the form

$$u(r) = Ar^{-\gamma}, \qquad v(r) = Br^{-\xi} \quad (A, B > 0,\ \gamma, \xi \in \mathbb{R}), \tag{1.8}$$

whenever

$$\delta = pq - (1-t)(1-s) \neq 0, \tag{1.9}$$

with

$$\begin{cases} \gamma = ((b+2)p + (a+2)(1-t))/\delta, \\ \xi = ((a+2)q + (b+2)(1-s))/\delta, \end{cases} \tag{1.10}$$

$$\begin{cases} A = (\gamma(N-2-\gamma)^{(1-t)/\delta}(\xi(N-2-\xi))^{p/\delta}, \\ B = (\gamma(N-2-\gamma)^{q/\delta}(\xi(N-2-\xi))^{(1-s)/\delta}, \end{cases} \tag{1.11}$$

under the condition

$$0 < \min(\gamma, \xi) \leq \max(\gamma, \xi) < N-2. \tag{1.12}$$

The values of γ and ξ and the condition (1.12) play a great part in the study. Another tool is a classical result of Brézis and Lions [4]. Since u, v are superharmonic in $\bar{\Omega}\backslash\{0\}$, we have $|x|^a u^s v^p, |x|^a u^q v^t \in L^1(\Omega)$, and there exist $\alpha, \beta \geq 0$ such that

$$\begin{cases} -\Delta u = |x|^a u^s v^p + \alpha\ \delta_0, \\ -\Delta v = |x|^a u^q v^t + \beta\ \delta_0, \end{cases} \tag{1.13}$$

in $\mathcal{D}'(\Omega)$, where δ_0 is the Dirac mass at the origin. Using a scalar inequality satisfied by a combination of the two functions u and v, we obtain existence conditions for system (1.1). They depend on γ and ξ, and the sign of the discriminant δ of the system. By applying for example to the case $a = b$, we prove in particular the following.

Theorem 1.1 *Assume $a = b$, and $\delta > 0$ or $s, t > 1$. If system (1.1) admits nontrivial solutions in $\bar{B}_1\backslash\{0\}$, then*

$$a + 2 > 0. \tag{1.14}$$

If it admits nontrivial solutions in $\mathbb{R}^N/B_1$, then

$$p + s > (N + a)/(N - 2) \text{ and } q + t > (N + a)/(N - 2), \quad \text{if } s, t > 1, \tag{1.15}$$

$$\max(\gamma, \xi) \leq N - 2 \text{ and } (\gamma, \xi) \neq (N - 2, N - 2), \quad \text{if } s, t \leq 1, \tag{1.16}$$

$$p + s \geq (N + a)/(N - 2) \text{ and } \gamma \leq N - 2, \quad \text{if } t \leq 1 < s, \tag{1.17}$$

$$q + t \geq (N + a)/(N - 2) \text{ and } \xi \leq N - 2, \quad \text{if } s \leq 1 < t, \tag{1.18}$$

and one inequality is strict in (1.17), (1.18).

In **Section 3** we prove our main result of comparison between the two solutions u and v under some conditions on a and b. We suppose in this section that

$$\max(p + 1 - t, q + 1 - s) = q + 1 - s > 0, \tag{1.19}$$

condition which is automatically satisfied when $\delta > 0$. Assuming for simplification $a = b$, we prove the following, where

$$c_N = 1/(N - 2)\left|S^{N-1}\right| \quad \text{and} \quad S^{N-1} = \partial B_1.$$

Theorem 1.2 *Let $(u, v) \in (C^2(\bar{\Omega}\backslash\{0\}))^2$ be any nonnegative nontrivial solution of (1.1), with (1.19) and $a = b$. Then*

i) If $p + 1 - t > 0$ and $q > s$, there exists $M \geq 0$ such that

$$u(x) \leq c_N \alpha |x|^{2-N} + D v(x)^{(p+1-t)/(q+1-s)} + M \tag{1.20}$$

in $\bar{\Omega}\backslash\{0\}$, where $D = ((q + 1 - s)/(p + 1 - t))^{1/(q+1-s)}$; and $M = 0$ when $u = 0$ on $\partial\Omega$.

ii) If $p + 1 - t > 0$ and $q < s$, then for any $\varepsilon > 0$ there exist $D_\varepsilon > 0$, $M_\varepsilon \geq 0$ such that

$$u(x) \leq c_N \alpha(1 + \varepsilon) |x|^{2-N} + D_\varepsilon v(x)^{(p+1-t)/(q+1-s)} + M_\varepsilon \tag{1.21}$$

in $\bar{\Omega}\backslash\{0\}$; *and* $D_\varepsilon = 1/(p+1-t)$ *when* $\alpha = 0$ *and* $u = 0$ *on* $\partial\Omega$.

iii) If $p+1-t \leq 0$, *then for any* $d > 0$ *and any open set* ω *such that* $\bar{\omega} \subset \Omega$, *there exist* $D_{d,\omega} > 0$, $M_{d,\omega} \geq 0$ *such that*

$$u(x) \leq c_N \alpha |x|^{2-N} + D_{d,\omega} v(x)^d + M_{d,\omega} \tag{1.22}$$

in $\bar{\omega}\backslash\{0\}$.

When $p+1-t > 0$, this implies in particular that any solution (u,v) of (1.1) satisfies the inequality of the form

$$u(x)^{q+1-s} \leq C v(x)^{p+1-t} \tag{1.23}$$

in $\bar{\Omega}\backslash\{0\}$, whenever $\alpha = 0$ and $u = 0$ on $\partial\Omega$. This property is remarkable, since the particular solution (u,v) defined in (1.9) satisfies the relation $u^{q+1-s} \equiv A^{q+1-s} B^{p+1-t} v^{p+1-t}$. It applies in particular to the regular Dirichlet problem in Ω.

In **Section 4** we extend to the system the well known results of [27], [19], relative to the local behaviour of the solutions of equation

$$-\Delta f = |x|^a f^\eta \tag{1.24}$$

in the (superlinear) first undercritical case where

$$1 < \eta < \min(N, N+a)/(N-2) = (N-a^-)/(N-2). \tag{1.25}$$

System (1.1) will be called *superlinear* whenever $\delta > 0$ or $s,t > 1$. Under this assumption, the *first undercritical case* will be defined by the conditions

$$\left\{ \begin{array}{c} \max(\gamma, \xi) > N-2, \\ \max(2(p+1-t)/\delta, 2(q+1-s)/\delta) > N-2, \end{array} \right. \quad \text{if } s,t < 1, \tag{1.26}$$

and

$$\left\{ \begin{array}{lll} q+t < (N-b^-)/(N-2) & \text{if} & t > 1, \\ p+s < (N-a^-)/(N-2) & \text{if} & s > 1. \end{array} \right. \tag{1.27}$$

The case $p+s, q+t > 1$ appears to be the easiest one. In the general case, we mainly use our comparison result. Our idea is to study the inequality satisfied by function v by plugging inequality (1.20) or (1.21) into the second line of (1.1). Then we prove that v satisfies the Harnack inequality, in order to obtain good estimates of u and v. For that purpose we use two well-known results. Let $w \in C^2(\bar{B}_1\backslash\{0\})$ be a nonnegative solution of an equation

$$-\Delta w = hw$$

with $h \in C(\bar{B}_1\backslash\{0\})$. If $h \in L^\tau(B_{1/2})$ for some $\tau > N/2$, then either the function $|x|^{2-N} w$ is bounded from above and from below near the origin, or w can be extended as a C^2 function in $\bar{B}_1$, from Serrin [27]. If h only satisfies an inequality

$$\int_{B_{|x_0|/2}(x_0)} h^\tau \leq C \;|x_0|^{N-2\tau} \qquad \text{for any } x_0 \in B'_{1/4},$$

for some $\tau > N/2$, where C does not depend on x_0, then w satisfies Harnack inequality, from Trudinger [32], see also [17]. Our results are optimal whenever $s, t > 1$, or

$$0 \leq b - a \leq \left\{ \begin{array}{ll} q + t - p - s, & \text{if } p + 1 - t > 0, \\ q + 1 - s, & \text{if } p + 1 - t \leq 0. \end{array} \right. \tag{1.28}$$

By applying for example to system (1.4) with $a = b$ near 0, we obtain the following theorem, which extends in particular the radial results of [15] to the nonradial case.

Theorem 1.3 *Let $(u, v) \in (C^2(\bar{B}_1 \backslash \{0\}))^2$ be any nonnegative nontrivial solution of (1.4) with $a = b$. Assume that*

$$\max((2 - a^-)(p + 1)/(pq - 1), (2 - a^-)(q + 1)/(pq - 1)) > N - 2. \tag{1.29}$$

Then either $(u, v) \in (C(\bar{B}_1))^2$, or

$$\lim_{|x| \to 0} |x|^{N-2} u(x) = c_N \alpha > 0, \qquad \lim_{|x| \to 0} |x|^{N-2} v(x) = c_N \beta > 0, \tag{1.30}$$

or (up to a change from u, p into v, q),

- *either $q > (a + 2)/(N - 2)$, and*

$$\left\{ \begin{array}{l} \lim_{|x| \to 0} |x|^{N-2} u(x) = c_N \alpha > 0, \\ \lim_{|x| \to 0} |x|^{(N-2)q-2-a} v(x) = (c_N \alpha)^q / ((N - 2)q - a - 2)(N + a - (N - 2)q), \end{array} \right. \tag{1.31}$$

- *or $q < (a + 2)/(N - 2)$ and*

$$\lim_{|x| \to 0} |x|^{N-2} u(x) = c_N \alpha > 0, \qquad \lim_{|x| \to 0} v(x) = C > 0, \tag{1.32}$$

- *or $q = (a + 2)/(N - 2)$ and*

$$\lim_{|x| \to 0} |x|^{N-2} u(x) = c_N \alpha > 0, \qquad \lim_{|x| \to 0} |Ln\, |x||^{-1} v(x) = (c_N \alpha)^q / (N - 2). \tag{1.33}$$

And $q < (N + a)/(N - 2)$ if $p \leq q$ and $\alpha > 0$.

Section 5 gives the complete local behaviour and the behaviour at infinity of any solution of (1.1) in the *sublinear* case where $\delta < 0$ and $s, t < 1$, under condition (1.28). It applies in particular to system (1.4) when $pq < 1$. This extends the results of [25], [2] for equation (1.24) when $0 < \eta < 1$. The "*linear*" case $\delta = 0$ and $s, t < 1$, also considered, appears to be delicate.

Our results beyond the first critical case are still partial in the nonradial case, and they will not be mentioned. Another paper will follow, concerning the parabolic system associated to (1.1):

$$\left\{ \begin{array}{rcl} \partial u/\partial t - \Delta u & = & |x|^a u^s v^p, \\ \partial v/\partial t - \Delta v & = & |x|^b u^q v^t, \end{array} \right. \tag{1.34}$$

which has still been studied in [13], see also [12], [7].

In the **Appendix** we summarize some results of existence and local behaviour of supersolutions or subsolutions of scalar equations of the type

$$-\Delta f = |x|^{\sigma} |Ln\,|x||^{\tau} f^{\eta}, \tag{1.35}$$

where $\sigma, \tau, \eta \in \mathbb{R}$, and $\eta > 0$. They play a crucial part in the study, and present an own interest. Some of them have still been used in [2], [25]. They are a consequence of [27] and an Osserman type estimate for the equation with the opposite sign.

2 Existence conditions and first properties

First let us give elementary results about system (1.1). We shall say that a solution $(u, v) \in (C^2(\bar{\Omega}\backslash\{0\}))^2$ of (1.1), or only one of the two components, is *regular* if it can be extended as a continuous function in whole $\bar{\Omega}$, and *singular* if not. In the sequel, the letters C, C_1, C_2.. denote different positive constants which may depend on u, v, but not on $x \in \bar{B}_{1/2}\backslash\{0\}$.

Proposition 2.1 *Assume that system (1.1) admits nontrivial solutions in $\bar{\Omega}\backslash\{0\}$. Then*

i) $\min\,(N + a, N + b) > 0$.

ii) If there exist regular solutions, then $\min(a + 2, b + 2) > 0$.

iii) If $\alpha > 0$ (resp. $\beta > 0$), then $s < (N + a)/(N - 2)$ (resp. $t < (N + b)/(N - 2)$).

iv) If $\alpha, \beta > 0$, then $p + s < (N + a)/(N - 2)$ and $q + t < (N + b)/(N - 2)$.

v) If $\max\,(\gamma, \xi) \leq N - 2$ *and $s < 1$ (resp. $t < 1$), then $\beta = 0$ (resp. $\alpha = 0$).*

If $\max\,(\gamma, \xi) < N - 2$ *and $s = 1$ (resp. $t = 1$), then $\beta = 0$ (resp. $\alpha = 0$).*

Proof We have seen that $|x|^a u^s v^p, |x|^b u^q v^t \in L^1(\Omega)$ from the Brézis-Lions theorem [4]. Moreover, from the strict maximum principle, there exists $C > 0$ such that

$$u(x) \geq C, \quad v(x) \geq C \tag{2.1}$$

in $\bar{B}_{1/2}\backslash\{0\}$, see for example [34]. If $\alpha > 0$ (resp. $\beta > 0$), then there exists $C > 0$ such that

$$u(x) \geq C\,|x|^{2-N} \quad (\text{resp. } v(x) \geq C\,|x|^{2-N}) \tag{2.2}$$

in $\bar{B}_{1/2}\backslash\{0\}$, hence i) iii) iv) hold. And ii) is immediate from the maximum principle. Now assume $\max(\gamma, \xi) \leq N - 2$ and $\beta > 0$. Then

$$-\Delta u(x) \geq C\,|x|^{a-(N-2)p} u^s$$

in $\bar{B}_{1/2}\backslash\{0\}$. If $0 \leq s < 1$, this implies that

$$u(x) \geq C\,|x|^{(a+2-(N-2)p)/(1-s)}$$

from the maximum principle when $s=0$, and from Lemma 6.2 when $0<s<1$. Then

$$|x|^b u^q v^t \geq C\,|x|^{b+q(a+2-(N-2)p)/(1-s)-(N-2)t}.$$

This is impossible because $|x|^b u^q v^t \in L^1(\Omega)$ and $\xi \leq N-2$. When $s=1$, then $a+2-(N-2)p \geq 0$ from Lemma 6.2, and $\xi=(a+2)/p$, hence a contradiction holds when $\max(\gamma,\xi)<N-2$.■

Remark 2.1 When $\max(\gamma,\xi)<N-2$ and $s\geq 1$, there can actually exist some solutions with $\beta>0$. That means that $v(x)\geq C\,|x|^{2-N}$ near the origin. Their existence is proved in [24] in case of system (1.3).

Remark 2.2 In the radial case any solution (u,v) satisfies the estimates

$$u^{s-1}(r)v^p(r) \leq Cr^{-(a+2)}, \qquad u^q(r)v^{t-1}(r) \leq Cr^{-(b+2)} \tag{2.3}$$

in $(0,1/2]$; see [15], [28] for system (1.4) and [23] for similar estimates in $\mathbb{R}^+$ of the ground states of system (1.1). Indeed u,v are nonincreasing near 0, and there is a constant $C>0$ such that

$$ru_r(r)+Cu(r)\geq 0, \qquad rv_r(r)+Cv(r)\geq 0$$

near 0, from [15]. By integration of the radial equations of (1.1), there holds

$$-r^{N-1}u_r(r) \geq C_1 r^{N+a}u^s(r)v^p(r), \qquad -r^{N-1}v_r(r) \geq C_2 r^{N+b}u^q(r)v^t(r)$$

for small r, hence the result. If $s\leq 1$ or $t\leq 1$, then (2.3) in turn implies

$$u(r)\leq Cr^{-\gamma}, \quad v(r)\leq Cr^{-\xi} \tag{2.4}$$

in $(0,1/2]$.

