Singularities in elliptic systems with absorption terms

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Abstract

We study the limit behaviour near the origin of nonnegative solutions of the semilinear elliptic system

$$\begin{cases} -\Delta u + |x|^a v^p = 0, \\ -\Delta v + |x|^b u^q = 0, \end{cases} \quad \text{in } \mathbb{R}^N \ (N \geq 3),$$

where $p, q, a, b \in \mathbb{R}$, with $p, q > 0$, $pq \neq 1$. Our main results are a priori estimates in the superlinear case $pq > 1$ and the sublinear one $pq < 1$. They essentially relie on fine properties of subharmonic functions . We also point out that the behaviour of the solutions is most often anisotropic.

Contents

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1 Introduction

This paper deals with the nonnegative solutions u, v of the semilinear elliptic system in $\mathbb{R}^N$ $(N \geq 3)$ with absorption terms:

$$\begin{cases} -\Delta u + |x|^a v^p = 0, \\ -\Delta v + |x|^b u^q = 0, \end{cases} \tag{1.1}$$

where $p, q, a, b \in \mathbb{R}$ with $p, q > 0$, and $pq \neq 1$. We study the behaviour of the solutions near an isolated singularity $x = 0$. This also provides the behaviour at infinity by Kelvin transform. Our results apply in particular to the nonnegative subharmonic solutions of the biharmonic equation

$$-\Delta^2 u + |x|^b u^q = 0, \tag{1.2}$$

with $q \neq 1$, by taking $p = 1$ and $a = 0$. In the sequel, we suppose that u, v are defined in $B' = B \setminus \{0\}$, where $B = \left\{x \in \mathbb{R}^N \,||x| \leq 1\right\}$.

Our study extends the results relative to the scalar case of the nonnegative solutions of equation

$$-\Delta w + |x|^\sigma w^Q = 0 \tag{1.3}$$

where $Q > 0$, $Q \neq 1$. Equation (1.3) was studied in detail in the superlinear case $Q > 1$ in [20], [21], [8], [24], and more recently in the sublinear case $Q < 1$ in [5], and in [4] when $N = 2$. For any $Q \neq 1$, defining

$$\Gamma = (\sigma + 2)/(Q - 1), \tag{1.4}$$

it admits a particular radial solution :

$$w^*(x) = C^* |x|^{-\Gamma}, \qquad C^* = (\Gamma(\Gamma - N + 2))^{1/(Q-1)}, \tag{1.5}$$

whenever $C^* > 0$, which is a guide-line of the study. This nonlinear effect fights with the linear one, due to the Laplacian. In the superlinear case $Q > 1$, all the subsolutions satisfy the Keller-Osserman estimate near the origin

$$w(x) \leq C \ |x|^{-\Gamma}, \tag{1.6}$$

where $C = C(N, Q, \sigma)$. And the solutions are asymptotically radial. When $Q \geq (N + \sigma)/(N - 2)$, then w^* does not exist, and the singularity is removable, which means that the solutions stay bounded near the origin. In the sublinear case $Q < 1$, the linear effect can dominate the nonlinear one. The solutions, and more generally the subharmonic supersolutions, of (1.3) satisfy the estimate

$$w(x) \leq \begin{cases} C \ \max(|x|^{-\Gamma}, |x|^{2-N})) & \text{if } \ Q \neq (N+\sigma)/(N-2), \\ C \ |x|^{2-N} \left|\ln|x|\right|^{1/(1-Q)}) & \text{if } \ Q = (N+\sigma)/(N-2), \end{cases} \tag{1.7}$$

for some $C > 0$. Moreover the solutions may present an anisotropic behaviour.

The case of the system appears to be quite more complicated: for example, it will be shown that the behaviour of one of the functions u, v can be of linear type, and the behaviour of the other one of nonlinear type. Moreover, the anisotropic character of the solutions is much more frequent. Technically, the maximum principle no longer holds. Thus the construction of supersolutions, essential in [8], is no more available. But the fundamental property of subharmonicity of the solutions is preserved. It will be the essential tool of our proofs. As in the scalar case, our study is governed by the existence of a radial particular solution (u^*, v^*) given by

$$u^*(x) = A^* |x|^{-\gamma}, \ v^*(x) = B^* |x|^{-\xi}, \tag{1.8}$$

where

$$\gamma = [(b+2)p + a + 2]/(pq-1), \quad \xi = [(a+2)q + b + 2]/(pq-1), \tag{1.9}$$

and

$$\begin{cases} A^* = [\gamma(\gamma+2-N)(\xi(\xi+2-N))^p]^{1/(pq-1)}, \\ B^* = [\xi(\xi+2-N)(\gamma(\gamma+2-N))^q]^{1/(pq-1)}, \end{cases} \tag{1.10}$$

whenever $\gamma(\gamma+2-N) > 0$ and $\xi(\xi+2-N) > 0$. Notice the relations

$$\gamma + a + 2 = p\xi, \qquad \xi + b + 2 = q\gamma. \tag{1.11}$$

We shall distinguish between the *superlinear case* $pq > 1$, and the *sublinear case* $pq < 1$. In the sequel, the same letter C denotes some positive constants which may depend on u, v, unless otherwise stated.

We give in Section 2 the key lemmas of our paper. For any function $w \in C^2(B')$, we denote by

$$\overline{w}(r) = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} w(r, \theta) \, d\theta \tag{1.12}$$

its mean value on the sphere of center 0 and radius r. In order to establish a priori estimates for system (1.1), a simple idea is to obtain first the corresponding estimates for the mean values $\overline{u}, \overline{v}$, by using the Jensen inequality:

$$\overline{w^Q} \geq \overline{w}^Q, \ \text{ if } Q > 1, \qquad \overline{w^Q} \leq \overline{w}^Q, \ \text{ if } Q < 1. \tag{1.13}$$

Then analogous estimates follow for u, v by using subharmonicity, as for example in the scalar sublinear case in [5]. This method rapidly fails when for example $p > 1$ and $q < 1$. Our first argument relies in a finer property of the mean-value of the subharmonic functions. We compare the value $w(x)$ in some point $x \in B'$ to the mean value $\overline{w}\left[|x|(1 \pm \varepsilon)\right]$ at some radius close to $|x|$. This allows us to cover the cases where the Jensen inequality is no longer valid. Thus we are reduced to a system of inequalities for $\overline{u}, \overline{v}$, involving the variables r and $r(1 \pm \varepsilon)$, which we call *shifted inequalities.* The second argument of our proofs is a delicate technique of bootstrap as ε tends to 0, in order to treat the shifted radial system as a non-shifted one.

In **Section 3** we give the a priori estimates in the superlinear case. Some recent results of [25] give sufficient conditions of removability for the solutions, under the restrictive assumption $p \geq 1$ and $q \geq 1$. Our main result is an extension of Keller-Osserman estimates to system (1.1) when $pq > 1$, *without any other restriction .* We prove the following.

Theorem 1.1 *Let us assume $pq > 1$. Let $u, v \in C^2(B')$ be any nonnegative subsolutions of (1.1) , that is*

$$\begin{cases} -\Delta u + |x|^a v^p \leq 0, \\ -\Delta v + |x|^b u^q \leq 0. \end{cases} \tag{1.14}$$

Then

$$u(x) \leq C \ |x|^{-\gamma}, \qquad v(x) \leq C \ |x|^{-\xi}, \qquad \textit{near the origin,} \tag{1.15}$$

where $C = C(a, b, p, q, N)$.

With these estimates , we can follow again and extend to the general case the removability results of [25] .

Corollary 1.2 *Under the assumptions of theorem (1.1), if*

$$\begin{cases} \textit{either} \qquad \max(\gamma, \xi) \leq N - 2, \\ \textit{or} \qquad [\gamma \leq N - 2 \quad \textit{and} \quad p \geq (a+2)/(N-2)], \\ \textit{or} \qquad [\xi \leq N - 2 \quad \textit{and} \; q \geq (b+2)/(N-2)], \end{cases} \tag{1.16}$$

then u and v are bounded near the origin.

In **Section 4** we give the a priori estimates in the sublinear case. As in the scalar case, the situation appears to be richer.

Theorem 1.3 *Let us assume $pq < 1$. Let $u, v \in C^2(B')$ be any nonnegative subharmonic supersolutions of solutions of (1.1), that is*

$$\begin{cases} 0 \leq \Delta u \leq |x|^a v^p, \\ 0 \leq \Delta v \leq |x|^b u^q. \end{cases} \tag{1.17}$$

Then, up to the change from u, p, a into v, q, b,

i) if $\min(\gamma, \xi) > N - 2$, *then*

$$u(x) \leq C \ |x|^{-\gamma}, \qquad v(x) \leq C \ |x|^{-\xi}, \tag{1.18}$$

ii) if $\xi < N - 2$ *and* $p > (N + a)/(N - 2)$, *then*

$$u(x) \leq C \ |x|^{a+2-(N-2)p}, \qquad v(x) \leq C \ |x|^{2-N}, \tag{1.19}$$

iii) if $p < (N + a)/(N - 2)$ *and* $q < (N + b)/(N - 2)$, *then*

$$u(x) + v(x) \leq C \ |x|^{2-N}, \tag{1.20}$$

and in the critical cases,

iv) if $p = (N + a)/(N - 2)$ *and* $q < (N + b)/(N - 2)$, *then*

$$u(x) \leq C \ |x|^{2-N} |\ln |x||, \qquad v(x) \leq C \ |x|^{2-N}. \tag{1.21}$$

v) if $\xi = N - 2 < \gamma$, *then*

$$u(x) \leq C \ |x|^{a+2-(N-2)p} |\ln |x||^{p/(1-pq)}, \quad v(x) \leq C \ |x|^{2-N} |\ln |x||^{1/(1-pq)}, \tag{1.22}$$

vi) if $\xi = N - 2 = \gamma$, *then*

$$u(x) \leq C \ |x|^{2-N} |\ln |x||^{(p+1)/(1-pq)}, \quad v(x) \leq C \ |x|^{2-N} |\ln |x||^{(q+1)/(1-pq)}, \tag{1.23}$$

In **Section 5**, we look for particular solutions of the system (1.1) under the form

$$u(x) = |x|^{-\gamma}\mathbf{U}(\theta), \quad v(x) = |x|^{-\xi}\mathbf{V}(\theta), \qquad \theta \in S^{N-1}. \tag{1.24}$$

It leads to the stationary system

$$\left\{ \begin{array}{l} \Delta_{S^{N-1}}\mathbf{U} + \gamma(\gamma + 2 - N)\mathbf{U} - \mathbf{V}^p = 0, \\ \Delta_{S^{N-1}}\mathbf{V} + \xi(\xi + 2 - N)\mathbf{V} - \mathbf{U}^q = 0. \end{array} \right. \tag{1.25}$$

We show that system (1.25) can admit nonconstant positive solutions $\mathbf{U}, \mathbf{V}$, in addition to the constant ones A^*, B^*, *even in the superlinear case* .

Theorem 1.4 *Assume that* $\alpha = \gamma(\gamma + 2 - N) > 0$ *and* $\beta = \xi(\xi + 2 - N) > 0$. *Let* λ_1, λ_2 *be the two roots of equation*

$$\lambda^2 - (\alpha + \beta)\lambda - (pq - 1)\alpha\beta = 0, \tag{1.26}$$

with $\lambda_1 < \lambda_2$. *Then for fixed* α *a branch of bifurcation* $(\mathbf{U}(\boldsymbol{\beta}), \mathbf{V}(\boldsymbol{\beta}))$ *appears near* (A^*, B^*) *in system (1.25), at each time* λ_2 *crosses a nonzero eigenvalue of* $-\Delta_{S^{N-1}}$ *if* $pq > 1$, *at each time* λ_1 *or else* λ_2 *crosses such an eigenvalue if* $pq < 1$.

Hence system (1.1) can admit *anisotropic positive solutions*. This phenomenon is new in the superlinear case, and Theorem 1.4 shows that anisotropy is still more commun in the sublinear one.

In **Section 6**, we take up the delicate question of precising the behaviour of the solutions near 0. We show the great complexity of the possible behaviours. Excluding for the sake of simplicity the critical cases, they can be divided into three categories:

(*i*) $\left(|x|^{-\gamma}, |x|^{-\xi}\right)$;
(*ii*) $\left(|x|^{a+2-(N-2)p}, |x|^{2-N}\right), \left(|x|^{a+2}, 1\right), \left(|x|^{2-N}, |x|^{b+2-(N-2)q},\right), \left(1, |x|^{b+2}\right)$;
(*iii*) $\left(|x|^{2-N}, |x|^{2-N}\right), \left(1, |x|^{2-N}\right), \left(|x|^{2-N}, 1\right)$;

The solutions of type (*i*) can be both anisotropic, and the question of convergence is still open. The solutions of type (*ii*) can present system a new form of anisotropy, where *only one function is anisotropic*. Here we can prove the convergence, by using the analyticity results of [19]. The solutions of type (*iii*) are isotropic.

In **Section 7**, we give extensions of our results to multipower systems of the form

$$\left\{\begin{array}{l} -\Delta u+|x|^a u^s v^p=0, \\ -\Delta v+|x|^b u^q v^t=0, \end{array}\right. \tag{1.27}$$

where $p, q, s, t, a, b \in \mathbb{R}$, with $p, q>0$. We cover the corresponding sublinear case $pq<(1-s)(1-t)$, with $s, t \in(0,1)$.

This article complements the results relative to the system with the other signs

$$\left\{\begin{array}{l} \Delta u+|x|^a v^p=0, \\ \Delta v+|x|^b u^q=0; \end{array}\right. \tag{1.28}$$

and more generally

$$\left\{\begin{array}{l} \Delta u+|x|^a u^s v^p=0, \\ \Delta v+|x|^b u^q v^t=0. \end{array}\right. \tag{1.29}$$

We refer to [3] for a detailed study of the singularities of system (1.29). It covers the sublinear case, and in the superlinear one up to a first critical condition. In case $s=q+1$, $t=p+1$, the study is carried on in [6] up to the second critical condition $p+q+1<(N+2)/(N-2)$. See also [9], [16], [17], [18] for studies in whole $\mathbb{R}^N$, and [10], [23] for the regular Dirichlet problem, and [11] for the singular one in the radial case.

2 The key tools

First we give a property of subharmonic nonnegative functions, essential in our study. Let us denote $B(x, r)=\left\{y \in \mathbb{R}^N \mid |y-x| \leq r\right\}$, for any $x \in \mathbb{R}^N$ and $r>0$.

Lemma 2.1 *Let $w \in C^2(B')$ be any nonnegative subharmonic function nonconstant near the origin. Then $\overline{w}$ is strictly monotone for small r (either increasing and bounded, or decreasing with $\lim_{r\to 0} r^{N-2}\overline{w}(r) > 0$). Moreover there exists a constant $C(N)$ such that for any $\varepsilon \in (0,1/2]$,*

$$w(x) \leq C(N)\, \varepsilon^{1-N}\, \overline{w}\left[|x|\,(1 \pm \varepsilon)\right] \qquad \text{near } 0, \tag{2.1}$$

with the sign $+$ if $\overline{w}$ is increasing, and the sign $-$ if $\overline{w}$ is decreasing. Finally, for small r, and for any $Q > 1$,

$$\overline{w}^Q(r) \leq \overline{w^Q}(r) \leq (C(N)\, \varepsilon^{1-N}\,)^Q\, \overline{w}^Q\left[r(1 \pm \varepsilon)\right], \tag{2.2}$$

and for any $Q \in (0,1)$,

$$\overline{w}^Q(r) \geq \overline{w^Q}(r) \geq (C(N)\, \varepsilon^{1-N}\,)^{Q-1}\, \overline{w}^{Q-1}\left[r(1 \pm \varepsilon)\right]\ \overline{w}(r). \tag{2.3}$$

Proof. By hypothesis, $(r^{N-1}\overline{w}_r)_r \geq 0$, hence either $r^{N-1}\overline{w}_r$ has a nonnegative limit. Then there is some $\rho \in (0,1/2)$ such that $\overline{w}$ is either increasing on $(0,\rho]$, hence bounded, or decreasing on $(0,\rho]$, with $\lim_{r\to 0} r^{N-2}\,\overline{w}(r) = l \in (0,+\infty]$. Let $x \in B(0,2\rho/3)$, and $\varepsilon \in (0,1/2]$. Then from the mean value inequality of subharmonic functions,

$$w(x) \leq \frac{1}{\varepsilon^N\, |x|^N\, |B|} \int_{B(x,\varepsilon|x|)} w(y)\, dy. \tag{2.4}$$

Hence denoting $\mathcal{C}_\varepsilon = \left\{ y \in \mathbb{R}^N\, \middle|\, |x|\,(1-\varepsilon) \leq |y| \leq |x|\,(1+\varepsilon) \right\}$,

$$w(x) \leq \frac{1}{\varepsilon^N\, |x|^N\, |B|} \int_{\mathcal{C}_\varepsilon} w(y)\, dy \leq \frac{N}{\varepsilon^N\, |x|^N} \int_{|x|(1-\varepsilon)}^{|x|(1+\varepsilon)} r^{N-1}\overline{w}(r)\, dr. \tag{2.5}$$

Since $\overline{w}$ is monotone, it implies

$$w(x) \leq \varepsilon^{-N}\left[(1+\varepsilon)^N - (1-\varepsilon)^N\right]\ \overline{w}(|x|\,(1 \pm \varepsilon)), \tag{2.6}$$

with the sign $+$ if $\overline{w}$ is increasing, and the sign $-$ if $\overline{w}$ is decreasing. Then (2.1) follows with $C(N) = 2N(3/2)^{N-1}$. Taking the Q - power for any x with $|x| = r < 2\rho/3$, and integrating on the sphere $|x| = r$, we deduce that, for any $Q > 0$,

$$\overline{w^Q}(r) \leq (C(N)\, \varepsilon^{1-N}\,)^Q\, \overline{w}^Q\left[r(1 \pm \varepsilon)\right], \tag{2.7}$$

hence (2.2) if $Q > 1$. If $Q \in (0,1)$ we take the $(1-Q)$ - power in (2.1) we first get

$$w(x) \leq \left[C(N)\, \varepsilon^{1-N}\, \overline{w}(|x|\,(1 \pm \varepsilon))\,\right]^{1-Q} w(x)^Q. \tag{2.8}$$

Then we integrate again on $|x| = r$, and obtain (2.3). ∎

Remark 2.1 Lemma 2.1 implies the following weaker property, still used in [25] and in [5], [4]: let $w \in C^2(B')$ be any nonnegative subharmonic function, such that $\overline{w}$ satisfies an estimate of the form

$$\overline{w}(r) = O(|\ln r|^b \, r^a) \text{ as } r \to 0 \tag{2.9}$$

for some $a, b \in \mathbb{R}$. Then w satisfies the corresponding estimate

$$w(x) = O(|\ln |x||^b \, |x|^a) \text{ as } x \to 0. \tag{2.10}$$

In particular, if $\overline{w}(r) = o(r^{2-N})$, then $\overline{w}(r) = O(1)$, hence $w(x) = O(1)$ near 0.

