Asymptotic behaviour of elliptic systems with mixed absorption and source terms

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Abstract. We study the limit behaviour near the origin of the nonnegative solutions of the semilinear elliptic system

$$\begin{cases} -\Delta u + |x|^a v^p = 0, \\ \Delta v + |x|^b u^q = 0, \end{cases} \quad \text{in } \mathbb{R}^N \ (N \geqslant 3),$$

where $p, q, a, b \in \mathbb{R}$, with $p, q > 0$, $pq \neq 1$. We give a priori estimates without any restriction on the values of p and q.

1. Introduction

Here we study the nonnegative solutions u, v of the semilinear elliptic system in $\mathbb{R}^N$ ($N \geqslant 3$) with mixed absorption and source terms:

$$\begin{cases} -\Delta u + |x|^a v^p = 0, \\ \Delta v + |x|^b u^q = 0, \end{cases} \tag{1.1}$$

where $p, q, a, b \in \mathbb{R}$ with $p, q > 0$ and $pq \neq 1$. We describe the asymptotic behaviour of the solutions near the origin. We suppose that u, v are defined in $B_1 \setminus \{0\}$, where $B_r = B(0, r)$ and $B(y, r) = \{x \in \mathbb{R}^N \mid |x - y| < r\}$ for any $r > 0$ and $y \in \mathbb{R}^N$. A Kelvin transform would give the behaviour near infinity. In particular, we cover the case of the biharmonic equation

$$\Delta^2 w + |x|^\sigma w^Q = 0, \tag{1.2}$$

for given reals σ, Q with $Q > 0$, $Q \neq 1$: we give the behaviour of the subharmonic or superharmonic nonnegative solutions of (1.2), by taking $p = 1$, $a = 0$, $b = \sigma$ or $q = 1$, $a = \sigma$, $b = 0$ in (1.1). This article complements the preceeding works relative to the system with absorption terms

$$\begin{cases} -\Delta u + |x|^a v^p = 0, \\ -\Delta v + |x|^b u^q = 0, \end{cases} \quad \text{see [5],} \tag{1.3}$$

and to the system with source terms

$$\begin{cases} \Delta u + |x|^a v^p = 0, \\ \Delta v + |x|^b u^q = 0, \end{cases} \quad \text{see [2].} \tag{1.4}$$

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For a better understanding of system (1.1), let us recall the behaviour of the nonnegative solutions of the two equations

$$-\Delta w + |x|^\sigma w^Q = 0, \tag{1.5}$$

and

$$\Delta w + |x|^\sigma w^Q = 0, \tag{1.6}$$

for given reals σ, Q with $Q > 0$, $Q \neq 1$.

Usually, (1.5) is called equation "with the good sign", because the maximum principle applies. Notice that the solutions are subharmonic, hence they satisfy the mean value inequality

$$w(x) \leqslant \frac{1}{|B(x,r)|} \int_{B(x,r)} w(x) \, \mathrm{d}x, \tag{1.7}$$

for any ball $\overline{B}(x,r) \subset B_1 \setminus \{0\}$. As a consequence, any estimate of the spherical mean value

$$\overline{w}(r) = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} w(r,\theta) \, \mathrm{d}\theta$$

near $r = 0$ implies an analogous estimate of w, see [4,22]. In such a way the obtention of a priori estimates reduces to the study of an ordinary differential inequality. Defining

$$\Gamma = (\sigma + 2)/(Q - 1) \tag{1.8}$$

for any $Q \neq 1$, the radial function

$$w^*(x) = C^*|x|^{-\Gamma}, \quad C^* = \left(\Gamma(\Gamma - N + 2)\right)^{1/(Q-1)}, \tag{1.9}$$

is a solution of (1.5) whenever $\Gamma(\Gamma - N + 2) > 0$. When $Q > 1$, any solution w satisfies the Keller–Osserman estimate near the origin

$$w(x) \leqslant C |x|^{-\Gamma}, \tag{1.10}$$

where $C = C(N, Q, \sigma)$. When $Q \geqslant (N + \sigma)/(N - 2)$, then w^* does not exist, and the singularity is removable, which means that w is bounded near 0, see [9,19–21]. The behaviour of the solutions is isotropic, that is, asymptotically radial. When $Q < 1$, then (1.10) is no longer true and it is replaced by the estimate

$$w(x) = \begin{cases} \mathrm{O}(\max(|x|^{-\Gamma}, |x|^{2-N})) & \text{if } Q \neq (N + \sigma)/(N - 2), \\ \mathrm{O}(|x|^{2-N}|\ln|x||^{1/(1-Q)}) & \text{if } Q = (N + \sigma)/(N - 2). \end{cases} \tag{1.11}$$

Moreover, some anisotropic solutions can occur, see [3,4].

The behaviour of the equation "with the bad sign" (1.6) is not completely known. It cannot be reduced to a radial problem, because now w is superharmonic, hence for any ball $\overline{B}(x,r) \subset B_1 \setminus 0$,

$$w(x) \geqslant \frac{1}{|B(x,r)|} \int_{B(x,r)} w(x)\,\mathrm{d}x.$$

Equation (1.6) still admits a particular radial solution

$$w_*(x) = C_* |x|^{-\Gamma}, \quad C_* = \left(\Gamma(N-2-\Gamma)\right)^{1/(Q-1)},$$

if $\Gamma(N-2-\Gamma) > 0$. When $Q > 1$ and (1.6) admits a nontrivial solution, then $\Gamma > 0$, which means $\sigma + 2 > 0$. And any solution w satisfies the estimate

$$w(x) = \mathrm{O}\!\left(\min(|x|^{-\Gamma}, |x|^{2-N})\right), \tag{1.12}$$

whenever $Q \leqslant (N+2)/(N-2)$ (with $Q \neq (N+2+2\sigma)/(N-2)$, if $\sigma \neq 0$). Consequently, w satisfies the Harnack inequality, and its behaviour is isotropic, see, for example, [1,10,13]. Beyond $(N+2)/(N-2)$, some anisotropic solutions can occur, for example, when $Q = (N+1)/(N-3)$ and $\sigma = 0$, see [7], and the a priori estimate is not known, see also [23]. When $Q < 1$, the solutions only exist when $Q < (N+\sigma)/(N-2)$, which means $\Gamma < N-2$. Then any solution satisfies

$$w(x) = \begin{cases} \mathrm{O}(\max(|x|^{-\Gamma}, 1)) & \text{if } \Gamma \neq 0, \\ \mathrm{O}(|\ln|x||^{1/(1-Q)}) & \text{if } \Gamma = 0, \end{cases} \tag{1.13}$$

and its behaviour is still isotropic, see [14].

Now let us return to system (1.1). It involves both subharmonic and superharmonic functions, and one may expect a mixed type behaviour. In Section 2, we give the main tools of our study: we essentially use fine properties of comparison of functions with their spherical mean value, in addition to classical tools, namely the maximum principle and the Brezis–Lions lemma [8].

In Section 3, we establish a priori estimates for the solutions of system (1.1), for any $p, q > 0$, such that $pq \neq 1$. In that case it admits a particular solution

$$u^*(x) = A^* |x|^{-\gamma}, \qquad v^*(x) = B^* |x|^{-\xi}, \tag{1.14}$$

where

$$\gamma = ((b+2)p + a + 2)/(pq-1), \qquad \xi = ((a+2)q + b + 2)/(pq-1), \tag{1.15}$$

and

$$A^* = \left[\gamma(\gamma+2-N)\left(\xi(N-2-\xi)\right)^p\right]^{1/(pq-1)},$$
$$B^* = \left[\xi(N-2-\xi)\left(\gamma(\gamma+2-N)\right)^q\right]^{1/(pq-1)}, \tag{1.16}$$

whenever $\gamma(\gamma+2-N) > 0$ and $\xi(N-2-\xi) > 0$. In the superlinear case $pq > 1$, we get the following estimates:

Theorem 1.1. *Assume $pq > 1$. Let $u, v \in C^2(B_1 \setminus \{0\})$ be any nonnegative solutions of* (1.1)*, with (u, v) $\neq (0, 0)$. Then $\xi \geqslant 0$, and*

$$u(x) = O(|x|^{-\gamma}), \quad v(x) = O\big(\min(|x|^{-\xi}, |x|^{2-N})\big), \quad \text{near } 0. \tag{1.17}$$

Moreover, if $\gamma \leqslant N - 2$, then u is bounded near 0.

This result shows a perfect behaviour of mixed type: the subharmonic function u satisfies an estimate of type (1.10), with an eventual removability, and the superharmonic function an estimate of type (1.12). In the sublinear case $pq < 1$, we get the following:

Theorem 1.2. *Assume $pq < 1$. Let $u, v \in C^2(B_1 \setminus \{0\})$ be any nonnegative solutions of* (1.1)*, with (u, v) $\neq (0, 0)$. Then $\xi < N - 2$, and*

$$v(x) = O(|x|^{2-N}), \tag{1.18}$$

$$u(x) = \begin{cases} O(\max(|x|^{a+2-(N-2)p}, |x|^{2-N})) & \text{if } p \neq (a + N)/(N - 2), \\ O(|x|^{2-N}|\ln|x||) & \text{if } p = (a + N)/(N - 2). \end{cases} \tag{1.19}$$

We notice that the estimates for v differ from the estimates of the scalar case (1.13): here v can admit a behaviour in $|x|^{2-N}$, whereas any solution w of (1.6) satisfies $w(x) = o(|x|^{2-N})$ when $Q < 1$.

Observe that, contrary to the case of Eq. (1.6), we have *no upper restriction on pq* in the superlinear case. Our proofs lead to the following main conclusion: *the fact that one of the solutions of the system is subharmonic implies a remarkable regularizing effect on the other one*. In particular, *the superharmonic function v always satisfies Harnack inequality*.

In Section 4, we give the precise convergence results for the solutions and study the possible existence of anisotropic solutions. As in [6] and [4], the behaviour of the system presents many possibilities. The study is uneasy, in particular in the critical cases $\gamma, \xi = 0$ or $N - 2$, since we have to combine the techniques of the two signs. In [4], we had noticed that the anisotropy is more frequent for system (1.3) than for system (1.4). Here we show that, for system (1.1), *the anisotropy is more frequent for u than for v*.

2. The key tools

Our main tools consist in precise comparisons between the two functions, either subharmonic or superharmonic, with their spherical mean values. In the sequel, the same letter C denotes some positive constants which may depend on u, v, unless otherwise stated.

2.1. Inequalities for superharmonic functions

Concerning the superharmonic functions, let us begin by a simple result.

Lemma 2.1. *Let $w \in C^2(B_1 \setminus \{0\})$ be any nonnegative superharmonic function, and $f = -\Delta w$. Then $\overline{w}$ is monotonous for small r, and there is a constant $C(N) > 0$ such that, for any $r \in (0, 1/2)$ and any $\varepsilon \in (0, 1/2]$,*

$$\overline{w}(r) \geqslant C(N)\varepsilon^2 r^2 \min_{s \in [r(1-\varepsilon), r(1+\varepsilon)]} \overline{f}(s). \tag{2.1}$$

Proof. Indeed, we have

$$-\left(r^{N-1}\overline{w}_r\right)_r = r^{N-1}\overline{f},$$

which obviously implies the monotonicity near 0. Integrating from $r/(1+\varepsilon)$ to r, we get

$$(1+\varepsilon)^{1-N}\overline{w}_r\left(r/(1+\varepsilon)\right) - \overline{w}_r(r) \geqslant r^{1-N} \int_{r/(1+\varepsilon)}^{r} s^{N-1}\overline{f}(s)\,\mathrm{d}s.$$

Integrating from r to $r(1+\varepsilon)$,

$$\left(1 + (1+\varepsilon)^{2-N}\right)\overline{w}(r) \geqslant \overline{w}\left(r(1+\varepsilon)\right) + (1+\varepsilon)^{2-N}\overline{w}\left(r/(1+\varepsilon)\right)$$
$$+ \int_{r}^{r(1+\varepsilon)} \tau^{1-N} \int_{\tau/(1+\varepsilon)}^{\tau} s^{N-1}\overline{f}(s)\,\mathrm{d}s, \tag{2.2}$$

and, in particular,

$$\overline{w}(r) \geqslant \frac{1}{2} \int_{r}^{r(1+\varepsilon)} \tau^{1-N} \int_{\tau/(1+\varepsilon)}^{\tau} s^{N-1}\overline{f}(s)\,\mathrm{d}s,$$

which implies (2.1). $\square$

Remark 2.1. Notice that, from (2.2),

$$\left(1 + (1+\varepsilon)^{2-N}\right)\overline{w}(r) \geqslant \overline{w}\left(r(1+\varepsilon)\right) + (1+\varepsilon)^{2-N}\overline{w}\left(r/(1+\varepsilon)\right),$$

hence any radial superharmonic positive function in $B_1 \setminus \{0\}$ satisfies the following form of the Harnack inequality: for any $r \in (0, 1/2)$ and any $\varepsilon \in (0, 1/2]$,

$$2^{1-N}\overline{w}(r) \leqslant \overline{w}\left(r(1+\varepsilon)\right) \leqslant 2\overline{w}(r). \tag{2.3}$$

Now we deduce a spherical form of the mean value inequality for superharmonic functions. We did not find any reference of it in the literature, so we give here a simple proof.

