Asymptotic behaviour of the solutions of sublinear elliptic equations with a potential

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Abstract

We study the limit behaviour of nonnegative solutions of the semilinear elliptic equation with a sublinear nonlinearity and a potential

$$-\Delta u - c\frac{u}{|x|^2} + |x|^\sigma u^q = 0 \text{ in } \mathbb{R}^N \ (N \geq 3),$$

where $q \in (0,1)$, and $c, \sigma \in \mathbb{R}$. The estimates lie upon a mean value inequality for the Helmoltz operator $u \mapsto \Delta u + k^2 u$ $(k > 0)$ in case $c > 0$. The behaviour of the solutions is essentially anisotropic, with possible dead cores.

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1 Introduction

Here we consider the solutions of the semilinear elliptic equation in $\mathbb{R}^N$ $(N \geq 3)$:

$$-\Delta u - c\,\frac{u}{|x|^2} + |x|^{\sigma}\,|u|^{q-1}\,u = 0, \tag{1.1}$$

where $q, \sigma, c \in \mathbb{R}$, in the sublinear case $q \in (0,1)$. We study the behaviour of the solutions near the singular point $x = 0$. The introduction of the power σ allows to give the behaviour at infinity by using Kelvin transform, which leads to an equation of the same type, where σ is replaced by $\sigma' = (N-2)q - (N+2+\sigma)$. In the sequel, we can suppose that u is defined in $B' = B \setminus \{0\}$, where $B = \left\{x \in \mathbb{R}^N \mid |x| \leq 1\right\}$. We denote by (r, θ) the spherical coordinates of x, with $r = |x|$ and $\theta \in S^{N-1}$.

In this equation, there are two effects which tend to cancel each other: the diffusion one, coming from the Laplacian with a potential $V(x) = -c/\,|x|^2$, and the absorption one, coming from the sublinear term $u \mapsto |x|^{\sigma}\,|u|^{q-1}\,u$, which actually is non-Lipschitz. For any $q \neq 1$, defining

$$\gamma = (\sigma + 2)/(1 - q), \tag{1.2}$$

equation 1.1 admits a particular radial solution:

$$u_0(x) = A_0\,|x|^{\gamma}\,, \qquad A_0 = (\gamma^2 + (N-2)\gamma + c)^{1/(q-1)}, \tag{1.3}$$

whenever $A_0 > 0$, which is the expression of the nonlinear effect.

In the superlinear case $q > 1$, the behaviour of the solutions is well known. The essential works concern the equation without potential ($c = 0$)

$$-\Delta u + |x|^{\sigma} |u|^{q-1} u = 0, \tag{1.4}$$

see [19], [20], [7], [21]. The extension to 1.1 is carried out in [13], at least when $\sigma = 0$. All the solutions of equation 1.1 with $q > 1$ satisfy the Keller-Osserman estimate near the origin

$$u(x) \leq C |x|^{\gamma}, \tag{1.5}$$

where $C = C(N, q, \sigma)$, and the nonnegative ones present an isotropic behaviour, i.e. asymptotically radial. The behaviour can be anisotropic only in case of changing sign solutions. In the sublinear case $0 < q < 1$, up to now, no estimates had been given in the nonradial case, even in the simpler case of the equation 1.4.

In Section 2 we give a priori estimates for the nonnegative solutions. In case of equation 1.4, the estimates follow easily from corresponding estimates for the mean value

$$\bar{u}(r) = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} u(r, \theta)\, d\theta \tag{1.6}$$

of u on the sphere of radius r, because the solution u is subharmonic, and consequently satisfies the mean value inequality . In the general case of equation 1.1 this property is no longer true, as soon as $c > 0$. Then the idea is to use a local mean value inequality for the Helmoltz operator $\Delta u + k^2\, u\, (k \in \mathbb{R})$. It implies that for any subsolution of equation 1.7, any a priori estimate of the mean value implies the same estimate for the function. By this way the study reduces to an ordinary differential inequality. We show that the uniform a priori estimate 1.5is *not always true* in the sublinear case. Indeed the linear effect can get the upper hand over the nonlinear one, and the solutions can behave as the radial solutions of the linear equation

$$-\Delta u - c\frac{u}{|x|^2} = 0 \tag{1.7}$$

when such solutions exist, that is when $c \leq c^*$, where

$$c^* = (N-2)^2/4. \tag{1.8}$$

Observe that the particular solution u_0 tends to 0 at the origin whenever $\sigma + 2 > 0$.

Besides the lack of uniform apriori estimate, a second point of interest of equation 1.1 is the existence of anisotropic solutions, in particular *nonnegative ones*. Their study is the objective of Section 3. The classical change of variables

$$u(x) = |x|^{\gamma} U(t, \theta), \quad t = -Ln\, r,\ \theta \in S^{N-1}, \tag{1.9}$$

leads to the equation on $[0, +\infty) \times S^{N-1}$

$$U_{tt} + (N - 2 + 2\gamma)U_t + \Delta_{S^{N-1}} U + (\gamma(\gamma + N - 2) + c)U - |U|^{q-1} U = 0. \tag{1.10}$$

Searching anisotropic solutions independent on t reduces to finding nonconstant solutions of the equation

$$\Delta_{S^{N-1}} U + (\gamma(\gamma + N - 2) + c)U - |U|^{q-1} U = 0 \tag{1.11}$$

on the sphere S^{N-1}. Thus we show in Theorems 3.3 and 3.2 that equation 1.1, and even equation 1.4, can admit anisotropic changing sign solutions, and also positive ones, which differs from the superlinear case. Moreover we can exhibit anisotropic solutions presenting *dead cores*. This phenomenon was known in the radial case, see [14], where by Kelvin transform, it reduces to a property of compact support of the solutions, proper to sublinear problems. At last in the case $q = (N + 2 + 2\sigma)/(N - 2)$, some types of travelling waves may exist.

In Section 4 we study the precise convergence of the solutions, according to the different values of q, σ, N, c. They use some general results of convergence in a cylinder, proved in the Appendix . The behaviour of the solutions, given in Theorems 4.1 and 4.2 presents a great diversity. Going back to equation 1.4, it shows that three types of behaviour are possible when $\sigma + 2 > 0$:

Theorem 1.1 *Let $u \in C^2(B')$ be any nonnegative solution of equation 1.4.*

i) Assume $\sigma + 2 > 0$ (hence $q < 1 < (N + \sigma)/(N - 2)$). Then

$$\lim_{x \to 0} |x|^{N-2} u(x) = C_2 \geq 0. \tag{1.12}$$

If $C_2 = 0$, then

$$\lim_{x \to 0} u(x) = C_1 \geq 0, \tag{1.13}$$

hence $u \in C^0(B)$. If $C_1 = 0$, then

$$u(x) = O(|x|^\gamma). \tag{1.14}$$

ii) Assume $q < (N+\sigma)/(N-2) \leq 1$. Then either 1.12 holds, or $u \equiv 0$ near the origin.

iii) Assume $(N+\sigma)/(N-2) < q < 1$. Then 1.14 holds.

iv) Assume $q = (N+\sigma)/(N-2) < 1$. Then either

$$\lim_{x \to 0} |x|^{N-2} |Ln\,|x||^{-1/(1-q)} u(x) = ((1-q)/(N-2))^{1/(1-q)}, \tag{1.15}$$

or $u \equiv 0$ near the origin.

Concerning the solutions of 1.1 or 1.4 which satisfy $u(x) = O(|x|^\gamma)$, we prove that the limit set in $C^2(\ S^{N-1})$ of the trajectories $r^{-\gamma}u(r,.)$ as r goes to 0 is contained in the set of stationary solutions of 1.10. If moreover $\lim r_n^{-\gamma}(\sup_{|x|=r_n} u(x)) = 0$ for some sequence $r_n \to 0$, then $u \equiv 0$ near the origin. This last property was also obtained very recently and independently in [11]. As in [20] for the superlinear case, the question of exact convergence to one of these solutions is still open in case of changing sign solutions, or solutions with dead cores. It can be solved in dimension 2 when the limit-set is one-dimensional, by using Jordan curve theorem. We refer to [5] for the study of convergence for equation 1.4 in $\mathbb{R}^2$, and to [12] for a detailed study of 1.1 in dimension 2, and for the behaviour at infinity in the general case.