Now we derive our second tool, which is a bootstrap result, allowing to transform a shifted inequality into an ordinary one.

Lemma 2.2 *Let $d, h, \ell \in \mathbb{R}$ with $d \in (0,1)$ and y, Φ be two continuous positive functions on some interval $(0, R]$. Assume that there exist some $C, M > 0$ and $\varepsilon_0 \in (0, 1/2]$ such that, for any $\varepsilon \in (0, \varepsilon_0]$,*

$$y(r) \leq C\,\varepsilon^{-h}\Phi(r)\, y^d\,[r(1-\varepsilon)] \qquad \textit{and} \quad \max_{\tau \in [r/2, r]} \Phi(\tau) \leq M\;\Phi(r), \tag{2.11}$$

or else

$$y(r) \leq C\,\varepsilon^{-h}\Phi(r)\, y^d\,[r(1+\varepsilon)] \qquad \textit{and} \quad \max_{\tau \in [r, 3r/2]} \Phi(\tau) \leq M\;\Phi(r), \tag{2.12}$$

for any $r \in (0, R/2]$. Then there exists another $C > 0$ such that

$$y(r) \leq C\;\Phi(r)^{1/(1-d)} \tag{2.13}$$

on $(0, R/2]$.

Proof. The result is obvious when $h \leq 0$, so we can suppose $h > 0$.

i) First assume (2.11). Consider the sequence $\varepsilon_m = \varepsilon_0/2^m$ ($m \in \mathbb{N}$). Then for any $r \in (0, R]$ and any $m \geq 1$, denoting $P_m = (1-\varepsilon_1)..(1-\varepsilon_m)$,

$$y(rP_{m-1}) \leq C\;\varepsilon_m^{-h}\;\Phi(rP_{m-1})\;y^d(rP_m).$$

In particular ,

$$\begin{cases} y(r) \leq C\;\varepsilon_1^{-h}\;\Phi(r)\;y^d(rP_1), \\ y^d(rP_1) \leq C^d\;\varepsilon_2^{-hd}\;\Phi^d(rP_1)\;y^{d^2}(rP_2), \\ \ldots \\ y^{d^{m-1}}(rP_{m-1}) \leq C^{d^{m-1}}\varepsilon_m^{-hd^{m-1}}\Phi^{d^{m-1}}(rP_{m-1})\;y^{d^m}(rP_m). \end{cases}$$

By the assumption on Φ, this implies

$$y(r) \leq C^{1+d+..+d^{m-1}}\varepsilon_1^{-h}\varepsilon_2^{-hd}..\varepsilon_m^{-hd^{m-1}}\Phi(r)\;\Phi^d(rP_1)..\Phi^{d^{m-1}}(rP_{m-1})\;y^{d^m}(rP_m)$$

for any $m \geq 1$. Hence

$$y(r) \leq (C\varepsilon_0^{-h})^{1+d+..+d^{m-1}} 2^{k(1+2d+..+md^{m-1})}$$

$$\times M^{d+2d^2+..+(m-1)d^{m-1}} \Phi(r)^{1+d+..+d^{m-1}} y^{d^m}(rP_m). \tag{2.14}$$

Let us go to the limit as m tends to $+\infty$, for any fixed $r \in (0,R]$: the sequence (P_m) has a finite limit $P > 0$, since the series $\sum_{i=1}^{\infty} \varepsilon_i$ is convergent, hence $\lim y^{d^m}(rP_m) = 1$, because $d < 1$, and

$$y(r) \leq (C\varepsilon_0^{-h})^{1/(1-d)} \, 2^{k/(1-d)^2} M^{d/(1-d)^2} \Phi(r)^{1/(1-d)}, \tag{2.15}$$

and (2.13) holds.

ii) Assume (2.12), and denote now $P_m = (1+\varepsilon_1)..(1+\varepsilon_m)$. Then (P_m) still has a finite limit $P > 0$, and more precisely $P \leq e$, because $\ln P_m \leq \sum_{i=1}^{m} \varepsilon_i \leq 2\varepsilon_0 \leq 1$. Then inequality (2.14) is still available for any $r \in (0, R/2e]$, hence also (2.15). This again implies (2.13). ■

Remark 2.2 This lemma shows that the solutions of the shifted inequality (2.11) or (2.12) behave exactly as the solutions of the ordinary inequality

$$y(r) \leq C \, \Phi(r) \, y^d(r) \tag{2.16}$$

relative to $\varepsilon = 0$. *This result is not evident and quite surprising in case* $h > 0$, since $\lim_{\varepsilon \to 0} \varepsilon^{-h} = +\infty$. Notice that the conditions on Φ are obviously satisfied by power functions $\Phi(r) = r^{\omega}$ ($\omega \in \mathbb{R}$) or logarithmical ones $\Phi(r) = |\ln r|^{\omega}$, or $\Phi(r) = \ln|\ln r|$, .., or by products of this functions.

We complete this section by two simple integration results, which are complementary.

Lemma 2.3 *Let $\sigma, k \in \mathbb{R}$, and let $y \in C^2((0,1])$ be nonnegative, such that*

$$\Delta y(r) \leq C \, r^{\sigma} \, |\ln r|^k \tag{2.17}$$

on $(0,1]$, for some $C > 0$. Then there is another $C > 0$ such that, near the origin,

$$y(r) \leq C \begin{cases} r^{\sigma+2} |\ln r|^k & \text{if} \quad \sigma + N < 0; \\ r^{2-N} |\ln r|^{k+1} & \text{if} \quad \sigma + N = 0 \quad \text{and} \quad k > -1; \\ r^{2-N} |\ln|\ln r|| & \text{if} \quad \sigma + N = 0 \quad \text{and} \quad k = -1; \\ r^{2-N} & \text{if} \quad \sigma + N > 0 \quad \text{or} \quad \sigma + N = 0 \quad \text{and} \quad k < -1. \end{cases}$$

If moreover $\lim_{r \to 0} y(r) = \lim_{r \to 0} r^{N-1} y_r(r) = 0$, then

$$y(r) \leq C \begin{cases} r^{\sigma+2} |\ln r|^k & \text{if} \quad \sigma + 2 > 0; \\ |\ln r|^{k+1} & \text{if} \quad \sigma + 2 = 0 \quad \text{and} \quad k < -1. \end{cases}$$

Proof. Let us define

$$y(r) = r^{2-N}\mathbf{y}(r), \tag{2.18}$$

and

$$\lambda = -(\sigma + N). \tag{2.19}$$

Then

$$(r^{3-N}\mathbf{y}_r)_r \le C\, r^{1-N-\lambda} \left|\ln r\right|^k. \tag{2.20}$$

Integrating twice over $[r, r_0]$,with $r < r_0 \le 1$, we get successively

$$r_0^{3-N}\mathbf{y}_r(r_0) - r^{3-N}\mathbf{y}_r(r) \le C \int_r^{r_0} s^{1-N-\lambda} \left|\ln s\right|^k ds,$$

$$\mathbf{y}(r) \le \mathbf{y}(r_0) - r_0^{3-N}\mathbf{y}_r(r_0) \int_r^{r_0} s^{N-3} ds + C\, I(r, r_0, \lambda) \le C + C\, I(r, r_0, \lambda), \tag{2.21}$$

where

$$I(r, r_0, \lambda, k) = \int_r^{r_0} \tau^{N-3} \int_\tau^{r_0} s^{1-N-\lambda} \left|\ln s\right|^k ds. \tag{2.22}$$

Now as r goes to 0,

$$I(r, r_0, \lambda, k) = \begin{cases} \frac{1}{\lambda(\lambda+N-2)}\, r^{-\lambda} \left|\ln r\right|^k (1 + o(1)), & \text{if } \lambda > 0, \\ \frac{1}{(N-2)(k+1)} \left|\ln r\right|^{k+1} (1 + o(1)), & \text{if } \lambda = 0 \text{ and } k > -1, \\ \frac{1}{(N-2)} \left|\ln\left|\ln r\right|\right| (1 + o(1)), & \text{if } \lambda = 0, \text{and } k = -1, \\ C_0(1 + o(1)), & \text{if } \lambda = 0 \text{ and } k < -1, \text{or } \lambda < 0, \end{cases} \tag{2.23}$$

with $C_0 = C(r_0, \lambda, k, N) > 0$. Hence we get the results by returning to y. Now assume that $\lim_{r\to 0} y(r) = \lim_{r\to 0} r^{N-1}y_r(r) = 0$. Then we integrate twice the inequality

$$(r^{N-1}y_r)_r \le C\, r^{N-1+\sigma} \left|\ln r\right|^k, \tag{2.24}$$

over $(0, r)$ and get the conclusions. ■

Lemma 2.4 *Let $\sigma, k \in \mathbb{R}$, and $y \in C^2((0,1])$ be nonnegative, such that*

$$\Delta y(r) \ge C\, r^\sigma \left|Lnr\right|^k \tag{2.25}$$

on $(0,1]$, for some $C > 0$. Then there is another $C > 0$ such that, near the origin,

$$y(r) \ge C \begin{cases} r^{\sigma+2} \left|\ln r\right|^k & \text{if } \sigma + N < 0; \\ r^{2-N} \left|\ln r\right|^{k+1} & \text{if } \sigma + N = 0 \text{ and } k > -1; \\ r^{2-N} \left|\ln\left|\ln r\right|\right| & \text{if } \sigma + N = 0 \text{ and } k = -1; \\ r^{2-N} & \text{if } \sigma + N = 0 \text{ and } k < -1, \text{ or} -N < \sigma < -2, \\ & \text{or } \sigma + 2 = 0 \text{ and } k > -1. \end{cases} \tag{2.26}$$

In particular, if y is bounded, then $\sigma+2\geq 0$, and $\sigma+2>0$ if $k>-1$. Moreover if $\lim_{r\to 0} y(r)=\lim_{r\to 0} r^{N-1}y_r(r)=0$, then

$$y(r)\geq C\left\{\begin{array}{ll} r^{\sigma+2}\left|\ln r\right|^{k} & \text{if} \quad \sigma+2>0; \\ \left|\ln r\right|^{k+1} & \text{if} \quad \sigma+2=0 \quad \text{and} \quad k<-1. \end{array}\right. \tag{2.27}$$

Proof. Here

$$(r^{3-N}\mathbf{y}_r)_r\geq C\, r^{\sigma+1}\left|\ln r\right|^{k}, \tag{2.28}$$

hence

$$r_0^{3-N}\mathbf{y}_r(r_0)-r^{3-N}\mathbf{y}_r(r)\geq C\int_r^{r_0} s^{1+\sigma}\left|\ln s\right|^{k} ds, \tag{2.29}$$

$$\mathbf{y}(r)\geq\mathbf{y}(r_0)-r_0^{3-N}\mathbf{y}_r(r_0)\int_r^{r_0} s^{N-3}ds+C\int_r^{r_0}\tau^{N-3}\int_\tau^{r_0} s^{1+\sigma}\left|\ln s\right|^{k} ds,$$

$$\geq -C+C\int_r^{r_0}\tau^{N-3}\int_\tau^{r_0} s^{1+\sigma}\left|\ln s\right|^{k} ds. \tag{2.30}$$

Thus the conclusions follow from (2.23) in the first three cases, because the integral is divergent. Moreover $\Delta y(r)>0$, that is

$$(r^{3-N}\mathbf{y}_r)_r(r)=r^{3-N}(\mathbf{y}_{rr}(r)+(3-N)\mathbf{y}_r(r)/r)>0,$$

hence $\mathbf{y}$ is strictly monotone for $r\leq r_0$ small enough. If $\sigma+2<0$, or $\sigma+2=0$ and $k>-1$, then $\mathbf{y}$ is decreasing, from (2.29), and $\mathbf{y}(r)\geq C>0$ from (2.30), and $y(r)\geq Cr^{2-N}$. We get (2.27) as in Lemma 2.3. ■

3 Estimates in the superlinear case

Here we give the proofs of Theorem 1.1, and Corollary 1.2.

In the case $p=q>1,\ a=b$, the proof of Theorem 1.1 is simple. Indeed system (1.1) admits particular solutions (w,w) , where w is any solution of equation (1.3) with $Q=p=q$ and $\sigma=a=b$. Here Theorem 1.1 reduces to the Osserman estimate (1.6) for the two functions u and v. The conclusion follows by observing that function $(u+v)/2$ is then a subsolution of equation (1.3).

Now let us come to the general case $p,q>0$ and $pq>1$. Here we present a first proof, which uses the main arguments of Section 2, and a second proof, which is shorter but restricted to the case $p>1$ and $q>1$, $p\neq q$. One can also find in [13] a variant of the first proof, which is restricted to the case $p\geq 1$ and $q\geq 1$, where the bootstrap technique is replaced by an energy argument.

3.1 Proof of Theorem 1.1 (general case $pq > 1$).

Let $u, v \in C^2(B')$ satisfying (1.14). Then the mean values satisfy the system in $(0, 1]$

$$(r^{N-1}\,\overline{u}_r)_r \geq r^{a+N-1}\,\overline{v^p}, \tag{3.1}$$

$$(r^{N-1}\,\overline{v}_r)_r \geq r^{b+N-1}\,\overline{u^q}. \tag{3.2}$$

From Lemma 2.1 we are reduced to get estimates for $\overline{u}, \overline{v}$. If $\overline{u}$ or $\overline{v}$ is constant near 0, then $u \equiv v \equiv 0$. In the general case each of these functions is subharmonic, hence strictly monotone on some interval $(0, \rho]$, either bounded with $\overline{u}_r > 0$ (resp. $\overline{v}_r > 0$), or unbounded with $\overline{u}_r < 0$ and $\overline{u}(r) > C\,r^{2-N}$ (resp. $\overline{v}(r) > C\,r^{2-N}$). Let $\varepsilon \in (0, 1/8]$ be fixed. We set

$$I_\varepsilon(r) = \int_r^{r(1+\varepsilon)} \int_\tau^{\tau(1+\varepsilon)} \overline{u^q}(s)dsd\tau, \qquad J_\varepsilon(r) = \int_r^{r(1+\varepsilon)} \int_\tau^{\tau(1+\varepsilon)} \overline{v^p}(s)dsd\tau, \tag{3.3}$$

for any $r \in (0, \rho/2]$. First integrate (3.2) over $[r, r(1+\varepsilon)]$. If $\overline{v}$ is decreasing, then

$$\begin{aligned} -r^{N-1}\overline{v}_r(r) &\geq -\left[r(1+\varepsilon)\right]^{N-1}\overline{v}_r\left[r(1+\varepsilon)\right] + \int_r^{r(1+\varepsilon)} s^{b+N-1}\overline{u^q}(s)ds, \\ &\geq \int_r^{r(1+\varepsilon)} s^{b+N-1}\overline{u^q}(s)ds, \end{aligned}$$

and a new integration gives

$$\overline{v}(r) \geq \overline{v}\left[r(1+\varepsilon)\right] + \int_r^{r(1+\varepsilon)} \tau^{1-N} \int_\tau^{\tau(1+\varepsilon)} s^{b+N-1}\overline{u^q}(s)dsd\tau,$$

hence

$$\overline{v}(r) \geq C\,r^b I_\varepsilon(r). \tag{3.4}$$

If $\overline{v}$ is increasing, we find

$$\left[r(1+\varepsilon)\right]^{N-1}\overline{v}_r\left[r(1+\varepsilon)\right] \geq \int_r^{r(1+\varepsilon)} s^{b+N-1}\overline{u^q}(s)ds,$$

hence

$$\overline{v}(r(1+\varepsilon)^2) \geq \overline{v}\left[r(1+\varepsilon)\right] + C\int_r^{r(1+\varepsilon)} \tau^{1-N} \int_\tau^{\tau(1+\varepsilon)} s^{b+N-1}\overline{u^q}(s)dsd\tau,$$

which now implies

$$\overline{v}(r) \geq C\,r^b I_\varepsilon\left[r/(1+\varepsilon)^2\right]. \tag{3.5}$$

Similarly

$$\overline{u}(r) \geq C\, r^{a} \times \begin{cases} J_\varepsilon(r), & \text{if} \quad \overline{u}_r < 0, \\ J_\varepsilon\left[r/(1+\varepsilon)^2\right] & \text{if} \quad \overline{u}_r > 0. \end{cases}$$

Without loss of generality, can assume $p \leq q$, hence $q > 1$. Then the Jensen inequality applies, since $q \geq 1$, and

$$I_\varepsilon(r) \geq \int_r^{r(1+\varepsilon)} \int_\tau^{\tau(1+\varepsilon)} \overline{u}^q(s) ds d\tau \geq \begin{cases} \varepsilon^2\, r^2\, \overline{u}^q\left[r(1+\varepsilon)^2\right], & \text{if} \quad \overline{u}_r < 0, \\ \varepsilon^2\, r^2\, \overline{u}^q(r), & \text{if} \quad \overline{u}_r > 0. \end{cases} \tag{3.6}$$

Hence we arrive to a first shifted inequality between $\overline{u}$ and $\overline{v}$:

$$\overline{v}(r) \geq C\, \varepsilon^2\, r^{b+2} \times \begin{cases} \overline{u}^q\left[r(1+\varepsilon)^2\right], & \text{if} \quad \overline{u}_r < 0, \overline{v}_r < 0, \\ \overline{u}^q(r), & \text{if} \quad \overline{u}_r \overline{v}_r < 0, \\ \overline{u}^q\left[r/(1+\varepsilon)^2\right], & \text{if} \quad \overline{u}_r > 0, \overline{v}_r > 0. \end{cases} \tag{3.7}$$

Now we argue according to the value of p.