Lemma 2.2. *Let $w \in C^2(B_1 \setminus \{0\})$ be any nonnegative superharmonic function. Then there exists a constant $C(N) > 0$ such that, for any $x \in B_{1/2} \setminus \{0\}$,*

$$w(x) \geqslant C(N)\overline{w}\left(|x|\right). \tag{2.4}$$

Proof. Let $x_0 \in B_{1/2} \setminus \{0\}$. We study the function w in the annulus $\mathcal{C}_{x_0} = \{y \in \mathbb{R}^N \mid |x_0|/2 \leqslant |y| \leqslant 3|x_0|/2\}$. We set

$$w(y) = W(z), \quad z = y/|x_0|, \ \forall y \in \mathcal{C}_{x_0},$$

and $z_0 = x_0/|x_0|$. Then the range of z is the annulus $\mathcal{C} = \{z \in \mathbb{R}^N \mid 1/2 \leqslant |z| \leqslant 3/2\}$. Let G be the Green function in $\mathcal{C}$ with Dirichlet conditions on $\partial\mathcal{C}$. Then we have the representation formula

$$W(z) = \int_{\mathcal{C}} G(z,\eta)(-\Delta W)(\eta)\,\mathrm{d}\eta + \int_{\partial\mathcal{C}} P(z,\lambda)W(\lambda)\,\mathrm{d}s(\lambda),$$

where $P(z,\lambda) = -\partial G(z,\lambda)/\partial\nu$ is the Poisson kernel in $\mathcal{C} \times \partial\mathcal{C}$. From [18] there exists a constant $K = K(\mathcal{C})$, hence $K = K(N)$ such that, for any $(z,\lambda) \in \mathcal{C} \times \partial\mathcal{C}$,

$$K\varrho(z)|z-\lambda|^{1-N} \leqslant P(z,\lambda) \leqslant 2K\varrho(z)|z-\lambda|^{1-N},$$

where $\varrho(z)$ is the distance from z to $\partial\mathcal{C}$. In particular, $P(z_0,\lambda) \geqslant 2^{-N}K$, hence

$$W(z_0) \geqslant 2^{-N}K \int_{\partial\mathcal{C}} W(\lambda)\,\mathrm{d}s(\lambda),$$

since W is superharmonic in $\mathcal{C}$. Returning to w, we get

$$w(x_0) \geqslant 2^{-N}K|x_0|^{1-N} \int_{\partial\mathcal{C}_{x_0}} w(y)\,\mathrm{d}s(y).$$

That means that there exists a constant $C(N)$ such that

$$w(x_0) \geqslant C(N)\big[\overline{w}(|x_0|/2) + \overline{w}(3|x_0|/2)\big].$$

But from the Harnack inequality (2.3), it implies that there is another constant $C(N)$ such that

$$w(x_0) \geqslant C(N)\overline{w}(|x_0|),$$

and the conclusion follows. $\quad\square$

Now we give an upper estimate which will play a crucial part in the sequel.

Lemma 2.3. *Let $w \in C^2(B_1 \setminus \{0\})$ be any nonnegative superharmonic function, and $f = -\Delta w$. Then there exists $C(N) > 0$ such that, for any $x \in B_{1/2} \setminus \{0\}$,*

$$w(x) \leqslant C(N)\Big[|x|^2 \max_{B(x,|x|/2)} f + \overline{w}(|x|)\Big]. \tag{2.5}$$

Proof. We start from the representation formula for any C^2 function w in a ball of center $\overline{B}(x, R)$ contained in $B_1 \setminus \{0\}$: for any $\rho \in (0, R]$,

$$w(x) = c_N \int_{B(x,\rho)} [|z - x|^{2-N} - \rho^{2-N}](-\Delta w)(z)\,\mathrm{d}z + \frac{1}{|\partial B(x,\rho)|} \int_{\partial B(x,\rho)} w(s)\,\mathrm{d}s, \tag{2.6}$$

where $c_N = 1/N(N-2)|B_1| = 1/(N-2)|S^{N-1}|$. It implies

$$\rho^{N-1} w(x) \leqslant c_N \rho^{N-1} \int_{B(x,\rho)} |z - x|^{2-N}(-\Delta w)(z)\,\mathrm{d}z + \frac{1}{|S^{N-1}|} \int_{\partial B(x,\rho)} w(s)\,\mathrm{d}s,$$

and, by integration from 0 to R,

$$w(x) \leqslant c_N \int_{B(x,R)} |z - x|^{2-N}(-\Delta w)(z)\,\mathrm{d}z + \frac{1}{|B(x,R)|} \int_{B(x,R)} w(z)\,\mathrm{d}z.$$

Hence, in particular, taking $R = |x|/2$ and replacing the ball by an annulus,

$$\begin{aligned}
w(x) &\leqslant \frac{c_N |S^{N-1}|}{8}|x|^2 \max_{B(x,|x|/2)}(-\Delta w) + \frac{1}{(|x|/2)^N |B_1|} \int_{|x|/2 \leqslant |y| \leqslant 3|x|/2} w(z)\,\mathrm{d}z \\
&\leqslant \frac{1}{8(N-2)}|x|^2 \max_{B(x,|x|/2)}(-\Delta w) + \frac{N}{(|x|/2)^N} \int_{|x|/2}^{3|x|/2} r^{N-1}\overline{w}(r)\,\mathrm{d}r \\
&\leqslant \frac{1}{8(N-2)}|x|^2 \max_{B(x,|x|/2)}(-\Delta w) + 3^N \max_{[|x|/2, 3|x|/2]} \overline{w},
\end{aligned}$$

hence (2.5) follows from (2.3). $\square$

2.2. Inequalities for subharmonic functions

Concerning the subharmonic functions, our main argument is a comparison between the value of the function at some point $x \in B_{1/2} \setminus \{0\}$ and the value of its mean value at some shifted radius $|x|(1 \pm \varepsilon)$, proved in [4].

Lemma 2.4. *Let $w \in C^2(x \in B_1 \setminus \{0\})$ be any nonnegative subharmonic function. Then $\overline{w}$ is monotonous for small r, either decreasing with $\lim_{r \to 0} r^{N-2}\overline{w}(r) > 0$, or nondecreasing and bounded. And there exists a constant $C(N)$ such that, for any $\varepsilon \in (0, 1/2]$,*

$$w(x) \leqslant C(N)\varepsilon^{1-N}\overline{w}(|x|(1 \pm \varepsilon)) \quad \text{near } 0, \tag{2.7}$$

with the sign $+$ if $\overline{w}$ is nondecreasing, and the sign $-$ if $\overline{w}$ is decreasing. Consequently, for small r and any $Q > 1$,

$$\overline{w}^Q(r) \leqslant \overline{w^Q}(r) \leqslant (C(N)\varepsilon^{1-N})^Q \overline{w}(r(1 \pm \varepsilon))^Q. \tag{2.8}$$

And for small r and any $Q \in (0, 1)$, if $w \neq 0$ near 0,

$$\overline{w}^Q(r) \geqslant \overline{w^Q}(r) \geqslant (C(N)\varepsilon^{1-N})^{Q-1}\overline{w}(r(1 \pm \varepsilon))^{Q-1}\overline{w}(r). \tag{2.9}$$

As a consequence, any estimate of $\overline{w}$ of the form

$$\overline{w}(r) = O(|\ln r|^b r^a) \quad \text{as } r \to 0 \tag{2.10}$$

for given reals a, b implies the corresponding estimate

$$w(x) = O(|\ln|x||^b |x|^a) \quad \text{as } x \to 0, \tag{2.11}$$

see also [3,22].

Property (2.1) of the superharmonic functions has to be compared with the following property, often used in [4].

Lemma 2.5. *Let $w \in C^2(B_1 \setminus \{0\})$ be any nonnegative subharmonic function, and $g = \Delta w$. Then there is a constant $C(N) > 0$ such that, for any $\varepsilon \in (0, 1/2]$ and r small enough,*

$$\overline{w}(r) \geqslant C(N)\varepsilon^2 r^2 \min_{s \in [r(1-\varepsilon), r(1+\varepsilon)]} \overline{g}(s). \tag{2.12}$$

Proof. Indeed, we have $(r^{N-1}\overline{w}_r)_r = r^{N-1}\overline{g}$. First, integrate over $[r, r(1+\varepsilon)^{1/2}]$ for r small enough. Either $\overline{w}$ is decreasing, then

$$-r^{N-1}\overline{w}_r(r) \geqslant \int_r^{r(1+\varepsilon)^{1/2}} s^{N-1}\overline{g}(s)\,\mathrm{d}s, \tag{2.13}$$

and a new integration gives

$$\overline{w}(r) \geqslant \int_r^{r(1+\varepsilon)^{1/2}} \tau^{1-N} \int_\tau^{\tau(1+\varepsilon)^{1/2}} s^{N-1}\overline{g}(s)\,\mathrm{d}s\,\mathrm{d}\tau,$$

hence

$$\overline{w}(r) \geqslant C\varepsilon^2 r^2 \min_{s \in [r, r(1+\varepsilon)]} \overline{g}(s). \tag{2.14}$$

Or $\overline{w}$ is nondecreasing, and we find

$$\left(r(1+\varepsilon)^{1/2}\right)^{N-1} \overline{w}_r\left(r(1+\varepsilon)^{1/2}\right) \geqslant \int_r^{r(1+\varepsilon)^{1/2}} s^{N-1}\overline{g}(s)\,\mathrm{d}s,$$

hence

$$\overline{w}\left(r(1+\varepsilon)\right) \geqslant C \int_r^{r(1+\varepsilon)^{1/2}} \tau^{1-N} \int_\tau^{\tau(1+\varepsilon)^{1/2}} s^{N-1}\overline{g}(s)\,\mathrm{d}s\,\mathrm{d}\tau,$$

which now implies

$$\overline{w}(r) \geqslant C\varepsilon^2 r^2 \min_{s \in [r(1-\varepsilon), r]} \overline{g}(s). \tag{2.15}$$

In any case, (2.12) follows. $\quad\square$

At last we recall some elementary properties given in [4].

Lemma 2.6. *Let $\sigma \in \mathbb{R}$, and let $y \in C^2((0,1))$ be nonnegative.*

(i) *Assume that*

$$\Delta y(r) := y_{rr}(r) + \frac{N-1}{r} y_r(r) \leqslant Cr^\sigma$$

on $(0,1)$, for some $C > 0$. If $\sigma + N < 0$, then $y(r) = \mathrm{O}(r^{\sigma+2})$. If $\sigma + N = 0$, then $y(r) = \mathrm{O}(r^{2-N}|\ln r|)$. If $\sigma + N > 0$, then $y(r) = \mathrm{O}(r^{2-N})$. If $\sigma + 2 > 0$ and $\lim_{r \to 0} y(r) = \lim_{r \to 0} r^{N-1} y_r(r) = 0$, then $y(r) = \mathrm{O}(r^{\sigma+2})$.

(ii) *Assume that*

$$\Delta y(r) \geqslant Cr^\sigma$$

on $(0,1)$, for some $C > 0$. If $\sigma + N < 0$, then $y(r) \geqslant Cr^{\sigma+2}$ for another $C > 0$. If $\sigma + N = 0$, then $y(r) \geqslant Cr^{2-N}|\ln r|$. If $-N < \sigma \leqslant -2$, then $y(r) \geqslant Cr^{2-N}$. If y is bounded, then $\sigma + 2 > 0$. If $\sigma + 2 > 0$ and $\lim_{r \to 0} y(r) = \lim_{r \to 0} r^{N-1} y_r(r) = 0$, then $y(r) \geqslant Cr^{\sigma+2}$.

2.3. Bootstrap result

Our third tool is a bootstrap result proved in [4], allowing to convert a shifted inequality into an ordinary one. Let us recall it for a better understanding.

Lemma 2.7. *Let $d, h, l \in \mathbb{R}$ with $d \in (0,1)$ and y, Φ be two continuous positive functions on some interval $(0, R]$. Assume that there exist some $C, M > 0$ and $\varepsilon_0 \in (0, 1/2]$ such that, for any $\varepsilon \in (0, \varepsilon_0]$,*

$$y(r) \leqslant C\varepsilon^{-h}\Phi(r)y^d\big(r(1-\varepsilon)\big) \quad and \quad \max_{\tau \in [r/2, r]} \Phi(\tau) \leqslant M\Phi(r), \tag{2.16}$$

or else,

$$y(r) \leqslant C\varepsilon^{-h}\Phi(r)y^d\big(r(1+\varepsilon)\big) \quad and \quad \max_{\tau \in [r, 3r/2]} \Phi(\tau) \leqslant M\Phi(r), \tag{2.17}$$

for any $r \in (0, R/2]$. Then there exists another $C > 0$ such that

$$y(r) \leqslant C\Phi(r)^{1/(1-d)} \tag{2.18}$$

on $(0, R/2]$.

3. A priori estimates

Let us return to system (1.1). First, notice that, if $v = 0$ in $B_1 \setminus \{0\}$, then $u = 0$. Excluding this case, there exists some $C > 0$ such that

$$v(x) \geqslant C \tag{3.1}$$

in $B_{1/2} \setminus \{0\}$, from the maximum principle. The function $\overline{v}$ always satisfies $\overline{v}(r) = \mathrm{O}(r^{2-N})$, since $r^{N-2}\overline{v}$ is concave near the origin. Moreover, from the Brezis–Lions lemma [8], $|x|^b u^q \in L^1_{\mathrm{loc}}(B_1)$, and there exists some $C_2 \geqslant 0$ such that

$$-\Delta v = |x|^b u^q + C_2 \delta_0 \quad \text{in } \mathcal{D}'(B_1), \tag{3.2}$$

where δ_0 is the Dirac mass at the origin. And this implies that

$$\lim_{r \to 0} r^{N-2}\overline{v}(r) = C_2. \tag{3.3}$$

Notice that u is positive in $B_{1/2} \setminus \{0\}$, since $u \in C^2(B_1 \setminus \{0\})$ and $\Delta u(x) \geqslant C|x|^a$ from (3.1). But u can eventually tend to 0 at the origin.