The following theorem extends a result of [28] for system (1.4). The proof is based upon a preceeding result of Souto [31], which was not optimal.

Theorem 2.2 *Assume that system (1.8) admits nontrivial solutions in* $\bar{B}_1\backslash\{0\}$.

i) If $s,t>1$, *then*

$$\min(a+2,b+2)\geq 0 \quad \textit{and} \quad (a+2,b+2)\neq(0,0). \tag{2.5}$$

ii) If $\delta>0$, *and* $s,t\leq 1$, *then*

$$\min(\gamma,\xi)\geq 0 \quad \textit{and} \quad (\gamma,\xi)\neq(0,0). \tag{2.6}$$

iii) If $\delta>0$, *and* $t\leq 1<s$ *(resp.* $s\leq 1<t$*), then*

$$a+2\geq 0 \quad \textit{and} \quad \gamma\geq 0 \quad (\textit{resp.}\ b+2\geq 0 \quad \textit{and} \quad \xi\geq 0), \tag{2.7}$$

and one inequality is strict.

iv) If $\delta<0$, *and* $s,t<1$, *then*

$$\max(\gamma,\xi)\leq N-2 \quad \textit{and} \quad (\gamma,\xi)\neq(N-2,N-2). \tag{2.8}$$

v) If $\delta=0$, *and* $s,t<1$, *then*

$$\min((b+2)p+(a+2)(1-t),(a+2)q+(b+2)(1-s))\geq 0. \tag{2.9}$$

Proof First observe that $a+2>0$ (resp. $b+2>0$) as soon as $s>1$ (resp. $t>1$). Indeed (2.1) implies

$$-\Delta u \geq C\left|x\right|^a u^s$$

in $\bar{B}_{1/2}\backslash\{0\}$, and the conclusion follows directly from Lemma 6.2. In particular (2.5) holds. Now come to the other cases. Let

$$f = u^m v^{1-m}, \quad \text{with} \quad 0 < m < 1.$$

Let us compute $-\Delta f$:

$$\begin{aligned} -\Delta f &= m(1-m)u^{m-2}v^{-1-m}\left|v\nabla u - u\nabla v\right|^2 \\ &\quad + m\left|x\right|^a u^{m-1+s}v^{1-m+p} + (1-m)\left|x\right|^b u^{m+q}v^{t-m}. \end{aligned}$$

Then for any $k>1$,

$$\begin{aligned} -\Delta f &\geq u^{m-1+s}v^{t-m}(m\left|x\right|^a v^{p+1-t} + \left|x\right|^b(1-m)u^{q+1-s}) \\ &\geq \min(m,1-m)\left|x\right|^{(a(k-1)+b)/k} u^{m-1+s+(q+1-s)/k}v^{t-m+(p+1-t)(k-1)/k}. \end{aligned}$$

from the Hölder inequality. If

$$(mp+(1-m)(1-s))(m(1-t)+(1-m)q) > 0, \tag{2.10}$$

one can choose

$$k = \frac{m(p+1-t)+(1-m)(q+1-s)}{mp+(1-m)(1-s)}.$$

This gives

$$-\Delta f \geq C_m\left|x\right|^\sigma f^\eta, \tag{2.11}$$

with $C_m = \min(m,1-m)$, and

$$\sigma = a + (b-a)/k, \tag{2.12}$$

$$\eta = 1 + \delta/((m(p+1-t)+(1-m)(q+1-s)). \tag{2.13}$$

Suppose that there exists a nontrivial solution (u,v) of system (1.1) in $\bar{B}_1\backslash\{0\}$. Then for any $m\in(0,1)$ satisfying (2.10), there exists a nontrivial solution of inequality (2.11) in $\bar{B}_1\backslash\{0\}$. If moreover $\eta>1$, then Lemma 6.2 implies that necessarily $\sigma>-2$, that is

$$m((b+2)p+(a+2)(1-t)) + (1-m)((a+2)q+(b+2)(1-s) > 0. \tag{2.14}$$

• If $\delta>0$ and $s,t\leq 1$, then (2.14) is satisfied for any $m\in(0,1)$ and (2.6) holds.
• If $\delta>0$ and $t\leq 1<s$, then (2.14) holds with $m=\theta(s-1)/(p+s-1)+1-\theta$, for any $\theta\in(0,1)$. Hence

$$\theta(a+2)/(p+s-1) + (1-\theta)((a+2)(1-t)+(b+2)p)/\delta > 0,$$

and (2.7) holds.

• If $\delta < 0$ and $s,t < 1$, then (2.10) is satisfied for any $m \in (0,1)$, and $0 < \eta < 1$, since $p+1-t+\delta = p(q+1)+s(1-t) > 0$, and $q+1-s+\delta = q(p+1)+t(1-s) > 0$.Then Lemma 6.2 implies $\eta < (N+\sigma)/(N-2)$. As a consequence

$$m((b+2)p+(a+2)(1-t))+(1-m)((a+2)q+(b+2)(1-s) > (N-2)\delta$$

for any $m \in (0,1)$, and (2.9) holds.
• If $\delta = 0$ and $s,t < 1$, then (2.14) is satisfied for any $m \in (0,1)$, and $\eta = 1$. This implies $\sigma \geq -2$ from Lemma 6.2, and (2.14) holds with possible equality. The conclusion follows.■

Remark 2.3 In particular this shows that the biharmonic problem

$$\Delta^2 u + |x|^b u^q = 0, \qquad \Delta u \leq 0,$$

studied in [30], has no nontrivial solution in $\bar{B}_1 \backslash \{0\}$ when $b \leq -4$.

Proof of Theorem 1.1 Let us make a Kelvin transform in system (1.1) when $x \in \mathbb{R}^N/B_1$. The functions

$$u_0(x) = |x|^{2-N} u(x/|x|^2), \qquad v_0(x) = |x|^{2-N} v(x/|x|^2), \tag{2.15}$$

satisfy in $\bar{B}_1 \backslash \{0\}$ the system

$$\begin{cases} \Delta u_0 + |x|^{a_0} u_0^s v_0^p = 0, \\ \Delta v_0 + |x|^{b_0} u_0^q v_0^t = 0, \end{cases} \tag{2.16}$$

where

$$a_0 = (N-2)(p+s)-(N+2+a), \qquad b_0 = (N-2)(q+t)-(N+2+b), \tag{2.17}$$

and, with obvious notations, $\gamma_0 = N-2-\gamma$, $\xi_0 = N-2-\xi$. By applying Theorem 2.2 to (u_0,v_0), we get analogous necessary conditions of existence near infinity. In the particular case $a=b$, and $\delta > 0$ or $s,t > 1$, this gives Theorem 1.1. In the same way, if $\delta < 0$ and $s,t < 1$, then $a < -2$. If $\delta = 0$ and $s,t < 1$, then necessarily $\min\,(p+s,q+t) \geq (N+a)/(N-2)$.■

3 Comparison properties

Here we prove fundamental comparison properties for system (1.1), under a condition on the difference $b-a$. In the sequel we use a weak form of maximum principle: any function $y \in C^2(\bar{\Omega}\backslash \{0\})$, such that

$$-\Delta y + Dy = g + \lambda\delta_0$$

in $\mathcal{D}'(\Omega)$, with $g \in L^1(\Omega)$, $g \geq 0$, $\lambda \geq 0$, and $|x|^{2-N} D \in L^1(\Omega)$, satisfies $y(x) \geq \min_{\partial\Omega} y$. It follows from the results of [35] and [2].

Theorem 3.1 *Let $(u,v) \in (C^2(\bar{\Omega}\backslash \{0\}))^2$ be any nonnegative nontrivial solution of (1.1) with (1.19).*

1) Assume that $p+1-t > 0$, and

$$b-a \leq (N-2)(q+t-p-s). \tag{3.1}$$

• *If $q \geq s$, then there exists $D > 0$ such that*

$$u(x) \leq c_N \alpha \left|x\right|^{2-N} + D \left|x\right|^{-(b-a)^+/(q+1-s)} v(x)^{(p+1-t)/(q+1-s)} + \max_{\partial\Omega} u \tag{3.2}$$

in $\bar{\Omega} \backslash \{0\}$, and $D = ((q+1-s)/(p+1-t))^{1/(q+1-s)}$ when $b-a \geq 0$.

• *If $q < s$, for any $\varepsilon > 0$ there exists $D_\varepsilon > 0$ such that*

$$u(x) \leq c_N \alpha (1+\varepsilon) \left|x\right|^{2-N} + D_\varepsilon \left|x\right|^{-(b-a)^+/(q+1-s)} v(x)^{(p+1-t)/(q+1-s)} + (1+\varepsilon) \max_{\partial\Omega} u \tag{3.3}$$

in $\bar{\Omega} \backslash \{0\}$. And $D_\varepsilon = 1/(p+1-t)$ when $b-a \geq 0$, $\alpha = 0$ and $\max_{\partial\Omega} u = 0$.

2) Assume that $p+1-t \leq 0$, and

$$b - a < (N-2)(q+1-s). \tag{3.4}$$

Then for any $d > 0$, and any open set ω such that $\bar{\omega} \subset \Omega$, if $q \geq s$, there exists $D_{d,\omega}$ such that

$$u(x) \leq c_N \alpha \left|x\right|^{2-N} + D_{d,\omega} \left|x\right|^{-(b-a)^+/(q+1-s)} v(x)^d + \max_{\partial\omega} u \tag{3.5}$$

in $\bar{\omega} \backslash \{0\}$. If $q < s$, then for any $\varepsilon > 0$ there exists $D_{\varepsilon,d,\omega} > 0$ such that

$$u(x) \leq c_N \alpha (1+\varepsilon) \left|x\right|^{2-N} + D_{\varepsilon,d,\omega} \left|x\right|^{-(b-a)^+/(q+1-s)} v(x)^d + (1+\varepsilon) \max_{\partial\omega} u \tag{3.6}$$

in $\bar{\omega} \backslash \{0\}$.

Since the proof is quite technical, we begin by the simple case of Hamiltonian system.

Proof (case of system (1.5) with $a = b = 0$). Here (1.19) reduces to $q \geq p$. Let

$$f = v^{(p+1)/(q+1)}.$$

Then

$$-\Delta f = K + ((p+1)/(q+1)) u^q f^{(p-q)/(p+1)},$$

where

$$K = ((p+1)(q-p)/(q+1)^2) v^{(p+1)/(q+1)-2} \left|\nabla v\right|^2 \geq 0.$$

Thus f is superharmonic in $\bar{\Omega} \backslash \{0\}$. Then $K \in L^1(\Omega)$ and there exists $\lambda \geq 0$ such that

$$-\Delta f = K + ((p+1)/(q+1)) u^q f^{(p-q)/(p+1)} + \lambda \delta_0$$

in $\mathcal{D}'(\Omega)$, from [4]. On the other hand, the first line of (1.13) can be expressed as

$$-\Delta u = f^q f^{(p-q)/(p+1)} + \alpha \delta_0.$$

Let $\ell = ((p+1)/(q+1))^{-1/(q+1)}$. Then by difference

$$-\Delta(\ell f - u) + H = \ell K + (\ell\lambda - \alpha)\delta_0,$$

where $H = \ell^{-q} f^{(p-q)/(p+1)}((\ell f)^q - u^q)$. Let ψ, φ be defined by

$$-\Delta\psi = 0 \text{ in } \Omega \quad \text{and } \psi = u \text{ on } \partial\Omega, \qquad -\Delta\varphi = \delta_0 \quad \text{and } \varphi = 0 \text{ on } \partial\Omega; \tag{3.7}$$

hence $\varphi(x) \leq c_N |x|^{2-N}$ in Ω. Then $u \geq \alpha\varphi + \psi$ in $\bar{\Omega}\backslash\{0\}$ from the maximum principle, and

$$-\Delta(\ell f - u + \alpha\varphi + \psi) + H = \ell K + \ell\lambda\delta_0$$

in $\mathcal{D}'(\Omega)$. And H is nonpositive on the set $\{u \geq \ell f + \alpha\varphi + \psi\}$. When $\lambda = 0$ it follows that

$$u \leq \ell f + \alpha\varphi + \psi, \tag{3.8}$$

in $\bar{\Omega}\backslash\{0\}$, from the Kato inequality. Hence

$$u(x) \leq c_N\alpha |x|^{2-N} + \ell v(x)^{(p+1)/(q+1)} + \max_{\partial\Omega} u. \tag{3.9}$$

If $\lambda > 0$, then $f(x) \geq C|x|^{2-N}$ in $\bar{B}_{1/2}\backslash\{0\}$, for some $C > 0$. But $\lim_{r\to 0} r^{N-2}\bar{v}(r) = c_N\beta$, hence necessarily $p = q$, $f = v$, $\ell = 1$, $\lambda = \beta > 0$, and

$$-\Delta(v - u + \alpha\varphi + \psi) + H' = K' + \beta\delta_0, \tag{3.10}$$

with a new nonnegative $K' \in L^1(\Omega)$, and

$$H' = v^q - (u - \alpha\varphi - \psi)^q.$$

We can write $H' = W \times (v - u + \alpha\varphi + \psi)$, where

$$0 \leq W \leq \max(q,1)(\max(u - \alpha\varphi - \psi, v))^{q-1}.$$

In any case it follows that $|x|^{2-N} W \in L^1_{loc}(\Omega)$. Indeed $v(x) \geq C|x|^{2-N}$ near 0 and $u^q, v^q \in L^1(\Omega)$ when $q > 1$, or $W \in L^\infty_{loc}(\Omega)$ when $q \leq 1$. Then for any open set ω such that $\bar{\omega} \subset \Omega$, we find

$$v - (u - \alpha\varphi - \psi) \geq \min_{\partial\omega}(v - (u - \alpha\varphi - \psi))$$

in $\bar{\omega}$, from the weak maximum principle. Using an increasing sequence (ω_n) recovering Ω, we get again (3.8), (3.9).■

Now let us come to the general case.