First case: $p \geq 1$. Then we get similarly

$$\overline{u}(r) \geq C\, \varepsilon^2\, r^{a+2} \times \begin{cases} \overline{v}^p\left[r(1+\varepsilon)^2\right], & \text{if} \quad \overline{u}_r < 0, \overline{v}_r < 0, \\ \overline{v}^p(r), & \text{if} \quad \overline{u}_r \overline{v}_r < 0, \\ \overline{v}^p\left[r/(1+\varepsilon)^2\right], & \text{if} \quad \overline{u}_r > 0, \overline{v}_r > 0. \end{cases} \tag{3.8}$$

Therefore

$$\overline{v}(r) \geq C\, \varepsilon^{2(q+1)}\, r^{(a+2)q+b+2} \times \begin{cases} \overline{v}^{pq}\left[r(1+\varepsilon)^2\right], & \text{if} \quad \overline{u}_r < 0, \overline{v}_r < 0, \\ \overline{v}^{pq}(r), & \text{if} \quad \overline{u}_r \overline{v}_r < 0, \\ \overline{v}^{pq}\left[r/(1+\varepsilon)^2\right], & \text{if} \quad \overline{u}_r > 0, \overline{v}_r > 0. \end{cases}$$

Changing ε into $\varepsilon/3$, this reduces to the estimates

$$\overline{v}(r) \leq C\, \varepsilon^{-2(q+1)/pq}\, r^{-[(a+2)q+b+2]/pq} \quad \begin{cases} \overline{v}^{1/pq}\left[r(1 \pm \varepsilon)\right], & \text{if} \quad \overline{u}_r \overline{v}_r > 0, \\ \overline{v}^{1/pq}(r), & \text{if} \quad \overline{u}_r \overline{v}_r < 0. \end{cases} \tag{3.9}$$

In case $\overline{u}_r \overline{v}_r < 0$, we immediately deduce the expected estimate of $\overline{v}$:

$$\overline{v}(r) \leq C\, r^{-\xi} \qquad \text{near } 0. \tag{3.10}$$

In case $\overline{u}_r \overline{v}_r > 0$, we are reduced to a shifted inequality of type (2.11) or (2.12). Thus we can apply Lemma 2.2 to $y = \overline{v}$, with $d = 1/pq < 1$, and get again (3.10). Taking $\varepsilon = 1/2$ in (3.7), it implies the corresponding estimate for $\overline{u}$:

$$\overline{u}(r) \leq C\, r^{-\gamma} \qquad \text{near } 0, \tag{3.11}$$

hence estimates (1.15) follow.

Second case: $p < 1$. Here we use the fundamental inequality (2.3) for function v :

$$\overline{v^p}(r) \geq (C(N)\, \varepsilon^{1-N}\,)^{p-1}\, \overline{v}^{p-1}\left[r(1 \pm \varepsilon)\right]\, \overline{v}(r), \tag{3.12}$$

with the sign $+$ if $\overline{v}_r > 0$ and $-$ if $\overline{v}_r < 0$. Then we find

$$\begin{aligned} J_\varepsilon(r) &\geq (C(N)\, \varepsilon^{1-N}\,)^{p-1} \int_r^{r(1+\varepsilon)} \int_\tau^{\tau(1+\varepsilon)} \overline{v}^{p-1}\left[s(1 \pm \varepsilon)\right]\, \overline{v}(s))dsd\tau \\ &\geq C\, \varepsilon^{N+1-(N-1)p}\, r^2 \times \begin{cases} \overline{v}^{p-1}\left[r(1-\varepsilon)\right]\, \overline{v}\left[r(1+\varepsilon)^2\right] & \text{if} \quad \overline{v}_r < 0, \\ \overline{v}^{p-1}\left[r(1+\varepsilon)^3\right]\, \overline{v}(r), & \text{if} \quad \overline{v}_r > 0. \end{cases} \end{aligned}$$

Hence

$$\overline{u}(r) \geq C\, \varepsilon^{N+1-(N-1)p}\, r^{a+2} \times \begin{cases} \overline{v}^{p-1}\left[r(1-\varepsilon)\right]\, \overline{v}\left[r(1+\varepsilon)^2\right], & \text{if } \overline{u}_r < 0, \overline{v}_r < 0, \\ \overline{v}^{p-1}\left[r(1+\varepsilon)^3\right]\, \overline{v}(r), & \text{if } \overline{u}_r < 0 < \overline{v}_r, \\ \overline{v}^{p-1}\left[r(1-\varepsilon)/(1+\varepsilon)^2\right]\, \overline{v}(r), & \text{if } \overline{v}_r < 0 < \overline{u}_r, \\ \overline{v}^{p-1}\left[r(1+\varepsilon)\right]\, \overline{v}\left[r/(1+\varepsilon)^2\right], & \text{if } \overline{u}_r > 0, \overline{v}_r > 0. \end{cases} \tag{3.13}$$

By reporting (3.13) in (3.7), it comes

$$\begin{aligned} \overline{v}(r) \quad &\geq \quad C\, \varepsilon^{2+(N+1-(N-1)p)q}\, r^{(a+2)q+b+2} \\ &\times \begin{cases} \overline{v}^{(p-1)q}(r)\, \overline{v}^q\left[r(1+\varepsilon)^4\right], & \text{if } \overline{u}_r < 0, \overline{v}_r < 0, \\ \overline{v}^{(p-1)q}\left[r(1+\varepsilon)^3\right]\, \overline{v}^q(r), & \text{if } \overline{u}_r < 0 < \overline{v}_r, \\ \overline{v}^{(p-1)q}\left[r(1-\varepsilon)^3\right]\, \overline{v}^q(r), & \text{if } \overline{v}_r < 0 < \overline{u}_r, \\ \overline{v}^{(p-1)q}(r)\, \overline{v}^q\left[r/(1+\varepsilon)^4\right], & \text{if } \overline{u}_r > 0, \overline{v}_r > 0, \end{cases} \end{aligned}$$

after noticing that

$$\begin{cases} \overline{v}^{(p-1)q}\left[r(1+\varepsilon)^2(1-\varepsilon)\right] \geq \overline{v}^{(p-1)q}(r) & \text{if } \overline{v}_r < 0, \\ \overline{v}^{(p-1)q}\left[r(1-\varepsilon)/(1+\varepsilon)^2\right] \geq \overline{v}^{(p-1)q}\left[r(1-\varepsilon)^3\right] & \text{if } \overline{v}_r < 0, \\ \overline{v}^{(p-1)q}\left[r/(1+\varepsilon)\right] \geq \overline{v}^{(p-1)q}(r) & \text{if } \overline{v}_r > 0. \end{cases}$$

Changing ε into $\varepsilon/6$, we finally get, for ε small enough,

$$\begin{aligned} \overline{v}(r) \quad &\leq \quad C\, \varepsilon^{-(2/q+N+1-(N-1)p)} \\ &\times \begin{cases} r^{-[(a+2)q+b+2]/q} \quad \overline{v}^{1-(pq-1)/q}\left[r(1 \pm \varepsilon)\right], & \text{if} \quad \overline{u}_r\overline{v}_r > 0, \\ r^{-[(a+2)q+b+2]/(q-1)}\, \overline{v}^{1-(pq-1)/(q-1)}\left[r(1 \pm \varepsilon)\right], & \text{if} \quad \overline{u}_r\overline{v}_r < 0. \end{cases} \end{aligned} \tag{3.14}$$

In any case we are still reduced to a shifted inequality. We can apply Lemma 2.2 to $y = \overline{v}$, with $d = 1 - (pq-1)/q < 1$, or $d = 1 - (pq-1)/(q-1) < 1$. Thus we get again estimate (3.10) and conclude as above. ■

3.2 Second Proof of Theorem 1.1 (case $q > p > 1$).

It relies directly on Keller-Osserman estimates for the scalar case, and is inspired by the methods of [3] relative to system (1.29). Let $x_0 \in B(0, 1/2)$ and $B_0 = B(x_0, |x_0|/2)$. Our proof consists in obtaining a suitable upper estimate of the minimum of the function u over B_0, and then the corresponding estimate for $u(x_0)$ by using the maximum principle. We can suppose that

$$m(x_0) = \min_{x \in B_0} u(x) > 0. \tag{3.15}$$

Recall that in case $p = q$, the function $(u+v)/2$ is a subsolution of equation (1.3). Here we assume that $q > p > 1$. Now notice that for any subharmonic positive function w and any $\delta > 1$, the function w^δ is still subharmonic. This leads to introduce the function in B_0

$$f = |x|^\tau u^\delta + v, \tag{3.16}$$

with $\delta = (q+1)/(p+1) > 1$ and $\tau = (b-a)/(p+1)$. Let us compute its Laplacian:

$$\begin{aligned} \Delta f &= \delta(\delta-1)\,|x|^\tau u^\delta \left| \frac{\nabla u}{u} + \frac{\tau}{\delta-1} \frac{\nabla r}{r} \right|^2 - \kappa\,|x|^{\tau-2} u^\delta \\ &\quad + \delta\,|x|^\tau u^{\delta-1} \Delta u + \Delta v, \end{aligned}$$

where $\kappa = \tau(\tau/(\delta-1)^2 + 2 - N - \tau)$. Hence from (1.17)

$$-\Delta f + |x|^{\tau+a} u^{\delta-1} (\delta v^p + |x|^{p\tau} u^{p\delta}) \leq \kappa\,|x|^{\tau-2} u^\delta$$

and consequently f appears as a subsolution of a problem of the form

$$-\Delta f + A(x)\, f^p \leq \kappa\,|x|^{-2} f,$$

for which we can apply Osserman-Keller estimates. But $A(x) = 2^{-p} |x|^{\tau+a} u^{\delta-1}(x)$ *depends on* f. Now we minorize A in terms of $m(x_0)$, and get

$$-\Delta f + 2\alpha(x_0)\, f^p \leq \beta(x_0)\, f,$$

with

$$\alpha(x_0) = 2^{-(p+1)} \min((1/2)^{\tau+a}, (3/2)^{\tau+a})\,|x_0|^{\tau+a} m(x_0)^{\delta-1}, \qquad \beta(x_0) = 4\kappa/9\,|x_0|^2.$$

Hence from Young inequality,

$$-\Delta f + \alpha(x_0)\, f^p \leq (\beta(x_0)^p/\alpha(x_0))^{1/(p-1)} \tag{3.17}$$

Then from Keller-Osserman estimates (see also [14]), we obtain

$$f(x) \leq C\ |x_0|^{-2} \alpha(x_0)^{-1/(p-1)} \leq C\ |x_0|^{-2-(\tau+a)/(p-1)} m(x_0)^{-(\delta-1)/(p-1)} \tag{3.18}$$

in B_0, with $C = C(N, p, q, a, b)$, in particular at x_0. But $f(x_0) \geq |x_0|^\tau m(x_0)^\delta$, hence we get the estimate

$$m(x_0) \leq C \ |x_0|^{-\gamma}. \tag{3.19}$$

The same estimate is also available for $\overline{u}$, since $\overline{u}, \overline{v}$ are also subsolutions of system (1.1), because $p, q > 1$. Let $r_0 = |x_0|$. Then there exists $s_0 \in [r_0/2, 3r_0/2]$ such that $\overline{u}(s_0) \leq C r_0^{-\gamma}$. By induction, defining $r_n = r_0/4^n$, for any $n \in \mathbb{N}$ there exists a decreasing sequence (s_n) such that $s_n \in [r_n/2, 3r_n/2]$ and

$$\overline{u}(s_n) \leq C \ r_n^{-\gamma} \leq (2/3)^{-\gamma} C \ s_n^{-\gamma}.$$

From the maximum principle in the annulus $\mathcal{C}_n = \left\{ y \in \mathbb{R}^N \mid s_{n+1} \leq |y| \leq s_n \right\}$, it follows that

$$\overline{u}(r) \leq (2/3)^{-\gamma} C \ s_{n+1}^{-\gamma} \leq (2/3)^{-\gamma} \max(1, 12^\gamma) \ C \ r^{-\gamma} \qquad \text{in } [s_{n+1}, s_n]$$

with $C = C(N, p, q, a, b)$, since $r \in [s_{n+1}, 12 s_{n+1}]$. Then, with new constants C,

$$\overline{u}(r) \leq C \ r^{-\gamma} \qquad \text{in } (0, r_0],$$

and from Lemma 2.1,

$$u(x) \leq C \ |x|^{-\gamma} \qquad \text{in } B'. \tag{3.20}$$

Now let $\Psi \in C^2(B')$ such that $-\Delta \Psi = 1$ and $\Psi = 0$ on $\partial B'$, and let $\varphi(x) = \Psi(2(x - x_0)/|x_0|)$. We multiply the first inequality of (1.14) by φ, integrate over B_0, and apply the Green formula. It follows easily that

$$\min_{x \in B_0} v(x) \leq C \ |x_0|^{-(a+2+\gamma)/p} = C \ |x_0|^{-\xi} \tag{3.21}$$

from (3.20). We get in the same way the estimate

$$v(x) \leq C \ |x|^{-\xi} \qquad \text{in } B', \tag{3.22}$$

which achieves the proof. ∎

3.3 Proof of Corollary 1.2

i) Let us prove that the condition $\gamma \leq N - 2$ implies that u is bounded . Assume that u is unbounded near 0. Then also $\overline{u}$ is unbounded, from Lemma 2.1, hence $\overline{u}(r) \geq C \ r^{2-N}$ for some $C > 0$, near 0. It implies $\gamma \geq N - 2$ from (1.15), and in fact $\gamma > N - 2$. Indeed if $\gamma = N - 2$, then

$$\Delta \overline{v}(r) \geq C \ r^{b - \gamma q} = C \ r^{-2-\xi},$$

hence $\overline{v}(r) \geq C \ r^{-\xi}$ from Lemma 2.4. But $\overline{v}(r) \leq C \ r^{-\xi}$ from (1.15). we report this estimate into (3.1). Then we get

$$\Delta \overline{u}(r) \geq C \ r^{a - p\xi} = C \ r^{-N},$$

from the Jensen inequality if $p \geq 1$, and from (3.12) if $p < 1$. We deduce $\overline{u}(r) \geq C\ r^{2-N} |\ln r|$, from Lemma 2.4, which contradicts (1.15). Similarly the condition $\xi \leq N-2$ implies that v is bounded. Hence the condition $\max(\gamma, \xi) \leq N-2$ implies that u and v are bounded.

ii) Assume $\xi \leq N-2$ (hence v is bounded), $q \geq (b+2)/(N-2)$ and suppose that u is unbounded. Then $\overline{u}(r) \geq C\ r^{2-N}$ near 0, hence

$$\Delta\overline{v}(r) \geq C\ r^{b-(N-2)q}.$$

This is impossible from Lemma 2.4, since v is bounded. Similarly after exchanging u and v. ■

4 Estimates in the sublinear case

Here also the estimates are simple in the case $p = q < 1$ and $a = b$. The system (1.1) still admits particular solutions (w, w), where w is any solution of equation (1.3) with $Q = p = q < 1$ and $\sigma = a = b$. Here Theorem 1.3 reduces to the estimates (1.7) for the two functions u and v. The conclusion follows by observing that function $(u+v)/2$ is a subharmonic supersolution of equation (1.3).

Now let us come to the general case. In this section and in the sequel of the study, we set

$$\ell_1 = (N-2)p - (N+a), \qquad \ell_2 = (N-2)q - (N+b), \tag{4.1}$$

and notice the relations

$$\ell_1 + p\ell_2 = (1-pq)(\gamma - (N-2)), \qquad q\ell_1 + \ell_2 = (1-pq)(\xi - (N-2)). \tag{4.2}$$

4.1 A sublinear shifted inequality

In order to prove Theorem 1.3, we first prove that the subharmonic supersolutions of a sublinear *shifted* inequality present the same behaviour as the supersolutions of the ordinary one .