3.1. Main estimates

In the next theorem, we give a first inequality for the mean value, which is essential for upper or lower estimates.

Theorem 3.1. *Let $u, v \in C^2(B_1 \setminus \{0\})$ be any nonnegative solutions of system* (1.1) *with $p, q > 0$. Then there exists some $C > 0$ such that, for any $r \in (0, 1/2)$,*

$$\overline{u}(r) \geqslant C r^{a+2} \min\big(\overline{v}^p(r), \overline{v^p}(r)\big), \tag{3.4}$$

$$\overline{v}(r) \geqslant C r^{b+2} \overline{u}^q(r). \tag{3.5}$$

Proof. We only need to prove the estimates for $r > 0$ small enough, from the continuity and the positivity of $\overline{u}, \overline{v}$. Hence we can assume that $\overline{u}, \overline{v}$ are monotonous. We first apply Lemma 2.5 to function $\overline{u}$ and get

$$\overline{u}(r) \geqslant C \varepsilon^2 r^{a+2} \min_{s \in [r(1-\varepsilon), r(1+\varepsilon)]} \overline{v^p}(s). \tag{3.6}$$

Then either $p \geqslant 1$, hence $\overline{v^p} \geqslant \overline{v}^p$, and (3.4) follows from (2.3). Or $p < 1$, hence v^p is still superharmonic, and (3.4) follows by applying (2.3) to v^p. Now we apply Lemma 2.1 to function $\overline{v}$. For any $\varepsilon \in (0, 1/2]$,

$$\overline{v}(r) \geqslant C \varepsilon^2 r^{b+2} \min_{s \in [r(1-\varepsilon), r(1+\varepsilon)]} \overline{u^q}(s). \tag{3.7}$$

First, assume that $q \geqslant 1$. Then $\overline{u^q} \geqslant \overline{u}^q$, hence

$$\overline{v}(r) \geqslant C \varepsilon^2 r^{b+2} \min_{s \in [r(1-\varepsilon), r(1+\varepsilon)]} \overline{u}^q(s).$$

In particular, from the monotonicity of $\overline{u}$,

$$\max\big(\overline{v}(4r/5), \overline{v}(4r/3)\big) \geqslant C r^{b+2} \overline{u}^q(r),$$

and (3.5) follows from (2.3).

Now assume that $q < 1$. Then

$$\overline{u^q}(r) \geqslant C \Big(\min_{s \in [r(1-\varepsilon), r(1+\varepsilon)]} \overline{u}^{q-1}(s) \Big) \overline{u}(r), \tag{3.8}$$

from Lemma 2.4. Reporting (3.8) into (3.7), we deduce that

$$\overline{v}(r) \geqslant C\varepsilon^2 r^{b+2} \min_{s \in [r(1-\varepsilon)^2, r(1+\varepsilon)^2]} \overline{u}^{(q-1)}(s) \min_{s \in [r(1-\varepsilon), r(1+\varepsilon)]} \overline{u}(s).$$

Hence

$$\overline{v}(r) \max_{s \in [r(1-\varepsilon)^2, r(1+\varepsilon)^2]} \overline{u}^{(1-q)}(s) \geqslant C\varepsilon^2 r^{b+2} \min_{s \in [r(1-\varepsilon), r(1+\varepsilon)]} \overline{u}(s).$$

It implies

$$\overline{v}\big(r/(1+\varepsilon)\big)\overline{u}^{(1-q)}\big(r(1-\varepsilon)^2/(1+\varepsilon)\big) \geqslant C\varepsilon^2 r^{b+2}\overline{u}(r)$$

if $\overline{u}$ is nonincreasing, and

$$\overline{v}\big(r/(1-\varepsilon)\big)\overline{u}^{(1-q)}\big(r(1+\varepsilon)^2/(1-\varepsilon)\big) \geqslant C\varepsilon^2 r^{b+2}\overline{u}(r)$$

if $\overline{u}$ is nondecreasing. In any case, we deduce from (2.3) the estimate

$$\overline{u}(r) \leqslant C\varepsilon^{-2} r^{-(b+2)}\overline{v}(r)\overline{u}^{(1-q)}\big(r(1 \pm \varepsilon)\big)$$

after an homothethy on ε. Now we can use our bootstrap technique and apply Lemma 2.7 with function $\Phi(r) = r^{-(b+2)}\overline{v}(r)$, because $\overline{v}$ satisfies (2.3). Hence we find

$$\overline{u}(r) \leqslant C r^{-(b+2)/q}\overline{v}^{1/q}(r),$$

and we get again (3.5). $\square$

Now we can prove an essential comparison property for the superharmonic component v, which shows the regularizing effect due to the subharmonic component u. In turn, it gives a remarkable punctual relation between u and v, which is valid for any $p, q > 0$.

Theorem 3.2. *Let $u, v \in C^2(B_1 \setminus \{0\})$ be any nonnegative solutions of system* (1.1) *with $p, q > 0$. Then there exists a constant $C > 0$ such that, for any $x \in B_{1/2} \setminus \{0\}$,*

$$C^{-1}\overline{v}\big(|x|\big) \leqslant v(x) \leqslant C\overline{v}\big(|x|\big), \tag{3.9}$$

and, consequently, v satisfies the Harnack inequality in $B_{1/2} \setminus \{0\}$. In particular, v always satisfies the estimate

$$v(x) = \mathrm{O}\big(|x|^{2-N}\big) \tag{3.10}$$

near 0*, and*

$$u(x) = \begin{cases} O(|x|^{a+2-(N-2)p}) & \text{if } p > (a+N)/(N-2), \\ O(|x|^{2-N}|\ln|x||) & \text{if } p = (a+N)/(N-2), \\ O(|x|^{2-N}) & \text{if } p < (a+N)/(N-2). \end{cases} \tag{3.11}$$

Moreover, there exists some $C > 0$ *such that*

$$u(x) \leqslant C\overline{u}(|x|) \tag{3.12}$$

in $B_{1/2} \setminus \{0\}$*, and, consequently,*

$$v(x) \geqslant C|x|^{b+2}u^q(x). \tag{3.13}$$

Proof. The minorization of v has been proved in Lemma 2.2. Here also we can suppose that $|x| > 0$ is small enough. Applying Lemma 2.3 to function v, we get

$$v(x) \leqslant C(N)\Big[|x|^{b+2}\Big(\max_{B(x,|x|/2)} u^q\Big) + \overline{v}(|x|)\Big].$$

But u is subharmonic. From Lemma 2.4 there exists another constant $C(N)$ such that

$$u(x) \leqslant C(N) \max_{[|x|/2,3|x|/2]} \overline{u}. \tag{3.14}$$

Then, from estimate (3.5),

$$v(x) \leqslant C\Big[|x|^{b+2} \max_{[|x|/4,9|x|/4]} \overline{u}^q + \overline{v}(|x|)\Big] \leqslant C\Big[\max_{[|x|/4,9|x|/4]} \overline{v} + \overline{v}(|x|)\Big].$$

Using (2.3), we finally deduce (3.9). It implies that v satisfies the Harnack inequality in $B_{1/2} \setminus \{0\}$. Clearly, (3.10) follows from (3.3), and (3.11) from (3.10), (3.14) and Lemma 2.6. From the Harnack inequality, there exist some constants $C_1, C_2 > 0$ such that

$$C_1\overline{v}^p(r) \leqslant \overline{v^p}(r) \leqslant C_2\overline{v}^p(r) \tag{3.15}$$

for $r \in (0, 1/2)$. As a consequence, $\overline{u}$ also satisfies the Harnack inequality. Indeed, we can write the equation for $\overline{u}$ under the form

$$\Delta\overline{u} = h\overline{u} \quad \text{with } h = r^a\overline{v^p}/\overline{u}.$$

Now, from (3.15) and (3.4),

$$h(r) \leqslant Cr^a\overline{v}^p(r)/\overline{u}(r) \leqslant Cr^{-2},$$

which, in turn, implies the Harnack inequality. Then, for any $r \in (0, 1/2)$ and any $\varepsilon \in (0, 1]$,

$$C^{-1}\overline{u}(r) \leqslant \overline{u}(r(1+\varepsilon)) \leqslant 2C\overline{u}(r), \tag{3.16}$$

and (3.12) follows from (3.14) and (3.16). Finally, we obtain (3.13) from (3.5), (3.9) and (3.12). $\square$

Remark 3.1. From (3.5), (3.4) and (3.15), we always have two symmetric relations in $(0, 1/2)$:

$$\overline{v}(r) \geqslant C r^{b+2} \overline{u}^q(r), \qquad \overline{u}(r) \geqslant C r^{a+2} \overline{v}^p(r). \tag{3.17}$$

Hence

$$\overline{u}(r) \geqslant C r^{a+2+(b+2)p} \overline{u}^{pq}(r), \qquad \overline{v}(r) \geqslant C r^{b+2+(a+2)q} \overline{v}^{pq}(r). \tag{3.18}$$

Notice also the inequalities for any $q > 0$,

$$C_1 \overline{u}^q(r) \leqslant \overline{u^q}(r) \leqslant C_2 \overline{u}^q(r), \tag{3.19}$$

for some other constants $C_1, C_2 > 0$. Indeed, this comes from (2.8) and (2.9), where we fix an ε and use (3.16).

Remark 3.2. On the one hand, inequality (3.13) implies that

$$\Delta u(x) \geqslant C |x|^{a+(b+2)p} u^{pq}(x) \tag{3.20}$$

in $B_{1/2} \setminus \{0\}$. That means that u is a subsolution of an equation of type (1.5), with still $Q = pq$, and now $\sigma = a + (b+2)p$. On the other hand, (3.19) and (3.17) imply that

$$-\Delta \overline{v}(r) = r^b \overline{u^q}(r) \geqslant C r^b \overline{u}^q(r) \geqslant C r^{b+(a+2)q} \overline{v}^{pq}(r) \tag{3.21}$$

in $(0, 1/2)$. That means that $\overline{v}$ is a supersolution of an equation of type (1.5), with $Q = pq$ and $\sigma = b + (a+2)q$.

Remark 3.3. If $q \geqslant 1$, we can prove the Harnack property for v in a shorter way. We apply (2.6) to the superharmonic function v and get

$$v(x) \geqslant c_N \int_{B(x,|x|/2)} \left[|z-x|^{2-N} - \left(|x|/2\right)^{2-N} \right] (-\Delta v)(z) \, \mathrm{d}z$$

$$\geqslant 2^{N-2} \left(2^{N-2} - 1 \right) c_N |x|^{2-N} \int_{B(x,|x|/4)} |z|^b u^q(z) \, \mathrm{d}z$$

$$\geqslant C |x|^{b+2-N} \int_{B(x,|x|/4)} u^q(z) \, \mathrm{d}z$$

in $B_{1/2} \setminus \{0\}$. But the function u^q is also subharmonic, since $q \geqslant 1$. Then also

$$u^q(x) \leqslant \frac{2^{-2N} |x|^{-N}}{|B|} \int_{B(x,|x|/4)} u^q(z) \, \mathrm{d}z \leqslant C |x|^{-(b+2)} v(x),$$

hence we find again (3.13). Then we write the equation satisfied by v under the form

$$-\Delta v = Hv \quad \text{with } H = |x|^b u^q / v,$$

and observe that $H(x) \leqslant C |x|^{-2}$. This implies the Harnack inequality in $B_{1/2} \setminus \{0\}$.

3.2. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. From (3.21), $\overline{v}$ is a supersolution of an equation of type (1.6), with $Q = pq > 1$ and $\sigma = b + (a+2)q$. Then $\sigma + 2 > 0$, from [2, Lemma A2], that is, $\xi > 0$. Here $pq > 1$, hence, from (3.18),

$$\overline{u}(r) \leqslant Cr^{-\gamma}, \qquad \overline{v}(r) \leqslant Cr^{-\xi}.$$

Then from (3.12) and (3.9), we get

$$u(x) \leqslant C|x|^{-\gamma}, \qquad v(x) \leqslant C|x|^{-\xi}, \tag{3.22}$$

and (1.17) follows from (3.10) and (3.22). Moreover, if $\gamma \leqslant N - 2$, then $\overline{u} = \mathrm{o}(r^{2-N})$, hence $\overline{u}$ is bounded, since it is subharmonic. And u is bounded from Lemma 2.4. $\square$

Proof of Theorem 1.2. From (3.21), $\overline{v}$ is a supersolution of an equation of type (1.6), with $Q = pq < 1$ and $\sigma = b + (a+2)q$. Then $Q < (N + \sigma)/(N-2)$, from [2, Lemma A2], that is, $\xi < N - 2$. And (1.18) and (1.19) follows directly from Theorem 3.2. $\square$

Remark 3.4. In the sublinear case $pq < 1$, relations (3.18) imply the estimates from below:

$$\overline{u}(r) \geqslant Cr^{-\gamma}, \qquad \overline{v}(r) \geqslant Cr^{-\xi}, \tag{3.23}$$

for $r \in (0, 1/2]$. From (3.9), we also deduce that

$$v(x) \geqslant C|x|^{-\xi} \tag{3.24}$$

in $B_{1/2} \setminus \{0\}$.