Proof of theorem 3.2 Let

$$f = |x|^{-m(1-d)(1-e)} u^e v^{d(1-e)},$$

where m, d, e are three parameters, with $d \in (0,1]$, $e \in [0,1)$ and $m \in [0, N-2]$. Let us compute $-\Delta f$:

$$\begin{aligned} -\Delta f &= K + e|x|^{a-m(1-d)(1-e)} u^{s-1+e} v^{p+d(1-e)} \\ &\quad + d(1-e)|x|^{b-m(1-d)(1-e)} u^{q+e} v^{t-1+d(1-e)}, \end{aligned}$$

where

$$K=(1-e)\left|x\right|^{-m(1-d)(1-e)}u^{e}v^{d(1-e)}(e\left|\frac{\nabla u}{u}-d\frac{\nabla v}{v}+m(1-d)\frac{\nabla\left|x\right|}{\left|x\right|}\right|^{2}$$

$$+d(1-d)\left|\frac{\nabla v}{v}+m\frac{\nabla\left|x\right|}{\left|x\right|}\right|^{2}+m(N-2-m)(1-d)\left|\frac{\nabla\left|x\right|}{\left|x\right|}\right|^{2})\geq 0.$$

Then f is superharmonic in $\bar{\Omega}\backslash\{0\}$, hence there exists $\lambda\geq 0$ such that

$$\begin{aligned}-\Delta f &= K+e\left|x\right|^{a-m(1-d)(1-e)}u^{s-1+e}v^{p+d(1-e)}\\ &\quad +d(1-e)\left|x\right|^{b-m(1-d)(1-e)}u^{q+e}v^{t-1+d(1-e)}+\lambda\delta_0\end{aligned}$$

in $\mathcal{D}'(\Omega)$. Then, for any $\ell>0$,

$$-\Delta(\ell f-u)+H=\ell K_1+(\ell\lambda-\alpha)\delta_0, \tag{3.11}$$

where

$$\begin{aligned}H &= \left|x\right|^{a}u^{s}v^{p}-d(1-e)\ell\left|x\right|^{b-m(1-d)(1-e)}u^{q+e}v^{t-1+d(1-e)}\\ &= \left|x\right|^{a+mp(1-d)d^{-1}}u^{s-pe/d(1-e)}f^{p/d(1-e)}\\ &\quad -d(1-e)\ell\left|x\right|^{b+m(t-1)(1-d)/d}u^{q+e(1-t)/d(1-e)}f^{1+(t-1)/d(1-e)},\end{aligned}$$

$$K_1=K+e\left|x\right|^{a-m(1-d)(1-e)}u^{s-1+e}v^{p+d(1-e)}.$$

Since $q+1-s>0$, one can choose $e\in[0,1)$ such that $q+e-s\geq 0$. We shall take

$$e=0\quad\text{if } q\geq s,\quad\text{ and } e=s-q\quad\text{if not.}$$

Then the function H is nonpositive on the set $\{u\geq\ell f\}$ as soon as

$$v^{d(q+1-s)-(p+1-t)}\geq\left(\ell^{(q+1-s)/(1-e)}d(1-e)\right)^{-1}\left|x\right|^{a-b+m(1-d)(q+1-s)}. \tag{3.12}$$

i) First assume that $p+1-t>0$, and choose $d=(p+1-t)/(q+1-s)$ and

$$m=(b-a)^{+}/(q+1-s)(1-d)\quad\text{if } d\neq 1,\qquad m=0\quad\text{if } d=1.$$

Indeed such an m satisfies $m\in[0,N-2]$ from (3.1). Then take

$$\begin{cases}\ell=(d(1-e))^{-(1-e)/(q+1-s)} & \text{if } b-a\geq 0,\\ \ell\geq((\text{diam }\Omega)^{a-b}/d(1-e))^{(1-e)/(q+1-s)} & \text{if } b-a<0.\end{cases}$$

Let ψ, φ be defined by (3.7). Then $u\geq\alpha\varphi+\psi$ in $\bar{\Omega}\backslash\{0\}$, and

$$-\Delta(\ell f-u+\alpha\varphi+\psi)+H=\ell K_1+\ell\lambda\delta_0$$

in $\mathcal{D}'(\Omega)$. And H is nonpositive on the set $\{u\geq\ell f+\alpha\varphi+\psi\}$.

• First suppose $\lambda=0$. From the Kato inequality, this implies

$$u\leq\ell f+\alpha\varphi+\psi\quad\text{in }\bar{\Omega}\backslash\{0\}, \tag{3.13}$$

hence

$$u(x) \leq c_N \alpha \left|x\right|^{2-N} + \ell \left|x\right|^{-(1-e)(b-a)^+/(q+1-s)} u(x)^e v(x)^{(1-e)(p+1-t)/(q+1-s)} + \max_{\partial\Omega} u. \qquad (3.14)$$

This implies (3.2), and (3.3) from the Hölder inequality.
Notice that function H can be written under the form

$$\begin{aligned} H &= \left|x\right|^{a+mp(1-d)/d} u^{s-pe/d(1-e)} f^{1+(t-1)/d(1-e)} \times \\ &\quad (f^{(q+e-s)/(1-e)} - \ell d(1-e) \left|x\right|^{b-a-m(q+t-p-s)} u^{(q+e-s)/(1-e)}). \end{aligned}$$

Hence $H = H_1 - H_2$, with

$$\begin{aligned} H_1 &= \ell^{-(q+e-s)/(1-e)} \left|x\right|^{a+mp(1-d)/d(1-e)} u^{s-pe/d(1-e)} f^{1+(t-1)/d(1-e)} \times \\ &\quad ((lf)^{(q+e-s)/(1-e)} - u^{(q+e-s)/(1-e)}) \end{aligned}$$

and nonnegative $H_2 \in L^1(\Omega)$.

• Now suppose $\lambda > 0$. Then $f(x) \geq C \left|x\right|^{2-N}$, and

$$u^e(x) v^{d(1-e)}(x) \geq C \left|x\right|^{2-N+m(1-d)(1-e)}$$

in $\bar{B}_{1/2} \backslash \{0\}$, for some $C > 0$. But $\lim_{r\to 0} r^{N-2}\overline{u}(r) = c_N\alpha$, $\lim_{r\to 0} r^{N-2}\overline{v}(r) = c_N\beta$, and

$$\overline{u^e v^{d(1-e)}}(r) \leq (\overline{u^{e+d(1-e)}} + \overline{v^{e+d(1-e)}})(r) \leq (\overline{u}(r) + \overline{v}(r))^{e+d(1-e)},$$

hence $m = N-2$ or $d = 1$; and $\alpha, \beta > 0$. On the other hand

$$-\Delta(\ell f - u + \alpha\varphi + \psi) + H' = K' + \ell\lambda\delta_0,$$

with a new nonnegative $K' \in L^1(\Omega)$, and

$$\begin{aligned} H' &= \ell^{-(q+e-s)/(1-e)} \left|x\right|^{a+mp(1-d)/d(1-e)} u^{s-pe/d(1-e)} f^{1+(t-1)/d(1-e)} \\ &\quad \times ((lf)^{(q+e-s)/(1-e)} - (u - \alpha\varphi - \psi)^{(q+e-s)/(1-e)}). \end{aligned}$$

If $q < s$, then $H' = 0$. If $q \geq s$, then function H' can be written under the form $H' = W \times (\ell f - u + \alpha\varphi + \psi)$, where

$$0 \leq W \leq \ell^{s-q} \max(q-s, 1) \left|x\right|^{a+mp(1-d)d^{-1}} u^s f^{1+(t-1)/d} (\max(u - \alpha\varphi - \psi, f))^{q-s-1}.$$

This implies that $\left|x\right|^{2-N} W \in L^1_{loc}(\Omega)$. Indeed $f(x) \geq C \left|x\right|^{2-N}$ near 0, and

$$\left|x\right|^{a+mp(1-d)d^{-1}} u^s f^{pd^{-1}} \in L^1(\Omega), \quad \text{and } \left|x\right|^{b+m(1-t)(1-d)d^{-1}} u^q f^{(t+d-1)d^{-1}} \in L^1(\Omega)$$

from (3.10) and [4], and

$$\max(u - \alpha\varphi - \psi, f)^{q-s-1} \leq f^{q-s-1} \quad \text{if } q - s < 1.$$

Then we get again (3.13), (3.14).

ii) Now assume that $p+1-t \leq 0$, and choose any $d \in (0, 1-(b-a)/(N-2)(q+1-s))$ and $m = (b-a)^+/(q+1-s)(1-d)$. Since the function v is superharmonic, it is positive in $\bar{\omega}$ from the strict maximum principle. Then one can find some $\ell_{d,\omega} > 0$ such that (3.12) is satisfied in $\bar{\omega}$ with $\ell = \ell_{d,\omega}$. Moreover $\lambda = 0$ and we find

$$u(x) \leq c_N \alpha \left|x\right|^{2-N} + \ell \left|x\right|^{-(b-a)^+/(q+1-s)} u(x)^e v(x)^{d(1-e)} + \max_{\partial\omega} u$$

in $\bar{\omega} \setminus \{0\}$, hence (3.5), (3.6) hold.■

Remark 3.1 Let $(u, v) \in (C^2(\bar{\Omega} \setminus \{0\}))^2$ be any nonnegative nontrivial solution of (1.13), with $\alpha \leq \beta$. Assume that $p+1-t > 0$, and

$$p + s = q + t, \quad \text{and} \quad a = b. \tag{3.15}$$

i) If $q \geq s$ and $\alpha \leq \beta$, then

$$u(x) \leq v(x) + \max_{\partial\Omega} u \tag{3.16}$$

in $\bar{\Omega} \setminus \{0\}$.

ii) If $q < s$ and $\alpha \leq (q+1-s)^{-1/(q+1-s)}\beta$, then

$$u(x) \leq (q+1-s)^{1/(q+1-s)} u^{s-q}(x) v^{q+1-s}(x) + \max_{\partial\Omega} u \tag{3.17}$$

in $\bar{\Omega} \setminus \{0\}$. In any case, there exists some constant $C > 0$ such that

$$u(x) \leq Cv(x) \quad \text{near the origin.} \tag{3.18}$$

Indeed the result comes from Theorem 1.2 when $\alpha = 0$. Now assume $0 < \alpha \leq \beta$. Here $f = u^e v^{1-e}$, and $\ell = 1$ whenever $q \geq s$, $\ell = 1/(q+1-s) > 1$ whenever $q < s$. Observe that $\ell\, \lambda \geq \alpha$, because $\lim_{r \to 0} r^{N-2} \bar{f}(r) = \lambda$ and for any $\varepsilon > 0$, $f(x) \geq (\alpha^e \beta^{1-e} - \varepsilon) \left|x\right|^{2-N}$ near 0. And $\alpha \leq \ell^{1/(1-e)}$ by hypothesis. Then (3.11) can be written under the form

$$-\Delta(\ell f - u + \psi) + H'' = K'' + (\ell\lambda - \alpha)\delta_0$$

with a new nonnegative $K'' \in L^1(\Omega)$, and

$$\begin{aligned} H'' &= \ell^{-(q+e-s)/(1-e)} \left|x\right|^a u^{s-pe/(1-e)} f^{1+(t-1)/(1-e)} \\ &\quad \times ((lf)^{(q+e-s)/(1-e)} - (u-\psi)^{(q+e-s)/(1-e)}). \end{aligned}$$

As above it follows that $u \leq \ell f + \psi$ in $\bar{\Omega} \setminus \{0\}$, hence we deduce (3.16) to (3.18).

Remark 3.2 Let us apply Theorem 1.2 to the case of the regular Dirichlet problem in Ω, with

$$q + 1 - s \geq p + 1 - t > 0.$$

Assume $a = b > -2$ for the sake of simplicity. Let $(u, v) \in (C^2(\bar{\Omega}))^2$ be any nonnegative nontrivial solution of (1.1) with $u = v = 0$ on $\partial\Omega$. Then

$$u(x)^{q+1-s} \leq Cv(x)^{p+1-t} \tag{3.19}$$

in $\bar{\Omega}$, where

$$C = \begin{cases} (q+1-s)/(p+1-t) & \text{if } q \geq s, \\ 1/(p+1-t)^{q+1-s} & \text{if } q < s. \end{cases} \tag{3.20}$$

In particular if $q - s = p - t \geq 0$, then $u \equiv v$, hence they are the solutions of the scalar equation

$$\Delta U + |x|^a U^{q+t} = 0 \tag{3.21}$$

in $\bar{\Omega}$.

Remark 3.3 Consider the case of radial ground states of system (1.1), with $a = b > -2$, and $s \leq 1$ or $t \leq 1$. If such ground states exist, they tend to 0 at infinity. Indeed by Kelvin transform, they also satisfy (2.4) at infinity, and in that case $\xi > \gamma > 0$. From (3.14), for any $\varepsilon > 0$, there exists some $R_\varepsilon > 0$ such that for any $R > R_\varepsilon$

$$u(x) \leq \ell u(x)^e v(x)^{e(p+1-t)/(q+1-s)} + \varepsilon \tag{3.22}$$

in $\bar{B}_R$. This implies that

$$u(x)^{q+1-s} \leq C v(x)^{p+1-t} \tag{3.23}$$

in $\mathbb{R}^N$, where C is given by (3.20). This propety deserves being compared to a result of [29] concerning the Hamiltonian system (1.4), where $s = t = 0$, $a = b = 0$. In the supercritical case, where (1.6) does not hold, they prove the existence of ground state solutions. They show that they form a family with one parameter: there exists a constant $c = c(N, p, q)$ such that (u, v) is a ground state if and only if

$$u(0)^{q+1} = c v(0)^{p+1},$$

and then (u, v) is unique. Here we find that necessarily $c \leq (q+1)/(p+1)$.
In the case $q - s = p - t \geq 0$, (3.23) implies that the ground states of (1.1) are given by $u \equiv v \equiv U$, where U is any ground state of equation (3.21). Then there exist such ground states when $q + t \geq (N + 2 + 2a)/(N - 2)$. This extends some results of [2].

4 The first undercritical case

In the case of the system, it is hard to define a notion of superlinearity, because the conditions $\delta > 0$, $s > 1$, $t > 1$ are not linked. Concerning the local behaviour, it appears to correspond to the case

$$\delta > 0 \quad \text{or } s, t > 1, \tag{4.1}$$

and we make this assumption in the whole section.

First let us give a simple result, which is the direct extension of Serrin's results and does not use comparison methods. So it does not require any condition on the coefficients a, b.

Theorem 4.1 *Let $(u, v) \in (C^2(\bar{B}_1 \backslash \{0\}))^2$ be any nonnegative nontrivial solution of (1.1). Assume that*

$$\begin{cases} 1 < q + t < (N - b^-)/(N - 2), \\ 1 < p + s < (N - a^-)/(N - 2), \end{cases} \tag{4.2}$$

(that implies (4.1)).Then

$$v(x) \leq C\,|x|^{2-N}\,, \qquad u(x) \leq C\,|x|^{2-N} \tag{4.3}$$

in $\bar{B}_{1/2}\backslash\{0\}$.

Proof Adding the two lines of (1.1), we get an equation relative to the sum $u+v$. It can be written under the form

$$-\Delta(u+v) = h(u+v),$$

where $0 \leq h \leq h_1 + h_2$, and

$$h_1 = |x|^a\,(u+v)^{s+p-1}, \qquad h_2 = |x|^b\,(u+v)^{q+t-1}.$$

Now $u+v \in M^{N/(N-2)}(B_1)$, hence $h_1, h_2 \in L^{\tau}_{loc}(B_{1/2})$ for some $\tau > N/2$, from (4.2) and Lemma 6.3. Then (4.3) holds from [27]. Moreover, either

$$C\,|x|^{2-N} \leq u(x) + v(x) \leq 2C\,|x|^{2-N}$$

in $\bar{B}_{1/2}\backslash\{0\}$, or $u+v$ is regular, and can be extended as a C^2 function in $\bar{B}_1$.■

Remark 4.1 In general the result is not optimal. For example in the case of Hamiltonian system (1.4) with $a = b = 0$, it only concerns the case of the square

$$1 \leq p, q < N/(N-2),$$

which does not cover the first undercritical region, where $pq > 1$ and (1.5) holds. However, the result appears to be optimal whenever $s > 1$ and $t > 1$. Indeed if for example $t > 1$ and $\alpha > 0$ in (1.13), one can expect that the function u will behave like $c_N\alpha\,|x|^{2-N}$, and the function v like a solution of equation

$$\Delta w + (c_N\alpha)^q\,|x|^{b-(N-2)q}\,w^t = 0.$$

In order to apply Serrin's results, the first condition of (4.2) is necessary to get the local behaviour in any case, and the second one when $s > 1$.