Theorem 4.1 *Let $Q, \sigma, h, k \in \mathbb{R}$, with $Q \in (0,1)$, $k \geq 0$, and let $y \in C^2((0,1])$. Assume there exists some $C > 0$ and $\varepsilon_0 \in (0, 1/2]$ such that, for any $\varepsilon \in (0, \varepsilon_0]$ and $r \in (0,1]$,*

$$0 \leq \Delta y(r) \leq C\ \varepsilon^{-h}\ r^{\sigma}\ \ |\ln r|^k\ \ y^Q\left[r(1 \pm \varepsilon)\right]. \tag{4.3}$$

Then y satisfies the same estimates as the solutions of inequality

$$0 \leq \Delta w(r) \leq C\ r^{\sigma}\ \ |\ln r|^k\ \ w^Q. \tag{4.4}$$

More precisely, with another $C>0$,

$$y(r)\leq C\left\{\begin{array}{l} r^{(2+\sigma)/(1-Q)}\left|\ln r\right|^{k/(1-Q)}, \qquad \textit{if} \quad Q>(N+\sigma)/(N-2), \\ r^{(2-N)}, \qquad \textit{if} \quad Q<(N+\sigma)/(N-2), \\ r^{(2-N)}L(r), \qquad \textit{if} \quad Q=(N+\sigma)/(N-2), \end{array}\right. \tag{4.5}$$

where $L(r)=\left|\ln r\right|^{(k+1)/(1-Q)}$ *if* $k>-1$, $\left|\ln\left|\ln r\right|\right|^{1/(1-Q)}$ *if* $k=-1$, 1 *if* $k<-1$.

Remark 4.1 When $k=0$ one finds again the estimates for equation (1.3) in the radial case. The following proof relies closely on the proof of the estimates for this equation, given in [5].

Proof. We can assume that $h\geq 0$, and y is nonidentically 0 near 0. Let us make the change of variables (2.18). It leads to the inequality in $(0,1]$

$$0\leq(r^{3-N}\mathbf{y}_r)_r(r)\leq C\,\varepsilon^{-h}r^{1-N-\ell}\left|\ln r\right|^k\ \mathbf{y}^Q\left[r(1\pm\varepsilon)\right], \tag{4.6}$$

where

$$\ell=(N-2)Q-(N+\sigma)=(1-Q)(\Gamma-N+2). \tag{4.7}$$

Then $\mathbf{y}$ is monotone and positive for $r\leq r_0$ small enough, since $(r^{3-N}\mathbf{y}_r)_r(r)\geq 0$. If $\mathbf{y}$ is bounded, then $y(r)=O(r^{2-N})$ and Theorem 4.1 is proved in any case. Now suppose that $\mathbf{y}$ is unbounded, then it is necessarily nonincreasing . Integrating over $[r,r_0]$, we get

$$-r^{3-N}\mathbf{y}_r(r)\leq C+C\,\varepsilon^{-h}\,\mathbf{y}^Q\left[r(1\pm\varepsilon)\right]\int_r^{r_0}s^{1-N-\ell}\left|\ln s\right|^k ds, \tag{4.8}$$

and by a new integration,

$$\mathbf{y}(r)\leq C+C\,\varepsilon^{-h}\,\mathbf{y}^Q\left[r(1\pm\varepsilon)\right]\ I(r,r_0,\ell,k), \tag{4.9}$$

where $I(r,r_0,\ell,k)$ is defined in (2.22). If $\ell>0$, this implies from (2.23) the shifted inequality

$$\mathbf{y}(r)\leq C\,\varepsilon^{-h}r^{-\ell}\left|\ln r\right|^k\mathbf{y}^Q\left[r(1\pm\varepsilon)\right].$$

Then we can apply Lemma 2.2 with $\Phi(r)=r^{-\ell}\left|\ln r\right|^k$, and deduce the first part of (4.5). If $\ell<0$, then we find from (2.23)

$$\mathbf{y}(r)\leq C\,\varepsilon^{-h}\,\mathbf{y}^Q\left[r(1\pm\varepsilon)\right],$$

hence $\mathbf{y}$ is bounded, from Lemma 2.2, hence a contradiction, and the second part of (4.5) follows. If $\ell=0$, it implies

$$\left\{\begin{array}{ll} \mathbf{y}(r)\leq C\,\varepsilon^{-h}\left|\ln r\right|^{k+1}\mathbf{y}^Q\left[r(1\pm\varepsilon)\right], & \text{if } k>-1, \\ \mathbf{y}(r)\leq C\,\varepsilon^{-h}\left|\ln(\left|\ln r\right|)\right|\mathbf{y}^Q\left[r(1\pm\varepsilon)\right], & \text{if } k=-1, \\ \mathbf{y}(r)\leq C\,\varepsilon^{-h}\,\mathbf{y}^Q\left[r(1\pm\varepsilon)\right], & \text{if } k<-1, \end{array}\right.$$

and the third part of (4.5) follows from Lemma 2.2. ■

4.2 Proof of Theorem 1.3

If $\overline{u} \equiv 0$ near 0, then v is harmonic, hence $v(x) \leq C\,|x|^{2-N}$, and the estimates are trivially satisfied. So we can assume that $\overline{u}, \overline{v}$ are positive near 0. Here we perform the change of variables

$$u(x) = |x|^{2-N}\,\mathbf{u}(x), \qquad v(x) = |x|^{2-N}\,\mathbf{v}(x), \tag{4.11}$$

It leads to a system of inequalities relative to $\overline{\mathbf{u}}, \overline{\mathbf{v}}$ in $(0,1]$:

$$0 \leq (r^{3-N}\overline{\mathbf{u}}_r)_r \leq r^{1-N-\ell_1}\,\overline{\mathbf{v}^p}, \tag{4.12}$$

$$0 \leq (r^{3-N}\overline{\mathbf{v}}_r)_r \leq r^{1-N-\ell_2}\,\overline{\mathbf{u}^q}. \tag{4.13}$$

It follows that $\overline{\mathbf{u}}$ and $\overline{\mathbf{v}}$ are monotone and positive for $r \leq r_0$ small enough. In case of (1.20), we have $\ell_1 < 0$ and $\ell_2 < 0$. In case of (1.23) and (1.21) we have $\ell_1 = 0$ and $\ell_2 \leq 0$. First assume that $\overline{\mathbf{v}}$ is bounded . Then $\mathbf{v}$ is also bounded, from Lemma 2.1. That means $v(x) \leq C\ \ |x|^{2-N}$, which implies $\Delta\overline{u}(r) \leq C\ r^{a-(N-2)p}$. From Lemmas 2.3 and 2.1, it follows that

$$u(x) \leq \begin{cases} C\ \ |x|^{a+2-(N-2)p}, & \text{if } \ \ell_1 > 0, \\ C\ \ |x|^{2-N}\,|\ln|x||\,, & \text{if } \ \ell_1 = 0, \\ C\ \ |x|^{2-N}, & \text{if } \ \ell_1 < 0. \end{cases} \tag{4.14}$$

This implies (1.19), (1.20) and (1.23); and also (1.18), (1.22) since $|x|^{a+2-(N-2)p} \leq |x|^{-\gamma}$ as soon as $\xi \geq N-2$; and at last (1.21), because $|\ln|x|| \leq |\ln|x||^{(p+1)/(1-pq)}$. Then we can assume that $\overline{\mathbf{v}}$ is unbounded . Then $\overline{\mathbf{v}}$ is decreasing. Using (4.12) we get from Lemma 2.1

$$(r^{3-N}\overline{\mathbf{u}}_r)_r(r) \leq C\,\varepsilon^{-(N-1)p}\,r^{1-N-\ell_1}\,\overline{\mathbf{v}}^p\left[r(1-\varepsilon)\right]. \tag{4.15}$$

Integrating over $[r, r_0]$,we get

$$-r^{3-N}\overline{\mathbf{u}}_r(r) \leq C + C\,\varepsilon^{-(N-1)p}\,\overline{\mathbf{v}}^p\left[r(1-\varepsilon)\right]\int_r^{r_0} s^{1-N-\ell_1}\,ds,$$

since $\overline{\mathbf{v}}$ is decreasing. A new integration gives

$$\overline{\mathbf{u}}(r) \leq C + C\,\varepsilon^{-(N-1)p}\,\overline{\mathbf{v}}^p\left[r(1-\varepsilon)\right]\ I(r, r_0, \ell_1, 0). \tag{4.16}$$

First step: Proof of (1.18), (1.19) and (1.22)

Under the assumptions of (1.19) or (1.22), we have $\ell_1 > 0$. In the case of (1.18), we find $\ell_1 > 0$ or $\ell_2 > 0$, from (4.2). After exchanging u into v, we can still assume that $\ell_1 > 0$. Then $I(r, r_0, \ell_1, 0) = O(r^{-\ell_1})$, from (2.23), hence from (4.16)

$$\overline{\mathbf{u}}(r) \leq C + C\,\varepsilon^{-(N-1)p}\,r^{-\ell_1}\,\overline{\mathbf{v}}^p\left[r(1-\varepsilon)\right] \leq C\,\varepsilon^{-(N-1)p}\,r^{-\ell_1}\,\overline{\mathbf{v}}^p\left[r(1-\varepsilon)\right]. \tag{4.17}$$

Using (4.13) and Lemma 2.1, we get in the same way

$$(r^{3-N}\overline{\mathbf{v}}_r)_r(r) \leq C\,\varepsilon^{-(N-1)q}\,r^{1-N-\ell_2}\,\overline{\mathbf{u}}^q\left[r(1\pm\varepsilon)\right], \tag{4.18}$$

Reporting (4.17) into (4.18) and changing ε into $\varepsilon/2$ if necessary, we find

$$(r^{3-N}\overline{\mathbf{v}}_r)_r(r) \leq C\,\varepsilon^{-(N-1)(p+1)q}\,r^{1-N-(q\ell_1+\ell_2)}\,\overline{\mathbf{v}}^{pq}\left[r(1-\varepsilon)\right]. \tag{4.19}$$

That means that function $\overline{v}$ satisfies the shifted inequality

$$0 \leq \Delta\overline{v}(r) \leq C\,\varepsilon^{-h}r^{\sigma}\,\overline{v}^{Q}\left[r(1-\varepsilon)\right], \tag{4.20}$$

of the form (4.3), with $Q = pq < 1$, and $h = (N-1)(p+1)q$; and σ is given by

$$(N-2)Q - (N+\sigma) = q\ell_1 + \ell_2 = (1-pq)(\xi - (N-2)),$$

from (4.2), thus $\sigma = (a+2)q + b$. Then we can apply Theorem 4.1. Under the assumption of (1.18), we have $\xi > (N-2)$, hence $Q > (N+\sigma)/(N-2)$. Then from (4.5),

$$\overline{v}(r) \leq C\,r^{(2+\sigma)/(1-Q)} = C\,r^{-\xi}, \tag{4.21}$$

and from (4.17),

$$\overline{u}(r) \leq C\,r^{(N-2)(p-1)-\ell_1-p\xi} = C\,r^{-\gamma}. \tag{4.22}$$

It implies (1.18) from Lemma 2.1. Under the assumption of (1.19), we have $\xi < (N-2)$, hence $Q < (N+\sigma)/(N-2)$. Then from (4.5), $\overline{v}(r) \leq C\ r^{2-N}$, which contradits our assumption on $\overline{\mathbf{v}}$. In the case of (1.22), we have $\xi = (N-2)$, thus $Q = (N+\sigma)/(N-2)$, then from (4.5) ,

$$\overline{v}(r) \leq C\,r^{2-N}\left|\ln r\right|^{1/(1-pq)}, \tag{4.23}$$

and from (4.17)

$$\overline{u}(r) \leq C\,r^{2-N-\ell_1}\left|\ln r\right|^{p/(1-pq)} = C\,r^{a+2-(N-2)p}\left|\ln r\right|^{p/(1-pq)}, \tag{4.24}$$

and (1.22) follows from Lemma 2.1.

Second step : Proof of (1.20)

Here $\ell_1 < 0$ and $\ell_2 < 0$. Then $I(r, r_0, \ell_1, 0) = O(1)$, from (2.23), and from (4.16)

$$\overline{\mathbf{u}}(r) \leq C\,\varepsilon^{-(N-1)p}\,\overline{\mathbf{v}}^p\left[r(1-\varepsilon)\right],$$

From symmetry we can also assume that $\overline{\mathbf{u}}$ is unbounded, hence in the same way

$$\overline{\mathbf{v}}(r) \leq C\,\varepsilon^{-(N-1)q}\,\overline{\mathbf{u}}^q\left[r(1-\varepsilon)\right].$$

Thus with a new $\varepsilon > 0$,

$$\overline{\mathbf{v}}(r) \leq C\ \varepsilon^{-(N-1)(p+1)q}\ \overline{\mathbf{v}}^{pq}\left[r(1-\varepsilon)\right],$$

and $\overline{\mathbf{v}}$ is bounded from Lemma 2.2, which is a contradiction. Thus (1.20) follows.

Third step *:* **Proof of (1.21)and (1.23)**

Here $\ell_1 = 0$ and $\ell_2 \leq 0$. Then $I(r, r_0, \ell_1, 0) = O(|Lnr|)$, hence

$$\overline{\mathbf{u}}(r) \leq C\ \varepsilon^{-(N-1)p}\left|\ln r\right|\ \overline{\mathbf{v}}^{p}\left[r(1-\varepsilon)\right]. \tag{4.25}$$

First suppose $\ell_2 < 0$. By reporting (4.25) into (4.13), and we find with a new $\varepsilon > 0$,

$$(r^{3-N}\overline{\mathbf{v}}_r)_r(r) \leq C\ \varepsilon^{-(N-1)(p+1)q}\ r^{1-N-\ell_2}\left|\ln r\right|^{q}\ \overline{\mathbf{v}}^{pq}\left[r(1-\varepsilon)\right].$$

This implies in particular

$$(r^{3-N}\overline{\mathbf{v}}_r)_r(r) \leq C\ \varepsilon^{-(N-1)(p+1)q}\ r^{1-N-\ell_2/2}\,\overline{\mathbf{v}}^{pq}\left[r(1-\varepsilon)\right].$$

We can apply Theorem 4.1, with σ defined by $\ell_2/2 = (N-2)pq - (N+\sigma)$. Thus $\overline{\mathbf{v}}$ is bounded from (4.5), hence a contradiction holds. Now suppose $\ell_2 = 0$. Then we can assume that $\overline{\mathbf{u}}$ is unbounded, and similarly

$$\overline{\mathbf{v}}(r) \leq C\ \varepsilon^{-(N-1)q}\left|\ln r\right|\ \overline{\mathbf{u}}^{q}\left[r(1-\varepsilon)\right],$$

then with a new ε,

$$\overline{\mathbf{v}}(r) \leq C\ \varepsilon^{-(N-1)(p+1)q}\left|\ln r\right|^{q+1}\ \overline{\mathbf{v}}^{pq}\left[r(1-\varepsilon)\right],$$

hence from Lemma 2.2 and (4.25)

$$\overline{\mathbf{v}}(r) \leq C\ \left|\ln r\right|^{(q+1)/(1-pq)}, \qquad \overline{\mathbf{u}}(r) \leq C\ \left|\ln r\right|^{(p+1)/(1-pq)},$$

hence (1.23) is proved. ■

5 Existence of anisotropic solutions

First recall the results relative to the scalar case of equation (1.3) for any $Q \neq 1$. If we look for particular solutions of the form

$$w(x) = |x|^{-\Gamma}\mathbf{W}(\theta), \qquad \theta \in S^{N-1}, \tag{5.1}$$

where Γ is given by (1.4), we are leaded to the equation on S^{N-1}

$$\Delta_{S^{N-1}}\mathbf{W} + \rho\mathbf{W} - \mathbf{W}^{Q} = 0. \tag{5.2}$$

with $\rho = \Gamma(\Gamma + 2 - N)$. It has no positive solution if $\rho \leq 0$, that means $Q \geq (N+\sigma)/(N-2) \geq 1$ or $Q \leq (N+\sigma)/(N-2) \leq 1$. This comes by multiplication by W and integration over S^{N-1}. Now assume that $\rho > 0$. In the superlinear case, it admits only one positive solution, the constant $\rho^{1/(Q-1)}$, see [22]. Hence equation (1.3) has no positive nonradial solutions. In the sublinear case, if $\rho(1-Q) \leq N-1$ it admits only the constant positive solution. If not, it can admit nonconstant solutions: let $(\mu_k)_{k\in\mathbb{N}}$ be the sequence of eigenvalues of $-\Delta_{S^{N-1}}$ on S^{N-1}, given by

$$\mu_k = k(k+N-2), \qquad \forall k \in \mathbb{N}.$$

Then equation (5.2) admits a continuum of solutions for any ρ in the neighborhood of $\mu_k/(1-Q)$, obtained by bifurcation, see [5]. Moreover it can admit many solutions with dead cores, which are not obtained by bifurcation, see [5] and [4]. Hence equation (1.3) can admit nonradial positive solutions.

Now let us return to the case of system (1.1). Searching solutions of the form (1.24), we are lead to system (1.25). Here we prove the theorem 1.4, showing that system (1.1) can admit nonradial positive solutions even in the superlinear case.