2 A priori estimates

2.1 Statement of the results

In this section, we give the a priori estimates for equation 1.1. In fact they are still available for subharmonic supersolutions of the equation. Their behaviour is the result of the superposition of the nonlinear effect and the linear one. When $c \leq c^*$, equation 1.7 admits two fundamental solutions ψ_1, ψ_2, given by

$$\begin{cases} \psi_1(x) = |x|^{k_1}, & \psi_2(x) = |x|^{k_2}, & \text{if } \; c < c^*, \\ \psi_1(x) = |x|^{(2-N)/2}, & \psi_2(x) = |x|^{(2-N)/2} \left| Ln \, |x| \right|, & \text{if } \; c = c^*, \end{cases} \tag{2.1}$$

where

$$\begin{cases} k_1 = -(N-2-\sqrt{D})/2, \quad k_2 = -(N-2+\sqrt{D})/2 < 0, \\ D = (N-2)^2 - 4c \end{cases} \tag{2.2}$$

(in case of equation 1.4, $\psi_1(x) = 1$, $\psi_2(x) = |x|^{2-N}$). When they do exist, functions ψ_2 and u_0 interact. When they have the same blow-up rate at 0, new logarithmic damping factors appear. Our main result is the following.

Theorem 2.1 *Let $u \in C^2(B')$ be any nonnegative solution of the inequality*

$$0 \leq \Delta u + c \frac{u}{|x|^2} \leq |x|^\sigma u^q. \tag{2.3}$$

in B'
i) If $c \leq c^*$ and $\gamma \neq k_2$, then

$$u(x) = O(\max(|x|^\gamma, \psi_2), \tag{2.4}$$

ii) If $c < c^*$ and $\gamma = k_2$, then

$$u(x) = O(|x|^{k_2} \left| Ln \, |x| \right|^{1/(1-q)}), \tag{2.5}$$

iii) If $c = c^*$ and $\gamma = k_2$, then

$$u(x) = O(|x|^{(2-N)/2} \left| Ln \, |x| \right|^{2/(1-q)}, \tag{2.6}$$

iv) If $c > c^*$, then

$$u(x) = O(|x|^\gamma). \tag{2.7}$$

In the case $c > 0$, this result is based on a local mean value inequality for the Helmoltz operator.

2.2 Subsolutions of Helmoltz equation

The following representation formula is given in 2.14 in case $N = 3$, $f = 0$. We did not find any reference in the general case, so we give here an elementary proof. It is inspired by the analogous result of [9] for the operator $-\Delta + k^2$.

Proposition 2.1 *Let Ω be a domain of $\mathbb{R}^N$ $(N \geq 2)$. Let $u \in C^2(\Omega)$, and $f = \Delta u + k^2 u$ with $k > 0$.*

i) Then the following representation formula holds, for any $x_0 \in \Omega$ and any ball $B_\rho = B(x_0, \rho) \subset \Omega$,

$$F_N(\rho)\ u(x_0) = \int_{B_\rho} f(x)\ \ [F_N(\rho)\ E_N(|x - x_0|) - E_N(\rho)\ F_N(|x - x_0|)]\ dx$$

$$+\frac{1}{|\partial B_\rho|} \int_{\partial B_\rho} u(x)\ d\sigma(x), \tag{2.8}$$

where

$$E_N(r) = \frac{\sqrt{\pi}\ \Gamma((N+1)/2)}{(N{-}1)!\ \ |S^{N-1}|} \left(\frac{2k}{r}\right)^{N/2-1} Y_{N/2-1}(kr), \tag{2.9}$$

$$F_N(r) = \Gamma(N/2)\ \left(\frac{2}{kr}\right)^{N/2-1} J_{N/2-1}(kr), \tag{2.10}$$

and J_ν, Y_ν are the Bessel and Weber functions of order ν.

ii) In particular if $f \geq 0$ in Ω, then for any $\varepsilon > 0$ there exists some $\delta = \delta(N, \varepsilon) > 0$ such that for any $x_0 \in \Omega$, and any $\rho > 0$ such that $k\ \rho \leq \delta$,

$$u(x_0) \leq \frac{1+\varepsilon}{|B_\rho|} \int_{B_\rho} u(x)\, dx. \tag{2.11}$$

Proof. i) From [8], function E_N is the fundamental solution of Helmoltz operator $\Delta + k^2$, and F_N is a particular solution of its kernel. In case $N = 3$, they have a simple expression:

$$E_3(r) = -(\cos kr)/4\pi r, \qquad F_3(r) = (\sin kr)/kr. \tag{2.12}$$

In the general case, F_N is regular in 0, with

$$\lim_{r\to 0} F_N(r) = 1, \qquad \lim_{r\to 0} r^{-1}F_N'(r) = -k^2/N, \tag{2.13}$$

by computation from the classical development of J_ν in Taylor series

$$J_\nu(r) = (\frac{r}{2})^\nu \sum_{s=0}^{+\infty} \frac{(-1)^s(r^2/4)^s}{s!\;\Gamma(\nu+s+1)}, \tag{2.14}$$

see for example [17] (p 56). Let $x_0 \in \Omega$ and $\rho > 0$ such that $B_\rho = B(x_0,\rho) \subset \Omega$,and $\alpha \in (0,\rho)$. Applying Green's formula in B_ρ, we first get

$$\int_{B_\rho} f(x)\; F_N(|x-x_0|)\; dx \;=\; \int_{\partial B_\rho} \left[\frac{\partial u}{\partial \nu}(x)\; F_N(\rho) - F_N'(\rho)\; u(x)\right]\; d\sigma(x) \tag{2.15}$$

Applying again Green's formula in the annulus $\Omega_{\alpha,\rho} = B_\rho \backslash \overline{B_\alpha}$, we get

$$\int_{\Omega_{\alpha,\rho}} f(x)\; E_N(|x-x_0|)\; dx \;=\; \int_{\partial B_\rho} \left[\frac{\partial u}{\partial \nu}(x)\; E_N(\rho) - E_N'(\rho)\; u(x)\right]\; d\sigma$$

$$+E_N'(\alpha)\int_{\partial B_\alpha} u\; d\sigma - E_N(\alpha)\int_{B_\alpha} \Delta u(x)\; dx \tag{2.16}$$

Now from [8],

$$\lim_{r\to 0} r^{N-2}E_N(r) = -1/(N-2)\left|S^{N-1}\right|, \qquad \lim_{r\to 0} r^{N-1}E_N'(r) = 1/\left|S^{N-1}\right|. \tag{2.17}$$

Hence let α go to 0, we deduce the representation formula

$$u(x_0) = \int_{B_\rho} f(x)\; E_N(|x-x_0|)\; dx \;+ \int_{\partial B_\rho} \left[E_N'(\rho)\; u(x) - \frac{\partial u}{\partial \nu}(x)\; E_N(\rho)\right] d\sigma(x). \tag{2.18}$$

Then from 2.15 and 2.18,

$$F_N(\rho)\; u(x_0) = \int_{B_\rho} f(x)\;\; [F_N(\rho)\; E_N(|x-x_0|) - E_N(\rho)\; F_N(|x-x_0|)]\;\; dx$$

$$+(E'_N\, F_N - E_N\, F'_N)(\rho) \int_{\partial B_\rho} u(x)\; d\sigma(x). \tag{2.19}$$