Theorem 4.1 covers in particular the case

$$\max(p+1-t, q+1-s) \leq 0. \tag{4.4}$$

Now let us come to the other case, where (1.19) holds. In order to apply our comparison result, we suppose that a, b satisfy the condition

$$0 \leq b-a \leq \begin{cases} (N-2)(q+t-p-s), & \text{if } p+1-t > 0, \\ (N-2)(q+1-s), & \text{if } p+1-t \leq 0. \end{cases} \tag{4.5}$$

Our assumptions of first undercriticallity are the following:

$$\begin{cases} \max\ (\gamma, \xi)\ > N-2, \\ \max\ (2(p+1-t)/\delta,\ 2(q+1-s)/\delta)\ > N-2, \end{cases} \tag{4.6}$$

if $s,t\leq 1$, and

$$\left\{\begin{array}{lll} q+t & < & (N-b^{-})/(N-2), \quad \text{if } t>1, \\ p+s & < & (N-a^{-})/(N-2), \quad \text{if } s>1. \end{array}\right. \tag{4.7}$$

Remark 4.2 i) In the sequel, some simple relations between γ and ξ are useful:

$$b+2=q\gamma-(1-t)\xi, \qquad a+2=p\xi-(1-s)\gamma. \tag{4.8}$$

Notice that each condition of (4.7) implies (4.6). Indeed if for example $t>1$ and $\max(\gamma,\xi)\leq N-2$, then $q+t\geq (N+b)/(N-2)$ from (4.8).

ii) Assume $p+1-t>0$. Then (4.5) implies

$$(p+1-t)\xi\leq (q+1-s)\gamma\leq (p+1-t)\xi+(N-2)(q+t-p-s)$$

from (4.8). Thus the condition $\xi\leq N-2$ implies $\gamma\leq N-2$. Hence (4.6) is equivalent to

$$\xi>N-2 \quad \text{and} \quad 2(q+1-s)/\delta>N-2. \tag{4.9}$$

Moreover our assumptions (4.5) to (4.7) imply

$$p+s<(N-a^{-})/(N-2), \tag{4.10}$$

for any $s,t\geq 0$. Indeed if $s\leq 1$ and $\xi>N-2$, and (4.10) does not hold, then $q+t<(N-b^{-})/(N-2)$, since $(1-s)(b+N-(N-2)(q+t)>q((N-2)(p+s)-(a+N))$. But this in turn implies in any case (4.10) from (4.5).

Theorem 4.2 *Let $(u,v)\in (C^2(\bar{B}_1\backslash\{0\}))^2$ be any nonnegative nontrivial solution of (1.1) with $\delta>0$. Assume (1.19) and (4.5) to (4.7). Then v satisfies the Harnack inequality, and*

$$v(x)\leq C\left|x\right|^{2-N}, \qquad u(x)\leq C\left|x\right|^{2-N} \tag{4.11}$$

in $\bar{B}_{1/2}\backslash\{0\}$.

Proof First assume $p+1-t>0$. From (3.3), (3.4), it follows that

$$u(x)\leq C(\alpha\left|x\right|^{2-N}+\left|x\right|^{(a-b)//(q+1-s)}v(x)^{(p+1-t)/(q+1-s)}) \tag{4.12}$$

in $\bar{B}_{1/2}\backslash\{0\}$, since $v(x)\geq C>0$ in $\bar{B}_{1/2}\backslash\{0\}$, and $b-a\geq 0$. Hence

$$-\Delta v(x)\leq C\left(\alpha^q\left|x\right|^{b-(N-2)q}v^t+\left|x\right|^{(aq+b(1-s))/(q+1-s)}v(x)^{(\delta/(q+1-s))+1}\right) \tag{4.13}$$

in $\bar{B}_{1/2}\backslash\{0\}$. Notice that

$$(\delta/(q+1-s))+1<(N-((aq+b(1-s))/(q+1-s))^{-})/(N-2) \tag{4.14}$$

from (4.9). Now assume $p+1-t\leq 0$. In the same way for any $d>0$,

$$u(x)\leq C(\alpha\left|x\right|^{2-N}+\left|x\right|^{(a-b)//(q+1-s)}v(x)^{d}) \tag{4.15}$$

in $\bar{B}_{1/2}\backslash\{0\}$, from (3.3). Hence

$$-\Delta v(x) \leq C(\alpha^q |x|^{b-(N-2)q} v^t + |x|^{(aq+b(1-s))/(q+1-s)} v(x)^{t+qd}) \tag{4.16}$$

in $\bar{B}_{1/2}\backslash\{0\}$. And observe that (4.5), (4.7) imply

$$t < (N - ((aq + b(1-s))/(q+1-s))^-)/(N-2). \tag{4.17}$$

Then, in any case, if $\alpha = 0$, the function v is a subsolution of an equation of the form (1.24) *in the first undercritical case*. In the general case $\alpha \geq 0$, the equation (4.7) can be written under the form

$$-\Delta v = hv, \tag{4.18}$$

where

$$0 \leq h = |x|^b u^q v^{t-1} \leq C(h_1 + h_2), \tag{4.19}$$

with

$$h_1 = \alpha^q |x|^{b-(N-2)q} v^{t-1}, \qquad h_2 = |x|^{(aq+b(1-s))/(q+1-s)} v^\lambda, \tag{4.20}$$

and

$$\lambda = \begin{cases} \delta/(q+1-s) & \text{if } p+1-t > 0, \\ t-1+qd & \text{if } p+1-t \leq 0. \end{cases}$$

Choosing d small enough whenever $p+1-t \leq 0$, we find $h_2 \in L^\tau(B_{1/2})$ for some $\tau > N/2$ from Lemma 6.3.

We claim that v satisfies Harnack inequality in $B_1\backslash\{0\}$. It is immediate when $\alpha = 0$ from [27]. When $\alpha > 0$, then

$$u(x) \geq C |x|^{2-N}$$

in $\bar{B}_{1/2}\backslash\{0\}$ from (2.2), and

$$-\Delta v(x) \geq C |x|^{b-(N-2)q} v^t. \tag{4.21}$$

This implies that $q < (b+2)/(N-2)$ if $t > 1$, $q \leq (b+2)/(N-2)$ if $t = 1$, from Lemma 6.2. When $0 \leq t < 1$, then $q + t < (N+b)/(N-2)$ and

$$v(x) \geq C |x|^{(b+2-(N-2)q)/(1-t)} \tag{4.22}$$

in $\bar{B}_{1/2}\backslash\{0\}$. If $t > 1$, then Lemma 6.3 implies that $h_1 \in L^\tau(B_{1/2})$ from (4.7). Hence $h \in L^\tau(B_{1/2})$, and the Harnack inequality holds.
If $0 \leq t \leq 1$, then from (4.22) or obviously if $t = 1$,

$$|x|^{b-(N-2)q} v^{t-1} \leq C |x|^{-2} \tag{4.23}$$

in $\bar{B}_{1/2}\backslash\{0\}$. Hence for any $x_0 \in B_{1/4}$, and any $\tau > 1$,

$$\int_{B_{|x_0|/2}(x_0)} h^\tau \leq C |x_0|^{N-2\tau},$$

where C does not depend on x_0, and the Harnack inequality follows again. In any case we deduce the estimate (4.11) for function v, because

$$\lim_{r\to 0} r^{N-2}\bar{v}(r) = c_N\beta,$$

Then we get the estimate of u from (4.5), (4.12) or (4.15) with d small enough.■

Remark 4.3 We do not know if we can relax the assumption (4.5) ofTheorem 4.2. This condition occurs in a very precise manner in our proof of this theorem, which lies upon the comparison method. See for example the proofs of (4.14), (4.17).

With these estimates we can deduce precise convergence results. In the following we describe exhaustively all the possible behaviours of the solutions. The most interesting results concern the case $s \leq 1$ or $t \leq 1$. Notice that the situation is not symmetrical with respect to u, p, s and v, q, t, because of the assumption (4.5).

Theorem 4.3 *Under the assumptions of Theorem 4.2,*

1) Either one of the following eventualities holds, up to a change from u, p, s into v, q, t:

i) (u, v) is regular (and then $a + 2 > 0$, $b + 2 > 0$):

$$\lim_{|x|\to 0} u(x) = C_1 > 0, \qquad \lim_{|x|\to 0} v(x) = C_2 > 0. \tag{4.24}$$

ii) $q + t < (N + b)/(N - 2)$, and

$$\lim_{|x|\to 0} |x|^{N-2} u(x) = c_N\alpha > 0, \qquad \lim_{|x|\to 0} |x|^{N-2} v(x) = c_N\beta > 0. \tag{4.25}$$

iii) $q < (b + 2)/(N - 2)$, and

$$\lim_{|x|\to 0} |x|^{N-2} u(x) = c_N\alpha > 0, \qquad \lim_{|x|\to 0} v(x) = C > 0, \tag{4.26}$$

iv) $0 \leq t < 1$, $q > (b + 2)/(N - 2)$ and $q + t < (N + b)/(N - 2)$, and

$$\left\{\begin{array}{c} \lim_{|x|\to 0} |x|^{N-2} u(x) = c_N\alpha > 0, \\ \lim_{|x|\to 0} |x|^{(b+2-(N-2)q)/(t-1)} v(x) = \frac{((N-2)q-b-2)(N+b-(N-2)(q+t))}{(1-t)^2(c_N\alpha)^q)^{1/(t-1)}}. \end{array}\right. \tag{4.27}$$

v) $0 \leq t < 1$, and $q = (b + 2)/(N - 2)$, and

$$\left\{\begin{array}{c} \lim_{|x|\to 0} |x|^{N-2} u(x) = c_N\alpha > 0, \\ \lim_{x\to 0} |Ln\,|x||^{1/(t-1)} v(x) = ((N-2)/(1-t)(c_N\alpha)^q)^{1/(t-1)}. \end{array}\right. \tag{4.28}$$

2) Or one of the other following eventualities holds, with no change from u, p, s into v, q, t:

vi) $0 \leq s < 1$ and $a < -2 < b$, and

$$\left\{\begin{array}{c} \lim_{x\to 0} |x|^{(a+2)/(s-1)} u(x) = C^p(1-s)^2/|a+2|\,(N+a-(N-2)s), \\ \lim_{|x|\to 0} v(x) = C > 0. \end{array}\right. \tag{4.29}$$

vii) $0 \leq s < 1$ *and* $a = -2 < b$,

$$\left\{ \begin{array}{c} \lim_{x \to 0} |Ln\,|x||^{1/(s-1)}\, u(x) = (C^p(1-s)/(N-2))^{1/(1-s)}, \\ \lim_{|x| \to 0} v(x) = C > 0. \end{array} \right. \tag{4.30}$$

viii) $t = 1$, $q = (b+2)/(N-2)$, *and* $(c_N\alpha)^q < (N-2)^2/4$,

$$\left\{ \begin{array}{c} \lim_{|x| \to 0} |x|^{N-2}\, u(x) = c_N\alpha > 0, \\ \lim_{x \to 0} |x|^{(N-2+((N-2)^2-4(c_N\alpha)^q)^{1/2}/2}\, v(x) = C > 0 \\ or \quad \lim_{x \to 0} |x|^{(N-2-((N-2)^2-4(c_N\alpha)^q)^{1/2}/2}\, v(x) = C > 0. \end{array} \right. \tag{4.31}$$

ix) $t = 1$, $q = (b+2)/(N-2)$ *and* $(c_N\alpha)^q = (N-2)^2/4$,

$$\left\{ \begin{array}{c} \lim_{|x| \to 0} |x|^{N-2}\, u(x) = c_N\alpha > 0, \\ \lim_{x \to 0} |x|^{(N-2)/2}\, |Ln\,|x||^{-1}\, v(x) = C > 0 \quad or \quad \lim_{x \to 0} |x|^{(N-2)/2}\, v(x) = C > 0. \end{array} \right. \tag{4.32}$$

Under the assumptions of Theorem 4.1, then 1) holds.

A part of the results comes from the following lemma of convergence, which does not require the assumptions (4.1), (4.5) to (4.7), and will be also useful in the sublinear case.

Lemma 4.4 *Let* $(u, v) \in (C^2(\bar{B}_1 \setminus \{0\}))^2$ *be any nonnegative nontrivial solution of system (1.13) such that*

$$u(x) + v(x) = O(|x|^{2-N})$$

near the origin.

i) If $\alpha > 0$ *and* $p + s < (N+a)/(N-2)$, *then*

$$\lim_{|x| \to 0} |x|^{N-2}\, u(x) = c_N\alpha > 0. \tag{4.33}$$

ii) If $\alpha > 0$ *and* $\beta > 0$, *then*

$$\lim_{|x| \to 0} u(x) = C_1 > 0, \qquad \lim_{|x| \to 0} v(x) = C_2 > 0. \tag{4.34}$$

iii) If $\alpha > 0$ *and* $\beta = 0$ *and* $0 \leq t < 1$, *then* $q + t < (N+b)/(N-2)$, *and*

- *either* $q < (b+2)/(N-2)$ *and*

$$\lim_{|x| \to 0} |x|^{N-2}\, u(x) = c_N\alpha > 0, \qquad \lim_{|x| \to 0} v(x) = C > 0, \tag{4.35}$$

- *or* $q > (b+2)/(N-2)$ *and*

$$\left\{ \begin{array}{c} \lim_{|x| \to 0} |x|^{N-2}\, u(x) = c_N\alpha > 0, \\ \lim_{|x| \to 0} |x|^{(b+2-(N-2)q)/(t-1)}\, v(x) = \frac{((N-2)q-b-2)(N+b-(N-2)(q+t))}{(1-t)^2(c_N\alpha)^q)^{1/(t-1)}}, \end{array} \right. \tag{4.36}$$

- *or* $q = (b+2)/(N-2)$ *and*

$$\left\{ \begin{array}{c} \lim_{|x| \to 0} |x|^{N-2}\, u(x) = c_N\alpha > 0, \\ \lim_{x \to 0} |Ln\,|x||^{1/(t-1)}\, v(x) = ((N-2)/(1-t)(c_N\alpha)^q)^{1/(t-1)}. \end{array} \right. \tag{4.37}$$

Proof of Lemma 4.1 i) The estimate on (u,v) implies

$$0 \leq -\Delta(u - c_N\alpha \left|x\right|^{2-N}) \leq C \left|x\right|^{a-(N-2)(p+s)}$$

in $\bar{B}_{1/2}\backslash\{0\}$. From the maximum principle, if $p+s < (a+2)/(N-2)$, then $u - c_N\alpha\left|x\right|^{2-N}$ is bounded in $\bar{B}_{1/2}$. If $p+s > (a+2)/(N-2)$, then

$$-C \leq u(x) - c_N\alpha \left|x\right|^{2-N} \leq C\left|x\right|^{a+2-(N-2)(p+s)} \leq C\left|x\right|^{(2-N+\varepsilon)}$$

for some $\varepsilon > 0$, because $p+s < (N+a)/(N-2)$. If $p+s = (a+2)/(N-2)$, then

$$-C \leq u(x) - c_N\alpha\left|x\right|^{2-N} \leq C\left|Ln\left|x\right|\right|.$$