Proof of Theorem 1.4

We consider more generally the system on S^{N-1}

$$\left\{ \begin{array}{l} \Delta_{S^{N-1}}\mathbf{U} + \alpha\,\mathbf{U} - \mathbf{V}^p = 0, \\ \Delta_{S^{N-1}}\mathbf{V} + \beta\,\mathbf{V} - \mathbf{U}^q = 0, \end{array} \right. \tag{5.3}$$

for any $\alpha, \beta > 0$. We look for bifurcation branches around the constant solutions (A, B), with

$$A = (\alpha\beta^p)^{1/(pq-1)}, \qquad B = (\alpha^q\beta)^{1/(pq-1)}.$$

Here we follow the proof given in the scalar sublinear case in [5]. In order to avoid the question of multiplicity of the eigenvalues μ_k, we look for solutions $\mathbf{U}, \mathbf{V}$ which are radially symmetric by respect to some diameter. In other words they depend only on some polar angle $\phi \in (0, \pi)$. The system reduces to

$$\left\{ \begin{array}{l} L\mathbf{U} + \alpha\,\mathbf{U} - \mathbf{V}^p = 0, \\ L\mathbf{V} + \beta\,\mathbf{V} - \mathbf{U}^q = 0, \end{array} \right. \tag{5.4}$$

where

$$L\omega(\phi) = \sin^{2-N}\phi\,\left[(\sin^{N-2}\phi)\,\omega_\phi\right]_\phi, \qquad \forall \phi \in (0, \pi)\,.$$

We know that $(I-L)^{-1}$ is a compact self-adjoint operator in the weighted space

$$L^2_*\left[(0,\pi)\right]) = \left\{ \omega \in \mathcal{D}'\left[(0,\pi)\right] \,\middle|\, \int_0^\pi \omega^2(\phi)\sin^{N-2}\phi\, d\phi < +\infty \right\}.$$

And $-L$ and $-\Delta_{S^{N-1}}$ have the same spectrum and each eigenspace of $-L$ is one-dimensional, see [2],[7]. Denoting

$$\mathbf{U}(\phi) = A + \mathbf{H}(\phi), \qquad \mathbf{V}(\phi) = B + \mathbf{K}(\phi),$$

system (5.4) takes the matricial form

$$L\begin{pmatrix}\mathbf{H}\\ \mathbf{K}\end{pmatrix}+M\begin{pmatrix}\mathbf{H}\\ \mathbf{K}\end{pmatrix}-\begin{pmatrix}T(\mathbf{K},\alpha,\beta)\\ S(\mathbf{H},\alpha,\beta)\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},$$

where

$$\begin{aligned}S(\mathbf{H},\alpha,\beta)&=(A+\mathbf{H})^q-A^q-qA^{q-1}\mathbf{H},\\ T(\mathbf{K},\alpha,\beta)&=(B+\mathbf{K})^p-B^p-pB^{p-1}\mathbf{K},\\ M&=\begin{pmatrix}\alpha & -pB^{p-1}\\ -qA^{q-1} & \beta\end{pmatrix}.\end{aligned}$$

The matrix M is invertible, since $\det M=\alpha\beta(1-pq)\neq 0$. Its eigenvalues are the two distinct roots $\lambda_1<\lambda_2$ of equation

$$\lambda^2-(\alpha+\beta)\lambda-(pq-1)\alpha\beta=0.$$

Observe that $\lambda_1<0<\lambda_2$ if $pq>1$, and $0<\lambda_1<\lambda_2$ if $pq<1$. We reduce the system to the diagonal form by setting

$$\begin{pmatrix}\mathbf{H}\\ \mathbf{K}\end{pmatrix}=R\begin{pmatrix}\mathbf{H}'\\ \mathbf{K}'\end{pmatrix},\qquad R=\begin{pmatrix}pB^{p-1} & \beta-\lambda_2\\ \alpha-\lambda_1 & qA^{q-1}\end{pmatrix},$$

and obtain

$$\begin{cases}L\mathbf{H}'+\lambda_1\,\mathbf{H}'-T'(\mathbf{H}',\mathbf{K}',\alpha,\beta)=0,\\ L\mathbf{K}'+\lambda_2\,\mathbf{K}'-\mathbf{S}'(\mathbf{H}',\mathbf{K}',\alpha,\beta)=0,\end{cases}$$

with

$$\begin{aligned}T'(\mathbf{H}',\mathbf{K}',\alpha,\beta)&=(\det R)^{-1}\left[(\beta-\lambda_2)\,S(\mathbf{H},\alpha,\beta)+qA^{q-1}\,T(\mathbf{K},\alpha,\beta)\right],\\ \mathbf{S}'(\mathbf{H}',\mathbf{K}',\alpha,\beta)&=(\det R)^{-1}\left[pB^{p-1}\,S(\mathbf{H},\alpha,\beta)+(\alpha-\lambda_1)\,T(\mathbf{K},\alpha,\beta)\right].\end{aligned}$$

Let μ_k be an eigenvalue of $-\Delta_{S^{N-1}}$. Let us fix $\alpha>0$, such that $\mu_k>\alpha$. We apply the local bifurcation theorem by respect to the second parameter β. Notice that the function $\lambda_2(\alpha,.)$ is increasing. Then there exists a unique $\beta_k>0$ such that $\mu_k=\lambda_2(\alpha,\beta_k)$. Let us assume that $\lambda_1(\alpha,\beta_k)$ is not an eigenvalue of $-\Delta_{S^{N-1}}$ if $pq>1$. We set

$$X=\left\{v\in C^2([0,\pi])\mid v_\phi(0)=v_\phi(\pi)=0\right\},\qquad Y=C([0,\pi]).$$

Let $S=(\beta_k-\rho,\beta_k+\rho)$, with $\rho<\beta_k/2$ small enough such that $\lambda_2(\alpha,S)$ belongs to $(\mu_k/2,3\mu_k/2)$. Consider a closed ball $\mathcal{B}$ of X, of center 0 and radius $\eta>0$ small enough such that T',S' are well-defined and smooth for $(\mathbf{H}',\mathbf{K}')\in\mathcal{B}$. One can take $\eta\leq\min\left[(2^{-p}\alpha\beta_k^p)^{1/(pq-1)},(2^{-1}\alpha^q\beta_k)^{1/(pq-1)}\right]$. Then the local bifurcation theorem applies to the function

$$f(\beta,\mathbf{H}',\mathbf{K}')=$$
$$\left[L\mathbf{H}'+\lambda_1(\alpha,\beta)\mathbf{H}'-T'(\mathbf{H}',\mathbf{K}',\alpha,\beta),\ L\mathbf{K}'+\lambda_2(\alpha,\beta)\mathbf{K}'-S'(\mathbf{H}',\mathbf{K}',\alpha,\beta)\right]$$

from $S \times \mathcal{B}$ into $Y \times Y$. Indeed the operators

$$\mathcal{L}_0 = D_2 f(\beta_k, 0, 0), \qquad \mathcal{L}_1 = D_1 D_2 f(\beta_k, 0, 0),$$

are given for any $(\mathbf{H}', \mathbf{K}') \in \mathcal{B}$ by

$$\mathcal{L}_0(\mathbf{H}', \mathbf{K}') = \left[L\mathbf{H}' + \lambda_1(\alpha, \beta_k)\mathbf{H}',\ L\mathbf{K}' + \lambda_2(\alpha, \beta_k)\mathbf{K}'\right],$$

$$\mathcal{L}_1(\mathbf{H}', \mathbf{K}') = \left[\frac{\partial\lambda_1(\alpha, \beta_k)}{\partial\beta}\mathbf{H}',\ \frac{\partial\lambda_2(\alpha, \beta_k)}{\partial\beta}\mathbf{K}'\right].$$

Then

$$Ker\mathcal{L}_0 = \{0\} \times Ker(L + \lambda_2(\alpha, \beta_k)\, I),$$

since $\lambda_1(\alpha, \beta_k)$ is not an eigenvalue of $-\Delta_{S^{N-1}}$. Hence $Ker\ \mathcal{L}_0$ is one-dimensional, generated by $(0, w_k)$, where w_k is an eigenvector of $-L$ for $\lambda_2(\alpha, \beta_k)$. And the image

$$R\mathcal{L}_0 = Y \times R(L + \beta_k\, I),$$

hence it has a codimension 1 in $Y \times Y$. At last $\mathcal{L}_1(0, w_k) \notin R\mathcal{L}_0$, since $\partial\lambda_1(\alpha, \beta_k)/\partial\beta \neq 0$. Hence a branch of bifurcation appears at β_k, i.e. at each time λ_2 crosses an eigenvalue of $-\Delta_{S^{N-1}}$ and λ_1 is not such an eigenvalue. Now if $pq < 1$ and $\alpha\ (1 - pq)/2 > \mu_k$, then there exists a unique $\widetilde{\beta_k}$ such that $\mu_k = \lambda_1(\alpha, \widetilde{\beta_k})$, since the function $\lambda_1(\alpha, .)$ is increasing. We prove in the same way that a bifurcation occurs when λ_1 crosses μ_k. ■

Remark 5.1 This theorem gives one case of existence of nonconstant solutions of system (1.25). In fact the situation can be quite more intricated, at least in the sublinear case. Suppose for example that $p = q < 1$, and $a = b$. Then the system admits solutions $(\mathbf{W}, \mathbf{W})$ where $\mathbf{W}$ satisfies (5.2), with $Q = p = q$. Then a bifurcation occurs in system (1.25) at each time $\lambda_1 = \alpha(1 - q)$ crosses an eigenvalue of $\Delta_{S^{N-1}}$, from [5], even if $\lambda_2 = \alpha(1 + q)$ is also an eigenvalue of $\Delta_{S^{N-1}}$. Moreover system (1.25) can admit many solutions with dead cores. Hence system (1.1) can admit *anisotropic solutions with dead cores.* In the general case $pq < 1$, the most simple example is given when $a = b = 0$. Then γ, ξ are negative, and system (1.1) admits solutions with support in $(\mathbb{R}^N)^+$:

$$u^*(x) = A_1^*\ \left[(x_n)^+\right]^{-\gamma},\ v^*(x) = B_1^*\ \left[(x_n)^+\right]^{-\xi},$$

with $x = (x_1, x_2, .., x_N)$, and

$$A_1^* = [\gamma(\gamma + 1)(\xi(\xi + 1))^p]^{1/(pq-1)}, \qquad B_1^* = [\xi(\xi + 1)(\gamma(\gamma + 1))^q]^{1/(pq-1)}.$$

Otherwise we shall also see other types of anisotropy in the next section.

6 Convergence results

6.1 The scalar case

First recall the precise results in the scalar case.

Theorem 6.1 *([20],[24]) Let $w \in C^2(B')$ be any nonnegative solution of equation (1.3), with $Q > 1$.*
i) If $Q \geq (N+\sigma)/(N-2)$ (i.e. $\Gamma \geq N-2$), then

$$\lim_{x \to 0} u(x) = C \geq 0.$$

ii) If $Q < (N+\sigma)/(N-2)$ (i.e. $\Gamma > N-2$), then

$$\text{either } \lim_{x \to 0} |x|^{\Gamma} u(x) = C*, \text{ or } \lim_{x \to 0} |x|^{N-2} u(x) = C > 0, \text{ or } \lim_{x \to 0} u(x) = C \geq 0.$$

Theorem 6.2 *([5]) Let $w \in C^2(B')$ be any nonnegative solution of equation (1.3), with $Q < 1$.*
i) If $Q > (N+\sigma)/(N-2)$ (i.e. $\Gamma > N-2$), then

$$w(x) \leq C \ |x|^{-\Gamma}).$$

ii) If $Q < (N+\sigma)/(N-2) \leq 1$ (i.e. $0 \leq \Gamma < N-2$), then

$$\lim_{x \to 0} |x|^{N-2} w(x) = C > 0, \text{ or } \ w \equiv 0 \text{ near } 0.$$

iii) If $Q < 1 < (N+\sigma)/(N-2)$ (i.e. $\Gamma < 0$), then

$$\text{either } \lim_{x \to 0} |x|^{N-2} w(x) = C > 0, \text{ or } \lim_{x \to 0} w(x) = C' > 0, \text{ or } w(x) = O(|x|^{-\Gamma}).$$

iv) If $Q = (N+\sigma)/(N-2)$ (i.e. $\Gamma = N-2$), then

$$\lim_{x \to 0} |x|^{N-2} |\ln|x||^{-1/(1-Q)} w(x) = (\frac{1-Q}{N-2})^{1/(1-Q)}, \text{ or } \ u \equiv 0 \text{ near } 0.$$

Remark 6.1 Moreover if $Q < 1$ and $w(x) = O(|x|^{-\Gamma})$, setting

$$w(x) = |x|^{-\Gamma} W(t, \theta), \qquad t = -\ln r, \ \ \theta \in S^{N-1}, \tag{6.1}$$

then the limit set of $W(t,.)$ in $C^2(S^{N-1})$ as $t \to +\infty$ is contained in the set of stationary solutions of equation (5.2). If 0 is in the limit set, then $w \equiv 0$ near 0, see [5].

6.2 Convergence Lemmas

Let u, v be any nonnegative solutions of system (1.1). We want to give a precise behaviour of u and v near 0. At each time when we have an upper estimate of the form

$$u(x) \leq C \ |x|^{-\eta}, \qquad v(x) \leq C \ |x|^{-\zeta}, \qquad \text{near the origin,} \tag{6.2}$$

we use the change of variables

$$u(x) = |x|^{-\eta} U(t,\theta), \quad v(x) = |x|^{-\zeta} V(t,\theta), \qquad t = -\ln r, \ \theta \in S^{N-1}. \tag{6.3}$$

It leads to a system in the cylinder $(0, +\infty) \times S^{N-1}$:

$$\left\{ \begin{array}{l} U_{tt} - (N-2-2\eta)U_t + \Delta_{S^{N-1}}U + \eta(\eta+2-N)U - e^{-(\eta+a+2-\zeta p)t}V^p = 0, \\ V_{tt} - (N-2-2\zeta)V_t + \Delta_{S^{N-1}}V + \zeta(\zeta+2-N)V - e^{-(\zeta+b+2-\eta q)t}U^q = 0, \end{array} \right. \tag{6.4}$$

where U and V are bounded for large t. Then the idea is the following: *if one exponential is negative,* for example $\eta + a + 2 - \zeta p > 0$, then we can obtain a result of convergence to a solution of the equation

$$\Delta_{S^{N-1}}U + \eta(\eta+2-N)U = 0.$$

Then reporting it in the second equation, and get in turn a second result of convergence for V . Both of them rely upon a result of [6]. Let us recall it for a better understanding.

Proposition 6.3 *Let $Y \in C^2((0, +\infty) \times S^{N-1})$ be a bounded solution of equation*

$$Y_{tt} - (a+b)Y_t + ab\, Y + \Delta_{S^{N-1}}Y + \varphi = 0,$$

with given reals $a < b$, with $ab \leq 0$.
i) If $\|\varphi(t,.)\|_{C(S^{N-1})} = O(t^{-1})$ at $+\infty$, then $\left\|Y(t,.) - \overline{Y}(t)\right\|_{C(S^{N-1})} = O(t^{-1/2})$.
ii) If $\|\varphi(t,.)\|_{C(S^{N-1})} = O(t^{-\lambda})$ with $\lambda > 1$, then $Y(t,.)$ converges in $C^1(S^{N-1})$ to a constant C ($C = 0$ if $ab \neq 0$), and

$$\|(|Y-C| + |Y_t| + |\nabla Y|)(t,.)\|_{C(S^{N-1})} = \left\{ \begin{array}{ll} O(t^{-\lambda}) & \text{if } ab \neq 0, \\ O(t^{1-\lambda}), & \text{if } ab = 0. \end{array} \right.$$

ii) If $\|\varphi(t,.)\|_{C(S^{N-1})} = O(e^{-\ell t})$ at $+\infty$, with $\ell > 0$, then

$$\|(|Y-C| + |Y_t| + |\nabla Y|)(t,.)\|_{C(S^{N-1})} = \left\{ \begin{array}{cl} O(e^{-\ell t}), & \text{if } a = 0. \\ O(e^{-\ell t}) + O(e^{at}), & \text{if } a < 0, a \neq -\ell. \\ O(te^{-\ell t}), & \text{if } a = -\ell. \end{array} \right.$$

The application of this Proposition provides several results of convergence.