3.3. Further results in the superlinear case

Estimate (3.13) can be written under the equivalent form

$$|x|^{\gamma}u(x) \leqslant C\big(|x|^{\xi}v(x)\big)^{1/q}. \tag{3.25}$$

Let us give another way to obtain relations of the same type. As in [2], we look for a direct comparison between the two functions u and v. In [2], one uses a product of the solutions in order to get some nonexistence results. Here the same method applies with a quotient of the solutions and gives again the estimate (3.22):

Proposition 3.3. *Let $u, v \in C^2(B')$ be any nonnegative solutions of system* (1.1) *with $pq > 1$. Then for any $d \in (0, 1)$ with $d \leqslant 1/q$, there exists a constant $C_d > 0$, such that*

$$|x|^{\gamma}u(x) \leqslant C_d\big(|x|^{\xi}v(x)\big)^d \tag{3.26}$$

in $B_{1/2} \setminus \{0\}$. As a consequence, we find again the estimate

$$u(x) \leqslant C|x|^{-\gamma}.$$

Proof. Let us consider the function $f = u^m v^{1-m}$, for some $m > 1$, and compute its Laplacian in $B_{1/2} \setminus \{0\}$:

$$\Delta f = m(m-1)u^{m-2}v^{-1-m}|v\nabla u - u\nabla v|^2 + m|x|^a u^{m-1}v^{1-m+p} + (m-1)|x|^b u^{m+q}v^{-m}.$$

Then for any $k > 1$,

$$\Delta f \geqslant u^{m-1}v^{-m}\big(m|x|^a v^{p+1} + (m-1)|x|^b u^{q+1}\big)$$

$$\geqslant (m-1)|x|^{(a(k-1)+b)/k}u^{m-1+(q+1)/k}v^{-m+(p+1)(k-1)/k},$$

from the Hölder inequality. Let $d = (m-1)/m \in (0,1)$. If $d < 1/q < p$, we can choose

$$k = 1 + \frac{1 - dq}{p - d} = \frac{p + 1 - d(q+1)}{p - d},$$

which gives

$$\Delta f \geqslant \frac{d}{1-d}|x|^{a+(b-a)/k}f^\eta, \tag{3.27}$$

with

$$\eta = 1 + (pq - 1)(1 - d)/\big(p + 1 - d(q + 1)\big). \tag{3.28}$$

Then $\eta > 1$, and from the Osserman–Keller estimate,

$$f(x) = u^{1/(1-d)}(x)v^{-d/(1-d)}(x) \leqslant C|x|^{-(a+2+(b-a)/k)/(\eta-1)} = C|x|^{(d\xi-\gamma)/(1-d)}$$

in $B_{1/2} \setminus \{0\}$, where $C = C(N, p, q, a, b, d)$. And (3.26) holds for any $d < 1/q$ and, by continuity, also for $d = 1/q$. It implies that

$$\Delta u(x) \geqslant C|x|^{a-p\xi+p\gamma/d}u^{p/d}.$$

As $p/d \geqslant pq > 1$, we again deduce that

$$u(x) \leqslant C|x|^{-[a+2-p\xi+p\gamma/d]/(p/d-1)} = C|x|^{-\gamma}$$

in $B_{1/2} \setminus \{0\}$, from the Osserman–Keller estimate. $\square$

4. The convergences

4.1. Possible behaviours

Here we try to give the precise behaviour of the solutions according to the different values of the parameters. Let us define, as in [4],

$$l_1 = (N - 2)p - (a + N), \qquad l_2 = (N - 2)q - (b + N). \tag{4.1}$$

Notice the relations

$$l_1 + pl_2 = (pq - 1)(N - 2 - \gamma), \qquad l_2 + ql_1 = (pq - 1)(N - 2 - \xi). \tag{4.2}$$

The study will show that the behaviour of the couple (u, v) can present various types which can be divided in five categories when $\gamma, \xi \neq 0, N - 2$:

   (I) $(|x|^{-\gamma}, |x|^{-\xi})$;
   (II) $(|x|^{a+2-(N-2)p}, |x|^{2-N})$, $(|x|^{a+2}, 1)$;
   (III) $(|x|^{2-N}, |x|^{b+2-(N-2)q})$, $(1, |x|^{b+2})$;
   (IV) $(|x|^{2-N}, |x|^{2-N})$, $(1, |x|^{2-N})$, $(|x|^{2-N}, 1)$;
   (V) $(|x|^{2-N}|\ln|x||, |x|^{2-N})$, $(|x|^{2-N}|\ln|x||, 1)$.

As in [4], the system can admit *anisotropic solutions*, which makes difficult the question of convergences.

More precisely, the solutions u, v of type (I) can be both anisotropic. In that case, the problem of the convergences is still open. Consider any solution (u, v) satisfying an upper estimate $u(x) = \mathrm{O}(|x|^{-\gamma})$, $v(x) = \mathrm{O}(|x|^{-\xi})$. Let $(r, \theta) \in (0, +\infty) \times S^{N-1}$ be the spherical coordinates in $\mathbb{R}^N \backslash \{0\}$. The change of variables

$$u(x) = |x|^{-\gamma} U(t, \theta), \quad v(x) = |x|^{-\xi} V(t, \theta), \quad r = |x|, \ t = -\ln r, \tag{4.3}$$

leads to the autonous system in the cylinder $(0, +\infty) \times S^{N-1}$:

$$\begin{cases} U_{tt} - (N - 2 - 2\gamma)U_t + \Delta_{S^{N-1}}U + \gamma(\gamma - N + 2)U - V^p = 0, \\ V_{tt} - (N - 2 - 2\xi)V_t + \Delta_{S^{N-1}}V - \xi(N - 2 - \xi)V + U^q = 0. \end{cases} \tag{4.4}$$

We look at its behaviour when t tends to $+\infty$. As in [4], the stationary system associated to (4.4),

$$\begin{cases} \Delta_{S^{N-1}}\mathbf{U} + \alpha\mathbf{U} - \mathbf{V}^p = 0, \\ \Delta_{S^{N-1}}\mathbf{V} - \beta\mathbf{V} + \mathbf{U}^q = 0, \end{cases} \tag{4.5}$$

with $\alpha = \gamma(\gamma - N + 2)$ and $\beta = \xi(N - 2 - \xi)$, can admit nonconstant solutions for suitable positive values of α and β. We conjecture that the limit set G at infinity of the trajectories of (U, V) in $C^2(S^{N-1})$ is contained in the set of stationary solutions; and that, if $0 \in G$, then $G = \{0\}$, hence $u(x) = \mathrm{o}(|x|^{-\gamma})$, $v(x) = \mathrm{o}(|x|^{-\xi})$.

Concerning the solutions of type (II) and (III), *the situation is not symmetric by respect to u and v*. The following lemma shows that the behaviour of u is often more anisotropic than the behaviour of v.

Lemma 4.1. *Let $u, v \in C^2(B_1 \backslash \{0\})$ be any nonnegative solutions of system* (1.1)*, with $pq \neq 1$.*

   (i) *Assume that $u(x) = \mathrm{O}(|x|^{a+2-(N-2)p})$, $v(x) = \mathrm{O}(|x|^{2-N})$, and $(\xi - N + 2)(pq - 1) > 0$ and $\rho = [(N - 2)p - (a + 2)][(N - 2)p - (a + N)] \geqslant 0$. Then*

$$\lim_{x \to 0} |x|^{N-2} v(x) = C_2 \geqslant 0, \tag{4.6}$$

and, if $\rho > 0$, then

$$\lim_{x \to 0} \left[|x|^{(N-2)p-(a+2)} u(|x|, .) - \rho^{-1} C_2^p \right] \tag{4.7}$$

exists (in the uniform convergence topology on S^{N-1}), and it belongs to $\ker(\Delta_{S^{N-1}} + \rho I)$.

(ii) *Assume that* $u(x) = O(|x|^{2-N})$, $v(x) = O(|x|^{b+2-(N-2)q})$, *and* $(\gamma - N + 2)(pq - 1) > 0$ *and* $\eta = -((N-2)q - (b+2))((N-2)q - (b+N)) \geqslant 0$. *Then*

$$\lim_{x \to 0} |x|^{N-2} u(x) = C_1 \geqslant 0, \tag{4.8}$$

and, if $\eta > 0$, *then*

$$\lim_{x \to 0} |x|^{(N-2)q-(b+2)} v(x) = \eta^{-1} C_1^q. \tag{4.9}$$

(iii) *Assume that* $u(x) = O(|x|^{a+2})$, $v(x) = O(1)$, *and* $\nu = (a+2)(a+N) \geqslant 0$ *and* $\xi(pq-1) > 0$. *Then*

$$\lim_{x \to 0} v(x) = C_2' > 0, \tag{4.10}$$

and, if $\nu > 0$, *then*

$$\lim_{x \to 0} \left[|x|^{-(a+2)} u(|x|, .) - \nu^{-1} C_2' \right] \tag{4.11}$$

exists (in the uniform convergence topology on S^{N-1}*), and it belongs to* $\ker(\Delta_{S^{N-1}} + \nu I)$. *And* $v(x) - C_2' = O(|x|^{\xi(pq-1)})$.

(iv) *Assume that* $u(x) = O(1)$, $v(x) = O(|x|^{b+2})$, *and* $\mu = -(b+2)(b+N) \geqslant 0$ *and* $\gamma(pq-1) > 0$. *Then*

$$\lim_{x \to 0} u(x) = C_1' \geqslant 0, \tag{4.12}$$

and, if $\mu > 0$, *then*

$$\lim_{x \to 0} |x|^{-(b+2)} v(x) = \mu^{-1} C_1^q, \tag{4.13}$$

and $u(x) - C_1' = O(|x|^{\gamma(pq-1)})$.

Proof. In case (i), the proof of [4, Lemma 6.4] adapts: we define

$$u(x) = |x|^{a+2-(N-2)p} U'(t, \theta), \qquad v(x) = |x|^{2-N} V'(t, \theta), \tag{4.14}$$

and get

$$\begin{cases} U_{tt}' - [N - 2 - 2((N-2)p - (a+2))]U_t' + \Delta_{S^{N-1}} U' + \rho U' - V'^p = 0, \\ V_{tt}' + (N-2)V_t' + \Delta_{S^{N-1}} V' + e^{-(\xi-N+2)(pq-1)t} U'^q = 0, \end{cases} \tag{4.15}$$

and the exponential is negative. From [6], there is a constant $C_2 \geqslant 0$ such that $\|V'(t, .) - C_2\|_{C(S^{N-1})} = O(e^{-\alpha t})$ for some $\alpha > 0$. Then the function $W'(t, \theta) = U'(t, \theta) - \rho^{-1} C_2^p$ satisfies an equation of the form

$$W_{tt}' - [N - 2 - 2((N-2)p - (a+2))]W_t' + \Delta_{S^{N-1}} W' + \rho W' = \psi,$$

where $\|\psi(t,.)\|_{C(S^{N-1})} = \mathrm{O}(\mathrm{e}^{-\beta t})$ for some $\beta > 0$. Then we can apply the Simon theorem as in [6, Theorem 4.1], see also [7,17]. It implies that the function $W'(t,.)$ precisely converges to a solution of the stationary equation

$$\Delta_{S^{N-1}}\varpi + \rho\varpi = 0,$$

that means an element of $\ker(\Delta_{S^{N-1}} + \rho I)$.

In case of (ii), we define

$$u(x) = |x|^{2-N}U''(t,\theta), \qquad v(x) = |x|^{b+2-(N-2)q}V''(t,\theta),$$

and now get

$$\begin{cases} U''_{tt} + (N-2)U''_t + \Delta_{S^{N-1}}U'' - \mathrm{e}^{-(\gamma-N+2)(pq-1)t}V''^p = 0, \\ V''_{tt} - [N-2-2((N-2)q-(b+2))]V''_t + \Delta_{S^{N-1}}V'' - \rho V'' + U''^q = 0. \end{cases}$$

Then there is a constant $C_1 \geqslant 0$ such that $\|U''(t,.) - C_1\|_{C(S^{N-1})} = \mathrm{O}(\mathrm{e}^{-\alpha t})$ for some $\alpha > 0$. But now the function $W''(t,\theta) = V''(t,\theta) - \rho^{-1}C_1^q$ converges to a solution of the stationary equation

$$\Delta_{S^{N-1}}\varpi - \rho\varpi = 0,$$

that is 0. We get (iii) and (iv) in a similar way. $\quad\square$

In case of types (III), (IV) or (V), the two solutions are isotropic. In those cases, we shall use the results of [4], which adapt with no difficulty.

Lemma 4.2. *Let $u, v \in C^2(B_1 \setminus \{0\})$ be any nonnegative solutions of system* (1.1), *with $pq \neq 1$.*

(i) *Assume that $u(x) + v(x) = \mathrm{O}(|x|^{2-N})$ near 0, and $p < (N+a)/(N-2)$ or $q < (b+N)/(N-2)$. Then*

$$\lim_{x\to 0} |x|^{N-2}u(x) = C_1 \geqslant 0, \tag{4.16}$$

or

$$\lim_{x\to 0} |x|^{N-2}v(x) = C_2 \geqslant 0. \tag{4.17}$$

(ii) *Assume that $u(x) = \mathrm{O}(|x|^{2-N})$, $v(x) = \mathrm{O}(1)$, and $a + N > 0$ and $(N-2)q - (b+2) < 0$. Then*

$$\lim_{x\to 0} |x|^{N-2}u(x) = C_1 \geqslant 0, \qquad \lim_{x\to 0} v(x) = C_2' > 0, \tag{4.18}$$

and $v(x) - C_2' = \mathrm{O}(|x|^{b+2-(N-2)q})$.

(iii) *Assume that $u(x) = \mathrm{O}(1)$, $v(x) = \mathrm{O}(|x|^{2-N})$, and $b + N > 0$ and $(N-2)p - (a+2) < 0$. Then*

$$\lim_{x \to 0} u(x) = C_1' \geqslant 0, \qquad \lim_{x \to 0} |x|^{N-2} v(x) = C_2 \geqslant 0, \tag{4.19}$$

and $u(x) - C_1' = \mathrm{O}(|x|^{a+2-(N-2)p})$.