Now the function $r \longmapsto r^{N-1}(E_N\, F'_N - E'_N\, F_N)(r)$ is constant, because E_N and F_N are in the kernel of $\Delta + k^2$. Therefore from 2.13 and 2.17, we have

$$r^{N-1}(E'_N\, F_N - E_N\, F'_N)(r) = \frac{1}{|S^{N-1}|}, \tag{2.20}$$

and 2.8 follows.

ii) Assume that $f \geq 0$ in Ω. Let $\delta_0 = \delta_0(N)$ the first zero of the function $J_{N/2-1}$. If $k\, \rho < \delta_0$, then $F_N(\rho) > 0$, and

$$F_N(\rho)E_N(r) - E_N(\rho)F_N(r) \leq 0 \qquad \text{on } [0, \rho]\,,$$

because E_N/F_N is nondecreasing, from 2.20. Hence we obtain the inequality

$$F_N(\rho)\; u(x_0) \leq \frac{1}{|\partial B_\rho|} \int_{\partial B_\rho} u(x)\; d\sigma(x), \qquad \text{if } \; k\, \rho < \delta_0. \tag{2.21}$$

Moreover for any $\varepsilon > 0$ there exists some $\delta = \delta(N, \varepsilon) \in (0, \delta_0)$ such that $F_N(\rho) \geq 1/(1+\varepsilon)$ for any $\rho < \delta/k$, hence 2.11 holds. ∎

2.3 Reduction to the radial case

This mean value inequality allows us to extend to the subsolutions of equation 1.7 a well-known property of subharmonic functions, still used in [22] and [5].

Proposition 2.2 *Let $u \in C^2(B')$ be any nonnegative solution of the inequality*

$$0 \leq \Delta u + c\frac{u}{|x|^2} \tag{2.22}$$

in B', with $c \in \mathbb{R}$. Suppose that the mean value $\overline{u}$ of u satisfies an estimate of the form

$$\overline{u}(r) = O(|Lnr|^b\, r^a) \quad \textit{as } r \to 0, \tag{2.23}$$

for some $a, b \in \mathbb{R}$. Then u satisfies the corresponding estimate

$$u(x) = O(|Ln\,|x||^b\, |x|^a) \quad \textit{as } x \to 0. \tag{2.24}$$

Proof. Let $x_0 \in \mathbb{R}^N$ such that $2x_0 \in B'$. Then $B(x_0, |x_0)| /m) \subset B'$ for any $m \geq 2$. Then

$$0 \leq \Delta u + k^2 \, u \qquad \text{in } B(x_0, |x_0)| /m), \tag{2.25}$$

with

$$k = m\sqrt{c^+}/(m-1) \, |x_0| \, . \tag{2.26}$$

From Proposition 2.1 if $c > 0$, and from the classical mean value inequality if $c \leq 0$, there exists $\delta = \delta(N)$ such that for any $\rho \in (0, \delta/k)$,

$$u(x_0) \leq \frac{2}{|B(x_0, \rho)|} \int_{B(x_0,\rho)} u(x) \, dx. \tag{2.27}$$

Choosing m large enough, precisely $m > 1 + \delta^{-1}\sqrt{c^+}$, we deduce that

$$u(x_0) \leq \frac{2m^N}{|x_0|^N \, |B|} \int_{B(x_0,|x_0)|/m)} u(x) \; dx. \tag{2.28}$$

Hence denoting $C_m = \left\{ x \in \mathbb{R}^N \, |(m-1) \, |x_0| \, /m \leq |x| \leq (m+1) \, |x_0| \, /m \right\}$,

$$u(x_0) \leq \frac{2m^N}{|x_0|^N \, |B|} \int_{C_m} u(x) \; dx$$

$$\leq \frac{2m^N N}{|x_0|^N} \int_{(m-1)|x_0|/m}^{(m+1)|x_0|/m} r^{N-1} \overline{u}(r) \; dr \leq \frac{C}{|x_0|} \int_{(m-1)|x_0|/m}^{(m+1)|x_0|/m} \overline{u}(r) \; dr, \tag{2.29}$$

where $C = C(m, N)$. The result follows by integration. ■

2.4 Proof of the estimates

Proof of Theorem 2.1. From Jensen inequality, since $q < 1$, we observe that $\overline{u}$ also satisfies 2.3. From Proposition 2.2, we are reduced to estimate $\overline{u}$. In the sequel, the letter C denotes some constants which may depend on u, q, σ, N, c , but not on $x \in B$.

Step 1: $c \leq c^*$. Let us make the change of variables

$$\overline{u}(r) = r^{k_2} w(r), \tag{2.30}$$

which leads to the inequality

$$0 \leq w_{rr} + (N-1+2k_2)\frac{w_r}{r} \leq r^{\sigma - k_2(1-q)} w^q, \tag{2.31}$$

for $0 < r \leq 1$. This implies that w is monotonous for $r \leq r_0$ small enough. Denoting

$$h_2 = k_2(1-q) - (2+\sigma) = (1-q)(k_2 - \gamma), \tag{2.32}$$

then 2.31 reduces to

$$0 \leq (r^{N-1+2k_2} w_r)_r \leq r^{N-3+2k_2-h_2} w^q. \tag{2.33}$$

If w is bounded, then $\overline{u}(r) \leq Cr^{k_2}$ in $(0,1]$, and 2.4, 2.5, or 2.6 holds. Now suppose that w is unbounded, then it is necessarily nonincreasing.

i) *First case :* $\gamma < k_2$ (i.e. $h_2 > 0$). Integrating over $[r, r_0]$, we get from monotonicity

$$-r^{N-1+2k_2} w_r(r) \leq C + w^q(r) \int_r^{r_0} s^{N-3+2k_2-h_2} ds, \tag{2.34}$$

where $N-3+2k_2-h_2 < -1$. This implies

$$-w_r(r) \leq C\, r^{-1-h_2} w^q(r), \tag{2.35}$$

and by a new integration, since $h_2 > 0$,

$$w(r) \leq C + w^q(r) \int_r^{r_0} s^{-1-h_2} ds \leq C\, r^{-h_2} w^q(r). \tag{2.36}$$

Then $\overline{u}(r) \leq C\, r^{\gamma}$ by returning to $\overline{u}$, and 2.4 follows.

ii) *Second case :* $\gamma > k_2$ (i.e. $h_2 < 0$). As above, 2.34 holds . If $N-3+2k_2-h_2 < -1$, then 2.35 holds, with $h_2 < 0$. Integrating again 2.35, we deduce that w is bounded, which is a contradiction. If $N-3+2k_2-h_2 = -1$, then

$$-w_r(r) \leq C\, r^{-1-h_2} \left| Lnr \right| w^q(r) \leq C\, r^{-1-h/2} w^q(r), \tag{2.37}$$

and by a new integration, w is still bounded. Now consider the case $N-3+2k_2-h_2>-1$. Then 2.34 implies

$$-w_r(r) \leq C\, r^{1-N-2k_2} w^q(r). \tag{2.38}$$

Integrating 2.38 gives

$$w(r) \leq C + w^q(r) \int_r^{r_0} s^{1-N-2k_2} ds. \tag{2.39}$$

If $c \neq c^*$, then $1-N-2k_2>-1$, w is still bounded. If $c=c*$, then 2.38 leads to the first estimate

$$w(r) \leq C \ \left| Lnr \right|^{1/(1-q)}. \tag{2.40}$$

By report in 2.33,

$$(rw_r)_r(r) \leq C\, r^{-1-h_2} \left| Lnr \right|^{1/(1-q)} \leq C\, r^{-1-h/2}, \tag{2.41}$$

hence $rw_r(r) \leq C$, and

$$w(r) \leq C \ \left| Lnr \right|, \tag{2.42}$$

which in turn implies 2.4.

iii) *Third case :* $\gamma = k_2$ (i.e. $h_2 = 0$). If $c \neq c^*$, then $N-3+2k_2<-1$, hence 2.35 holds with $h=0$, hence 2.40 follows by integration, and 2.5 holds. If $c=c^*$, then 2.34 implies

$$-w_r(r) \leq C\, r^{-1} \left| Lnr \right| w^q(r) \tag{2.43}$$

and by integration

$$w(r) \leq C \ \left| Lnr \right|^{2/(1-q)}, \tag{2.44}$$

and 2.6 holds.