In any case, one gets the new estimate near the origin

$$u(x) - c_N\alpha\left|x\right|^{2-N} = O(\left|x\right|^{2-N+\varepsilon})$$

for some $\varepsilon > 0$. Since $\alpha > 0$, (4.33) follows.

ii) If $\alpha > 0$ and $\beta > 0$, then $p+s < (N+a)/(N-2)$ and $q+t < (N+b)/(N-2)$ from Proposition 2.1, hence (4.34) follows.

iii) Assume $\alpha > 0$ and $\beta = 0$ and $0 \leq t < 1$. Then v satisfies an inequality

$$C\left|x\right|^{b-(N-2)q} v^t \leq -\Delta v \leq 2C\left|x\right|^{b-(N-2)q} v^t$$

in $\bar{B}_{1/2}\backslash\{0\}$. From Lemma 6.5, this again implies $q+t < (N+b)/(N-2)$, and v is regular if $q < (b+2)/(N-2)$, hence (4.35) holds. If $q > (b+2)/(N-2)$, then

$$C\left|x\right|^{(2+b-(N-2)q)/(1-t)} \leq v(x) \leq 2C\left|x\right|^{(2+b-(N-2)q)/(1-t)}, \tag{4.38}$$

and $q = (b+2)/(N-2)$, then

$$C\left|Ln\left|x\right|\right|^{1/(1-t)} \leq v(x) \leq 2C\left|Ln\left|x\right|\right|^{1/(1-t)}. \tag{4.39}$$

Now let us prove the precise convergences given in (4.36), (4.37).When $t = 0$, they come from the maximum principle. In the general case, we follow the proof of [2], theorem 5.4, and use the change of variables

$$u(x) = r^{2-N}U(T,\theta), \qquad v(x) = r^{-k}V(T,\theta),$$

where

$$T = -Lnr, \theta \in S^{N-1}, k = ((N-2)q - b - 2)/(1-t) \in [0, N-2).$$

It leads to the following system:

$$\begin{cases} U_{TT} + (N-2)U_T + \Delta_{S^{N-1}}U + e^{-\delta(1-t)^{-1}(\gamma-N+2)T}U^sV^p = 0, \\ V_{TT} - (N-2-2k)V_T + \Delta_{S^{N-1}}V - k(N-2-k)V + U^qV^t = 0. \end{cases}$$

We observe that the condition $q+t < (N+b)/(N-2)$ implies $\delta\ (\gamma - N + 2) > 0$, because $t < 1$. Then $U(T,.)$ converges exponentially to $c_N\alpha$ in $C(S^{N-1})$, from [2], proposition 4.1, hence

$\lim_{|x|\to 0} |x|^{N-2} u(x) = c_N\alpha$. First assume $k > 0$, hence $C \leq V(T,.) \leq 2\ C$ near infinity. Then $V(T,.)$ converges exponentially in $C(S^{N-1})$ to a positive solution $\omega(.)$ of the equation

$$\Delta_{S^{N-1}}\omega - k(N-2-k)\omega + (c_N\alpha)^q\omega^t = 0$$

on S^{N-1}, from [2], theorem 4.1. But the only positive solution is the constant function $(k(N-2-k)(c_N\alpha)^{-q})^{1/(t-1)}$. Indeed if $\omega, \hat{\omega}$ are two solutions, we obtain $\omega = \hat{\omega}$ by dividing by $\omega, \hat{\omega}$ and multiplying by $\omega^2 - \hat{\omega}^2$, since the function $\omega \longmapsto \omega^t/\omega$ is nonincreasing, see [5]. Hence (4.36) holds. Now assume $k = 0$. Then $V(T,.)$ converges to 0 from (4.38). The new change of variables

$$V(T,.) = T^{-1/(t-1)}\hat{V}(T,.),$$

leads to the equation

$$\hat{V}_{TT} - (N-2-\frac{2T^{-1}}{1-t})\hat{V}_T + \Delta_{S^{N-1}}\hat{V} + T^{-1}(U^q\hat{V}^t - (\frac{N-2}{1-t} - \frac{T^{-1}}{(1-t)^2})\hat{V}) = 0.$$

And $C \leq \hat{V}(T,.) \leq 2\ C$ near infinity, from (4.38). We deduce that $\hat{V}(T,.)$ converges to $((N-2)/(1-t)(c_N\alpha)^q)^{1/(t-1)}$ from [2], corollary 4.2. Hence (4.37) holds. ■

Proof of Theorem 4.3 First notice that $p+s < (N+a)/(N-2)$, from (4.2) and Remark 4.2. But it may occur that $q+t \geq (N+b)/(N-2)$ under the assumptions of Theorem 4.2, provided $q \leq 1$.

i) Case $\alpha > 0$, $\beta > 0$. Then (4.24) follows from Lemma 4.4.

ii) Case $\alpha > 0$, $\beta = 0$.

•If $0 \leq t < 1$, then $q+t < (N+b)/(N-2)$ and (4.26) to (4.28) hold from Lemma 4.4.

•If $t > 1$, then $q+t < (N+b)/(N-2)$ from (4.7). And v is regular from [27]. Indeed the function defined by $-\Delta v = hv$ satisfies $h \in L^\tau(B_{1/2})$, for some $\tau > N/2$, from (4.3), (4.11). Then (4.26) holds.

• If $t = 1$, then $q+1 \leq (N+b)/(N-2)$ in case of Theorem 4.2, and $q+1 < (N+b)/(N-2)$ in case of Theorem 4.1. When the inequality is strict, that is $q < (b+2)/(N-2)$, then (4.26) holds as in the case $t > 1$. At last consider the limit case $t = 1$ and $q = (b+2)/(N-2)$, which is delicate. Here we use the change of variables

$$u(x) = r^{2-N}U(T,\theta), \qquad v(x) = V(T,\theta).$$

It leads to the system

$$\begin{cases} U_{TT} + (N-2)U_T + \Delta_{S^{N-1}}U + e^{-p\xi T}U^sV^p = 0, \\ V_{TT} - (N-2)V_T + \Delta_{S^{N-1}}V + U^qV = 0. \end{cases}$$

Adapting carefully the proof of [2], theorem 5.5, we obtain (4.31), (4.32).

iii) Case $\alpha = 0$, $\beta = 0$. Under the assumptions of Theorem 4.1, the sum $u+v$ is regular. Under the assumptions of Theorem 4.2, the function h defined in (4.18) satisfies $h = h_2 \in L^\tau(B_{1/2})$, and v is regular since $\beta = 0$.

• If $s \geq 1$, then from (4.3), (4.11), the function $H = |x|^a u^{s-1} v^p$ defined by $-\Delta u = Hu$ satisfies $H \in L^\tau(B_{1/2})$, for some $\tau > N/2$, because $p + s < (N+a)/(N-2)$. Then u is regular since $\alpha = 0$, hence (u, v) is regular, which implies $a + 2 > 0$, $b + 2 > 0$.

• If $0 \leq s < 1$, then

$$C\,|x|^a\,u^s \leq -\Delta u \leq 2C\,|x|^a\,u^s$$

in $\bar{B}_{1/2} \backslash \{0\}$. Then $s < (N+a)/(N-2)$ from Lemma 6.5. Either $a + 2 > 0$ and u is regular. Or $a + 2 < 0$ and

$$u(x) \leq 2C\,|x|^{(a+2)/(1-s)}$$

in $\bar{B}_{1/2} \backslash \{0\}$. Or $a + 2 = 0$ and

$$u(x) \leq 2C\,|Ln\,|x||^{1/(1-s)}.$$

In the first case, (4.24) holds. In the two other cases $a + 2 \leq 0$ implies $b + 2 > 0$, because $\xi > 0$. In order to get the convergences, we set

$$u(x) = r^{(a+2)/(1-s)} U(T, \theta), \qquad v(x) = V(T, \theta),$$

and obtain

$$\begin{cases} U_{TT} - (N - 2 + 2\frac{a+2}{1-s})U_T + \Delta_{S^{N-1}} U + \frac{a+2}{1-s}(N - 2 + \frac{a+2}{1-s})U + U^s V^p = 0, \\ V_{TT} - (N-2)V_T + \Delta_{S^{N-1}} V + e^{-(\xi\delta/(1-s))T} U^q V^t = 0. \end{cases}$$

When $a + 2 < 0$, we deduce that $V(T, .)$ converges exponentially in $C(S^{N-1})$ to a positive constant C, from [2], proposition 4.1, because $(\xi\delta/(1-s)) > 0$. Then $U(T, .)$ converges exponentially to $C^p(1-s)^2/\,|a+2|\,(N + a - (N-2)s)$. This proves (4.29). When $a + 2 = 0$, we set

$$U(T, .) = T^{-1/(t-1)} \hat{U}(T, .).$$

It leads to the equation

$$\hat{U}_{TT} - (N - 2 - \frac{2T^{-1}}{1-s})\hat{U}_T + \Delta_{S^{N-1}} \hat{U} + T^{-1}(\hat{U}^s V^p - (\frac{N-2}{1-s} - \frac{sT^{-1}}{(1-s)^2})\hat{U}) = 0,$$

and $\hat{U}(T, .)$ converges to $(C^p(1-s)/(N-2))^{1-s}$. This proves (4.30).

iv) Case $\alpha = 0$, $\beta > 0$.

• If $s \geq 1$, u is regular as in the third case. Notice that (4.31), (4.32) have no equivalent when $s = 1$, because of the strict inequality $p + 1 < (N+a)/(N-2)$.

• If $0 \leq s < 1$, we use Lemma 4.4 and deduce (4.26) to (4.28), up to the change from u, q, t into v, p, s. ■

Remark 4.4 By Kelvin transform, Theorem 4.1 gives the behaviour of system (1.1) at infinity, whenever

$$1 < q + t < N/(N-2), \quad 1 < p + s < N/(N-2) \quad \text{and} \quad \max(a+2, b+2) < 0.$$

Otherwise when $p+1-t>0$, condition (4.5) is invariant by Kelvin transform, since $a_0-b_0+a-b=(N-2)(q+t-p-s)$ from (2.17). Thus Theorem 4.2 gives also the behaviour at infinity. Since $\gamma_0=N-2-\gamma$, $\xi_0=N-2-\xi$, conditions (4.6) and (4.7) are replaced by

$$\min(\gamma,\xi)<0, \quad \max(2(p+1-t)/\delta,(q+1-s)/\delta)>N-2, \quad \text{if } s,t\leq 1,$$

$$\begin{cases} b+2<0, & q+t<N/(N-2) \quad \text{if } t>1, \\ a+2<0, & p+s<N/(N-2) \quad \text{if } s>1. \end{cases}$$

Applying for example to the case of Hamiltonian system (1.4), the following holds.

Theorem 4.5 *Let $(u,v)\in(C^2(\mathbb{R}^N/B_1))^2$ be any nonnegative nontrivial solution of (1.4) with $a=b$. Assume that (1.5) holds, and $a<-2$. Then*
i) either u,v are regular at infinity:

$$\lim_{|x|\to+\infty}|x|^{N-2}u(x)=C_1>0, \lim_{|x|\to+\infty}|x|^{N-2}v(x)=C_2>0, \tag{4.40}$$

ii) or

$$\lim_{|x|\to+\infty}u(x)=c_N\alpha>0, \lim_{|x|\to+\infty}v(x)=c_N\beta>0, \tag{4.41}$$

iii) or (up to a change from u,p into v,q),

- *if $a+N>0$, then*

$$\begin{cases} \lim\limits_{|x|\to+\infty}u(x)=c_N\alpha>0, \\ \lim\limits_{|x|\to+\infty}|x|^{-(a+2)}v(x)=(c_N\alpha)^q/(N+a)\left|a+2\right|, \end{cases} \tag{4.42}$$

- *if $a+N<0$, then*

$$\begin{cases} \lim_{|x|\to+\infty}u(x)=c_N\alpha>0, \\ \lim_{|x|\to+\infty}|x|^{N-2}v(x)=C_2>0, \end{cases} \tag{4.43}$$

- *if $a+N=0$, then*

$$\begin{cases} \lim_{|x|\to+\infty}u(x)=c_N\alpha>0, \\ \lim_{|x|\to+\infty}|x|^{N-2}\left|Ln\left|x\right|\right|^{-1}v(x)=(c_N\alpha)^q/(N-2). \end{cases} \tag{4.44}$$

Remark 4.5 The assumptions (4.6), (4.7) can be compared with the sufficient conditions of nonexistence of positive solutions for the exterior problem, given by Theorem 3.1 and Kelvin transform:

$$p+s\leq(N+a)/(N-2) \quad \text{or } q+t\leq(N+b)/(N-2), \quad \text{if } s,t>1, \tag{4.45}$$

$$\max(\gamma,\xi)>N-2 \quad \text{or } \gamma=\xi=N-2, \quad \text{if } \delta>0 \quad \text{and } s,t\leq 1, \tag{4.46}$$

$$p+s\leq(N+a)/(N-2) \quad \text{or } \gamma>N-2, \quad \text{if } \delta>0 \quad \text{and } t\leq 1<s, \tag{4.47}$$

$$q+t\leq(N+b)/(N-2) \quad \text{or } \xi>N-2, \quad \text{if } \delta>0 \quad \text{and } s\leq 1<t. \tag{4.48}$$

In any case (4.6),(4.7) imply (4.45) to (4.48). They are equivalent when $s,t\leq 1$ and $a=b=0$.

5 The sublinear and linear cases

a) The sublinear case.

Here we assume

$$\delta < 0, \quad \text{with} \quad s,t < 1, \tag{5.1}$$

$$0 \leq b-a \leq (N-2)(q+t-p-s). \tag{5.2}$$

Our purpose is to extend to system (1.1) the results of [25], [2] relative to the equation (1.24) when $0 < \eta < 1$. This case appears to be very rich. Indeed (1.12) may be satisfied, and we actually find particular solutions given by (1.8). Let us recall the necessary condition of existence given in (2.8):

$$\max(\gamma,\xi) \leq N-2 \quad \text{ and } (\gamma,\xi) \neq (N-2,N-2).$$

That means that $\max\,(\gamma,\xi) \leq N-2$ and $\gamma \neq N-2$, from (4.8), (5.2).