Lemma 6.4 *Let u, v be any nonnegative solutions of system (1.1), with $pq \neq 1$.*

i) If $u(x) = o(|x|^{2-N})$ near 0, then $u(x) = O(1)$, similarly for v.

ii) Assume that $u(x) + v(x) = O(|x|^{2-N})$ near 0, and $p < (N+a)/(N-2)$ or $q < (N+b)/(N-2)$. Then

$$\lim_{x \to 0} |x|^{N-2} u(x) = C_1 \geq 0, \qquad \text{or } \lim_{x \to 0} |x|^{N-2} v(x) = C_2 \geq 0, \tag{6.5}$$

and

$$u(x) - C_1 |x|^{2-N} = \begin{cases} O(|x|^{a+2-(N-2)p}) + O(1) & \text{if } p \neq (a+2)/(N-2), \\ O(|Ln\,|x||) & \text{if } p = (a+2)/(N-2), \end{cases} \tag{6.6}$$

or

$$v(x) - C_2 |x|^{2-N} = \begin{cases} O(|x|^{b+2-(N-2)q}) + O(1) & \text{if } q \neq (b+2)/(N-2), \\ O(|\ln |x||) & \text{if } q = (b+2)/(N-2). \end{cases} \tag{6.7}$$

iii) Assume that $u(x) = O(|x|^{2-N})$, $v(x) = O(1)$, and $a + N > 0$ and $(N-2)q - (b+2) < 0$. Then

$$\lim_{x \to 0} |x|^{N-2} u(x) = C_1 \geq 0, \qquad \lim_{x \to 0} v(x) = C_2' \geq 0, \tag{6.8}$$

and $v(x) - C_2' = O(|x|^{b+2-(N-2)q})$ and

$$u(x) - C_1 |x|^{2-N} = \begin{cases} O(|x|^{a+2}) + O(1) & \text{if } a+2 \neq 0, \\ O(|\ln |x||) & \text{if } a+2 = 0. \end{cases} \tag{6.9}$$

If $C_1 = 0$, then $u(x) = O(1)$.

iv) Assume that $u(x) + v(x) = O(1)$, and $a + 2 > 0$ (or $b + 2 > 0$) .Then

$$\lim_{x \to 0} u(x) = C_1' \geq 0 \ (\text{or } \lim_{x \to 0} v(x) = C_2' \geq 0), \tag{6.10}$$

and $u(x) - C_1' = O(|x|^{a+2})$ (or $v(x) - C_2' = O(|x|^{b+2})$).

Proof. i) It comes directly from Remark 2.1.

ii) Assume for example $p < (N+a)/(N-2)$. Here we perform the change of variables (6.3) with $\eta = \zeta = N-2$, and get

$$\begin{cases} U_{tt} + (N-2)U_t + \Delta_{S^{N-1}} U - e^{\ell_1 t} V^p = 0, \\ V_{tt} + (N-2)V_t + \Delta_{S^{N-1}} V - e^{\ell_2 t} U^q = 0, \end{cases} \tag{6.11}$$

with ℓ_1, ℓ_2 given by (4.2), hence $\ell_1 < 0$. Then there exist some $C_1 \geq 0$ such that

$$\|U(t,.) - C_1\|_{C(S^{N-1})} = \begin{cases} O(e^{-(N-2)t}) + O(e^{\ell_1 t}), & \text{if } \ell_1 \neq 2-N \\ O(te^{-(N-2)t}), & \text{if } \ell_1 = 2-N, \end{cases}$$

from Proposition 6.3. Hence the results hold for u and similarly for v. If $C_1 = 0$, then u is bounded, because it is subharmonic, similarly for v.

iii) Here we use the transformation (6.3) with $\eta = N-2$ and $\zeta = 0$ and obtain

$$\begin{cases} U_{tt} + (N-2)U_t + \Delta_{S^{N-1}} U - e^{-(a+N)t} V^p = 0, \\ V_{tt} - (N-2)V_t + \Delta_{S^{N-1}} V - e^{-(b+2-(N-2)q)t} U^q = 0, \end{cases} \tag{6.12}$$

with negative exponentials. Then there exist constants $C_1, C_2' \geq 0$ such that

$$\|U(t,.) - C_1\|_{C(S^{N-1})} = \begin{cases} O(e^{-(N-2)t}) + O(e^{-(a+N)t}), & \text{if } a+2 \neq 0, \\ O(te^{-(N-2)t}), & \text{if } a+2 = 0, \end{cases}$$

and $\|V(t,.) - C_2'\|_{C(S^{N-1})} = O(e^{((N-2)q-b-2)t})$, from Proposition 6.3, which proves (6.9).

iv) Here we use the transformation (6.3) with $\eta = \zeta = 0$, which gives

$$\begin{cases} U_{tt} - (N-2)U_t + \Delta_{S^{N-1}} U - e^{-(a+2)t} V^p = 0, \\ V_{tt} - (N-2)V_t + \Delta_{S^{N-1}} V - e^{-(b+2)t} U^q = 0. \end{cases} \tag{6.13}$$

If for example $a+2 > 0$, then in the same way there exists a constants $C_1' \geq 0$ such that $\|U(t,.) - C_1'\|_{C(S^{N-1})} = O(e^{-(a+2)t})$, hence the result. $\blacksquare$.

The next lemma essentially shows that a *new form of anisotropy* can occur in system (1.1), where one and only one of the functions u, v presents an asymptotically nonradial behaviour.

Lemma 6.5 *Let u, v be any nonnegative solutions of system (1.1), with $pq \neq 1$.*
i) Assume that $u(x) = O(|x|^{a+2-(N-2)p})$, $v(x) = O(|x|^{2-N})$, and $(\xi - N+2)(pq-1) > 0$ and $\rho = [(N-2)p - (a+2)]\,[(N-2)p - (a+N)] > 0$. Then

$$\lim_{x \to 0} |x|^{N-2}\, v(x) = C_2 \geq 0, \tag{6.14}$$

and

$$\lim_{x \to 0} \left[|x|^{(N-2)p-(a+2)}\, u(|x|,.) - \rho^{-1} C_2^p \right] \text{ exists} \tag{6.15}$$

in $C(S^{N-1})$, and it belongs to $\ker(\Delta_{S^{N-1}} + \rho\, I)$.

ii) Assume that $u(x) = O(|x|^{a+2})$, $v(x) = O(1)$, and $\xi(pq-1) > 0$ and $\nu = (a+2)(a+N) > 0$. Then

$$\lim_{x \to 0} v(x) = C_2' \geq 0, \tag{6.16}$$

$$\lim_{x \to 0} \left[|x|^{-(a+2)}\, u(|x|,.) - \nu^{-1} C_2' \right] = 0 \text{ exists} \tag{6.17}$$

in $C(S^{N-1})$, and it belongs to $\ker(\Delta_{S^{N-1}} + \nu\, I)$.

Proof. i) Here we use the transformation (6.3) with $\eta = (N-2)p - a - 2$ and $\zeta = N - 2$, and get

$$\begin{cases} U_{tt} - [N-2-2((N-2)p-(a+2)]\,U_t + \Delta_{S^{N-1}}U + \rho U - V^p = 0, \\ V_{tt} + (N-2)V_t + \Delta_{S^{N-1}}V - e^{-(\xi-N+2)(pq-1)t}U^q = 0, \end{cases} \tag{6.18}$$

and the exponential is negative. Then Proposition 6.3 still applies: there is a constant $C_2 \geq 0$ such that $\|V(t,.) - C_2\|_{C(S^{N-1})} = O(e^{-\alpha t})$, with $\alpha = (\xi - N+2)(pq-1) > 0$. Now the function

$$W(t,\theta) = U(t,\theta) - \rho^{-1}C_2^p \tag{6.19}$$

satisfies an equation of the form

$$W_{tt} - [N-2-2((N-2)p-(a+2)]\,W_t + \Delta_{S^{N-1}}W + \rho W = \psi, \tag{6.20}$$

where $\|\psi(t,.)\|_{C(S^{N-1})} = \|V'^p(t,.) - C_2^p\|_{C(S^{N-1})} = O(e^{-\beta t})$ for some $\beta > 0$. And the coefficient of W_t is different from 0. Then we can apply Simon's theorem (see [19],[7]) as in [6](Theorem 4.1). It implies that the function $W(t,.)$ precisely converges to a solution of the stationary equation

$$\Delta_{S^{N-1}}\varpi + \rho\varpi = 0$$

hence the conclusion follows.

ii) Now we use the transformation (6.3) with $\eta = -a-2$ and $\zeta = 0$, and get

$$\begin{cases} U_{tt} - (N+2+2a)U_t + \Delta_{S^{N-1}}U + \nu U - V^p = 0, \\ V_{tt} - (N-2)V_t + \Delta_{S^{N-1}}V - e^{-\xi(pq-1)t}U^q = 0, \end{cases} \tag{6.21}$$

with again a negative exponential. There exists a constant $C_2' \geq 0$ such that $\|V(t,.) - C_2'\|_{C(S^{N-1})} = O(e^{-\xi(pq-1)t})$, from Proposition 6.3. Then (6.16) and (6.17) follows as above, since the coefficient of U_t is different from 0. ■

6.3 The open problems

The question of convergence is partly open in the case u, v satisfy one of the estimates (1.15), (1.18). Indeed the change of variables (6.3) with $\eta = \gamma$ and $\zeta = \xi$ now gives

$$\begin{cases} U_{tt} - (N-2-2\gamma)U_t + \Delta_{S^{N-1}}U + \gamma(\gamma+2-N)U - V^p = 0. \\ V_{tt} + (N-2+2\xi)V_t + \Delta_{S^{N-1}}V + \xi(\xi+2-N)V - U^q = 0. \end{cases} \tag{6.22}$$

This system has no negative exponential: it is autonomous. Denote by $\mathbf{E}$ the set of solutions $(\mathbf{U}, \mathbf{V})$ of system (1.25), which is the stationary system associated to (6.22).

Unlike in the scalar case, we miss a suitable energy function in order to prove that the limit set

$$\Gamma(U,V)=\bigcap_{t\geq 0}\overline{\bigcup_{\tau\geq t}(U(\tau,.),V(\tau,.)}^{C^2(S^{N-1})} \tag{6.23}$$

is contained in $\mathbf{E}$, and the problem is open. We conjecture that it is true, and moreover that if $0\in\Gamma(U,V)$, then $\Gamma(U,V)=\{0\}$, which implies that $u(x)=o(|x|^{-\gamma})$ and $v(x)=o(|x|^{-\xi})$. We also conjecture that in that case $u\equiv v\equiv 0$ near 0, if $pq<1$.

Remark 6.2 In the radial case, $\Gamma(U,V)$ is a singleton, hence $0\in\Gamma(U,V)$ implies $\Gamma(U,V)=\{0\}$ from connectedness .

6.4 The superlinear case

The question of convergence is not easy, since precisely (1.15) holds. The case $\min(\gamma,\xi)>N-2$ is the most delicate, since the particular solutions u^*,v^* do exist. Here we search the behaviour of solutions that

$$u(x)=o(|x|^{-\gamma})\quad\text{and}\quad v(x)=o(|x|^{-\xi}).$$

First look at the radial case, with $p>1$ and $q>1$. Then the linearization of system (6.22) is possible, and gives the estimates $U(t)+V(t)=O(\max(e^{(N-2-\gamma)t},e^{(N-2-\xi)t}))$. They imply that $u(r)=O(r^{2-N})$, or $v(r)=O(r^{2-N})$. In the general case we extend this result and describe the behaviour, under an additional assumption on u and v.

Proposition 6.6 *Assume $pq>1$ and $\min(\gamma,\xi)>N-2$. Let $u,v\in C^2(B')$ be any nonnegative solutions of system (1.1). Assume that*

$$u(x)=O(|x|^{-\gamma+\varepsilon}),\quad \textit{or}\quad v(x)=O(|x|^{-\xi+\varepsilon}),\quad \textit{for some } \varepsilon>0.$$

Then, up to the change from u,p,a into v,q,b, we have $q<(b+N)/(N-2)$,

$$\lim_{x\to 0}|x|^{N-2}v(x)=C_2\geq 0; \tag{6.24}$$

and

i) either $p>(a+N)/(N-2)$, and

$$\lim_{x\to 0}\left[|x|^{(N-2)p-(a+2)}u(|x|,.)-(\ell_1(\ell_1+N-2))^{-1}C_2^p\right]\ \textit{exists} \tag{6.25}$$

and it belongs to $\ker(\Delta_{S^{N-1}}+\ell_1(\ell_1+N-2)\,I)$.

ii) or $p<(a+N)/(N-2)$, and

$$\lim_{x\to 0}|x|^{N-2}u(x)=C_1\geq 0, \tag{6.26}$$

iii) or $p=(a+N)/(N-2)$, and

$$\lim|x|^{N-2}|\ln|x||\,u(x)=C_2^p/(N-2). \tag{6.27}$$

If $C_2=0$, then v is bounded. If $C_1=0$, then u is bounded.

Proof. The assumption $\min(\gamma,\xi) > N-2$ reduces to

$$\ell_1 + p\ell_2 < 0, \qquad q\ell_1 + \ell_2 < 0. \tag{6.28}$$

It implies $\ell_1 < 0$ or $\ell_2 < 0$. By symmetry we can suppose $\ell_2 < 0$, hence $q < (b+N)/(N-2)$. Now notice that the assumption $u(x) = O(|x|^{-\gamma+\varepsilon})$ implies

$$\Delta\overline{v}(r) \leq C\, r^{b-\gamma q+q\varepsilon} = C\, r^{-2-\xi+q\varepsilon},$$

hence $\overline{v}(r) = O(r^{-\xi+q\varepsilon})$, and $v(x) = O(|x|^{-\xi+q\varepsilon})$, till $q\varepsilon < \xi - N + 2$, from Lemmas 2.3 and 2.1. Reciprocally any estimate on v implies an analogous one on u. Hence we can start from the assumption $u(x) = O(|x|^{-\gamma+\varepsilon})$, with ε small enough. Consider $\varepsilon_0 = \varepsilon$ and $\varepsilon_0' = q\varepsilon$, and define $\varepsilon_n = p\varepsilon_{n-1}'$, and $\varepsilon_n' = q\,\varepsilon_n$. Then by induction

$$\overline{u}(r) = O(r^{-\gamma+\varepsilon_n}), \qquad \overline{v}(r) = O(r^{-\xi+\varepsilon_n'}),$$

till $q\varepsilon_n < \xi - N + 2$, and $p\varepsilon_n' < \gamma - N + 2$. But $\varepsilon_n = pq\,\varepsilon_{n-1}$, hence $\lim \varepsilon_n = +\infty$. Hence there is a first integer n_0 such that $q\varepsilon_{n_0} \geq \xi - N + 2$ or $\varepsilon_{n_0} = p\varepsilon_{n_0-1}' \geq \gamma - N + 2$. Then from Lemmas 2.3 and 2.1,

$$u(x) + v(x) = O(|x|^{2-N}\,|\ln|x||). \tag{6.29}$$

It implies

$$\Delta\overline{v}(r) \leq C_\varepsilon\, r^{b-(N-2)q-\varepsilon},$$

for any $\varepsilon > 0$. But the condition $\ell_2 < 0$ implies $b + N - (N-2)q > 0$. Hence in fact

$$v(x) = O(|x|^{2-N}) \tag{6.30}$$

from Lemma 2.3 and 2.1. Then

$$0 \leq \Delta\overline{u}(r) \leq C\, r^{a-(N-2)p}.$$

Applying Lemma 2.3, we discuss according to the sign of $a + N - (N-2)p = -\ell_1$.

i) Case $\ell_1 > 0$. Then $u(x) = O(|x|^{a+2-(N-2)p})$ from Lemmas 2.3 and 2.1. Now we can apply Lemma 6.5, because $\xi > N-2$. Hence (6.24) and (6.25) follow.

ii) Case $\ell_1 < 0$. Then $u(x)+v(x) = O(|x|^{2-N})$. Then (6.24) and (6.26) follow from Lemma 6.4, ii). Moreover $u(x) - C_1\,|x|^{2-N} = O(|x|^{a+2-(N-2)p})$ and $v(x) - C_2\,|x|^{2-N} = O(|x|^{b+2-(N-2)q})$. Now assume that $C_2 = 0$, hence v is bounded . Observe that our assumptions implies $a + N > 0$, and $b + N > 0$.

iii) Case $\ell_1 = 0$. Then $u(x) = O(|x|^{2-N}\,|\ln|x||)$, $v(x) = O(|x|^{2-N})$. The transformation (6.3) with $\eta = \zeta = N-2$ gives

$$\begin{cases} U_{tt} + (N-2)U_t + \Delta_{S^{N-1}}U - V^p = 0, \\ V_{tt} + (N-2)V_t + \Delta_{S^{N-1}}V - e^{\ell_2 t}\,U^q = 0. \end{cases} \tag{6.31}$$

And $\|U(t,.)\|_{C(S^{N-1})} = O(t)$. Then $\|V(t,.) - C_2'\|_{C(S^{N-1})} = O(e^{-\alpha t})$, for some $\alpha > 0$, from Proposition 6.3, since $\ell_2 < 0$. Consequently,

$$\overline{U}_{tt} + (N-2)\overline{U}_t = C_2'^p + O(e^{-\alpha t}),$$

hence by integration,

$$\overline{U}(t) = \frac{C_2'^p}{N-2}\, t + O(1). \tag{6.32}$$

Now setting

$$U(t,.) = t\, W(t,.), \tag{6.33}$$

we find

$$W_{tt} + (N-2+\frac{2}{t})W_t + \Delta_{S^{N-1}}W + \frac{1}{t}\,[(N-2)W - V^p] = 0. \tag{6.34}$$

In particular

$$W_{tt} + (N-2)W_t + \Delta_{S^{N-1}}W = \Psi, \tag{6.35}$$

with $\|\Psi(t,.)\|_{C(S^{N-1})} = O(1/t)$. Then $\left\|W(t,.) - \overline{W}(t)\right\|_{C(S^{N-1})} = O(t^{-1/2})$, from Proposition 6.3. We deduce (6.24) and (6.27). In any case, if $C_2 = 0$, or $C_1 = 0$, then v or u is bounded. ■

Remark 6.3 Contrary to the scalar superlinear case, we observe that some logarithmical behaviours can occur, and they are isotropic.

Proposition 6.7 *Assume $pq > 1$ and $\xi \leq N-2$. Let $u, v \in C^2(B')$ be any nonnegative solutions of system (1.1). Then v is bounded.*

Proof. This comes from the proof of Corollary 1.2. If moreover $q > (b+2)/(N-2)$, then u is also bounded. ■

Remark 6.4 The behaviour of the system in the case where one solution is bounded will be given in paragraph 6.6.