(iv) *Assume that $u(x) + v(x) = \mathrm{O}(1)$, and $a + 2 > 0$, or $b + 2 > 0$. Then*

$$\lim_{x \to 0} u(x) = C_1' \geqslant 0, \tag{4.20}$$

or

$$\lim_{x \to 0} v(x) = C_2' > 0, \tag{4.21}$$

and $u(x) - C_1' = \mathrm{O}(|x|^{a+2})$ (or $v(x) - C_2' = \mathrm{O}(|x|^{b+2})$).

(v) *Assume $u(x) = \mathrm{O}(|x|^{2-N}|\ln|x||)$, $v(x) = \mathrm{O}(|x|^{2-N})$, and $p = (a+N)/(N-2)$ and $q < (b+N)/(N-2)$, then*

$$\lim_{x \to 0} |x|^{N-2} v(x) = C_2 \geqslant 0, \qquad \lim_{x \to 0} |x|^{N-2}|\ln|x||^{-1} u(x) = C_2^p/(N-2), \tag{4.22}$$

and $u(x) - C_2^p/(N-2)|x|^{2-N}|\ln|x|| = \mathrm{O}(|x|^{2-N})$.

(vi) *Assume $u(x) = \mathrm{O}(|x|^{2-N}|\ln|x||)$, $v(x) = \mathrm{O}(1)$, and $a + N = 0$ and $q < (b+2)/(N-2)$, then*

$$\lim_{x \to 0} v(x) = C_2' > 0, \qquad \lim_{x \to 0} |x|^{N-2}|\ln|x||^{-1} u(x) = C_2'^p/(N-2). \tag{4.23}$$

(vii) *Assume $u(x) = \mathrm{O}(|x|^{2-N})$, $v(x) = \mathrm{O}(|\ln|x||)$, and $a + N > 0$ and $q = (b+2)/(N-2)$, then*

$$\lim_{x \to 0} |x|^{N-2} u(x) = C_1 \geqslant 0, \qquad \lim_{x \to 0} |\ln|x||^{-1} v(x) = C_1^q/(N-2). \tag{4.24}$$

(viii) *Assume $u(x) = \mathrm{O}(1)$, $v(x) = \mathrm{O}(|\ln|x||)$, and $a + 2 > 0$ and $b + 2 = 0$, then*

$$\lim_{x \to 0} u(x) = C_1 \geqslant 0, \qquad \lim_{x \to 0} |\ln|x||^{-1} v(x) = C_1^q/(N-2), \tag{4.25}$$

and $v(x) - (C_1^q/(N-2))|\ln|x|| = \mathrm{O}(1)$.

4.2. The superlinear case

First, we give some general properties of convergence:

Proposition 4.3. *Let $u, v \in C^2(B_1 \setminus \{0\})$ be any nonnegative solutions of system (1.1), with $pq > 1$. Assume that $q < (b+N)/(N-2)$ and $\gamma > N - 2$.*

(i) *Suppose $q > (b + 2)/(N - 2)$ and*

$$v(x) = O(|x|^{b+2-(N-2)q}).$$

Hence $a + N > 0$, and either (4.8) and (4.9) hold with $C_1 > 0$, or $u(x) = O(1)$.

(ii) *Suppose $q = (b + 2)/(N - 2)$ and*

$$v(x) = O(|\ln|x||).$$

Hence again $a + N > 0$. Then (4.25) holds with $C_1 > 0$, or $u(x) = O(1)$.

(iii) *Now suppose $q < (b + 2)/(N - 2)$ and*

$$v(x) = O(1).$$

If $a + N < 0$, then (4.10) and (4.11) hold. If $a + N = 0$, then (4.23) holds. If $a + N > 0$, then (4.18) holds; if $C_1 = 0$, then $a + 2 > 0$, and (4.20) and (4.21) hold with $C_1' > 0$, or (4.10) and (4.11) hold.

Proof. (i) $q > (b+2)/(N-2)$. Then $a+N > 0$; indeed, $\xi > 0$, hence $(a+2)q > -(b+2) \geqslant -(N-2)q$. Now

$$\Delta\overline{u}(r) \leqslant Cr^{a+(b+2)p-(N-2)pq}$$

in $(0, 1/2)$, and $a + (b + 2)p - (N - 2)pq + N = (pq - 1)(\gamma - N + 2) > 0$. Then $u(x) = O(|x|^{2-N})$ from Lemma 2.6 and (3.12). Now Lemma 4.1(ii) applies since $\gamma > N - 2$ and (4.8) follows. If $C_1 > 0$, then Lemma 4.1(ii) gives (4.9). If $C_1 = 0$, then $\overline{u}$ is bounded, hence also u from (3.12).

(ii) $q = (b + 2)/(N - 2)$. Here again we get $a + N > 0$. Then

$$\Delta\overline{u}(r) \leqslant Cr^a|\ln r|^p \leqslant C_\varepsilon r^{a-\varepsilon}$$

in $(0, 1/2)$, for any $\varepsilon > 0$, hence again $u(x) = O(|x|^{2-N})$ from Lemma 2.6 and from (3.12). Then (4.25) follows from Lemma 4.2(vii). If $C_1 = 0$, then u is bounded as above.

(iii) $q < (b + 2)/(N - 2)$. Here

$$\Delta\overline{u}(r) \leqslant Cr^a$$

in $(0, 1/2)$. From Lemma 2.6, we distinguish three cases:

- Either $a + N < 0$; then $u(x) = O(|x|^{a+2})$. Then (4.10) and (4.11) follow from Lemma 4.1(iii).
- Or $a + N = 0$; then $u(x) = O(|x|^{2-N}|\ln|x||)$, and we get (4.23) from Lemma 4.2(vi).
- Or $a + N > 0$; then $u(x) = O(|x|^{2-N})$. Then (4.18) holds from Lemma 4.2(ii). If $C_1 = 0$, then $u(x) = O(1)$. It implies $a + 2 > 0$ from Lemma 2.6. And we get (4.20) and (4.21) from Lemma 4.2(iv), with $u(x) = C_1' + O(|x|^{a+2})$. If $C_1' = 0$, then (4.10) and (4.11) hold from Lemma 4.1(iii). $\square$

Proposition 4.4. *Let $u, v \in C^2(B_1 \setminus \{0\})$ be any nonnegative solutions of system* (1.1)*. Assume*

$$u(x) = \mathrm{O}(1), \qquad v(x) = \mathrm{o}(|x|^{2-N}),$$

and $b + N > 0$ and $\gamma > 0$. Then, either

   (i) $b + 2 < 0$ *and* (4.12) *and* (4.13) *hold; or*
  (ii) $b + 2 = 0$ *and* (4.25) *holds; or*
 (iii) $b + 2 > 0$ *and* (4.20) *and* (4.21) *hold, and if $C_1' = 0$, then* (4.10) *and* (4.11) *hold.*

Proof. (i) $b + 2 < 0$. Hence $a + 2 > 0$, since $\gamma > 0$. Then $\mu = -(b+2)(b+N) \geqslant 0$, and Lemma 4.1(iv) applies and (4.12) follows. We deduce (4.13). If $C_1' = 0$, then $u(x) = \mathrm{O}(|x|^{\varepsilon_0})$, with $\varepsilon_0 = \gamma(pq - 1) > 0$. But any estimate $u(x) = \mathrm{O}(|x|^\varepsilon)$ implies the estimate

$$-\Delta v(x) \leqslant C |x|^{b + q\varepsilon}$$

in $\mathcal{D}'(B_{1/2})$, because $|x|^{b+q\varepsilon} \in L^1(B_{1/2})$, since $b + N > 0$. Then $v(x) = \mathrm{O}(|x|^{b+2+q\varepsilon})$ if $b + 2 + \varepsilon q < 0$, and $v(x) = \mathrm{O}(|\ln|x||)$ if $b + 2 + \varepsilon q = 0$, and $v(x) = \mathrm{O}(1)$ if $b + 2 + \varepsilon q > 0$, from the maximum principle. Till $b + 2 + \varepsilon q < 0$, it gives

$$\Delta \overline{u}(r) \leqslant C r^{a + (b+2)p + pq\varepsilon}$$

in $(0, 1/2)$, hence $u(x) = \mathrm{O}(|x|^{a+2+(b+2)p+pq\varepsilon}) = \mathrm{O}(|x|^{\varepsilon_0 + pq\varepsilon})$, from Lemma 2.6, since $\varepsilon_0 + pq\varepsilon > 0$. Now observe that the sequence defined by $\varepsilon_n = \varepsilon_0 + pq\varepsilon_{n-1}$, satisfies $\lim_{n \to +\infty} \varepsilon_n = +\infty$. Then by modifying slightly ε_0 if necessary, we deduce that $v(x) = \mathrm{O}(1)$ after a finite number of steps. Then

$$\Delta \overline{u}(r) \leqslant C r^a$$

in $(0, 1/2)$, hence $u(x) = \mathrm{O}(|x|^{a+2})$ from Lemma 2.6. At last, (4.10) and (4.11) hold from Lemma 4.1(iii).
   (ii) $b + 2 = 0$. Hence again $a + 2 > 0$. Then Lemma 4.2(viii) gives (4.25). Moreover, if $C_1 = 0$, then $v(x) = \mathrm{O}(1)$, and $u(x) = \mathrm{O}(|x|^{a+2})$ from Lemma 2.6. Then (4.10) and (4.11) hold as above.
   (iii) $b + 2 > 0$. Then $u(x) + v(x) = \mathrm{O}(1)$. It implies $a + 2 > 0$ from Lemma 2.6. And we get (4.20), (4.21) from Lemma 4.2(iv), with $u(x) = C_1' + \mathrm{O}(|x|^{a+2})$. If $C_1' = 0$, then (4.10) and (4.11) hold from Lemma 4.1(iii). $\quad\square$

Now we can give the different types of behaviour according to the values of γ and ξ. First, we look at the solutions which have a power upper estimate strictly smaller than the one of the particular solution given in (1.14).

Proposition 4.5. *Assume $pq > 1$ and $0 < \xi < N - 2$, and $\gamma > N - 2$. Let $u, v \in C^2(B_1 \setminus \{0\})$ be any nonnegative solutions of system* (1.1)*, such that*

$$u(x) = \mathrm{O}(|x|^{-\gamma+\varepsilon}), \quad or \quad v(x) = \mathrm{O}(|x|^{-\xi+\varepsilon}), \quad for some \ \varepsilon > 0. \tag{4.26}$$

Then $q < (b + N)/(N - 2)$ and $p > (a + N)/(N - 2)$, and Proposition 4.3 applies. Moreover, if $q \geqslant (b + 2)/(N - 2)$ and u is bounded, then Proposition 4.4 applies.

Proof. From (4.2), we have $l_1 + pl_2 < 0 < l_2 + ql_1$, hence $l_2 < 0 < l_1$, that is, $q < (b + N)/(N - 2)$ and $p > (a + N)/(N - 2)$. From (1.17) we have $v(x) = \mathrm{O}(|x|^{-\xi})$, hence $v(x) = \mathrm{o}(|x|^{2-N})$. Then the constant C_2 defined in (3.2) is zero. Now notice that the assumption $u(x) = \mathrm{O}(|x|^{-\gamma+\varepsilon})$ implies

$$-\Delta v(x) \leqslant C|x|^{b-\gamma q+q\varepsilon} = C|x|^{-2-\xi+q\varepsilon}$$

in $\mathcal{D}'(B_{1/2})$, because $|x|^{-2-\xi+q\varepsilon} \in L^1(B_{1/2})$, since $N - 2 - \xi > 0$. Hence

$$v(x) = \begin{cases} \mathrm{O}(|x|^{-\xi+q\varepsilon}) & \text{if } \xi > q\varepsilon, \\ \mathrm{O}(|\ln|x||) & \text{if } \xi = q\varepsilon, \\ \mathrm{O}(1) & \text{if } \xi < q\varepsilon, \end{cases}$$

from the maximum principle. And any estimate $v(x) = \mathrm{O}(|x|^{-\xi+\varepsilon'})$ implies

$$\Delta\overline{u}(r) \leqslant Cr^{a-\xi p+p\varepsilon'} = Cr^{-2-\gamma+p\varepsilon'}$$

in $(0, 1/2)$. Consequently,

$$u(x) = \begin{cases} \mathrm{O}(|x|^{-\gamma+p\varepsilon'}) & \text{if } p\varepsilon' < \gamma - N + 2, \\ \mathrm{O}(|x|^{2-N}|\ln|x||) & \text{if } p\varepsilon' = \gamma - N + 2, \\ \mathrm{O}(|x|^{2-N}) & \text{if } p\varepsilon' > \gamma - N + 2, \end{cases}$$

from Lemma 2.6 and (3.12). We can start from the assumption $u(x) = \mathrm{O}(|x|^{-\gamma+\varepsilon})$, with ε small enough. Consider $\varepsilon_0 = \varepsilon$ and $\varepsilon_0' = q\varepsilon$, and define $\varepsilon_n = p\varepsilon_{n-1}'$ and $\varepsilon_n' = q\varepsilon_n$. Then, by induction, $u(x) = \mathrm{O}(|x|^{-\gamma+\varepsilon_n})$ and $v(x) = \mathrm{O}(|x|^{-\xi+\varepsilon_n'})$, till $p\varepsilon_n' < \gamma - N + 2$, and $\xi > q\varepsilon_n$. But $\varepsilon_n = pq\varepsilon_{n-1}$, hence $\lim_{n\to+\infty}\varepsilon_n = +\infty$. Then by modifying sligthly ε_0 if necessary, we find after a finite number of steps that either $u(x) = \mathrm{O}(|x|^{2-N})$, or $v(x) = \mathrm{O}(1)$. Now if $u(x) = \mathrm{O}(|x|^{2-N})$, then

$$-\Delta v(x) \leqslant C|x|^{b-(N-2)q}$$

in $\mathcal{D}'(B_{1/2})$, because $|x|^{b-(N-2)q} \in L^1(B_{1/2})$, since $q < (b + N)/(N - 2)$. Hence in any case we have the estimate

$$v(x) = \mathrm{O}(|x|^{b+2-(N-2)q}) + \mathrm{O}(|\ln|x||), \tag{4.27}$$

and Proposition 4.3 applies. When $q \geqslant (b + 2)/(N - 2)$ and u is bounded, then Proposition 4.4 applies, since $b + N > 0$ and $\gamma > N - 2 > 0$. $\quad\square$

Proposition 4.6. *Assume $pq > 1$ and $0 < \xi < N - 2$, and $\gamma < 0$. Let $u, v \in C^2(B_1 \setminus \{0\})$ be any nonnegative solutions of system* (1.1). *Assume that*

$$u(x) = \mathrm{O}(|x|^{-\gamma+\varepsilon}), \quad or \quad v(x) = \mathrm{O}(|x|^{-\xi+\varepsilon}), \quad \text{for some small } \varepsilon > 0. \tag{4.28}$$

Then $b + 2 < 0$, $a + 2 > 0$, and (4.10) and (4.11) hold.