Theorem 5.1 *Let $(u,v) \in (C^2(\,\bar{B}_1\backslash\{0\}))^2$ be any nonnegative nontrivial solution of (1.1) with (5.1), (5.2). Then*

1) Either one of the following eventualities holds, up to a change from u,p,s into v,q,t:

i) $q+t < (N+b)/(N-2)$, $p+s < (N+a)/(N-2)$, and

$$\lim_{|x|\to 0} |x|^{N-2} u(x) = c_N\alpha > 0, \qquad \lim_{|x|\to 0} |x|^{N-2} v(x) = c_N\beta > 0. \tag{5.3}$$

ii) $q < (b+2)/(N-2)$ and

$$\lim_{|x|\to 0} |x|^{N-2} u(x) = c_N\alpha > 0, \quad \text{and} \quad v \text{ is regular.} \tag{5.4}$$

iii) $q = (b+2)/(N-2)$ and

$$\left\{ \begin{array}{c} \lim\limits_{|x|\to 0} |x|^{N-2} u(x) = c_N\alpha > 0, \\ \lim\limits_{x\to 0} |Ln\,|x||^{1/(t-1)} v(x) = ((N-2)/(1-t)(c_N\alpha)^q)^{1/(t-1)}. \end{array} \right. \tag{5.5}$$

iv) $q > (b+2)/(N-2)$ and

$$\left\{ \begin{array}{c} \lim\limits_{|x|\to 0} |x|^{N-2} u(x) = c_N\alpha > 0, \\ \lim\limits_{|x|\to 0} |x|^{(b+2-(N-2)q)/(t-1)} v(x) = \frac{((N-2)q-b-2)(N+b-(N-2)(q+t))}{(1-t)^2(c_N\alpha)^q)^{1/(t-1)}}. \end{array} \right. \tag{5.6}$$

2) Or one of the other following eventualities holds, with no change from u,p,s into v,q,t:

v) $\min(\xi,\gamma) > 0$ (hence $a+2<0$) and

$$\left\{ \begin{array}{c} C\,|x|^{-\gamma} \leq u(x) \leq 2C\,|x|^{-\gamma}, \\ C\,|x|^{-\xi} \leq v(x) \leq 2C\,|x|^{-\xi} \end{array} \right. \tag{5.7}$$

in $\bar{B}_{1/2}\backslash\{0\}$.

vi) $\min(\xi,\gamma)<0$, *that is* $\xi<0$ *(hence* $0<b+2$*) and*

- *either* $a+2<0$, *and*

$$\begin{cases} \lim_{x\to 0}|x|^{(a+2)/(s-1)}u(x)=C^p(1-s)^2/|a+2|(N+a-(N-2)s), \\ \lim_{|x|\to 0}v(x)=C>0, \end{cases} \tag{5.8}$$

- *or* $a+2=0$, *and*

$$\begin{cases} \lim_{x\to 0}|Ln|x||^{1/(s-1)}u(x)=(C^p(1-s)/(N-2))^{1/(1-s)}, \\ \lim_{|x|\to 0}v(x)=C>0, \end{cases} \tag{5.9}$$

- *or* $a+2>0$, *and*

$$(u,v) \text{ is regular.} \tag{5.10}$$

vii) $\xi=\gamma=0$ *(that is* $a=b=-2$*) and*

$$\begin{cases} C|Ln|x||^{-(p+1-t)/\delta}\le u(x)\le 2C|Ln|x||^{-(p+1-t)/\delta}, \\ C|Ln|x||^{-(q+1-s)/\delta}\le v(x)\le 2C|Ln|x||^{-(q+1-s)/\delta}, \end{cases} \tag{5.11}$$

in $\bar{B}_{1/2}\backslash\{0\}$.

viii) $\min(\xi,\gamma)=0$, *and* $(\gamma,\xi)\neq(0,0)$, *that is* $\xi=0<\gamma$ *(hence* $a+2<0<b+2$*) and*

$$\begin{cases} C_\varepsilon|x|^{-\gamma}|Ln|x||^{-p/\delta-\varepsilon}\le u(x)\le 2C|x|^{-\gamma}|Ln|x||^{-(p+1-t)/\delta}, \\ C_\varepsilon|Ln|x||^{-(1-s)/\delta-\varepsilon}\le v(x)\le 2C|Ln|x||^{-(q+1-s)/\delta}, \end{cases} \tag{5.12}$$

in $\bar{B}_{1/2}\backslash\{0\}$.

Proof 1st step: v satisfies Harnack inequality. One finds again (4.12), (4.13). Let us write the second line of (1.1) under the form $-\Delta v=hv$, and define h_1,h_2 as in (4.19), (4.20), with $\lambda=\delta/(q+1-s)$. If $\alpha=0$, then (4.12) implies the inequality

$$-\Delta u\ge|x|^{(bp+a(1-t))/(p+1-t)}u^{1+\delta/(p+1-t)}=|x|^{-2+\gamma\delta/(p+1-t)}u^{1+\delta/(p+1-t)}. \tag{5.13}$$

In turn it implies

$$u(x)\ge C|x|^{-\gamma}, \qquad v(x)\ge C|x|^{-\xi} \tag{5.14}$$

in $\bar{B}_{1/2}\backslash\{0\}$, from Lemma 6.2 and (4.8), (4.12), since $1+\delta/(p+1-t)\in(0,1)$. Now $h_1=0$ and $h_2=|x|^{-2+\xi\delta/(q+1-s)}v^{\delta/(q+1-s)}$, hence

$$h_2(x)\le C|x|^{-2}.$$

Therefore v satisfies the Harnack inequality. If $\alpha>0$, then as in (4.21),

$$-\Delta v\ge C|x|^{b-(N-2)q}v^t$$

in $\bar{B}_{1/2}\backslash\{0\}$. As in (4.22),

$$v(x) \geq C\,|x|^{(2+b-(N-2)q)/(1-t)} \tag{5.15}$$

from Lemma 6.2. In the same way, if $\beta > 0$, then

$$u(x) \geq C\,|x|^{(2+a-(N-2)p)/(1-s)}.$$

As in (4.23), this implies

$$h_1(x) \leq C\,|x|^{-2}, \qquad h_2(x) \leq C\,|x|^{-2+q\delta(\gamma-(N-2))/(q+1-s)(1-t)} \leq C\,|x|^{-2},$$

because $\gamma \leq N-2$ from (2.8). Then again v satisfies the Harnack inequality . As in (4.11), this implies

$$v(x) \leq C\,|x|^{2-N}, \qquad u(x) \leq C\,|x|^{2-N}$$

in $\bar{B}_{1/2}\backslash\{0\}$, from (5.2), (4.12).

2nd step: the convergences.

i) Case $(\alpha,\beta) \neq (0,0)$. Then Lemma 4.4 applies and we deduce (5.3) if $\alpha > 0$, $\beta > 0$, and (5.4) to (5.6) if $\alpha > 0$, $\beta = 0$; and similarly if $\alpha = 0$, $\beta > 0$, after exchanging u,q,t into v,p,s.

ii) Case $\alpha = \beta = 0$. Then

$$C_1\,|x|^{b-\gamma q}\,v^t \leq -\Delta v \leq C_2\,|x|^{(aq+b(1-s))/(q+1-s)}\,v^{1+\delta/(q+1-s)},$$

from (4.12) and (5.14). That means, from (4.8),

$$C_1\,|x|^{-2-(1-t)\xi}\,v^t \leq -\Delta v \leq C_2\,|x|^{-2+\xi\delta/(q+1-s)}\,v^{1+\delta/(q+1-s)}. \tag{5.16}$$

• First suppose $\xi > 0$. Then $\gamma > 0$, from Remark 4.2. We deduce the estimate

$$C\,|x|^{-\xi} \leq v(x) \leq 2C\,|x|^{-\xi},$$

¿From (5.14) and Lemma 6.4, and using the Harnack inequality. In turn it implies

$$C\,|x|^{a-p\xi}\,u^s \leq -\Delta u \leq 2C\,|x|^{a-p\xi}\,u^s.$$

Hence

$$C\,|x|^{-\gamma} \leq u(x) \leq 2C\,|x|^{-\gamma},$$

from Lemma 6.5, and (5.7) holds.

• Then assume $\xi < 0$. Since $h = h_2 = |x|^{-2+\xi\delta/(q+1-s)}\,v^{\delta/(q+1-s)}$, it follows that $h \in L^\tau(B_{1/2})$ for some $\tau > N/2$, and v is regular. We conclude to (5.8) to (5.10) as in Theorem 4.2, because here also $(\xi\delta/(1-s)) > 0$.

•At last assume $\xi = 0$. Then (5.16) becomes

$$C_1\,|x|^{-2}\,v^t \leq -\Delta v \leq C_2\,|x|^{-2}\,v^{1+\delta/(q+1-s)}.$$

It implies

$$C\left|Ln\left|x\right|\right|^{1/(1-t)} \leq v(x) \leq 2C\left|Ln\left|x\right|\right|^{-(q+1-s)/\delta} \tag{5.17}$$

from Lemmas 6.2, 6.4 and the Harnack inequality. Then (4.8), (4.12) in turn imply

$$u(x) \leq 2C\left|x\right|^{-\gamma}\left|Ln\left|x\right|\right|^{-(p+1-t)/\delta}. \tag{5.18}$$

First suppose $\xi = \gamma = 0$. Then from (5.13) and Lemma 6.2 we obtain

$$C\left|Ln\left|x\right|\right|^{-(p+1-t)/\delta} \leq u(x), \tag{5.19}$$

hence (5.11) holds from (5.18), (5.19), and (4.12), (5.17). Now suppose $\xi = 0 < \gamma$. For any $k > 0$ such that

$$C\left|Ln\left|x\right|\right|^{k} \leq v(x),$$

we get

$$C\left|x\right|^{-\gamma}\left|Ln\left|x\right|\right|^{-kp/(1-s)} \leq u(x)$$

from (1.1) and Lemma 6.2, and similarly

$$C\left|Ln\left|x\right|\right|^{(kpq+1-s)/(1-s)(1-t)} \leq v(x).$$

Defining $k_0 = 1/(1-t)$ and $k_{n+1} = (k_n pq + 1 - s)/(1-s)(1-t)$, the sequence converges to $-(1-s)/\delta$, hence (5.12) follows.■

Remark 5.1 i) In case $\min(\xi,\gamma) > 0$, we conjecture that

$$\lim_{|x|\to 0} \left|x\right|^{\gamma} u(x) = A, \qquad \lim_{|x|\to 0} \left|x\right|^{\xi} v(x) = B,$$

where A, B are given by (1.11). The change of variables

$$u(x) = r^{-\gamma}U(T,\theta), \qquad v(x) = r^{-\xi}V(T,\theta)$$

leads to the system

$$\begin{cases} U_{TT} - (N-2-2\gamma)U_T + \Delta_{S^{N-1}}U - \gamma(N-2-\gamma)U + U^s V^p = 0, \\ V_{TT} - (N-2-2\xi)V_T + \Delta_{S^{N-1}}V - \xi(N-2-\xi)V + U^q V^t = 0. \end{cases}$$

One cannot conclude to the convergence because of a lack of energy function for this system, unless $p+s = q+t$, $a = b$ and $q \geq s$. In that case, using the fact that $\left\|U(t,.) - V(t,.)\right\|_{C(S^{N-1})}$ is exponentially small from (3.15), the convergence is proved as in [2].

ii) Now consider the case $\xi = \gamma = 0$, and make the change of variables

$$u(x) = T^{-(p+1-t)/\delta}\hat{U}(T,\theta), \qquad v(x) = T^{-(q+1-s)/\delta}\hat{V}(T,\theta).$$

It leads to the new system

$$\begin{cases} \hat{U}_{TT} - (N-2-2\frac{c_1}{T})\hat{U}_T + \Delta_{S^{N-1}}\hat{U} - \frac{1}{T}(c_1(N-2+\frac{1-c_2}{T})\hat{U} - \hat{U}^s\hat{V}^p) = 0, \\ \hat{V}_{TT} - (N-2-2\frac{c_2}{T})\hat{V}_T + \Delta_{S^{N-1}}\hat{V} - \frac{1}{T}(c_2(N-2+\frac{1-c_2}{T})\hat{V} - \hat{U}^q\hat{V}^t) = 0, \end{cases}$$

where $c_1 = -(p+1-t)/\delta$, $c_2 = -(q+1-s)/\delta$. Then we can conjecture that

$$\begin{cases} \lim_{x\to 0} |Ln\,|x||^{(p+1-t)/\delta}\, u(x) = ((N-2)^{p+1-t} c_1^{1-t} c_2^{p})^{1/\delta}, \\ \lim_{x\to 0} |Ln\,|x||^{(q+1-s)/\delta}\, v(x) = ((N-2)^{q+1-s} c_1^{q} c_2^{1-s})^{1/\delta}. \end{cases}$$

iii) Finally, in case $\xi = 0 < \gamma$, the estimates (5.12) are not optimal. We cannot prove the Harnack inequality for function u in the general case. Assuming moreover that $q \leq 1$ or u, v are radial, then one gets a more precise estimate

$$\begin{cases} 0 < C_\varepsilon\, |x|^{-\gamma}\, |Ln\,|x||^{-p/\delta-\varepsilon} \leq u(x) \leq 2C_\varepsilon\, |x|^{-\gamma}\, |Ln\,|x||^{-p/\delta+\varepsilon} \\ 0 < C_\varepsilon\, |Ln\,|x||^{-(1-s)/\delta-\varepsilon} \leq v(x) \leq 2C_\varepsilon\, |Ln\,|x||^{-(1-s)/\delta+\varepsilon}, \end{cases}$$

in $\bar{B}_{1/2}\backslash\{0\}$, by using Lemma 6.4 instead of Lemma 6.2, and the Harnack inequality for function v. The change of variables

$$u(x) = r^{-\gamma} T^{-p/\delta} \hat{U}(T,\theta), \qquad v(x) = T^{-(1-s)/\delta} \hat{V}(T,\theta)$$

now leads to the system

$$\begin{cases} \hat{U}_{TT} - (N-2-2\gamma - \frac{2a_1}{T})\hat{U}_T + \Delta_{S^{N-1}}\hat{U} \\ \qquad -(\gamma(N-2-\gamma) + \frac{(N-2-2\gamma)a_1}{T} + \frac{a_1(N-2-a_1)}{T^2})\hat{U} - \hat{U}^s \hat{V}^p) = 0, \\ \hat{V}_{TT} - (N-2-\frac{2a_2}{T})\hat{V}_T + \Delta_{S^{N-1}}\hat{V} - T^{-1}(a_2(N-2+\frac{1-a_2}{T})\hat{V} - \hat{U}^q \hat{V}^t) = 0, \end{cases}$$

where $a_1 = -p/\delta$, $a_2 = -(1-s)/\delta$. We conjecture that

$$\begin{aligned} \lim_{x\to 0} |x|^\gamma\, |Ln\,|x||^{p/\delta}\, u(x) &= ((N-2)a_2)^{p/\delta}(\gamma(N-2-\gamma))^{(1-t)/\delta}, \\ \lim_{x\to 0} |Ln\,|x||^{(1-s)/\delta}\, v(x) &= ((N-2)a_2)^{(1-s)/\delta}(\gamma(N-2-\gamma))^{q/\delta}. \end{aligned}$$

b) The linear case.