Step 2 : $c>c^*$. Here no change of variable of the form $\overline{u}(r) = r^\ell w(r)$ ($\ell \in \mathbb{R}$) can lead to a monotonous function. Therefore we set

$$\overline{u}(r) = r^\gamma Y(t), \quad t = -Ln\, r, \tag{2.45}$$

that is $Y(t) = \overline{U}(t)$, where U is given in 1.9 . Then Y satisfies

$$0 \leq Y_{tt} + (N - 2 + 2\gamma)Y_t + (\gamma(\gamma + N - 2) + c)Y \leq Y^q, \tag{2.46}$$

from Jensen inequality. From Young inequality, it implies, for any $\varepsilon > 0$,

$$Y_{tt} + (N - 2 - 2\gamma)Y_t + (\gamma(\gamma + N - 2) + c - \varepsilon)Y \leq \varepsilon^{-q/(1-q)}. \tag{2.47}$$

Now we can apply 5.1 with $A = N - 2 - 2\gamma$, and $B = \gamma(\gamma + N - 2) + c - \varepsilon = A^2/4 + c - c^* - \varepsilon > 0$ for ε small enough. Then Y is bounded, and 2.7 holds, which completes the proof. ■

3 Anisotropic solutions

Here we study the structure of the solutions of equation on S^{N-1}

$$\Delta_{S^{N-1}}\omega + \lambda\omega - |\omega|^{q-1}\omega = 0, \tag{3.1}$$

for any $\lambda > 0$, which governs the existence of anisotropic solutions of 1.1: to each solution ω of 3.1 corresponds a solution of equation 1.1, given by

$$u(x) = u(|x|, \theta) = |x|^\gamma \omega(\theta), \qquad \theta \in S^{N-1}, \tag{3.2}$$

if

$$\lambda = \gamma(\gamma + N - 2) + c. \tag{3.3}$$

The set E_λ of the solutions of equation 3.1 always contains the three constant solutions 0 and $\pm\lambda^{1/(q-1)}$.

3.1 Non-existence results

First we give the extremal values of λ for the existence of possibly nonconstant solutions. The first result is elementary, but the second one is far from being evident.

Theorem 3.1 *i) If $\lambda \leq N-1$, then equation 3.1 admits no nonconstant solution.*

ii) If $\lambda(1-q) \leq N-1$, then it admits no positive nonconstant solution.

Proof. i) Let $\omega \in E_\lambda$, and denote by $\overline{\omega}$ its average on S^{N-1}. Then following techniques of [19] , [20],

$$\Delta_{S^{N-1}}(\omega - \overline{\omega}) + \lambda(\omega - \overline{\omega}) - (|\omega|^{q-1}\omega - |\overline{\omega}|^{q-1}\overline{\omega}) = 0, \tag{3.4}$$

hence

$$\begin{aligned}\lambda \int_{S^{N-1}} (\omega - \overline{\omega})^2 &= \int_{S^{N-1}} \left[|\nabla(\omega - \overline{\omega})|^2 + (|\omega|^{q-1}\omega - |\overline{\omega}|^{q-1}\overline{\omega})(\omega - \overline{\omega}) \right] \\ &\geq \int_{S^{N-1}} \left[(N-1)(\omega - \overline{\omega})^2 + (|\omega|^{q-1}\omega - |\overline{\omega}|^{q-1}\overline{\omega})(\omega - \overline{\omega}) \right],\end{aligned} \tag{3.5}$$

since $N-1$ is the first nonzero eigenvalue of $-\Delta_{S^{N-1}}$. If $\lambda \leq N-1$, it implies $\omega = \overline{\omega}$.

ii) Here we use the proof of analogous result of [3](Theorem 6.1) in case of the equation with the other sign :

$$\Delta_{S^{N-1}}\omega - \lambda\omega + \omega^q = 0 \quad \text{on } S^{N-1}, \quad \text{with } q > 1. \tag{3.6}$$

The idea, which comes from Bernstein methods, is to get an estimate of $|\nabla\omega|^2$. Since $\omega > 0$, we can apply the Böchner-Weitzenböck formula on S^{N-1},

$$\frac{1}{2}\Delta_{S^{N-1}}(|\nabla v|^2) = |Hess\ v|^2 + < \nabla\Delta_{S^{N-1}}v, \nabla v > + (N-2)|\nabla v|^2 \tag{3.7}$$

to the function $v = \omega^{1-y}$, and multiply it by the function ω^δ, where y and δ are two real parameters, with $y \neq 1$. Then we integrate over S^{N-1} and integrate by parts several times as in [3]. After computations, using the inequality $|Hess\ v|^2 \geq \frac{1}{N-1}(\Delta_{S^{N-1}}v)^2$ and returning to the function ω, we derive the following inequality :

$$C_1 \int_{S^{N-1}} \omega^{\delta-2-2y} |\nabla\omega|^4 + C_2 \int_{S^{N-1}} \omega^{q-1-2y} |\nabla\omega|^2 + C_3 \int_{S^{N-1}} \omega^{-2y} |\nabla\omega|^2 \leq 0, \tag{3.8}$$

where

$$C_1 = -\frac{N-2}{N-1}y^2 + \delta y - \frac{1}{2}(\delta^2 - \delta), \tag{3.9}$$

$$C_2 = \frac{1}{N-1}\left[N-2)q - \frac{N+1}{2}\delta\right], \tag{3.10}$$

$$C_3 = N - 2 - \lambda\left[\frac{N-2}{N-1} - \frac{N+1}{2(N-1)}\delta\right]. \tag{3.11}$$

Now we take $\delta = 2(N-2)q/(N+1)$, so that

$$C_2 = 0, \qquad C_3 = (N-2)\left[1 - \lambda(1-q)/(N-1)\right] \geq 0 \tag{3.12}$$

because $\lambda(1-q) \leq N-1$. And we can choose $y \neq 1$ such that $C_1 > 0$. Indeed the discriminant of the trinomial relative to y is positive, since $q < 1$. Then $|\nabla\omega| = 0$ on S^{N-1}. ■

3.2 Positive solutions

Now we prove the existence of positive solutions for an infinite set of values of λ. Denote by $(\mu_j)_{j\in\mathbb{N}}$ the sequence of eigenvalues of $-\Delta_{S^{N-1}}$ on S^{N-1}, given by $\mu_j = j(j+N-2)$.