Proof. We have $\gamma < 0 < \xi$, hence $b + 2 < 0$ and $a + 2 > 0$. As above, the assumption $u(x) = O(|x|^{-\gamma+\varepsilon})$ implies $v(x) = O(|x|^{-\xi+q\varepsilon})$, till $q\varepsilon < \xi$. Now $u(x)$ tends to 0. From Lemma 2.6, the estimate $v(x) = O(|x|^{-\xi+\varepsilon'})$ still implies that $u(x) = O(|x|^{-\gamma+p\varepsilon'})$, since $-\gamma + p\varepsilon' > 0$. Consider $\varepsilon_0 = \varepsilon$ and $\varepsilon'_0 = q\varepsilon$, and $\varepsilon_n = p\varepsilon'_{n-1}$, $\varepsilon'_n = q\varepsilon_n$. Then $u(x) = O(|x|^{-\gamma+\varepsilon_n})$ and $v(x) = O(|x|^{-\xi+\varepsilon'_n})$, till $q\varepsilon_n < \xi$. But $\lim \varepsilon_n = +\infty$, hence we deduce that $v(x) = O(1)$ after a finite number of steps. Then again $u(x) = O(|x|^{a+2})$ from Lemma 2.6, since $a + 2 > 0$. As above, we find (4.10) and (4.11) from Lemma 4.1(iii). $\square$

Now we consider the cases where the particular solution (1.14) does not exist.

Proposition 4.7. *Assume $pq > 1$ and $\xi > N - 2$. Let $u, v \in C^2(B_1 \setminus \{0\})$ be any nonnegative solutions of system* (1.1).

   (i) *Assume $p > (a + N)/(N - 2)$. Then $q < (b + N)/(N - 2)$ and $\gamma > N - 2$. And either* (4.6) *and* (4.7) *hold with $C_2 > 0$, or Proposition 4.3 applies. And if $q \geqslant (b + 2)/(N - 2)$ and u is bounded, then Proposition 4.4 applies.*

  (ii) *Assume $p < (a + N)/(N - 2)$ and $q < (b + N)/(N - 2)$. Then* (4.16) *and* (4.17) *hold. If $C_2 > 0$ and $C_1 = 0$, then $p < (a + 2)/(N - 2)$, and* (4.19) *holds; and if $C'_1 = 0$, then* (4.6) *and* (4.7) *hold. If $C_2 = 0$, then Proposition 4.3 applies (with $a + N > 0$). And if $q \geqslant (b + 2)/(N - 2)$ and u is bounded, then Proposition 4.4 applies.*

 (iii) *Assume $p < (a + N)/(N - 2)$ and $q \geqslant (b + N)/(N - 2)$. Then either* (4.19) *holds for some $C_2 > 0$, and if $C'_1 = 0$, in particular, if $b + N \leqslant 0$, then* (4.6) *and* (4.7) *hold. Or $C_2 = 0$. Then either $b + N > 0$, and Proposition 4.4 applies. Or $b + N \leqslant 0$, and* (4.10) *and* (4.11) *hold.*

 (iv) *Assume $p = (a + N)/(N - 2)$. Then $q < (b + N)/(N - 2)$, and* (4.22) *holds. If $C_2 = 0$, then Proposition 4.3 applies (with $a + N > 0$). And if $q \geqslant (b + 2)/(N - 2)$ and u is bounded, then Proposition 4.4 applies.*

Proof. We have estimates (3.10) and (3.11) from Theorem 3.2.

   (i) $p > (a+N)/(N-2)$. Here $l_1 > 0$; and $l_2 + ql_1 < 0$, hence $l_2 < 0$, that is, $q < (b+N)/(N-2)$, and $l_1 + pl_2 < l_1(1 - pq) < 0$, that is, $\gamma > N - 2$. As $v(x) = O(|x|^{2-N})$, we have $u(x) = O(|x|^{a+2-(N-2)p})$. Then we find (4.6) and (4.7) from Lemma 4.1(i). If $C_2 = 0$, then

$$-\Delta v(x) \leqslant C|x|^{b+(a+2)q-(N-2)pq} = C|x|^{(\xi-N+2)(pq-1)-N}$$

in $\mathcal{D}'(B_{1/2})$, because $|x|^{(\xi-N+2)(pq-1)-N} \in L^1(B_{1/2})$. Applying the maximum principle, it follows that $v(x) = O(|x|^{2-N+\varepsilon_0}) + O(1)$, with $\varepsilon_0 = (\xi - N + 2)(pq - 1)$ if $\varepsilon_0 \neq N - 2$; and $v(x) = O(|\ln|x||)$ if $\varepsilon_0 = N - 2$. But from Lemma 2.6, any estimate $v(x) = O(|x|^{2-N+\varepsilon})$ implies

$$u(x) = \begin{cases} O(|x|^{a+2+p\varepsilon-(N-2)p}) & \text{if } p > (a + N + p\varepsilon)/(N - 2), \\ O(|x|^{2-N}) & \text{if } p < (a + N + p\varepsilon)/(N - 2), \\ O(|x|^{2-N}|\ln|x||) & \text{if } p = (a + N + p\varepsilon)/(N - 2). \end{cases}$$

In the first case,

$$-\Delta v(x) \leqslant C|x|^{b+(a+2)q-(N-2)pq+pq\varepsilon} = C|x|^{\varepsilon_0-N+pq\varepsilon} \tag{4.29}$$

in $\mathcal{D}'(B_{1/2})$, hence $v(x) = \mathrm{O}(|x|^{2-N+\varepsilon_0+pq\varepsilon}) + \mathrm{O}(1)$ if $\varepsilon_0 + pq\varepsilon \neq N - 2$. But the sequence defined from ε_0 by $\varepsilon_n = \varepsilon_0 + pq\varepsilon_{n-1}$ tends to $+\infty$. Hence, changing slightly ε_0 if necessary, after a finite number of steps we find that either $u(x) = \mathrm{O}(|x|^{2-N})$, or $v(x) = \mathrm{O}(1)$. In any case we have estimate (4.27). And we get the conclusions of Proposition 4.3, and of Proposition 4.4 in case $q \geqslant (b+2)/(N-2)$ and u is bounded.

(ii) $p < (a+N)/(N-2)$ and $q < (b+N)/(N-2)$. Then we have $u(x) + v(x) = \mathrm{O}(|x|^{2-N})$ from (3.10) and (3.11). First, notice that $l_1, l_2 < 0$, hence also $l_1 + pl_2 < 0$, so that $\gamma > N - 2$. Then we deduce (4.16) and (4.17) from Lemma 4.2(i). If $C_1 = 0$, then $u(x) = \mathrm{O}(1)$, since u is subharmonic. If $C_2 > 0$ and $C_1 = 0$, then

$$\Delta \overline{u}(r) \geqslant Cr^{a-(N-2)p}$$

in $(0, 1/2)$, hence $a + 2 - (N-2)p > 0$ from Lemma 2.6; and (4.19) holds from Lemma 4.2(ii), since $b + N > 0$. Moreover, if $C_1' = 0$, then $u(x) = \mathrm{O}(|x|^{a+2-(N-2)p})$ and (4.6) and (4.7) hold. If $C_2 = 0$, then

$$v(x) = \begin{cases} \mathrm{O}(|x|^{b+2-(N-2)q}) + \mathrm{O}(1) & \text{if } q \neq (b+2)/(N-2), \\ v(x) = \mathrm{O}(|\ln|x||) & \text{if } q = (b+2)/(N-2). \end{cases}$$

Then Proposition 4.3 applies after noticing that here $a + N > 0$. And Proposition 4.4 applies when $q \geqslant (b+2)/(N-2)$ and u is bounded.

(iii) $p < (a+N)/(N-2)$ and $q \geqslant (b+N)/(N-2)$. We still have $u(x) + v(x) = \mathrm{O}(|x|^{2-N})$. We know that $r^{2-N}\overline{u}(r)$ has a finite limit. It is necessary 0, since $|x|^b u^q \in L^1(B_{1/2})$, and $q \geqslant (b+N)/(N-2)$, and $r^b \overline{u}^q \geqslant Cr^b \overline{u}^q$ from (3.19). Hence $u(x) = \mathrm{O}(1)$ and $v(x) = \mathrm{O}(|x|^{2-N})$. Now v satisfies (3.2) for some $C_2 \geqslant 0$, and $\overline{u}$ has a finite limit $C_1' \geqslant 0$.

First, suppose that $C_2 > 0$. Then $a + 2 - (N-2)p > 0$ from Lemma 2.6. Either $b + N > 0$. Then (4.19) holds. If $C_1' = 0$, then $u(x) = \mathrm{O}(|x|^{a+2-(N-2)p})$ as above. And (4.6) and (4.7) hold from Lemma 4.1(i). Or $b + N \leqslant 0$. Then $C_1' = 0$. Indeed, if $C_1' > 0$, then $r^b \overline{u}^q \geqslant Cr^b \overline{u}^q \geqslant Cr^{-N}$; this is impossible because $|x|^b \overline{u}^q(|x|) \in L^1(B_{1/2})$. Then we obtain $u(x) = \mathrm{O}(|x|^{a+2-(N-2)p})$ from Lemma 2.6 and (3.12). And (4.6) and (4.7) hold again.

Now suppose that $C_2 = 0$. Either $b + N > 0$. Then

$$-\Delta v(x) \leqslant C|x|^b$$

in $\mathcal{D}'(B_{1/2})$, hence $v(x) = \mathrm{O}(|x|^{b+2}) + \mathrm{O}(|\ln|x||)$, so that $v(x) = \mathrm{o}(|x|^{2-N})$. Now $\gamma q = \xi + b + 2 > b + N$, hence $\gamma > 0$. Then Proposition 4.4 applies. Or $b + N \leqslant 0$, and $C_1' = 0$, as above. Then we still have $a + 2 - (N-2)p > 0$. Indeed, $\xi > (N-2)$ implies that $a + 2 - (N-2)p > -(b+N)/q$. It implies that $u(x) = \mathrm{O}(|x|^{a+2-(N-2)p})$ from Lemma 2.6. Let $\varepsilon_0 = a + 2 - (N-2)p$. If $u(x) = \mathrm{O}(|x|^{\varepsilon})$, for some $\varepsilon \geqslant \varepsilon_0$, then

$$-\Delta v(x) \leqslant C|x|^{b+\varepsilon q}$$

in $\mathcal{D}'(B_{1/2})$, because $b + N + \varepsilon q > 0$. Hence we have $v(x) = \mathrm{O}(|x|^{b+2+\varepsilon q})$, till $b + 2 + \varepsilon q < 0$. As a consequence,

$$\Delta \overline{u}(r) \leqslant Cr^{a+(b+2+\varepsilon q)p}$$

in $(0, 1/2)$. Observing that $a + 2 + (b + 2 + \varepsilon q)p \geqslant \gamma(pq - 1) > 0$, we deduce that $u(x) = O(|x|^{\varepsilon pq + \gamma(pq-1)})$ from Lemma 2.6. Defining from ε_0 the sequence $\varepsilon_n = \varepsilon_{n-1}pq + \gamma(pq - 1)$, we have $\lim \varepsilon_n = +\infty$. It follows that $v(x) = O(1)$ after a finite number of steps, and then $u(x) = O(|x|^{a+2})$, since $a + 2 > 0$. Now (4.10) and (4.11) follow from Lemma 4.1(iii).