Here we assume

$$\delta = 0, \quad \text{with} \quad s,t < 1, \tag{5.20}$$

under condition (5.2). From (2.9),

$$(a+2)q + (b+2)(1-s) = ((b+2)p + (a+2)(1-t))(1-s)/p \geq 0.$$

We intend to extend to system (1.1) the results of [24], [2] relative to the equation

$$-\Delta f = M\,|x|^a\, f^\eta \qquad (M > 0) \tag{5.21}$$

when $\eta = 1$. But the existence of the solutions of this equation depends on the constant $M > 0$, in the case $\sigma = -2$ (and only in that case). Then there exist nontrivial solutions if and only if $M \leq (N-2)^2/4$: for example radial power solutions

$$|x|^{\rho_i}\ (i = 1,2), \quad \text{with} \quad \rho_i = ((N-2) \pm ((N-2)^2 - 4M)^{1/2}),$$

and a logarithmical one $|x|^{(2-N)/2} |Ln |x||$ when $M = (N-2)^2/4$. The local behaviour of the solutions is governed by those radial solutions, see [24]. The same difficulty occurs for the system, when

$$(a+2)q + (b+2)(1-s) = 0. \tag{5.22}$$

As in the scalar case, the problem appears to be ill-posed. Consider the more general system

$$\begin{cases} \Delta u \ + \ M_1 \ |x|^a u^s v^p = 0, \\ \Delta v \ + \ M_2 \ |x|^b u^q v^t = 0, \end{cases} \tag{5.23}$$

where $M_1, M_2 > 0$. When $\delta \neq 0$, it can be reduced to system (1.1) by setting

$$u(x) = M_1^{-(1-t)/\delta} M_2^{-p/\delta} \widetilde{u}(x), \qquad v(x) = M_1^{-q/\delta} M_2^{-(1-s)/\delta} \widetilde{v}(x).$$

But it is no more the case when $\delta = 0$. It can only be reduced to the system

$$\begin{cases} \Delta u \ + \ M \ |x|^a u^s v^p = 0, \\ \Delta v \ + \ M \ |x|^b u^q v^t = 0, \end{cases} \tag{5.24}$$

where $M = M_1^{(1-t)/(p+1-t)} M_2^{p/(p+1-t)} = M_1^{q/(q+1-s)} M_2^{(1-s)/(q+1-s)}$, by setting

$$u(x) = (M_1/M_2)^{p/(p+1-t)} \widetilde{u}(x), \qquad v(x) = (M_1/M_2)^{-s/(p+1-t)} \widetilde{v}(x).$$

And the local behaviour will depend upon constant M when (5.22) holds.

A great part of the preceeding results are still available. System (5.24) can be written in $\mathcal{D}'(\Omega)$ under the form

$$\begin{cases} -\Delta u \ = \ M \ |x|^a u^s v^p + \alpha \ \delta_0, \\ -\Delta v \ = \ M \ |x|^b u^q v^t + \beta \ \delta_0, \end{cases} \tag{5.25}$$

with some new $\alpha, \beta \geq 0$.

Proposition 5.2 *Let $(u,v) \in (C^2(\bar{B}_1 \backslash \{0\}))^2$ be any nonnegative nontrivial solution of (5.25) with (5.2), (5.20). Then*

i) either $q+t < (N+b)/(N-2)$, $p+s < (N+a)/(N-2)$ (which implies $(a+2)q+(b+2)(1-s) > 0$), and (5.3) holds;

ii) or (5.4) or (5.5) or (5.6) holds (up to a change from u,p,s into v,q,t);

iii) or $\alpha = \beta = 0$, $(a+2)q + (b+2)(1-s) > 0$, and (u,v) is regular if $a+2 > 0$ and $b+2 > 0$, or (4.30) or (4.31) holds;

iv) or $\alpha = \beta = 0$, $(a+2)q + (b+2)(1-s) = 0$.

Proof As in the proof of Theorem 5.1, using (4.12), (4.13), and writing equation under the form (4.18), we find (4.19), (4.20) with $\lambda = 0$. Then

$$h_2(x) = |x|^{(aq+b(1-s))/(q+1-s)} \leq C |x|^{-2}, \tag{5.26}$$

and v satisfies Harnack inequality when $\alpha = 0$. When $\alpha > 0$, then (5.15) holds, and

$$h_1(x) \leq C\,|x|^{-2}\,, \tag{5.27}$$

hence again the Harnack inequality holds. The convergences follow from Lemma 4.4, as in Theorem 5.1, when $(\alpha, \beta) \neq (0,0)$. When $\alpha = \beta = 0$, and $(a+2)q + (b+2)(1-s) > 0$, then v is regular, because from (5.26), $h = h_2 \leq |x|^{-2+\varepsilon}$ for some $\varepsilon > 0$, and the conclusions follow as in Theorem 4.2.■

At last consider the critical case $(a+2)q + (b+2)(1-s) = 0$, when $\alpha = \beta = 0$. From (5.2), it implies $a+2 < 0 < b+2$, or $a = b = 0$.

Proposition 5.3 *Assume (5.22).*

i)Under the assumptions of Proposition 5.2, and $\alpha = \beta = 0$, if $q < (b+2)/(N-2)$, then (4.29) holds.

ii) If $q > (b+2)/(N-2)$, then there exists a constant M^* such that for any $M < M^*$ (resp. $M = M^*$), there exist at least two families (resp. exactly one family) of radial solutions of system (5.25) with $\alpha = \beta = 0$, under the form

$$u(x) = C\,|x|^{-\Gamma}\,, \qquad v(x) = (M^{-1}\Gamma(N-2-\Gamma)C^{1-s})^{1/p}\,|x|^{-(a+2+(1-s)\Gamma)}\,, \tag{5.28}$$

with $C > 0$, where Γ is a root of equation

$$(\Gamma(N-2-\Gamma))^{\frac{1-t}{p}}\,\frac{(a+2+(1-s)\Gamma)}{p}(N-2-\frac{(a+2+(1-s)\Gamma)}{p}) = M^{\frac{p+1-t}{p}} \tag{5.29}$$

such that

$$\Gamma \in I = ((b+2)/q, \min(N-2, ((N-2)p-(a+2))/(1-s)))\,. \tag{5.30}$$

Proof If $q < (b+2)/(N-2)$, then $h = |x|^b\,u^q v^{t-1} \leq C\,|x|^{2-\varepsilon}$ from (2.1), hence v is regular, and (4.29) follows. Now assume $q > (b+2)/(N-2)$. Let us look for solutions under the form

$$u(x) = C\,|x|^{-\Gamma}\,, \qquad v(x) = D\,|x|^{-\Sigma} \qquad (C, D > 0),$$

with $\Gamma, \Sigma < N-2$, in order to have $\alpha = \beta = 0$. We easily find the conditions

$$\Sigma = p^{-1}(a+2+(1-s)\Gamma) = (1-t)^{-1}(q\Gamma-(b+2))$$

and

$$MC^{s-1}D^p = \Gamma(N-2-\Gamma), \qquad MC^qD^{t-1} = \Sigma(N-2-\Sigma),$$

which imply (5.29), and limit p, q and Γ as mentionned above. The function $\Gamma \longmapsto M(\Gamma)$ defined by (5.29) is continuous on I and vanishes at the extremities, hence the result with $M^* = \max_{\Gamma \in I} M(\Gamma)$.■

Remark 5.2. When $a = b = -2$ and $p + s = q + t = 1$, then $M^* = (N-2)^2/4$ and there exist exactly two roots for any $M \leq M^*$, equal to ρ_1, ρ_2 as in the scalar case. Indeed the radial solutions are given by $u = v = f$, solution of equation $-\Delta f = M\;|x|^{-2}\,f$.

We can expect that system (5.25) does not admit solutions with $\alpha = \beta = 0$ when $q > (b+2)/(N-2)$ and $M > M^*$, and that the local behaviour of system (5.25) when $\alpha = \beta = 0$ is governed by the particular solutions as in the scalar case. The problem is open. It should be solved at least in the case of Remark 5.2, by following some ideas of [2], theorems 5.1, 5.5. Indeed in that case the difference $u - v$ is bounded in $\bar{B}_{1/2} \backslash \{0\}$, from (3.15).

6 Appendix

Lemma 6.1 *(Osserman type estimate). Let $g \in C^2(B_1 \backslash \{0\})$ be a nonnegative nontrivial solution of the inequality*

$$-\Delta g + |x|^\sigma |Ln\, |x||^\kappa g^Q \leq 0, \tag{6.1}$$

where $\sigma, \kappa, Q \in \mathbb{R}$, and $Q > 1$. If $\sigma \neq -2$, then

$$g(x) \leq C\, |x|^{-(2+\sigma)/(Q-1)} |Ln\, |x||^{-\kappa/(Q-1)} \tag{6.2}$$

in $\bar{B}_{1/2} \backslash \{0\}$. If $\sigma = -2$, then

$$g(x) \leq C\, |Ln\, |x||^{-(1+\kappa)/(Q-1)}. \tag{6.3}$$

Proof The classical Osserman's estimate implies that

$$g(x) \leq C_\varepsilon\, |x|^{-(2+\sigma)/(Q-1)-\varepsilon}$$

for any $\varepsilon > 0$, and $\varepsilon = 0$ if $\kappa \geq 0$. And $\bar{g}$ also satisfies (6.1) from the Jensen inequality. Let us make the change of variables

$$\bar{g}(r) = r^{-\Gamma} T^m G(T),$$

where $T = -Ln\ r$, $\Gamma = -(2+\sigma)/(Q-1)$, and m is a parameter. It leads to the following inequality:

$$\begin{aligned} &-G_{TT} + (N - 2 - 2\Gamma - \tfrac{2m}{T})\, G_T \\ &+(\Gamma(N-2-\Gamma) + (N-2-2\Gamma)\tfrac{m}{T} - \tfrac{m(m-1)}{T^2})\, G + T^{\kappa+m(Q-1)} G^Q \leq 0, \end{aligned} \tag{6.4}$$

with $G(T) = O(e^{\varepsilon T})$ for any $\varepsilon > 0$.

i) First assume $\sigma \neq -2$, and take $m = -\kappa/(Q-1)$. Then G is bounded near infinity. Indeed suppose it were not true. Then either there exists a sequence (T_n) with

$$\lim T_n = +\infty, G_T(T_n) = 0, G_{TT}(T_n) \leq 0, \quad \text{and} \quad \lim G(T_n) = +\infty.$$

This is impossible from (6.4), since $Q > 1$. Or G is increasing to infinity. Then there exist $B, C > 0$ such that

$$-G_{TT} - BG_T + CG \leq 0$$

for large T. But the corresponding equation admits solutions under the form $C_1 e^{\rho_1 T} + C_2 e^{\rho_2 T}$ with $\rho_1 > 0 > \rho_2$. Choosing $C_1 > 0$ arbitrary, it follows that $G(T) = O(e^{\rho_2 T})$ from the maximum principle, hence a contradiction. Since g is subharmonic, (6.2) follows from the estimate on $\bar{g}$, see for example [36].

ii) Now assume $\sigma = -2$ and take $m = -(1+\kappa)/(Q-1)$. Then G is still bounded near infinity. Indeed by contradiction G must be increasing to infinity. Then

$$-G_{TT} + (N-2)G_T + CT^{-1}G \leq 0$$

for some $C > 0$. But the corresponding equation admits supersolutions under the form C_1 $e^{(N-2)T} + C_2$. As above, it implies that G is bounded, hence a contradiction. Then (6.3) follows.■

Lemma 6.2 *Let $f \in C^2(B_1 \backslash \{0\})$ be a nonnegative nontrivial solution of the inequality*

$$-\Delta f \geq |x|^{\sigma} |Ln\,|x||^{\kappa} f^{\eta}, \tag{6.5}$$

where $\sigma, \kappa, \eta \in \mathbb{R}$, and $\eta \geq 0$.

i) If $\eta > 1$, then $\sigma > -2$, or $\sigma = -2$ and $\kappa < -1$. Moreover if $\sigma > -2$, then

$$\bar{f}(r) = O(r^{-(2+\sigma)/(\eta-1)} |Lnr|^{-\kappa/(\eta-1)}). \tag{6.6}$$

If $\sigma = -2$ and $\kappa < -1$, then

$$\bar{f}(r) = O(|Lnr|^{-(1+\kappa)/(\eta-1)}). \tag{6.7}$$

ii) If $\eta = 1$, then $\sigma \geq -2$.

iii) If $0 \leq \eta < 1$, then $\eta < (N+\sigma)/(N-2)$, or ($\eta = (N+\sigma)/(N-2)$ and $\kappa < (\sigma+2)/(N-2)$). If $\sigma \neq -2$, then for some $C > 0$,

$$f(x) \geq C\,|x|^{(2+\sigma)/(1-\eta)} |Ln\,|x||^{\kappa/(1-\eta)}, \tag{6.8}$$

in $\bar{B}_{1/2} \backslash \{0\}$. If $\sigma = -2$, then

$$f(x) \geq C\,|Ln\,|x||^{(1+\kappa)/(1-\eta)}. \tag{6.9}$$

Proof In any case $N + \sigma > 0$, because $|x|^{\sigma} |Ln\,|x||^{\kappa} f^{\eta} \in L^1_{loc}(B_1)$ from [4], and $f(x) \geq C > 0$ in $\bar{B}_{1/2} \backslash \{0\}$ from the strict maximum principle.

i) Case $\eta > 1$. The result is classical when $\tau = 0$ and one finds it again in the following proof. From the Jensen inequality, the average function $\bar{f}$ also satisfies (6.1), that is

$$(r^{N-1}\bar{f}_r)_r + r^{N-1+\sigma} |Lnr|^{\kappa} \bar{f}^{\eta} \leq 0 \tag{6.10}$$

in $(0,1)$. Then either $\lim_{r\to 0} r^{N-1}\bar{f}_r \in (0,+\infty]$ and $\lim_{r\to 0} \bar{f} = C \geq 0$, or $\bar{f}_r$ is negative near the origin. Integrating on $[r_0, r]$ and going to the limit when $r_0 \to 0$, we obtain

$$r^{N-1}\bar{f}_r + \bar{f}^{\eta} \int_0^r \vartheta^{N-1+\sigma} |Ln\vartheta|^{\kappa} d\vartheta \leq 0.$$

Integrating by parts we deduce

$$\bar{f}^{-\eta}\bar{f}_r + (2(N+\sigma))^{-1} r^{\sigma+1} |Lnr|^{\kappa} \leq 0$$

for small r. Integrating again, this implies $\sigma > -2$, or $\sigma = -2$ and $\kappa < -1$, and the conclusion holds.

ii) Case $\eta = 1$ **and** $\sigma < -2$**.** Integrating (6.10), we get an estimate $\bar{f}(r) \geq C_{\varepsilon} e^{r^{-\varepsilon}}$ near the origin, for some $\varepsilon > 0$ and $C_{\varepsilon} > 0$. This is impossible because the integral $\int_0^r \vartheta^{N-1+\sigma} |Ln\vartheta|^{\kappa} \bar{f}^{\eta}(\vartheta) d\vartheta$ exists.

iii) Case $0 \leq \eta < 1$**.** The results are proved in [2], [25], when $\kappa = 0$, $\eta \neq 0$, by applying Osserman's estimate to the inverse function $g = 1/f$. In the general case this function satisfies inequality (6.1) with $Q = 2 - \eta > 1$. Applying Lemma 6.1 to g one gets (6.8), (6.9) by returning to f. Iin case $\eta = 0$ the result comes more quickly from the maximum principle. And $\eta < (N+\sigma)/(N-2)$: indeed, it follows from (6.8) when $\sigma \neq -2$, because $|x|^{\sigma} |Ln|x||^{\kappa} f^{\eta} \in L^1_{loc}(B_1)$, and it is obvious when $\sigma = -2$.■