Theorem 3.2 *For any $j \geq 1$, equation 3.1 admits a continuum of positive solutions for any λ in a small neighborhood of $\lambda_j = \mu_j/(1-q)$.*

Proof. We look for bifurcation branches emanating from the constant solution $\omega_0 = \lambda^{1/(q-1)}$. In order to avoid the question of multiplicity of the eigenvalues of $-\Delta_{S^{N-1}}$, we consider solutions ω which are axially symmetric by respect to some diameter, that means they depend only on some polar angle $\phi \in (0,\pi)$. Then the equation reduces to

$$L\omega(\phi) = \sin^{2-N}\phi(\sin^{N-2}\phi\, \omega_\phi)_\phi = \omega^q - \lambda\omega \text{ on } (0,\pi)\,. \tag{3.13}$$

Now $(I-L)^{-1}$ is a compact self-adjoint operator in the weighted space

$$L^2_*((0,\pi)) = \left\{ \omega \in \mathcal{D}'((0,\pi)) \,\middle|\, \int_0^\pi \omega^2(\phi) \sin^{N-2}\phi \, d\phi < +\infty \right\}. \tag{3.14}$$

And $-L$ and $-\Delta_{S^{N-1}}$ have the same eigenvalues (see [2],[3]), and each eigenspace of $-L$ is one-dimensional. Setting $\omega = \omega_0 + v$, we write 3.13 under the form

$$f(\mu, v) = Lv + \mu v - \left[((\frac{\mu}{1-q})^{1/(q-1)} + v)^q - (\frac{\mu}{1-q})^{q/(q-1)} - q\frac{\mu}{1-q} v \right], \tag{3.15}$$

where $\mu = (1-q)\lambda$. Now the local bifurcation theorem applies to the function f in a neighborhood of $(\mu_k, 0)$ in $\mathbb{R} \times X$, with

$$X = \left\{ v \in C^2([(0,\pi)]) \,|\, v_\phi(0) = v_\phi(\pi) = 0 \right\}. \tag{3.16}$$

Hence a branch of bifurcation emanates from this point. ■

As a consequence, solving equation 3.3 by respect to γ, we get the following for equation 1.4.

Corollary 3.1 *For any integer* $j > \max(2+\sigma, -N-\sigma)$, *and for any* q *in a small neighborhood of*

$$q_j = 1 - ((2+\sigma)^2 / \left[j^2 + (N-2)j - (N-2)(2+\sigma) \right], \tag{3.17}$$

equation 1.4 admits anisotropic positive solutions of the form 3.2 in $\mathbb{R}^N \backslash \{0\}$.

3.3 Changing sign solutions

Now we prove the existence of changing sign solutions of equation 3.1 for any $\lambda > N-1$, which vanish on an equator of S^{N-1}. Let $(e_1, e_2, .., e_N)$ be the canonical basis in $\mathbb{R}^N$, and $(S^{N-1})^+ = S^{N-1} \cap (\mathbb{R}^{N-1} \times \mathbb{R}^+)$.

Theorem 3.3 *For any* $\lambda > N-1$, *equation 3.1 admits nontrivial changing sign solutions, which vanish on the equator, and are nonnegative on the half sphere* $(S^{N-1})^+$, *nonpositive on the complementary, and axially symmetric by respect to* e_N.

Proof. We consider the following problem of minimization with constraints in the space $V = H_0^1((S^{N-1})^+)$:

$$m = \inf_{w\in\Sigma} J(w) = \inf_{w\in\Sigma} \int_{(S^{N-1})^+} (|\nabla w|^2 - \lambda w^2), \tag{3.18}$$

where $\Sigma = \left\{ w \in V \,\middle|\, \int_{(S^{N-1})^+} |w|^{q+1} = 1 \right\}$. Let $(\rho_n)_{n\geq 1}$ be the sequence of eigenvalues of $-\Delta_{S^{N-1}}$ in V. Then $\rho_1 = N-1 < \lambda$ by hypothesis. First observe that $m < 0$: denoting by φ_1 the first positive eigenvector such that $\|\varphi_1\|_{L^2((S^{N-1})^+)} = 1$, and setting $t_1 = (\int_{(S^{N-1})^+} |\varphi_1|^{q+1})^{-1/(q+1)}$, then $t_1\varphi_1 \in \Sigma$ and $J(\ t_1\varphi_1) = t_1^2(\rho_1 - \lambda) < 0$. Let $k \geq 1$ such that $\rho_k \leq \lambda < \rho_{k+1}$, and denote by E_k the eigenspace relative to ρ_k and by E_k' its othogonal space in V. Consider a minimizing sequence (w_n), such that $J(w_n) \leq 0$. Then

$$w_n = v_n + v_n', \qquad v_n \in E_k, \quad v_n' \in E_k', \tag{3.19}$$

and

$$\begin{aligned} J(w_n) &= \textstyle\int_{(S^{N-1})^+} (|\nabla v_n|^2 - \lambda v_n^2 + \int_{(S^{N-1})^+} (|\nabla v_n'|^2 - \lambda v_n'^2) \\ &\geq -(\lambda - \rho_1) \|v_n\|^2_{L^2((S^{N-1})^+)} + (1 - \lambda/\rho_{k+1}) \|\nabla v_n'\|^2_{L^2((S^{N-1})^+)} \end{aligned} \tag{3.20}$$

Suppose that (w_n) is unbounded in V . Then (v_n) is unbounded in the finite dimensional space E_k , since $J(w_n) \leq 0$. After extraction, we can assume that $\|v_n\|_{E_k} \to +\infty$. Let

$$y_n = v_n / \|v_n\|_{E_k}, \qquad y_n' = v_n' / \|v_n\|_{E_k}. \tag{3.21}$$

Then $\|y_n\|_{E_k} = 1$, and y_n' is bounded in V from 3.20. After extraction, (y_n) converges to some $y \in E_k$, with $\|y\|_{E_k} = 1$, and (y_n') converges to some y' weakly in V and strongly in $L^{q+1}\left[S^{N-1})^+\right]$. But $(y_n + y_n')$ converges to 0 in $L^{q+1}\left[(S^{N-1})^+\right]$, since $w_n \in \Sigma$. Then $y + y' = 0$, and $y = y' = 0$, since $y' \in E_k'$. Hence we arrive to a contradiction. Therefore (w_n) is bounded in V. After extraction it converges weakly to some $w \in V$. Clearly m is finite and $J(w) = J(|w|) = m$. And $\omega = |m|^{1/(1-q)} |w|$ is a nonnegative solution of equation 3.1 on $(S^{N-1})^+$, which vanishes on the equator. By reflection, it gives a solution of the equation in whole S^{N-1}, which in fact belongs to $C^2(S^{N-1})$. Now the same proof applies in the subspace of V of the axially symmetric functions by respect to e_N, and gives the existence of an axially symmetric solution. ■

Remark 3.1 Here also the solution constructed only depends on the polar angle with e_N. In such a way we have proved the existence of a solution ω of equation 3.13 in space X , nonnegative on $[0,\pi/2)$, with $\omega(\pi-\phi)=\omega(\phi)$ on $[0,\pi]$.

Corollary 3.2 *If* $1>q>\min\left[(N+1+\sigma)/(N-1),-1-\sigma\right]$, *then equation 1.4 admits anisotropic solutions of the form 3.2 in* $\mathbb{R}^N\backslash\{0\}$, *positive in an half-space and negative in the other one.*

Remark 3.2. In the same way, following the idea of [13], we can construct solutions of 3.1, which change of sign on more general sets. Let G be any finite subgroup of $O(N)$ generated by reflections through hyperplanes containing 0, and S be a fundamental domain for G. Let $\rho_1(S)$ be the first eigenvalue of $-\Delta_{S^{N-1}}$ in $H_0^1(S)$. Then if $\lambda>\rho_1(S)$, we can construct a solution of 3.1 in $H_0^1(S)$, and then extend it by reflections to whole S^{N-1}.