6.5 The sublinear case

Here we can give a quite complete description of the behaviour of the system, from the estimates of Theorem 1.3. In case (1.18) we have conjectured the existence of a dead core for u and v. Now we study the cases (1.18) to (1.21) .

Proposition 6.8 *Assume $pq < 1$ and $\xi < N-2$ and $p > (N+a)/(N-2)$. Let $u, v \in C^2(B')$ be any nonnegative solutions of system (1.1). Then*

$$\lim_{x\to 0} |x|^{N-2}\, v(x) = C_2 \geq 0, \tag{6.36}$$

and

$$\lim_{x\to 0}\left[|x|^{(N-2)p-(a+2)}\,u(|x|\,,.)-(\ell_1(\ell_1+N-2))^{-1}C_2^p\right]\ \textit{exists} \tag{6.37}$$

and it belongs to $\ker(\Delta_{S^{N-1}}+\ell_1(\ell_1+N-2)\,I)$. *If* $C_2=0$, *then* v *is bounded.*

Proof. First notice that the assumptions can be written under the form

$$\ell_1>0, \qquad q\ell_1+\ell_2<0, \tag{6.38}$$

from (4.1) and (4.2), hence they imply $\ell_2<0$, that is $q<(b+N)/(N-2)$. From Theorem 1.3, we have the estimates $u(x)=O(|x|^{(a+2)-(N-2)p})$ and $v(x)=O(|x|^{2-N})$. Then Lemma 6.5 applies and gives (6.36) and (6.37). If $C_2=0$, then $v(x)=o(|x|^{2-N})$, hence v is bounded . ■

Proposition 6.9 *Assume* $pq<1$ *and* $q<(N+b)/(N-2)$. *Let* $u,v\in C^2(B')$ *be any nonnegative solutions of system (1.1).*

i) If $p<(N+a)/(N-2)$, *then*

$$\lim_{x\to 0}|x|^{N-2}\,u(x)=C_1\geq 0, \qquad \lim_{x\to 0}|x|^{N-2}\,v(x)=C_2\geq 0. \tag{6.39}$$

ii) If $p=(N+a)/(N-2)$, *then*

$$\lim_{x\to 0}|x|^{N-2}\,u(x)=C_2^p/(N-2)\geq 0, \qquad \lim_{x\to 0}|x|^{N-2}\,v(x)=C_2\geq 0.$$

iii) In any case, if $C_2=0$, *then* v *is bounded, and* $\gamma<N-2$. *If* $C_1=0$, *then* u *is bounded, and* $\xi<N-2$.

Proof. Here our assumptions resume to

$$\ell_1\leq 0, \qquad \ell_2<0, \tag{6.40}$$

hence they imply $\xi<N-2$ and $\gamma\leq N-2$ from (4.1) and (4.2). From Theorem 1.3, we have the estimates (1.20) if $\ell_1<0$ and (1.21) if $\ell_1=0$. Then we argue as in Proposition 6.6, ii) and iii). ■

Remark 6.4 Let us give attention on the critical cases (1.22) and (1.23), which are not completely described.

i) Assume $\xi=N-2<\gamma$. Setting

$$u(x)=|x|^{a+2-(N-2)p}\,|\ln|x||^{p\mu}\,X(t,\theta), \qquad v(x)=|x|^{2-N}\,|\ln|x||^{\mu}\,Y(t,\theta), \tag{6.41}$$

with $\mu=1/(1-pq)$, we get the system

$$\left\{\begin{array}{l} X_{tt}-\left[M-\frac{2p\mu}{t}\right]X_t+\Delta_{S^{N-1}}X+(\mu-\frac{p\mu M}{t}+\frac{p\mu(p\mu-1)}{t^2})X-Y^p=0,\\ Y_{tt}+(N-2+\frac{2\mu}{t})Y_t+\Delta_{S^{N-1}}Y+\frac{1}{t}\left[\mu(N-2+\frac{\mu-1}{t})Y-X^q\right]=0, \end{array}\right. \tag{6.42}$$

where $M = N-2-2((N-2)p-(a+2)$. Here also we miss a suitable energy function to conclude. We presume that an anisotropic behaviour of logarithmical type can appear. It means that Y behaves like $X^q/\mu(N-2)$, and X like one of the possibly nonconstant solutions of equation

$$\Delta_{S^{N-1}} X + \mu X - (\mu(N-2))^{-p} X^{pq} = 0,$$

or $X(t,.), Y(t,.)$ converge to 0.

ii) Assume $\xi = N-2 = \gamma$. Setting

$$u(x) = |x|^{2-N} |\ln |x||^{(p+1)\mu} R(t,\theta), \qquad v(x) = |x|^{2-N} |\ln |x||^{(q+1)\mu} S(t,\theta), \tag{6.43}$$

we now obtain

$$\begin{cases} R_{tt} + \left[N-2+\frac{2(p+1)\mu}{t}\right] R_t + \Delta_{S^{N-1}} R + \frac{(p+1)\mu}{t}\left[(N-2+\frac{(p+1)\mu-1}{t})R - S^p\right] = 0, \\ S_{tt} + (N-2+\frac{2\mu}{t})S_t + \Delta_{S^{N-1}} S + \frac{(p+1)\mu}{t}\left[(N-2+\frac{(q+1)\mu-1}{t})S - R^q\right] = 0. \end{cases} \tag{6.44}$$

Here we conjecture that the behaviour is isotropic, and $R(t,.), S(t,.)$ converge respectively to $[(N-2)(p+1)]^{1/(pq-1)}$, $[(N-2)(q+1)]^{1/(pq-1)}$, or to 0.

6.6 Behaviour of the bounded solutions

Here we study the behaviour of the system in the superlinear or the sublinear case, when at least one of the solutions, for example v, is bounded near 0. The question is not simple, *all the more since the solutions can tend to* 0. We distinguish three cases, according to the sign of $a+N$.

Proposition 6.10 *Assume $pq \neq 1$. Let u, v be any nonnegative solutions of system (1.1), with v bounded near 0, and $a+N<0$. Then $q < (b+2)/(N-2)$ and*

$$\lim_{x\to 0} v(x) = C_2' \geq 0, \tag{6.45}$$

$$\lim_{x\to 0}\left[|x|^{-(a+2)} u(|x|,.) - ((a+2)(a+N))^{-1} C_2'\right] \text{ exists} \tag{6.46}$$

and it belongs to $\ker(\Delta_{S^{N-1}} + (a+2)(a+N)\, I)$. *Moreover*

i) If $C_2' > 0$, then $\xi(pq-1) > 0$.

ii) If $C_2' = 0$, and $\xi(pq-1) > 0$, and $\gamma < N-2$ if $pq < 1$, then

$$\lim_{x\to 0} |x|^{N-2} u(x) = C_1 \geq 0, \tag{6.47}$$

$$\lim_{x\to 0}\left[|x|^{(N-2)q-(b+2)} v(|x|,.) - (\ell_2(\ell_2+N-2))^{-1} C_1^q\right] \text{ exists} \tag{6.48}$$

and it belongs to $\ker(\Delta_{S^{N-1}} + (\ell_2(\ell_2+N-2))\, I)$. *If $\gamma > N-2$ and $pq < 1$, then*

$$u(x) = O(|x|^{-\gamma}), \qquad v(x) = O(|x|^{-\xi}). \tag{6.49}$$

Proof. By hypothesis, v is bounded, hence $\overline{v}$ has a limit $C_2' \geq 0$.

Since $a + N < 0$, we have $u(x) = O(|x|^{a+2})$ from Lemma 2.3 and 2.1.

i) First suppose $C_2' > 0$. Then $v(x) \geq C > 0$, from (2.2) and (2.3). Then $\overline{u}(r) \geq C\ r^{a+2}$ from Lemma 2.4, hence

$$\Delta\overline{v}(r) \geq C\ r^{b+(a+2)q};$$

hence also $b+2+(a+2)q > 0$, because v is bounded. Now $b+2+(a+2)q = \xi(pq-1)$, so that Lemma 6.5 applies . We deduce (6.45) and (6.46), with $C_2' > 0$.

ii) Now suppose $C_2' = 0$. Then $\lim_{x\to 0} v(x) = 0$ from subharmonicity, hence (6.45) again holds. Now

$$\Delta\overline{v}(r) \leq C\ r^{b+(a+2)q}.$$

Under the assumption $\xi(pq-1) > 0$, it implies $v(x) \leq C\ \ |x|^{\widetilde{k}}$, with $\widetilde{k} = \xi(pq-1) = (a+2)q + b + 2$, from Lemma 2.3, hence

$$\Delta\overline{u}(r) \leq C^p r^{a+\widetilde{k}p}.$$

Observe that $0 < (a+2)q + (b+2) < b + 2 - (N-2)q$, and that $\gamma > N-2$ if $pq > 1$, from (1.11).

–First suppose $a + N + \widetilde{k}p > 0$. then $u(x) = O(|x|^{2-N})$, and $v(x) = O(|x|^{b+2-(N-2)q})$ from Lemmas 2.3 and 2.1. In the case $pq > 1$, we can apply Lemma 6.5 after exchanging u and v, and get (6.47) and (6.48). In the case $pq < 1$ and $\gamma > N-2$, it implies $u(x) = O(|x|^{-\gamma})$ and $v(x) = O(|x|^{-\xi})$, because $b + 2 - (N-2)q > -\xi$ from (1.11).

–Now suppose $a + N + \widetilde{k}p \leq 0$. Then any estimate $\overline{v}(r) \leq C_k\ r^k$ with $a + N + kp < 0$ implies

$$\Delta\overline{u}(r) \leq C_k^p\ r^{a+kp},$$

hence $\overline{u}(r) = O(r^{a+2+kp})$ from Lemma 2.3. More precisely, we get $\overline{u}(r) \leq C\ r^{2-N} + C\ C_k^p r^{a+2+kp}$, from 2.21 and 2.23, hence $\overline{u}(r) \leq C(1 +\ C_k^p)\ r^{a+2+kp}$. Now

$$\Delta\overline{v}(r) \leq C^q(1 +\ C_k^p)^q\ r^{b+(a+2)q+kpq},$$

and $b + 2 + (a+2)q + kpq > 0$. Hence $\overline{v}(r) \leq C'C^q(1 +\ C_k^p)^q r^{b+2+(a+2)q+kpq}$ from Lemma 2.3. Then

$$\overline{v}(r) \leq C_{k_n}\ r^{k_n},$$

with

$$k_0 = \widetilde{k}, \quad k_n = b + 2 + (a+2)q + k_{n-1}pq, \quad \text{and} \quad C_{k_n} = C(1 +\ C_{k_{n-1}}^{pq}),$$

with a new constant C, till $a + N + k_n p < 0$. If $pq > 1$, then $\lim k_n = +\infty$; if $pq < 1$, then $\lim k_n = -\xi$, and $a + N - p\xi = N - 2 - \gamma$. In the case $pq > 1$, or $pq < 1$ and $\gamma < N-2$, after a finite number n_0 of steps, we get $a + N + k_{n_0}p > 0$, by changing sligthly k_0 if necessary. Thus we find again $\overline{u}(r) = O(r^{2-N})$, hence $u(x) = O(|x|^{2-N})$, and $v(x) = O(|x|^{b+2-(N-2)q})$. We get (6.47) and (6.48) as above. In the case $pq < 1$ and $\gamma > N-2$, it follows that $\overline{v}(r) \leq C\, r^{-\xi}$, because the sequence (C_{k_n}) is convergent. Then

$$\Delta \overline{u}(r) \leq C\, r^{a-p\xi} = C\, r^{-\gamma-2},$$

hence $u(x) = O(|x|^{-\gamma})$ from Lemmas 2.3 and 2.1, because $\gamma > N-2$. ■

Proposition 6.11 *Assume $pq \neq 1$. Let u, v be any nonnegative solutions of system (1.1), with v bounded near 0, and $a + N > 0$. Then*

$$\lim_{x \to 0} v(x) = C_2' \geq 0, \tag{6.50}$$

$$\lim_{x \to 0} |x|^{N-2} u(x) = C_1 \geq 0. \tag{6.51}$$

Now we can distinguish different cases.

i) If $C_1 > 0$, then $q < (b+2)/(N-2)$. If moreover $C_2' = 0$ and $(\gamma - N + 2)(pq - 1) > 0$, then

$$\lim_{x \to 0} \left[|x|^{(N-2)q-(b+2)}\, v(|x|\,,.) - (\ell_2(\ell_2 + N - 2))^{-1} C_1^q \right] \text{ exists} \tag{6.52}$$

and it belongs to $\ker(\Delta_{S^{N-1}} + (\ell_2(\ell_2 + N - 2))\, I)$.

ii) If $C_1 = 0$, then

$$\lim_{x \to 0} u(x) = C_1' \geq 0, \qquad \lim_{x \to 0} v(x) = C_2' \geq 0. \tag{6.53}$$

If $C_1' = 0$, and $C_2' > 0$, then moreover $a + 2 > 0$, and $\xi(pq - 1) > 0$, and

$$\lim_{x \to 0} \left[|x|^{-(a+2)}\, u(|x|\,,.) - ((a+2)(a+N))^{-1} C_2' \right] = 0 \text{ exists} \tag{6.54}$$

and it belongs to $\ker(\Delta_{S^{N-1}} + (a+2)(a+N)\, I)$. *And similarly if $C_1' > 0$, and $C_2' = 0$, then $b + 2 > 0$, and $\gamma(pq - 1) > 0$, and*

$$\lim_{x \to 0} \left[|x|^{-(b+2)}\, v(|x|\,,.) - ((b+2)(b+N))^{-1} C_1' \right] = 0 \text{ exists} \tag{6.55}$$

and it belongs to $\ker(\Delta_{S^{N-1}} + (b+2)(b+N)\, I)$.

iii) If $C_1' = C_2' = 0$ and $pq > 1$ and $\max(\gamma, \xi) > 0$, then

$$u \equiv v \equiv 0 \qquad \text{near } 0. \tag{6.56}$$

If $C_1' = C_2' = 0$ and $pq < 1$ and $\max(\gamma, \xi) < 0$, then

$$u(x) = O(|x|^{-\gamma}), \qquad v(x) = O(|x|^{-\xi}). \tag{6.57}$$

Proof. Since $a+N>0$, we have $u(x)=O(|x|^{2-N})$ from Lemma 2.3, hence $r^{N-2}\overline{u}$ has a limit $C_1\geq 0$.

i) Assume $C_1>0$, then $|x|^{N-2}u(x)\geq C>0$, from (2.2) and (2.3). Consequently

$$\Delta\overline{v}(r)\geq C\ r^{b-(N-2)q},$$

hence $b+2-(N-2)q>0$ from Lemma 2.4. Then Lemma 6.4, iii) applies. We deduce (6.51), with $C_1>0$; and $v(x)-C_2'=O(|x|^{(b+2-(N-2)q)})$. If moreover $C_2'=0$ and $(\gamma-N+2)(pq-1)>0$, then Lemma 6.5 applies after exchanging u and v. And (6.52) follows.

ii) Now assume $C_1=0$. Then u and v are bounded, hence $\overline{u},\overline{v}$ admit some limits C_1', $C_2'\geq 0$. If $C_2'>0$, then

$$\Delta\overline{u}(r)\geq C\ r^{a},$$

hence $a+2>0$ because u is bounded. Hence $\lim_{x\to 0}u(x)=C_1'$ from Lemma 6.4, iii), and $u(x)-C_1'=O(|x|^{a+2})$. And also

$$\Delta\overline{v}(r)\geq C\ r^{b+(a+2)q},$$

hence $b+2+(a+2)q=\xi(pq-1)>0$. If $C_2'>0$ and $C_1'>0$, then similarly $b+2>0$ and $\lim_{x\to 0}v(x)=C_2'$, and (6.53) follows. If $C_2'>0$ and $C_1'=0$, then Lemma 6.5 gives (6.53) and (6.54), as in the case $a+N<0$.

iii) Suppose $C_1'=C_2'=0$. Then $\lim_{x\to 0}u(x)=\lim_{x\to 0}v(x)=0$ from subharmonicity. If moreover $pq>1$ and $\max(\gamma,\xi)>0$, or $pq<1$ and $\max(\gamma,\xi)<0$, then $a+2>0$ or $b+2>0$. We can suppose that $a+2>0$. Then $(a+2)q+b+2>0$. Here again from Lemma 2.3, $v(x)\leq C\ |x|^{\widetilde{k}}$, with $\widetilde{k}=\xi(pq-1)=(a+2)q+b+2$, hence

$$\Delta\overline{u}(r)\leq C^p r^{a+\widetilde{k}p}.$$

Then any estimate $\overline{v}(r)\leq C_k\, r^k$ with $k>0$ implies

$$\Delta\overline{u}(r)\leq C_k^p\ r^{a+kp}.$$

This in turn implies $\overline{u}(r)\leq CC_k^p\ r^{a+2+kp}$, hence

$$\Delta\overline{v}(r)\leq C^qC_k^{pq}\ r^{b+(a+2)q+kpq}.$$

If $pq<1$, we get in the same way $\overline{v}(r)\leq C\ r^{-\xi}$; and

$$\Delta\overline{u}(r)\leq C\ r^{a-p\xi}=C\ r^{-\gamma-2},$$

hence $\overline{u}(r)\leq C\ r^{-\gamma}$ from Lemma 2.3, because $\gamma<0$. If $pq>1$, using a sequence as above, we deduce that $v(x)=O(|x|^m)$ for any $m>0$, hence also $u(x)=O(|x|^m)$. We can find again these results and improve the last one by using the techniques of Section 2: the mean values $\overline{u},\overline{v}$ satisfy the system

$$0\leq(r^{N-1}\overline{u}_r)_r\leq r^{N-1+a}\ \overline{v^p}, \tag{6.58}$$

$$0 \leq (r^{N-1}\overline{v}_r)_r \leq r^{N-1+b}\,\overline{u^q}, \tag{6.59}$$

and $\overline{u}, \overline{v}$ are increasing for small r; and $\lim_{r\to 0} r^{N-1}\overline{u}_r(r) = \lim_{r\to 0} r^{N-1}\overline{v}_r(r) = 0$, because $\overline{u}, \overline{v}$ are subharmonic. From Lemma 2.1 ,

$$(r^{N-1}\overline{u}_r)_r(r) \leq C\,\varepsilon^{-(N-1)p}\,r^{N-1+a}\,\overline{v}^p\,[r(1+\varepsilon)]\,.$$