Lemma 6.3 *Let* $f \in M_{loc}^{N/(N-2)}(B_1)$ *be nonnegative, and* σ, η *be two reals such that*

$$1 < \eta < \min(N, N+\sigma)/(N-2).$$

Then there exists some $\tau > N/2$ *and some constant* $C > 0$ *such that for any* $x_0 \in \bar{B}_{1/4} \backslash \{0\}$*,*

$$\int\limits_{B_{|x_0|/2}(x_0)} (|x|^{\sigma} f^{\eta-1})^{\tau} \leq C |x_0|^{N-2\tau}, \tag{6.11}$$

and $|x|^{\sigma} f^{\eta-1} \in L^{\tau}_{loc}(B_1)$*.*

Proof For any $k, \tau > 1$, the Hölder inequality implies

$$\int\limits_G (|x|^{\sigma} f^{\eta-1})^{\tau} \leq \left(\int\limits_G f^{(\eta-1)\tau k} \right)^{1/k} \left(\int\limits_G |x|^{\sigma\tau k/(k-1)} \right)^{(k-1)/k},$$

for any open set G relatively compact in $B_1 \backslash \{0\}$. One can choose k, τ such that max

$$(N/2, N/k(\sigma+2)) < \tau < N/k(\eta-1)(N-2), \quad \text{and} \quad \tau\sigma^- < N(k-1)/k,$$

because $1 < \eta < (N - \sigma^-)/(N-2)$. The conclusions follow, since $f \in M_{loc}^{N/(N-2)}(B_1)$.■

Lemma 6.4 *Let* $f \in C^2(B_1 \backslash \{0\})$ *be a nonnegative solution of the inequality*

$$0 \leq -\Delta f \leq |x|^{\sigma} |Ln|x||^{\kappa} f^{\eta}, \tag{6.12}$$

where $\sigma, \kappa, \eta \in \mathbb{R}$*.*

i) Assume $1 < \eta < (N - \sigma^-)/(N-2)$, *or* $0 < \eta \leq 1 < (N+\sigma)/(N-2)$. *Then either* f *is regular, or*

$$C \leq |x|^{N-2} f(x) \leq 2C \tag{6.13}$$

in $\bar{B}_{1/2} \backslash \{0\}$, *for some* $C > 0$.

ii) Assume $0 < \eta < 1$, *and* $\eta < (N+\sigma)/(N-2) \leq 1$, *and* $\kappa \geq -1$ *when* $\sigma = -2$. *Then either*

$$\lim_{r \to 0} r^{N-2} \bar{f}(r) = \lambda > 0, \tag{6.14}$$

or

$$\bar{f}(r) = O(r^{(2+\sigma)/(1-\eta)} \left|Lnr\right|^{\kappa/(1-\eta)}), \qquad \textit{if } \sigma < -2, \tag{6.15}$$

$$\bar{f}(r) = O(\left|Lnr\right|^{(1+\kappa)/(1-\eta)}), \qquad \textit{if } \sigma = -2 \quad \textit{and } \kappa > -1, \tag{6.16}$$

$$\bar{f}(r) = O((Ln(|Lnr|))^{1/(1-\eta)}), \qquad \textit{if } \sigma = -2 \quad \textit{and } \kappa = -1. \tag{6.17}$$

Proof i) Let us write the inequality under the form $-\Delta f = Hf$, where

$$0 \leq H \leq |x|^{\sigma} \left|Ln\,|x|\right|^{\kappa} f^{\eta-1}$$

in $\bar{B}_{1/2} \backslash \{0\}$. Then $H \in L^{\tau}_{loc}(B_1)$ for some $\tau > N/2$, from Lemma 6.3 when $1 < \eta < (N - \sigma^-)/(N-2)$, or from the estimate $H \leq C\,|x|^{\sigma} \left|Ln\,|x|\right|^{\kappa}$ when $0 < \eta \leq 1 < (N+\sigma)/(N-2)$. The conclusion follows from [27].

ii) The results are known in case $\kappa = 0$, see [25]. Our proof is slightly different. Here also $\bar{f}$ also satisfies (6.1) from Jensen inequality. Let us define

$$w(\rho) = \bar{f}(\rho^{1/(2-N)}) \quad \text{and} \quad y(\rho) = w(\rho) - w(2^{N-2}).$$

Then y satisfies the inequality

$$0 \leq -y_{\rho\rho} \leq C\rho^{-(\sigma+2N-2)/(N-2)}(Ln\rho)^{\kappa} y^{\eta},$$

on $\left(2^{N-2}, +\infty\right)$, for some $C > 0$. Therefore y_ρ decreases to a limit $\lambda \geq 0$ at infinity, because w is concave. If $\lambda > 0$, then $\lim_{r \to 0} r^{N-2}\bar{f}(r) = \lambda$. Consider the case $\lambda = 0$. By integration we deduce that

$$y_\rho(\rho) \leq C \int_{\rho}^{+\infty} \vartheta^{-(\sigma+2N-2)/(N-2)}(Ln\vartheta)^{\kappa} y^{\eta} d\vartheta$$

on $\left(2^{N-2}, +\infty\right)$, for some $C > 0$. We have $y(\rho) = O(\rho)$ near infinity. Integrating twice we get in any case $y(\rho) = O(\rho^{1-\varepsilon_0})$, where $\varepsilon_0 \in (0,1)$, since $\eta < (N+\sigma)/(N-2)$. Now let us make the change of variables

$$\bar{f}(r) = r^{-\Gamma} T^m F(T),$$

where $T = -Ln\, r$, $\Gamma = (2+\sigma)/(\eta-1)$, and m is a parameter. It leads to the following inequality:

$$\begin{aligned} &F_{TT} - (N - 2 - 2\Gamma - \tfrac{2m}{T})\ F_T \\ &-(\Gamma(N-2-\Gamma) + (N-2-2\Gamma)\tfrac{m}{T} - \tfrac{m(m-1)}{T^2})\ F + T^{\kappa+m(\eta-1)} F^{\eta} \geq 0. \end{aligned} \tag{6.18}$$

• First assume $\sigma \neq -2$, and take $m = -\kappa/(\eta-1)$. Then F is bounded near infinity. Indeed suppose it is not true. Then either there exists a sequence (T_n) with

$$\lim T_n = +\infty, F_T(T_n) = 0, F_{TT}(T_n) \leq 0, \quad \text{and} \quad \lim F(T_n) = +\infty.$$

This is impossible from (6.18), since $\eta < 1$ and $\Gamma(N-2-\Gamma) > 0$. Or F is increasing to infinity. Then for any $\varepsilon > 0$, there exists some $T(\varepsilon)$ such that

$$-F_{TT} + (N-2-2\Gamma-\varepsilon)F_T + \Gamma(N-2-\Gamma-\varepsilon)F \leq 0$$

for any $T \geq T(\varepsilon)$. But the equation

$$\omega^2 - (N-2-2\Gamma-\varepsilon)\omega - \Gamma(N-2-\Gamma-\varepsilon) = 0$$

has two real roots $\omega_{1,\varepsilon}, \omega_{2,\varepsilon}$ such that $\lim_{\varepsilon\to 0}\omega_{1,\varepsilon} = -\Gamma < 0$ and $\lim_{\varepsilon\to 0}\omega_{2,\varepsilon} = N-2-\Gamma > 0$. And $F(T) = O(e^{(N-2-\Gamma-\varepsilon_1)T})$ for some $\varepsilon_1 \in (0,1)$. Choosing ε small enough, we deduce that $F(T) = O(e^{\omega_{1,\varepsilon}T})$, hence a contradiction.

• Now assume $\sigma = -2$, $\kappa > -1$, and take $m = -(1+\kappa)/(\eta-1)$. Then F is still bounded near infinity. Indeed by contradiction F must be increasing to infinity. Then for any $\varepsilon > 0$ we find

$$-F_{TT} + (N-2-\varepsilon)F_T \leq 0,$$

that is $(e^{-(N-2-\varepsilon)T}F_T)_T \geq 0$, for any $T \geq T(\varepsilon)$. Hence again a contradiction, because $F(T) = O(e^{(N-2-\varepsilon_1)T})$.

• At last assume $\sigma = -2$ and $\kappa = -1$. Then setting

$$\bar{f}(r) = (LnT)^{1/(1-\eta)}H(T)s$$

one finds

$$\begin{aligned} &H_{TT} - (N-2-\tfrac{1}{(1-\eta)TLnT})H_T \\ &+\tfrac{1}{TLnT}(H^\eta - \tfrac{1}{(1-\eta)}((N-2+\tfrac{1}{T} - \tfrac{\eta}{(1-\eta)TLnT})H) \geq 0. \end{aligned}$$

As above, if H is unbounded, then it is increasing near infinity. Then a contradiction holds as in the case $\kappa > -1$.■

Lemma 6.5 *Let $f \in C^2(B_1\backslash\{0\})$ be a nonnegative nontrivial solution of inequality*

$$C\,|x|^\sigma f^\eta \leq -\Delta f \leq 2C\,|x|^\sigma f^\eta, \tag{6.19}$$

where $\sigma, \eta, C \in \mathbb{R}$, with $C > 0$. Assume $0 \leq \eta < 1$. Then $\eta < (N+\sigma)/(N-2)$. If $\sigma < -2$, there exists $C > 0$ such that

$$C\,|x|^{(2+\sigma)/(1-\eta)} \leq f(x) \leq 2C\,|x|^{(2+\sigma)/(1-\eta)} \tag{6.20}$$

in $\bar{B}_{1/2}\backslash\{0\}$. If $\sigma = -2$, then

$$C\,|Ln\,|x||^{1/(1-\eta)} \leq f(x) \leq 2C\,|Ln\,|x||^{1/(1-\eta)}. \tag{6.21}$$

If $\sigma < -2$, then

$$C\,|x|^{2-N} \leq f(x) \leq 2C\,|x|^{2-N} \quad \text{or } f \text{ is regular.} \tag{6.22}$$

Proof When $\eta = 0$, the result follows elementary from [4] and the maximum principle. When $0 < \eta < 1$, it is a consequence of the preceeding results. From Lemma 6.2, $\eta < (N+\sigma)/(N-2)$ and f satisfies (6.8), (6.9). Writing $-\Delta f = Hf$, this implies $0 \leq H \leq C\,|x|^{-2}$ from (6.12). Then f satisfies the Harnack inequality. The estimates follow from Lemma 6.4.■

References

[1] M.F. Bidaut-Véron, *Local and global behavior of solutions of quasilinear equations of Emden-Fowler type,* Arc. Rat. Mech. and Anal., 107 (1989), 293-324.

[2] M.F. Bidaut-Véron and T. Raoux, *Asymptotics of solutions of some nonlinear elliptic systems,* Comm. Part. Diff. Equ., 21 (1996), 1035-1086.

[3] M.F. Bidaut-Véron and L. Véron, *Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations,* Invent. Math., 106 (1991), 489-539.

[4] H. Brézis and P.L. Lions, *A note on isolated singularities for linear elliptic equations,* Jl. Math. Anal. and Appl., 7A (1981), 263-266.

[5] H. Brézis and L. Oswald, *Remarks on sublinear elliptic equations,* Nonlinear Anal., 10 (1986), 55-64.

[6] L.A. Caffarelli, B. Gidas and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth,* Comm. Pure and Appl. Math., 42 (1989), 271-297.

[7] G. Caristi and E. Mitidieri, *Blow up estimates of positive solutions of a parabolic system,* Jl. Diff. Equ., 113 (1994), 265-271.

[8] P. Clément, G. De Figueiredo and E. Mitidieri, *Positive solutions of semilinear elliptic systems*, Comm. Part. Diff. Eq., 17 (1992), 923-940.

[9] P. Clément, R. Manasevitch, E. Mitidieri, *Positive solutions for a quasilinear system via blow-up,* Comm. Part. Diff. Eq. 18, 12 (1993), 2071-2106.

[10] D. De Figueiredo and P. Felmer, *On superquadratic elliptic systems,* Trans. Amer. Math. Soc., 343 (1994), 99-116.

[11] F. De Thélin and J. Vélin, *Existence et non existence de solutions non triviales pour des systèmes elliptiques non linéaires,* C.R. Acad. Sci. Paris, 314 (1991), 589-592.

[12] M. Escobedo and M.A. Herrero, *Boundedness and blow up for a semilinear reaction-diffusion system,* Jl. Diff. Equ.,89,(1991), 176-202.

[13] M. Escobedo and H.A. Levine, *Critical blow-up and global existence numbers for a weakly coupled system of reaction-diffusion equations,* Arc. Rat. Mech. and Anal., 129 (1995), 47-100.

[14] P. Felmer, R. Manasevitch, F. De Thélin, *Existence and uniqueness of positive solutions for certain quasilinear elliptic systems,* Comm. Part. Diff. Eq., 17 (1992), 2013-2029.

[15] M. Garcia, R. Manasevitch, E. Mitidieri, and C. Yarur, *Existence and nonexistence of singular positive solutions for a class of semilinear elliptic systems,* Arch. Rat. Mech. Anal. 140 (1997), 253-284.

[16] B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations,* Comm. Pure Applied Math. 34 (1981), 525-598.

[17] D. Gilbarg and N. Trudinger, Elliptic Partial Differential Equations of second order, 2nd Ed., Springer-Verlag, Berlin, New-York (1983).

[18] T. Kato, *Schrödinger operators with singular potentials,* Israël Jl. Math.,13 (1972), 135-148.

[19] P.L. Lions, *Isolated singularities in semilinear problems,* Jl. Diff. Eq., 38 (1980), 441-450.

[20] E. Mitidieri, *A Rellich type identity and applications,* Comm. Part. Diff. Eq., 18 (1993), 125-151.

[21] E. Mitidieri, *Non-existence of positive solutions of semilinear elliptic systems in* $\mathbb{R}^N$, Quaderno Matematici 285, University of Trieste (1992).

[22] W.M. Ni, *On a singular elliptic equation,* Proc. Am. Math. Soc., 88 (1983), 614-616.

[23] Y.W. Qi, *Global existence and uniqueness of a reaction-diffusion system via invariant solutions,* Nonlinear Anal., 23 (1994), 1277-1291.

[24] T. Raoux, *Singularités et comportement asymptotique pour des équations et des systèmes elliptiques semi-linéaires,* Thèse de doctorat, Tours (1995).

[25] T. Raoux, *Local and global behaviour for an elliptic equation with a sublinear nonlinearity,* Adv. in Math. Sci. and Appl., 7 (1997), 859-879.

[26] J. Serrin, *Local behavior of solutions of quasilinear equations,* Acta Math., 111 (1964), 247-302.

[27] J. Serrin, *Isolated singularities of solutions of quasilinear equations,* Acta Math., 113 (1965), 219-240.

[28] J. Serrin and H. Zou, *Nonexistence of positive solutions for the Lane-Emden system*, Diff. Int. Equ., 9 (1996), 635-653.

[29] J. Serrin and H. Zou, *Existence of positive solutions of the Lane-Emden system*, Atti Sem. Math. Univ. Modena (1997), to appear.

[30] R. Soranzo, *Isolated singularities of positive solutions of a superlinear biharmonic equation,* Quaderni Matematici, University of Trieste (1994), 1-32.

[31] M. Souto, *A priori estimates and existence of positive solutions of nonlinear cooperative elliptic systems,* Diff. and Int. Eq., to appear.

[32] N. Trudinger, *On Harnack type inequalities and their application to quasilinear equations*, Comm. Pure Applied Math., 20 (1967), 721-747.

[33] R. van der Vorst, *Variational problems with a strongly definite structure,* Thesis, Leiden University (1994).

[34] J.L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations,* Appl. Math. Optim. 12 (184), 191-202.

[35] J.L. Vázquez and C. Yarur, *Schrödinger equations with unique positive singularities,* Trans. Amer. Math. Soc., 315 (1989), 711-720.

[36] C. Yarur, *Nonexistence of positive solutions for a class of semilinear elliptic systems*, E. Journal of Diff. Equ., 8 (1996), 1-22.