3.4 Nonnegative solutions with possible zeros

For some values of λ, we find explicit solutions of 3.13 with double zeros, so we can exhibit some nonnegative anisotropic solutions of 1.1 with a nonempty set of zeros.

i) Solutions vanishing on a half axis.
If

$$\lambda=\widetilde{\lambda}=\frac{1}{1-q}(N-2+\frac{1}{1-q}) \tag{3.22}$$

then equation 3.13 admits a solution $\widetilde{\omega}$, positive on $[0,\pi)$, with $\widetilde{\omega}(\pi)=0:$

$$\widetilde{\omega}(\phi)=(2\widetilde{C}\ \cos^2\frac{\phi}{2})^{1/(1-q)},\qquad \text{with } \widetilde{C}=\frac{(1-q)^2}{N-1-(N-3)q}. \tag{3.23}$$

Consequently, when $\sigma=\widetilde{\sigma}$, associated to $\widetilde{\lambda}$ by 1.2 and 3.3, equation 1.1 admits a nonnegative solution with a zero set which is *an half axis*, given by

$$\widetilde{u}(x)=\left[\widetilde{C}\ |x|^{(\widetilde{\sigma}+1)}(|x|+x_n)\right]^{1/(1-q)}, \tag{3.24}$$

which vanishes on $\{ t\, e_N \mid t \leq 0 \}$ and is positive on the complementary part. In case of equation 1.4, it corresponds to the case $\sigma = \widetilde{\sigma} = -1$ of the equation

$$-\Delta u + |x|^{-1} u^q = 0 \tag{3.25}$$

which admits the solution $\widetilde{u}(x) = \left[\widetilde{C}\, (|x| + x_n) \right]^{1/(1-q)}$ in whole $\mathbb{R}^N$, or to the case $\sigma = \widetilde{\sigma}' = (N-2)q - (N+1)$ of the equation deduced by Kelvin transform.

ii) Solutions vanishing on an half space.
If

$$\lambda = \widehat{\lambda} = \frac{2}{1-q}(N - 2 + \frac{2}{1-q}) \tag{3.26}$$

then equation 3.13 admits a solution $\widehat{\omega}_1$, which only vanishes at $\pi/2$,

$$\widehat{\omega}_1(\phi) = \ (\widehat{C}_1 \cos^2 \phi)^{1/(1-q)}, \qquad \text{with } \widehat{C}_1 = \frac{(1-q)^2}{2(1+q)}, \tag{3.27}$$

and also the solution

$$\widehat{\omega}_1'(\phi) = \begin{cases} \widehat{\omega}_1(\phi), & \text{if } \phi \in [0, \pi/2) \\ 0, & \text{if } \phi \in [\pi/2, \pi]\,. \end{cases} \tag{3.28}$$

which presents a dead core on $[\pi/2, \pi]$, and of course $\widehat{\omega}_1''(\phi) = \widehat{\omega}_1'(\pi - \phi)$. It also admits a solution $\widehat{\omega}_2$, positive on $(0, \pi)$, with $\widehat{\omega}_2(0) = \widehat{\omega}_2(\pi) = 0$:

$$\widehat{\omega}_2(\phi) = (\widehat{C}_{N-1} \sin^2 \phi)^{1/(1-q)}, \qquad \text{with } \widehat{C}_{N-1} = \frac{(1-q)^2}{2(N-1-(N-3)q)}. \tag{3.29}$$

Consequently, when $\sigma = \widehat{\sigma}$, associated to $\widehat{\lambda}$, equation 1.1 admits nonnegative solutions with *an half space, an hyperplane* or *an axis* as a zero set, given by

$$\widehat{u}_1(x) = \left[\widehat{C}_1\ |x|^{\widehat{\sigma}} x_n^2 \right]^{1/(1-q)}, \qquad \widehat{u}_1'(x) = \left[\widehat{C}_1\ |x|^{\widehat{\sigma}} (x_n^+)^2 \right]^{1/(1-q)}, \tag{3.30}$$

$$\widehat{u}_2(x) = \left[\widehat{C}_2\ |x|^{\widehat{\sigma}}\ |(x_1, x_2, .., x_{n-1})|^2 \right]^{1/(1-q)}. \tag{3.31}$$

In case of equation 1.4, up to a Kelvin transform, it corresponds to the simple case $\sigma = \widehat{\sigma} = 0$ of the equation

$$-\Delta u + u^q = 0. \tag{3.32}$$

Here these solutions can be obtained elementarily, noticing that this equation admits the solutions

$$\widehat{u}_k(x) = \left[\widehat{C}_k \left|(x_1, x_2, .., x_k)\right|^2\right]^{1/(1-q)}, \qquad \widehat{C}_k = \frac{(1-q)^2}{2(k-(k-2)q)}, \tag{3.33}$$

for $k = 1, 2, .., N.$

iii) Comments.

We conjecture that the value $\widetilde{\lambda}$ coincides with the first appearance of solutions of equation 3.13 in X with a nonempty range of zeros, and that for any $\lambda \geq \widetilde{\lambda}$, there exists at least a solution vanishing on an interval $[a_\lambda, \pi]$. A numerical approach also suggests that it is unique if and only if $\lambda \in \left[\widetilde{\lambda}, \widehat{\lambda}\right)$, and then the function $\lambda \mapsto a_\lambda$ is decreasing . Such solutions would give solutions of equation 1.1 with *a cone* as a zero set.

This conjecture is true in dimension 2 (see [5]), where one can give a complete description of the solutions of equation 3.1, because it becomes autonomous . Recall that if $N = 2$, the function

$$\widehat{\omega}_{2,\lambda}(\phi) = \begin{cases} \left[\widehat{C}_\lambda \cos^2((1-q)\sqrt{\lambda}\phi/2)\right]^{1/(1-q)}, & \text{if } |\phi| \in \left[0, \pi/(1-q)\sqrt{\lambda}\right) \\ 0, & \text{if not.} \end{cases} \tag{3.34}$$

where $\widehat{C}_\lambda = 2/\lambda(1+q)$, is a solution of equation 3.1 on S^1 as soon as $\lambda > \widetilde{\lambda} = 1/(1-q)^2$. The lack of uniqueness occurs when $\lambda > \widehat{\lambda} = 4/(1-q)^2$, because of the possible superposition of translated of $\widehat{\omega}_{2,\lambda}$, and this since its least period becomes less than π.

3.5 The case $c > c^*$ and $q = q^*$

In this case many other anisotropic solutions appear. The value $q = q^* \in (0, 1)$, which supposes $\sigma + 2 \in ((2-N)/2, 0)$, corresponds to the case

$$\gamma = -(N-2)/2, \tag{3.35}$$

where equation 1.10 takes the form

$$U_{tt} + \Delta_{S^{N-1}}U + (c - c^*)U - |U|^{q-1}U = 0. \tag{3.36}$$

with $c - c^* > 0$. It is the delicate case of an equation without term in U_t, hence with conservation of the energy, and with nontrivial constant solutions. Here also the situation has to be compared with the case of the equation with the other sign , see [3], and [16]:

$$\Delta u + c\,\frac{u}{|x|^2} + |x|^\sigma |u|^{q-1}u = 0, \quad \text{with } q = q^* > 1, \tag{3.37}$$

when $c < c^*$ and $\sigma + 2 > 0$,which leads to the equation in the cylinder

$$U_{tt} + \Delta_{S^{N-1}}U - (c^* - c)U + |U|^{q-1}U = 0. \tag{3.38}$$

In addition to the stationary solutions still described above, equation 3.36 admits also solutions *independent of* $\theta \in S^{N-1}$, that means solutions of equation

$$U_{tt} + (c - c^*)U - |U|^{q-1}U = 0, \tag{3.39}$$

on $[0, +\infty)$. Those solutions (which include the solutions of 3.1 in dimension 2) are completly described in [5]. In particular, for any $\tau \geq \pi/(1-q)\sqrt{c - c^*}$, equation 3.39 admits the nonnegative solution $U_\tau(t) = \widehat{\omega}_{2,c-c^*}(t - \tau)$, which has compact support $\left[\tau - \pi/(1-q)\sqrt{c - c^*},\ \tau + \pi/(1-q)\sqrt{c - c^*},\right]$. And the possible superposition of such solutions gives the existence of *radial* solutions of equation 1.1 which vanish on a union of disjoint rings.

Moreover, as in [3], equation 3.36 can also admit elliptic waves as solutions, of the form $U(t, \theta) = \varpi(e^{t\mathcal{A}}\theta)$, where $\mathcal{A}$ is a skew-symmetric matrix of dimension N. Following the proof of Theorems 3.2 and 3.3 one can show the existence of positive or changing sign waves.