We can integrate twice (6.58) between 0 and r, and get successively

$$r^{N-1}\overline{u}_r(r) \leq C\,\varepsilon^{-(N-1)p}\int_0^r s^{a+N-1}\,\overline{v}^p\,[s(1+\varepsilon)]\,ds \leq C\,\varepsilon^{-(N-1)p} r^{N+a}\,\overline{v}^p\,[r(1+\varepsilon)]\,,$$

$$\overline{u}(r) \leq C\,\varepsilon^{-(N-1)p}\int_0^r s^{a+1}\,\overline{v}^p\,[s(1+\varepsilon)]\,ds \leq C\,\varepsilon^{-(N-1)p} r^{a+2}\,\overline{v}^p\,[r(1+\varepsilon)]\,, \tag{6.60}$$

since $a+2>0$. Plugging into (6.59) and using again Lemma 2.1, we find

$$(r^{N-1}\overline{v}_r)_r(r) \leq C\,\varepsilon^{-(N-1)(p+1)q}\,r^{N-1+(a+2)q+b}\,\;\overline{v}^{pq}\,[r(1+\varepsilon)]\,, \tag{6.61}$$

for a new $\varepsilon > 0$, where $(a+2)q+b+2 = \xi(pq-1) > 0$. Then similarly by a double integration,

$$\overline{v}(r) \leq C\,\varepsilon^{-(N-1)(p+1)q}\,r^{(a+2)q+b+2}\,\overline{v}^{pq}\,[r(1+\varepsilon)]\,. \tag{6.62}$$

If $pq<1$ we find again hence $\overline{v}(r) = O(\,r^{-\xi})$ from Lemma 2.2, and $\overline{u}(r) = O(\,r^{-\gamma})$ from (6.60); thus (6.57) follows. If $pq>1$ and v is non identically 0, then $\overline{v}$ is positive for small r, and

$$\overline{v}^{-1}(r) \leq C\,\varepsilon^{-(N-1)(p+1)q/pq}\,r^{((a+2)q+b+2)/pq}\,(\overline{v}^{-1})^{1/pq}(r/(1+\varepsilon)). \tag{6.63}$$

Then from Lemma 2.2

$$\overline{v}^{-1}(r) \leq C\,r^{\xi},$$

which is impossible, since $\xi > 0$. Hence $v \equiv 0$ near 0, and $u \equiv 0$ from (6.60). And (6.56) follows. ■

Proposition 6.12 *Assume $pq \neq 1$. Let u, v be any nonnegative solutions of system (1.1), with v bounded near 0, and $a+N=0$. Then*

$$\lim_{x\to 0} v(x) = C_2' \geq 0, \tag{6.64}$$

$$\lim_{x\to 0} |x|^{N-2}\,|\ln|x||^{-1}\,u(x) = C_2'^p/(N-2). \tag{6.65}$$

If $C_2' = 0$, and $b+2-(N-2)q>0$, then the results of Proposition 6.11 are still valid (with $C_2' = 0$).

Proof. Here $a+N=0$, hence $u(x)=O(|x|^{2-N}|\ln|x||)$ from Lemma 2.3.

i) If $C_2'>0$, then again $b+2+(a+2)q>0$, that is $b+2-(N-2)q>0$. The change of variables (6.3) with $\eta=N-2$ and $\zeta=0$ gives

$$\left\{\begin{array}{l} U_{tt}+(N-2)U_t+\Delta_{S^{N-1}}U-V^p=0,\\ V_{tt}-(N-2)V_t+\Delta_{S^{N-1}}V-e^{-(b+2-(N-2)q)t}U^q=0, \end{array}\right. \tag{6.66}$$

with $\|U(t,.)\|_{C(S^{N-1})}=O(t)$. Then $\|V(t,.)-C_2'\|_{C(S^{N-1})}=O(e^{((N-2)q-b-2+\varepsilon)t})$, for any $\varepsilon>0$ small enough, from Proposition 6.3. Using the same arguments as in Proposition 6.6, iii), we deduce that $\overline{U}(t)= C_2'^p t/(N-2)+O(1)$. Then (6.65) follows

ii) If $C_2'=0$, and $b+2-(N-2)q>0$, then $\overline{U}(t)=O(1)$, hence $u(x)=O(|x|^{2-N})$. We deduce (6.51) to (6.53) and (6.55) to (6.57) as in the case $a+N>0$. ■

7 Extensions to multipower systems

Our system (1.1) is Hamiltonian, i.e. of the form

$$\left\{\begin{array}{l} -\Delta u+\partial H/\partial v=0,\\ -\Delta v+\partial H/\partial u=0, \end{array}\right. \tag{7.1}$$

with

$$H(x,u,v)=|x|^a v^{p+1}/(p+1)+|x|^b u^{q+1}/(q+1). \tag{7.2}$$

But this fact did not interfere in our proofs. In fact they extend in some measure to the system (1.27). When $pq\neq(1-s)(1-t)$, this system still admits a particular solution under the form

$$u^*(x)=\widetilde{A^*}\,|x|^{-\widetilde{\gamma}},\qquad v*(x)=\widetilde{B^*}\,|x|^{-\widetilde{\xi}}, \tag{7.3}$$

whith new $\widetilde{\gamma}$ and $\widetilde{\xi}$, given by

$$\widetilde{\gamma}=((b+2)p+(a+2)(1-s))/D,\quad \widetilde{\xi}=((a+2)q+(b+2)(1-t))/D, \tag{7.4}$$

where

$$D=pq-(1-s)(1-t), \tag{7.5}$$

and new coefficients $\widetilde{A^*},\widetilde{B^*}$, whenever $\widetilde{\gamma}(\widetilde{\gamma}+2-N)>0$ and $\widetilde{\xi}(\widetilde{\xi}+2-N)>0$. Notice the relations which extend (1.11):

$$\widetilde{\gamma}(1-s)+a+2=p\widetilde{\xi},\qquad \widetilde{\xi}(1-t)+b+2=q\widetilde{\gamma}. \tag{7.6}$$

We can extend the a priori estimates of theorem 1.3 to the new sublinear case $D<0$. For simplicity we obmit the critical cases, and just give the ideas of the proof.

Theorem 7.1 *Let $p, q, s, t, a, b \in \mathbb{R}$ with $p, q > 0$, and $s, t \in (0,1)$. Assume $pq < (1-s)(1-t)$. Let $u, v \in C^2(B')$ be any nonnegative subharmonic supersolutions of system (1.27), that is*

$$\left\{ \begin{array}{l} 0 \leq \Delta u \leq |x|^a u^s v^p, \\ 0 \leq \Delta v \leq |x|^b u^q v^t. \end{array} \right. \tag{7.7}$$

Then, up to the change from u, p, a into v, q, b,

i) if $\min(\widetilde{\gamma}, \widetilde{\xi}) > N-2$, then

$$u(x) \leq C\,|x|^{-\widetilde{\gamma}}, \qquad v(x) \leq C\,|x|^{-\widetilde{\xi}}, \tag{7.8}$$

ii) if $\widetilde{\xi} < N-2$ and $p+s > (N+a)/(N-2)$, then

$$u(x) \leq C\,|x|^{(a+2-(N-2)p)/(1-s)}, \qquad v(x) \leq C\,|x|^{2-N}, \tag{7.9}$$

iii) if $p+s < (N+a)/(N-2)$ and $q+t < (N+b)/(N-2)$, then

$$u(x) + v(x) \leq C\ \ |x|^{2-N}. \tag{7.10}$$

Proof. Here the change of variables (4.11) leads to the system

$$0 \leq (r^{3-N}\overline{\mathbf{u}}_r)_r \leq r^{1-N-\widetilde{\ell_1}}\ \overline{\mathbf{u}^s\mathbf{v}^p}, \tag{7.11}$$

$$0 \leq (r^{3-N}\overline{\mathbf{v}}_r)_r \leq r^{1-N-\widetilde{\ell_2}}\ \overline{\mathbf{u}^q\mathbf{v}^t}, \tag{7.12}$$

with

$$\widetilde{\ell_1} = (N-2)(p+s) - (N+a), \qquad \widetilde{\ell_2} = (N-2)(q+t) - (N+b). \tag{7.13}$$

They satisfy the relations

$$(1-t)\widetilde{\ell_1} + p\widetilde{\ell_2} = -D\,(\widetilde{\gamma} - (N-2)), \qquad q\widetilde{\ell_1} + (1-s)\widetilde{\ell_2} = -D\,(\widetilde{\xi} - (N-2)). \tag{7.14}$$

Assume $\min(\widetilde{\gamma}, \widetilde{\xi}) > N-2$, hence for example $\widetilde{\ell_1} > 0$. First suppose that $\overline{\mathbf{v}}$ is bounded. Then

$$\Delta\overline{u}(r) \leq C\ r^{a-(N-2)p}\overline{u^s}(r) \leq C\ r^{a-(N-2)p}\overline{u}^s(r).$$

From (1.7), and Lemma 2.1 it follows that $u(x) = O(|x|^{(a+2-(N-2)p)/(1-s)}) = O(|x|^{-\widetilde{\gamma}})$, because $\widetilde{\ell_1} > 0$ and $\widetilde{\xi} \geq N-2$. Then (7.8) is proved. Now suppose that $\overline{\mathbf{v}}$ is unbounded. From Lemma 2.1, we have

$$\overline{\mathbf{u}^s\mathbf{v}^p}(r) \leq C\ \varepsilon^{-(N-1)p}\ \overline{\mathbf{u}^s}(r)\ \overline{\mathbf{v}}^p\left[r(1-\varepsilon)\right] \leq C\ \varepsilon^{-(N-1)p}\ \overline{\mathbf{u}}^s(r)\ \overline{\mathbf{v}}^p\left[r(1-\varepsilon)\right]\ ,$$

since $s<1$. Hence

$$(r^{3-N}\overline{\mathbf{u}}_r)_r(r) \leq C\,\varepsilon^{-(N-1)p} r^{1-N-\widetilde{\ell_1}}\,\overline{\mathbf{u}}^s(r)\,\overline{\mathbf{v}}^p\left[r(1-\varepsilon)\right].$$

Similarly we can assume that $\overline{\mathbf{u}}$ is unbounded. Then integrating twice over $[r, r_0]$,we get

$$-r^{3-N}\overline{\mathbf{u}}_r(r) \leq C\,\varepsilon^{-(N-1)p} r^{1-N-\widetilde{\ell_1}}\,\overline{\mathbf{u}}^s(r)\,\overline{\mathbf{v}}^p\left[r(1-\varepsilon)\right],$$

$$\overline{\mathbf{u}}(r) \leq C\,\varepsilon^{-(N-1)p} r^{-\widetilde{\ell_1}}\,\overline{\mathbf{u}}^s(r)\,\overline{\mathbf{v}}^p\left[r(1-\varepsilon)\right],$$

which gives a majorization of $\overline{\mathbf{u}}$, since $s<1$. In the same way, since $t<1$,

$$(r^{3-N}\overline{\mathbf{v}}_r)_r \leq C\,\varepsilon^{-(N-1)q} r^{1-N-\widetilde{\ell_2}}\,\overline{\mathbf{v}}^t(r)\,\overline{\mathbf{u}}^q([r(1-\varepsilon)],$$

then with a new ε,

$$(r^{3-N}\overline{\mathbf{v}}_r)_r(r) \leq C\,\varepsilon^{-(N-1)(p/(1-s)+1)q}\,r^{1-N-(q\widetilde{\ell_1}/(1-s)+\widetilde{\ell_2})}\,\overline{\mathbf{v}}^{t+pq/(1-s)}\left[r(1-\varepsilon)\right].$$

It means that function $\overline{v}$ satisfies the shifted inequality

$$0 \leq \Delta\overline{v}(r) \leq C\,\varepsilon^{-h} r^{\sigma}\,\overline{v}^d\left[r(1-\varepsilon)\right] \tag{7.15}$$

with $d = t + pq/(1-s) < 1$, and $(N-2)d - (N+\sigma) = q\widetilde{\ell_1}/(1-s) + \widetilde{\ell_2}$. Theorem 4.1 applies and gives (7.8); and (7.9) and (7.10) follow similarly. ■

Remark 7.1 The question of a priori estimates for system 1.27 is still open in the superlinear case, and also in case $s \geq 1$ or $t \geq 1$.

References

[1] P. Aviles, *On isolated singularities in some nonlinear partial differential equations*, Indiana Univ. Math. Jour., 35 (1983), 773-79.

[2] M. Berger, P. Gauduchon and P. Mazet, *Le spectre d'une variété riemannienne,* Lect. Notes Math., 194, Berlin, Heidelberg, New-York, Springer Ed. (1971).

[3] M-F. Bidaut-Véron, *Local behaviour of solutions of a class of nonlinear elliptic systems,* Adv. in Diff. Equ. (to appear).

[4] M-F. Bidaut-Véron, V. Galaktionov, P. Grillot and L. Véron, *Singularities for a 2-dimensional semilinear elliptic equation with a non-Lipschitz linearity,* Journal Diff. Equ. (to appear).

[5] M-F. Bidaut-Véron and P. Grillot, *Asymptotic behaviour of the solutions of sublinear elliptic equations with a potential* , Applicable Anal. (to appear).

[6] M-F. Bidaut-Véron and T. Raoux, *Asymptotics of solutions of some nonlinear elliptic systems,* Comm. Part. Diff. Equ., 21 (1996), 1035-1086.

[7] M-F. Bidaut-Véron and L. Véron, *Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations,* Invent. Math., 106 (1991), 489-539.

[8] H. Brézis and L. Véron, *Removable singularities of some nonlinear elliptic equations*, Arc. Rat. Mech. Anal., 75 (1980), 1-6.

[9] D. De Figueiredo and P. Felmer, *A Liouville-type theorem for elliptic systems*, Ann. Scu. Norm. Sup. Pisa, 21 (1994), 387-397.

[10] D. De Figueiredo and P. Felmer, *On superquadratic elliptic systems,* Trans. Amer. Math. Soc., 343 (1994), 99-116.

[11] M. Garcia, R. Manasevitch, E. Mitidieri, and C. Yarur, *Existence and nonexistence of singular positive solutions for a class of semilinear elliptic systems,* Arch. Rat. Mech. Anal. 140 (1997), 253-284.

[12] B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations,* Comm. Pure and Applied Math. 34 (1981), 525-598.

[13] P. Grillot, Thèse de Doctorat, Université de Tours (1997).

[14] B Guerch and L.Véron, *Local properties of stationary solutions of some nonlinear singular Schrödinger equations,* Rev. Mat. Iberoamericana, 7 (1991), 65-114.

[15] M. Herrero and J. Velazquez, *On the dynamics of a semilinear heat equation with absorption,* Comm. Part. Diff. Equ., 14 (1989),1653-1715.

[16] J.Hulshof and R. van der Vorst, *Asymptotic behaviour of ground states*, Proc. Amer. Math. Soc. 124 (1996), 2423-2431.

[17] J. Serrin and H. Zou, *Non-existence of positive solutions for the Lane-Emden system*, Diff. Int. Equ., 9 (1996), 635-653.

[18] J. Serrin and H. Zou, *Existence of positive solutions of the Lane-Emden system* , Atti Sem. Math. Univ. Modena (1997), to appear.

[19] L. Simon, *Isolated singularities of extrema of geometric variational problems*, in: Harmonic Mappings and Minimal Immersions, Lecture Notes Math., 1161 (1985), 206-277.

[20] L. Véron, *Comportement asymptotique des solutions d'équations elliptiques semilinéaires dans* $\mathbb{R}^N$, Annali Mat. Pura Appl., 127 (1981), 25-50.

[21] L. Véron, *Singular solutions of some nonlinear elliptic equations*, Nonlinear Anal. 5 (1981), 225-242.

[22] L. Véron, Singularities of solutions of second order quasilinear equations, Pitman Research Notes in Math. Series, Longman (1996).

[23] R. van der Vorst, *Variational problems with a strongly definite structure,* Thesis, Leiden University (1994).

[24] C. Yarur, *Singularidades de ecuaciones de Schrödinger estacionarias*, Tesis doctoral, Univ. Complutense, Madrid (1984).

[25] C. Yarur, *Nonexistence of positive solutions for a class of semilinear elliptic systems* , E. Journal of Diff. Equ., 8 (1996), 1-22.