(iv) $p = (a + N)/(N - 2)$. Then $l_1 = 0$, $l_2 + ql_1 < 0$, hence $l_2 < 0$, which means $q < (b + N)/(N - 2)$. Here we have $v(x) = O(|x|^{2-N})$ and $u(x) = O(|x|^{2-N} |\ln |x||)$. Then Lemma 4.2(v) applies and gives (4.22). If $C_2 = 0$, then $u(x) = O(|x|^{2-N})$, and we can apply Propositions 4.3 and 4.4 as in the second case. $\square$

Proposition 4.8. *Assume* $pq > 1$ *and* $0 < \gamma \leqslant N - 2$, $0 < \xi \leqslant N - 2$, *and* γ *or* $\xi \neq N - 2$. *Let* $u, v \in C^2(B_1 \setminus \{0\})$ *be any nonnegative solutions of system* (1.1). *Then* $b + N > 0$, *and Proposition* 4.4 *applies.*

Proof. We have $u(x) = O(|x|^{-\gamma})$. If $\gamma < N - 2$, hence $u(x) = O(1)$, since u is subharmonic. If $\gamma = N - 2$, then $\xi < N - 2$, hence $v(x) = O(|x|^{-\xi}) = o(|x|^{2-N})$ from Theorem 1.1. And $r^{2-N}\overline{u}(r)$ has a finite limit $C_1 \geqslant 0$. Let us prove that $C_1' = 0$. If $C_1' > 0$, then from (3.20),

$$\Delta \overline{u}(r) \geqslant Cr^{-N}$$

in $(0, 1/2)$. It implies that $\overline{u}(r) \geqslant Cr^{2-N}|\ln r|$ from Lemma 2.6, which is impossible. Then we get in any case $u(x) = O(1)$. And v satisfies (3.2) with $C_2 = 0$, since $v(x) = o(|x|^{2-N})$. In this case, $0 < \gamma q = b + 2 + \xi \leqslant b + N$, since $\xi \leqslant N - 2$, hence $b + N > 0$. Then we can apply Proposition 4.4. $\square$

Proposition 4.9. *Assume* $pq > 1$ *and* $\gamma = 0$, *and* $0 < \xi < N - 2$. *Let* $u, v \in C^2(B_1 \setminus \{0\})$ *be any nonnegative solutions of system* (1.1). *Then*

$$u(x) = O\big(|\ln |x||^{-1/(pq-1)}\big), \qquad v(x) = O\big(|x|^{-\xi}|\ln |x||^{-q/(pq-1)}\big). \tag{4.30}$$

Proof. Here again u is bounded, and $\overline{u}(r)$ has a finite limit $C_1' \geqslant 0$. The change of variables (4.3) gives

$$\begin{cases} U_{tt} - (N - 2)U_t + \Delta_{S^{N-1}}U - V^p = 0, \\ V_{tt} - (N - 2 - 2\xi)V_t + \Delta_{S^{N-1}}V - \xi(N - 2 - \xi)V + U^q = 0. \end{cases}$$

From (3.20), there exists a constant $C > 0$, such that, for large t,

$$-\overline{U}_{tt} + (N - 2)\overline{U}_t + C\overline{U}^{pq} \leqslant 0.$$

If $C_1' > 0$, there exists another constant $C > 0$ such that $e^{-(N-2)t}(\overline{U}_t(t) + C)$ is nondecreasing; it tends to 0, since $\overline{U}_t$ is bounded from Schauder estimates. Then $\overline{U}(t) + Ct$ is nonincreasing, which is impossible. Then $C_1' = 0$. Now the equation

$$-y_{tt} + (N - 2)y_t + Cy^{pq} = 0$$

admits a solution Y such that $Y(t) = ((N - 2)/C(pq - 1))^{1/(Q-1)}t^{-1/(pq-1)}(1 + o(1))$, see [19]. But for any $\varepsilon \in (0, 1]$, the function $\varepsilon\overline{U}$ is again a subsolution of this equation. Choosing ε small enough, we deduce that $\varepsilon\overline{U} \leqslant Y$. This proves that $\overline{U}(t) = O(t^{-1/(pq-1)})$, that is, $u(x) = O(|\ln |x||^{-1/(pq-1)})$. Then

$$-\Delta v(x) \leqslant C|x|^b|\ln |x||^{-q/(pq-1)} = C|x|^{-2-\xi}|\ln |x||^{-q/(pq-1)}$$

in $\mathcal{D}'(B_{1/2})$, because $N-2-\xi > 0$. It follows that $v(x) = \mathrm{O}(|x|^{-\xi}|\ln|x||^{-q/(pq-1)})$, from the maximum principle. $\quad\square$

Proposition 4.10. *Assume $pq > 1$ and $\xi = N - 2$, and $\gamma < 0$ or $\gamma > N - 2$. Let $u, v \in C^2(B_1 \setminus \{0\})$ be any nonnegative solutions of system* (1.1). *Then*

$$u(x) = \mathrm{O}(|x|^{-\gamma}|\ln|x||^{-p/(pq-1)}), \qquad v(x) = \mathrm{O}(|x|^{2-N}|\ln|x||^{-1/(pq-1)}). \tag{4.31}$$

Proof. The change of variables (4.3) gives

$$\begin{cases} U_{tt} - (N - 2 - 2\gamma)U_t + \Delta_{S^{N-1}}U - \gamma(N - 2 - \gamma)U - V^p = 0, \\ V_{tt} + (N - 2)V_t + \Delta_{S^{N-1}}V + U^q = 0. \end{cases}$$

Then $(\mathrm{e}^{(N-2)t}\overline{V}_t)_t \leqslant 0$, hence $\overline{V}_t \leqslant C\mathrm{e}^{-(N-2)t}$, and $\overline{V}$ has a finite limit. If it is positive, then $\overline{u}(r) \geqslant Cr^{-\gamma}$ from (3.4); in turn, $r^b\overline{u}^q(r) \geqslant Cr^{-2-\xi} = Cr^{-N}$, which is impossible. Hence $\overline{V}$ tends to 0, hence $v(x) = \mathrm{o}(|x|^{2-N})$. Consequently $u(x) = \mathrm{o}(|x|^{-\gamma})$ from (3.5) and (3.12). From (3.21), there exists a constant $C > 0$, such that, for large t,

$$\overline{V}_{tt} + (N - 2)\overline{V}_t + C\overline{V}^{pq} \leqslant 0.$$

This implies that $\overline{V}(t) = \mathrm{O}(t^{-1/(pq-1)})$ at infinity, see, for example, [6, Theorem 5.1]. We deduce that $v(x) = \mathrm{O}(|x|^{2-N}|\ln|x||^{-1/(pq-1)})$ from (3.9). Then

$$\Delta\overline{u}(r) \leqslant Cr^{a-p\xi}|\ln r|^{-p/(pq-1)} = Cr^{-2-\gamma}|\ln r|^{-p/(pq-1)}$$

in $(0, 1/2)$, hence $u(x) = \mathrm{O}(|x|^{-\gamma}|\ln|x||^{-p/(pq-1)})$, from (3.12) and [4, Lemma 2.3]. $\quad\square$

Remark 4.1. In the critical case $\xi = N - 2$, $\gamma = 0$, we find $u(x) = \mathrm{O}(|\ln|x||^{-1/(pq-1)})$ as in Proposition 4.9, and $v(x) = \mathrm{O}(|x|^{2-N}|\ln|x||^{-1/(pq-1)})$ as in Proposition 4.10. But these estimates are not optimal, as in [2, Theorem 5.1]. We conjecture that

$$u(x) = \mathrm{O}(|\ln|x||^{-(p+1)/(pq-1)}), \qquad v(x) = \mathrm{O}(|x|^{2-N}|\ln|x||^{-(q+1)/(pq-1)}).$$

4.3. The sublinear case

Now we describe the behaviour according to the value of $p - (a + N)/(N - 2)$.

Proposition 4.11. *Assume $pq < 1$ with $p > (a + N)/(N - 2)$, and $\gamma \neq N - 2$, $\xi \neq 0$. Let $u, v \in C^2(B_1 \setminus \{0\})$ be any nonnegative solutions of system* (1.1). *Then* (4.6) *and* (4.7) *hold. Now, if $C_2 = 0$, then*

(i) *either $\xi > 0$ and $\gamma > N - 2$, and*

$$C^{-1}r^{-\gamma} \leqslant \overline{u}(r) \leqslant Cr^{-\gamma}, \qquad C^{-1}|x|^{-\xi} \leqslant v(x) \leqslant C|x|^{-\xi}; \tag{4.32}$$

(ii) *or $\xi < 0$. Then either $a + N < 0$ and (4.10) and (4.11) hold. Or $a + N = 0$, and (4.23) holds. Or $a + N > 0$ and $\gamma < N - 2$ and (4.18) holds; if $C_1 > 0$, then $q < (b+2)/(N-2)$; if $C_1 = 0$, then $\gamma < 0$, $a + 2 > 0$, and (4.20) and (4.21) hold; if $C'_1 > 0$, then $b + 2 > 0$; if $C'_1 = 0$, then (4.10) and (4.11) hold;*

(iii) *or $\xi > 0$ and $\gamma < N - 2$, and either (4.8) and (4.9) hold with $C_1 > 0$. Or (4.20) and (4.21) hold as above, and if $C'_1 = 0$, then (4.10) and (4.11) hold.*

Proof. We have $v(x) = \mathrm{O}(|x|^{2-N})$ and $u(x) = \mathrm{O}(|x|^{a+2-(N-2)p})$ from Theorem 1.2. And $l_2 + ql_1 < 0$ and $l_1 > 0$ imply $l_2 < 0$, that is, $q < (b+N)/(N-2)$. Now Lemma 4.1(i) applies, because $\varepsilon_0 = (\xi - N + 2)(pq - 1) = -(l_2 + ql_1) > 0$. More precisely,

$$v(x) - C_2|x|^{2-N} = \begin{cases} \mathrm{O}(|x|^{2-N+\varepsilon_0}) + \mathrm{O}(1) & \text{if } \varepsilon_0 \neq N - 2, \\ \mathrm{O}(|\ln|x||) & \text{if } \varepsilon_0 = N - 2, \end{cases}$$

from [6]. Now suppose that $C_2 = 0$.

– Either $\varepsilon_0 > N - 2$, hence $\xi < 0$ and $v(x) = \mathrm{O}(1)$.
– Or $\varepsilon_0 < N - 2$, hence $v(x) = \mathrm{O}(|x|^{2-N+\varepsilon_0})$.
– Or $\varepsilon_0 = N - 2$, hence $v(x) = \mathrm{O}(|x|^{2-N+\varepsilon_0-\varepsilon'})$ for any $\varepsilon' > 0$. As in Proposition 4.7, any estimate $v(x) = \mathrm{O}(|x|^{2-N+\varepsilon_n})$ implies that

$$u(x) = \begin{cases} u(x) = \mathrm{O}(|x|^{a+2+p\varepsilon_n-(N-2)p}) & \text{if } \lambda_n = (N-2)p - (a+N+p\varepsilon_n) > 0, \\ \mathrm{O}(|x|^{2-N}) & \text{if } \lambda_n < 0, \\ \mathrm{O}(|x|^{2-N}|\ln|x||) & \text{if } \lambda_n = 0. \end{cases}$$

But here the sequence defined from ε_0 by $\varepsilon_n = \varepsilon_0 + pq\varepsilon_{n-1}$ tends to $\varepsilon_0/(1-pq)$. Hence $2 - N + \varepsilon_n$ tends to $-\xi$, and the sequence λ_n decreases to $\lambda = \gamma - N + 2$. As a consequence, if $\gamma < N - 2$ or $\xi < 0$, we find $v(x) = \mathrm{O}(1)$ or $u(x) = \mathrm{O}(|x|^{2-N})$. If $\gamma > N - 2$ and $\xi > 0$, then $v(x) = \mathrm{O}(|x|^{-\xi-\varepsilon})$ for any $\varepsilon > 0$.

(i) $\gamma > N - 2$ and $\xi > 0$. Then we have in fact $v(x) = \mathrm{O}(|x|^{-\xi})$. Indeed, any estimate $v(x) \leqslant C_{\varepsilon_n}|x|^{2-N+\varepsilon_n}$ in $B_{1/2} \setminus \{0\}$ implies more precisely

$$u(x) \leqslant C|x|^{2-N} + CC_n C_{\varepsilon_n}^p |x|^{a+2+p\varepsilon_n-(N-2)p}$$

with $C_n = 1/(\lambda_n(\lambda_n + N - 2)) \leqslant 1/\lambda^2$, see [4, Lemma 2.3]. Hence, with a new constant $C > 0$,

$$u(x) \leqslant C(1 + C_{\varepsilon_n}^p)|x|^{a+2+p\varepsilon_n-(N-2)p}.$$

And then

$$v(x) \leqslant C'_n C^q (1 + C_{\varepsilon_n}^p)^q |x|^{2-N+\varepsilon_0+pq\varepsilon_n} + C,$$

from the maximum principle, with $C'_n = 1/(\varepsilon_0 + pq\varepsilon_n)(N - 2 - \varepsilon_0 - pq\varepsilon_n) \leqslant 1/\varepsilon_0\xi$. Then

$$v(x) \leqslant C_{\varepsilon_n}|x|^{2-N+\varepsilon_n},$$

with $C_{\varepsilon_n} = C(1 + C_{\varepsilon_{n-1}}^{pq})$, for another C. It follows that $v(x) \leqslant C|x|^{-\xi}$, because the sequence (C_{ε_n}) is convergent. Then $u(x) = O(|x|^{-\gamma})$ from (3.5) and (3.12), and we deduce (4.32) from (3.23).

(ii) $\gamma < N - 2$ or $\xi < 0$ and $v(x) = O(1)$. Now we apply Lemma 2.6. Either $a + N < 0$, and then $u(x) = O(|x|^{a+2})$. Then Lemma 4.1(iii) applies, because $\xi(pq - 1) < 0$, and we get (4.10) and (4.11). Or $a + N \geqslant 0$, and $\gamma + 2 - N \leqslant \gamma + a + 2 = p\xi < 0$, hence $\gamma < N - 2$. If $a + N = 0$, then $u(x) = O(|x|^{2-N}|\ln|x||)$, and $q < (b+2)/(N-2)$, since $(a+2)q + b + 2 > 0$. Then we get (4.23) from Lemma 4.2(vi). Now consider the case $a + N > 0$; then $u(x) = O(|x|^{2-N})$.