Remark 3.3 In dimension 2, when $\sigma + 2 = 0$ and $c > 0$, we find in the same way solutions of the form $U(t, \theta) = \varpi(\theta + \alpha t)$ $(\alpha \in \mathbb{R})$. It reduces to the equation

$$(1 + \alpha^2)\,\varpi_{\theta\theta} + c\,\varpi - |\varpi|^{q-1}\varpi = 0 \tag{3.40}$$

on S^1, of the same form as 3.1 and 3.39. Thus we can also construct elliptic waves with dead cores: $U(t, \theta) = \widehat{\omega}_{2,c}\left[(\theta + \alpha t)/\sqrt{(1 + \alpha^2)}\right]$, whenever $c > (1 + \alpha^2)/(1 - q)^2$.

4 Convergence properties

4.1 Statement of the results

In this paragraph we give the precise behaviour of the nonnegative solutions of equation 1.1. It depends on the position of q by respect to some critical values depending on σ. Let us denote, in case $c \leq c^*$,

$$q_2 = 1 - \frac{\sigma + 2}{k_2}, \tag{4.1}$$

(hence $q_2 = (N + \sigma)/(N - 2)$ if $c = 0$), and

$$q_1 = 1 - \frac{\sigma + 2}{k_1}, \tag{4.2}$$

whenever $c \neq 0$. Observe that $k_1 > k_2$ and

$$\begin{cases} \gamma > k_2 \Longleftrightarrow q < q_2, \\ \gamma > k_1 \Longleftrightarrow c(q - q_1) < 0 \quad \text{or } (c = 0 \text{ and } \sigma + 2 > 0), \end{cases} \tag{4.3}$$

and

$$\sigma + 2 \geq 0 \Rightarrow q < q_2 \ \text{ and } \ (q < q_1 \ \text{ if } \ c > 0). \tag{4.4}$$

In case $c \geq c^*$, we define

$$q^* = (N + 2 + 2\sigma)/(N - 2). \tag{4.5}$$

Notice that $q_1 = q_2 = q*$ when $c = c^*$.

In the noncritical cases, we obtain the following.

Theorem 4.1 *Let $u \in C^2(B')$ be any nonnegative solution of equation 1.1 in B'.*
i) Assume $c < c^*$, and $c(q - q_1) < 0$ or $c = 0 < \sigma + 2$ (hence $q < q_2$). Then

$$\lim_{x \to 0} |x|^{-k_2} u(x) = C_2 \geq 0. \tag{4.6}$$

If $C_2 = 0$, then

$$\lim_{x \to 0} |x|^{-k_1} u(x) = C_1 \geq 0. \tag{4.7}$$

If $C_1 = 0$, *then*

$$u(x) = O(|x|^{\gamma}) \tag{4.8}$$

and the limit set in $C^2(\ S^{N-1})$ *of* $r^{-\gamma}u(r,.)$ *as* r *goes to 0 is contained in the set of solutions of equation*

$$\Delta_{S^{N-1}}\omega + (\gamma(\gamma+N-2)+c)\omega - \omega^q = 0. \tag{4.9}$$

If moreover $\lim r_n^{-\gamma}(\sup_{|x|=r_n} u(x)) = 0$ *for some sequence* $r_n \to 0$, *then* $u \equiv 0$ *near the origin.*

ii) Assume $c < c^*$, *and* $c(q-q_1) > 0$ *or* $c = 0 > \sigma + 2$, *and* $q < q_2$. *Then*

$$\lim_{x\to 0} |x|^{-k_2} u(x) = C_2 > 0, \quad \text{or } u \equiv 0 \text{ near the origin.} \tag{4.10}$$

iii) Assume $c < c^*$ *and* $q > q_2$. *Then 4.8 holds, and the behaviour is as above.*
iv) Assume $c > c*$ *and* $q \neq q^*$. *Then 4.8 still holds, and the behaviour is as above.*

Now we study the critical cases, except the case $c > c^*$, $q = q^*$.

Theorem 4.2 *Let* $u \in C^2(B')$ *be any nonnegative solution of equation 1.1 in* B'.
i) Assume $c < c^*$, *and* $q = q_2$. *Then*

$$\lim_{x\to 0} |x|^{-k_2} |Ln\,|x||^{-1/(1-q)} u(x) = ((1-q)/\sqrt{D})^{1/(1-q)}, \text{ or } u \equiv 0 \text{ near } 0. \tag{4.11}$$

ii) Assume $c < c^*$, *and* $q = q_1$, $c \neq 0$, *or* $c = 0 = \sigma + 2$. *Then 4.10 still holds.*

iii) Assume $c = c^*$ *and* $q < q^*$. *Then*

$$\lim_{x\to 0} |x|^{(N-2)/2} |Ln\,|x||^{-1} u(x) = C_2 \geq 0; \tag{4.12a}$$

if $C_2 = 0$, *then*

$$\lim_{x\to 0} |x|^{(N-2)/2} u(x) = C_1 \geq 0. \tag{4.13}$$

if $C_1 = 0$, then 4.8 still holds, and the behaviour is as above.

iv) Assume $c = c^*$ and $q = q^*$. Then

$$\lim_{x\to 0} |x|^{(N-2)/2} |Ln\,|x||^{-2/(1-q)} u(x) = \left(\frac{(1-q)^2}{2(1+q)}\right)^{1/(1-q)}, \text{ or } u \equiv 0 \text{ near } 0. \tag{4.14}$$

v) Assume $c = c^*$ and $q > q^*$. Then 4.8 holds, and the behaviour is as above.

4.2 Proofs

Proof of Theorem 4.1.

i) Here $c < c^*$ and $\gamma > k_1 > k_2$, hence $|x|^{k_2} > |x|^{k_1} > |x|^{\gamma}$ for $|x| < 1$. From 2.4 we get $u(x) = O(|x|^{k_2})$. Then we define

$$u(x) = |x|^{k_2} W(t,\theta), \quad t = -Ln\, r,\ \theta \in S^{N-1}, \tag{4.15}$$

so that W satisfies the equation

$$W_{tt} - (N-2+2k_2)W_t + \Delta_{S^{N-1}} W - e^{h_2 t}\, W^q = 0, \tag{4.16}$$

where

$$h_2 = k_2(1-q) - (2+\sigma) = (1-q)(k_2-\gamma) < 0, \tag{4.17}$$

still defined in 2.32 (observe that the changes of variable 2.30 and 4.15 are linked by $w(r) = \overline{W}(t)$). Then from [6] (Proposition 4.1), there exists some $C_2 \geq 0$ such that $\|W(t,.) - C_2\|_{C^0(S^{N-1})} = O(e^{-\delta_2 t})$ for some $\delta_2 > 0$, hence 4.6 holds. Suppose $C_2 = 0$. For any $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that $\|W(t,.)\|_{C^0(S^{N-1})} \leq \varepsilon$ on $[t_\varepsilon, +\infty)$. Then the function

$$t \mapsto W_\varepsilon(t) = \varepsilon + \|W(0,.)\|_{C^0(S^{N-1})}\ e^{(k_2-k_1)t} \tag{4.18}$$

is a supersolution of 4.16 , hence from the maximum principle $W(t,.) \leq W_\varepsilon(t)$ on $[1, t_\varepsilon]$, hence on $[1,+\infty)$, for any $\varepsilon > 0$, so that $u(x) = O(|x|^{k_1})$. In that case, we define

$$u(x) = |x|^{k_1} V(t,\theta). \tag{4.19}$$

Then V satisfies

$$V_{tt} + (N - 2 + 2k_1)V_t + \Delta_{S^{N-1}} V - e^{h_1 t} V^q = 0, \tag{4.20}$$