– Either $q < (b + 2)/(N - 2)$. Then Lemma 4.2(ii) applies and gives (4.18) with $v(x) - C_2' = O(|x|^{b+2-(N-2)q})$. Now suppose that $C_1 = 0$. Then $u(x) = O(1)$, hence $\gamma \leqslant 0$ from (3.23). It implies $a + 2 > 0$ from Lemma 2.6. Then $\gamma < 0$, since $\gamma = p\xi - (a + 2)$. Then there is a constant $C_1' \geqslant 0$ such that $u(x) = C_1' + O(|x|^{a+2})$, from Lemma 4.1(iv). If $C_1' > 0$, then $b + 2 > 0$ from (3.5), since v is bounded. In the same way there is a constant $C_2' > 0$ such that $v(x) = C_2' + O(|x|^{b+2})$, from Lemma 4.1(iv). If $C_1' = 0$, then $u(x) = O(|x|^{a+2})$, with $v(x) = O(1)$. We get again (4.10) and (4.11) from Lemma 4.1(iii), because $\xi(pq - 1) > 0$.

– Or $q \geqslant (b + 2)/(N - 2)$. We know that $r^{N-2}\,\overline{u}(r)$ has a finite limit C_1. Let us prove that $C_1 = 0$. If $C_1 > 0$, then

$$-\Delta\overline{v}(r) \geqslant Cr^{b-(N-2)q} \geqslant Cr^{-2}$$

in $(0, 1/2)$. This is impossible because v is bounded. Hence $u(x) = O(1)$, and we conclude as above.

(iii) $\gamma < N - 2$ or $\xi < 0$ and $u(x) = O(|x|^{2-N}|\ln|x||)$. Then $a + N \geqslant 0$, from (3.1) and (3.4).

– Either $a + N = 0$. Then $q < (b + 2)/(N - 2)$, and

$$-\Delta v(x) \leqslant C|x|^{b-(N-2)q-\varepsilon}$$

in $\mathcal{D}'(B_{1/2})$, for any $\varepsilon > 0$, since $q < (b + N)/(N - 2)$. Hence $v(x) = O(1)$ from the maximum principle, and we return to the preceeding case.

– Or $a + N > 0$, and $v(x) = O(|x|^{(b+2-(N-2)q-\varepsilon)}) + O(1)$ from the maximum principle. Either $q < (b+2)/(N-2)$, hence $v(x) = O(1)$, and we again return to the second case. Or $q > (b+2)/(N-2)$, then $v(x) = O(|x|^{(b+2-(N-2)q-\varepsilon)})$, and $u(x) = O(|x|^{2-N})$ from Lemma 2.6, since $\gamma < N - 2$. And (4.8) and (4.9) hold from Lemma 4.1(ii), because $(\gamma - N + 2)(pq - 1) > 0$. If $C_1 = 0$, then $u(x) = O(1)$, and we return to the second case. $\quad\square$

Remark 4.2. In the critical cases $\xi = 0$ or $\gamma = N - 2$, our proofs give the estimate $v(x) = O(|x|^{-\varepsilon})$ for any $\varepsilon > 0$, and, consequently, $u(x) = O(|x|^{-\gamma-\varepsilon})$ from (3.5).

(i) In the case $\xi = 0$, $\gamma > N - 2$, we also have the lower estimates

$$\overline{u}(r) \geqslant Cr^{-\gamma}|\ln r|^{p/(1-pq)}, \qquad v(x) \geqslant C|\ln|x||^{1/(1-pq)}. \tag{4.33}$$

Indeed, we have

$$-\Delta\overline{v}(r) \geqslant Cr^{-2}\overline{v}^{pq}(r)$$

in $(0, 1/2)$, from (3.21), hence $\overline{v}(r) \geqslant C|\ln r|^{1/(1-pq)}$ from [2, Lemma A2], and (4.33) follow from (3.4) and (3.9). We conjecture that the upper estimates

$$u(x) = \mathrm{O}(|x|^{-\gamma}|\ln|x||^{p/(1-pq)}), \qquad v(x) = \mathrm{O}(|\ln|x||^{1/(1-pq)})$$

are true.

(ii) In the case $\xi > 0$, $\gamma = N - 2$, we conjecture that

$$u(x) = \mathrm{O}(|x|^{2-N}|\ln|x||^{1/(1-pq)}), \qquad v(x) = \mathrm{O}(|x|^{-\xi}|\ln|x||^{q/(1-pq)}).$$

(iii) In the case $\xi = 0$, $\gamma = N - 2$, we conjecture that

$$u(x) = \mathrm{O}(|x|^{2-N}|\ln|x||^{(p+1)/(1-pq)}), \qquad v(x) = \mathrm{O}(|x|^{-\xi}|\ln|x||^{(q+1)/(1-pq)}).$$

Proposition 4.12. *Assume $pq < 1$ with $p < (a + N)/(N - 2)$, and $\xi \neq 0$ if $\gamma < 0$. Let $u, v \in C^2(B')$ be any nonnegative solutions of system (1.1). Then $\gamma < N - 2$.*

(i) *Suppose $q < (b + N)/(N - 2)$. Then (4.16) and (4.17) hold. If $C_2 > 0$ and $C_1 = 0$, then $\gamma < 0$, $p < (a + 2)/(N - 2)$, and (4.19) holds; if $C_1' = 0$, then (4.6) and (4.7) hold. If $C_1 > 0$ and $C_2 = 0$, then either $q > (b + 2)/(N - 2)$ and (4.8) and (4.9) hold, or $q < (b + 2)/(N - 2)$ and (4.18) holds. If $C_1 = C_2 = 0$, then $\gamma < 0$, $a + 2 > 0$, and:*

  – *Either $b + 2 < 0$, and (4.12) and (4.13) hold; if $C_1' = 0$, then either $\xi > 0$ and (4.32) holds, or (4.10) and (4.11) hold.*
  – *Either $b + 2 > 0$, and (4.20) and (4.21) hold; if $C_1' = 0$, then (4.10) and (4.11) hold.*
  – *Or $b + 2 = 0$, and (4.25) holds.*

(ii) *Suppose $q \geqslant (b + N)/(N - 2)$. If $b + N > 0$, then (4.19) holds. If $C_2 > 0$ and $C_1' = 0$, then (4.6) holds. If $C_2 = 0$, we conclude as above. If $b + N \leqslant 0$, then either $\xi > 0$ and (4.32) holds, or (4.10) and (4.11) hold.*

Proof. First, notice that here $\gamma < N-2$; indeed, $l_1 < 0$, $l_2 + ql_1 < 0$, hence $l_1 + pl_2 < l_1(1 - pq) < 0$. We have $u(x) + v(x) = \mathrm{O}(|x|^{2-N})$ from Theorem 1.2. Hence $r^{N-2}\overline{u}(r)$ has a finite limit $C_1 \geqslant 0$, $r^{N-2}\overline{v}(r)$ has a finite limit $C_2 \geqslant 0$, and v satisfies (3.2).

(i) $q < (b + N)/(N - 2)$. We get (4.16) and (4.17) from [4, Lemma 6.3]. Moreover,

$$u(x) - C_1|x|^{2-N} = \begin{cases} \mathrm{O}(|x|^{a+2-(N-2)p}) + \mathrm{O}(1) & \text{if } p \neq (a + 2)/(N - 2), \\ \mathrm{O}(|\ln|x||) & \text{if } p = (a + 2)/(N - 2), \end{cases} \tag{4.34}$$

and

$$v(x) - C_2|x|^{2-N} = \begin{cases} \mathrm{O}(|x|^{b+2-(N-2)q}) + \mathrm{O}(1) & \text{if } q \neq (b + 2)/(N - 2), \\ \mathrm{O}(|\ln|x||) & \text{if } q = (b + 2)/(N - 2). \end{cases} \tag{4.35}$$

Now suppose that $C_1 = 0$ or $C_2 = 0$. We consider each case separately.

– Either $C_2 > 0$ and $C_1 = 0$. Then u is bounded, hence $\gamma \leqslant 0$ from (3.23); in fact, $\gamma < 0$. Indeed, if $\gamma = 0$, then (4.32) hold, which contradicts $C_2 > 0$. And we obtain $p < (a+2)/(N-2)$ from Lemma 2.6. Then (4.19) holds from Lemma 4.1, and $u(x) - C_1' = \mathrm{O}(|x|^{a+2-(N-2)p})$. If $C_1' = 0$, then (4.6) and (4.7) hold, because $(\xi - N + 2)(pq - 1) > 0$.

– Or $C_1 > 0$ and $C_2 = 0$. If $q > (b+2)/(N-2)$, then $v(x) = \mathrm{O}(|x|^{b+2-(N-2)q})$ and we get (4.8), since $(\gamma - N + 2)(pq - 1) > 0$. If $q \leqslant (b+2)/(N-2)$, then $v(x) = \mathrm{O}(1)$. And (4.18) holds, because $b + N > 0$. If $q = (b+2)/(N-2)$, then

$$-\Delta\overline{v}(r) \geqslant Cr^{-2}$$

in $(0, 1/2)$, which is impossible.

– Or $C_1 = C_2 = 0$. Then $u(x) = \mathrm{O}(1)$, hence again $\gamma \leqslant 0$, and, in fact, $\gamma < 0$. Indeed, if $\gamma = 0$, then (4.32) hold, but then

$$\Delta\overline{u}(r) \geqslant Cr^{-2}$$

in $(0, 1/2)$, which contradicts Lemma 2.6. Then we get also $a + 2 > 0$ from Lemma 2.6. And

$$-\Delta v(x) \leqslant C|x|^b$$

in $\mathcal{D}'(B_{1/2})$, since $b + N > 0$, hence $v(x) = \mathrm{O}(|x|^{b+2}) + \mathrm{O}(|\ln|x||)$. First, suppose that $b + 2 < 0$, then $v(x) = \mathrm{O}(|x|^{b+2})$, and (4.12) and (4.13) hold, since $\gamma(pq - 1) > 0$. Moreover, $u(x) = C_1' + \mathrm{O}(|x|^{\gamma(pq-1)})$. If $C_1' = 0$, then $u(x) = \mathrm{O}(|x|^{\varepsilon_0})$ with $\varepsilon_0 = \gamma(pq - 1)$. But any estimate $u(x) = \mathrm{O}(|x|^{\varepsilon})$ again implies that

$$-\Delta v(x) \leqslant |x|^{b+\varepsilon q}$$

in $\mathcal{D}'(B_{1/2})$, hence $v(x) = \mathrm{O}(|x|^{b+2+\varepsilon q}) + \mathrm{O}(1)$. And any estimate $v(x) = \mathrm{O}(|x|^{b+2+\varepsilon q})$ in turn implies $u(x) = \mathrm{O}(|x|^{a+2+(b+2)p+\varepsilon pq})$ from Lemma 2.6, since $u(x)$ tends to 0. But the sequence defined from ε_0 by $\varepsilon_n = a + 2 + (b+2)p + \varepsilon_{n-1}pq$ tends to $-\gamma$. After a finite number of steps, we arrive to $u(x) = \mathrm{O}(|x|^{-\gamma+\varepsilon'})$ for any $\varepsilon' > 0$, or $v(x) = \mathrm{O}(1)$. In the first case, we can prove as in Proposition 4.11 that, in fact, $u(x) = \mathrm{O}(|x|^{-\gamma})$, since $\xi \neq 0$. This implies estimate (4.32), and necessarily $\xi > 0$. In the second case, we find again (4.10) and (4.11). Now assume that $b + 2 > 0$. Then we obtain (4.20) and 4.21, and (4.10) and (4.11) in case $C_1' = 0$. At last, assume that $b + 2 = 0$. Then we get (4.25) from Lemma 4.2(viii).

(ii) $q \geqslant (b+N)/(N-2)$. Then $C_1 = 0$, because $|x|^b u^q \in L^1(B_{1/2})$. Hence u is bounded. If $b + N > 0$, then (4.19) holds from [4, Lemma 6.3]. If $C_2 > 0$, and $C_1' = 0$, then (4.6) and (4.7) hold. If $C_2 = 0$, we conclude as above. If $b + N \leqslant 0$, then $l_2 > (N-2)q$, hence $l_1 < 2 - N$, since $l_2 + ql_1 < 0$. Then $\gamma < 0$, because $pl_2 + l_1 < (1 - pq)l_1 < (2 - N)(1 - pq)$. Necessarily $\lim_{x \to 0} u(x) = 0$, since $|x|^b u^q \in L^1(B_{1/2})$. We conclude as above that either $u(x) = \mathrm{O}(|x|^{-\gamma})$, hence $\xi > 0$, and (4.32) holds, or $v(x) = \mathrm{O}(1)$, and (4.10) and (4.11) hold. $\quad\square$

Proposition 4.13. *Assume $pq < 1$ with $p = (a+N)/(N-2)$. Let $u, v \in C^2(B_1 \backslash \{0\})$ be any nonnegative solutions of system* (1.1). *Then* (4.22) *holds. If $C_2 = 0$, then either $q > (b+2)/(N-2)$ and* (4.8) *and* (4.9) *hold, or $q < (b+2)/(N-2)$, and* (4.18) *holds, or $q = (b+2)/(N-2)$, and* (4.24) *holds.*

Proof. We have $u(x) = \mathrm{O}(|x|^{2-N}|\ln|x||)$ and $v(x) = \mathrm{O}(|x|^{2-N})$ from Theorem 1.2. As in Proposition 4.7, we conclude to (4.22). If $C_2 = 0$, then $u(x) = \mathrm{O}(|x|^{2-N})$, and we have $q < (b+N)/(N-2)$. If $q > (b+2)/(N-2)$, then $v(x) = \mathrm{O}(|x|^{b+2-(N-2)q})$, and we get (4.8) and (4.9) from Lemma 4.1(ii). If $q < (b+2)/(N-2)$, then $v(x) = \mathrm{O}(1)$, and we get (4.18) from Lemma 4.2(ii). If $q = (b+2)/(N-2)$, then $v(x) = \mathrm{O}(|\ln|x||)$, and (4.24) follows from Lemma 4.2(vii). $\quad\square$

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