where

$$h_1 = k_1(1 - q) - (2 + \sigma) = (1 - q)(k_1 - \gamma) < 0. \tag{4.21}$$

From [6] (Proposition 4.1), since $N - 2 + 2k_1 > 0$, there is some $C_1 \geq 0$ such that $\|V(t,.) - C_1\|_{C^0(S^{N-1})} + \|V_t(t,.)\|_{C^0(S^{N-1})} = O(e^{h_1 t})$, and 4.7 holds. Now assume $C_1 = 0$. Here we cannot conclude as above by using a supersolution, but we argue similarly to Theorem2.1 by estimating the mean value $\overline{V}(t)$, which satisfies

$$0 \leq (e^{-(N-2+2k_1)t}\, \overline{V_t})_t \leq e^{(-(N-2+2k_1)+h_1)t}\, \overline{V}^q, \tag{4.22}$$

with $\lim_{t\to+\infty} \overline{V}(t) = 0$, $\lim_{t\to+\infty} \overline{V_t}(t) = 0$. Moreover $\overline{V}$ is nonincreasing for large t. Integrating from t to $T > t$, and passing to the limit because $\overline{V}_t(t) = O(e^{h_1 t})$ and $N - 2 + 2k_1 - h_1 > 0$, we deduce that $-\overline{V_t}(t) \leq e^{h_1 t}\, \overline{V}^q(t)$. A new integration between t and $+\infty$ gives $\overline{V}(t) \leq e^{h_1 t}\, \overline{V}^q(t)$, since $\lim_{t\to+\infty} \overline{V}(t) = 0$. Then $\overline{u}(r) = O(r^{k_1 - h_1/(1-q)}) = O(r^\gamma)$, and 4.8 follows from Lemma 2.2. The precise behaviour of the solution is given by Theorem 5.1.

ii) Here $c < c^*$ and $k_1 > \gamma > k_2$, hence $|x|^{k_2} > |x|^\gamma > |x|^{k_1}$. Then 4.6 follows as above. In case $C_2 = 0$, we get similarly 4.7, which implies in particular 4.8. Hence from 5.1, the limit set of the solution $U(t,\theta)$ is contained in the set $\Gamma(U)$ of solutions of equation 4.9. But here

$$\gamma(\gamma + N - 2) + c = (\gamma - k_1)(\gamma - k_2) < 0, \tag{4.23}$$

which implies $\Gamma(U) = \{0\}$. Then U has a compact support, hence $u \equiv 0$ near the origin, from Theorem 5.1.

iii) Here $c < c^*$ and $k_2 > \gamma$, hence $|x|^\gamma > |x|^{k_2}$. Then 4.8 follows from 2.4, and the precise behaviour from Theorem 5.1.

iv) Here $c > c^*$and 4.8 follows from 2.7. We conclude as above. ■

Proof of Theorem 4.2.

i) In that case, $c < c^*$ and $\gamma = k_2$, hence $u(x) = O(|x|^{k_2} |Ln\,|x||^{1/(1-q)})$ from 2.5. Then we define

$$u(x) = |x|^{k_2} |Ln\,|x||^{1/(1-q)}\, Z(t,\theta), \tag{4.24}$$

and Z satisfies

$$Z_{tt} + (\sqrt{D} + \frac{2}{(1-q)t})\ Z_t + \Delta_{S^{N-1}} Z + \frac{1}{t}((\frac{\sqrt{D}}{1-q} + \frac{q}{(1-q)^2 t})\ Z\ - Z^q) = 0, \tag{4.25}$$

where $D \neq 0$. Then Theorem 5.2 applies and gives 4.11.

ii) Here $c < c^*$ and $k_1 = \gamma > k_2$. The proof given in Theorem 4.1,ii), is still available, with now $(\gamma - k_1)(\gamma - k_2) = 0$, and the 4.10 again follows.

iii) Here $c = c^*$, hence $k_1 = k_2 = (2-N)/2$, and $q < q^*$. From 4.3, that means $\gamma > k_1$, and $|x|^\gamma < |x|^{(2-N)/2}$. Then $u(x) = O(\ |x|^{(2-N)/2} |Ln\,|x||)$ from 2.4. Let us define

$$u(x) = |x|^{(2-N)/2} |Ln\,|x||\ H(t,\theta). \tag{4.26}$$

Then

$$H_{tt} + \frac{2}{t}\ H_t + \Delta_{S^{N-1}} H - t^{q-1} e^{mt} H^q = 0, \tag{4.27}$$

where

$$m = (1-q)(k_1 - \gamma) < 0. \tag{4.28}$$

Then $H_{tt} + \Delta_{S^{N-1}} H = O(t^{-1})$, hence $\left\| H(t,.) - \overline{H}(t) \right\|_{C^0(S^{N-1})} = O(t^{-1/2})$, from a slight variant of [6] (Proposition 4.1), see [12]. But from 4.27, $(t^2 \overline{H}_t)_t = O(e^{mt/2})$ at infinity, hence $\overline{H}$ has a finite limit $C_2 \geq 0$, and 4.12a holds. If $C_2 = 0$, we define

$$u(x) = |x|^{(2-N)/2} K(t,\theta), \tag{4.29}$$

and obtain

$$K_{tt} + \Delta_{S^{N-1}} K - e^{mt} K^q = 0. \tag{4.30}$$

That implies $0 \leq \overline{K}_{tt} \leq e^{mt} \overline{K}^q \leq C e^{mt/2}$, because $\overline{K}(t) = o(t)$. We deduce easily that $\overline{K}$ has a finite limit $C_1 \geq 0$, and $\left\| K(t,.) - \overline{K}(t) \right\|_{C^0(S^{N-1})} = O(t^{-1/2})$ as above, hence 4.13 follows. If $C_1 = 0$, then the nonincreasing function $t \mapsto \overline{K}^{-q} \overline{K}_t(t) + m^{-1} e^{mt}$ tends necessarily to 0, because $\overline{K}^{1-q}$ is bounded; hence it is nonnegative. Consequently, $\overline{K}(t) = O(e^{mt/(1-q)})$, hence $\overline{u}(r) = O(r^{\gamma})$ and 4.8 holds from 2.2.

iv) Here $c = c^*$, and $q = q^*$. Then $u(x) = O(\ |x|^{(2-N)/2}\,|Ln\,|x||^{2/(1-q)})$ from 2.6. Hence we set

$$u(x) = |x|^{(2-N)/2}\,|Ln\,|x||^{2/(1-q)}\,L(t,\theta), \tag{4.31}$$

and derive

$$L_{tt} + \frac{4}{(1-q)t}\,L_t + \Delta_{S^{N-1}}L + \frac{1}{t^2}(\frac{2(1+q)}{(1-q)^2}\,L\ - L^q) = 0, \tag{4.32}$$

Applying 5.2, we deduce 4.14.

iv) Here $c = c^*$, and $q > q^*$. Then the result follows directly from 2.4. ■

5 Appendix: sublinear equations in a cylinder

Here we give some general convergences properties for solutions of elliptic equations in a infinite cylinder.

Lemma 5.1 *Let $y \in C^2([0,+\infty))$ be nonnegative, and satisfying an inequality*

$$y_{tt}(t) + A\,y_t(t) + B\,y(t) \leq M \tag{5.1}$$

in $[0,+\infty)$, with given reals A, B, M $(M \geq 0)$. If $A \geq 0$ and $B > 0$, or $A < 0$ and $B - A^2/4 > 0$, then y is bounded .

Proof. i) *Case $A \geq 0$, $B > 0$.* The result is clear when y is nonincreasing for large t . If it is nondecreasing for large t, the energy function

$$E = y_t^2 + B\,y^2 - 2M\,y \tag{5.2}$$

is nonincreasing, hence again y is bounded. Suppose that y is not monotonous near infinity, and unbounded. Then there exist some sequences $(s_n), (t_n)$, tending to $+\infty$, such that $s_n \leq t_n \leq s_{n+1}, s_n$ is a minimal point of y, t_n is a maximal point, y is nondecreasing on $[s_n, t_n]$, and $\lim y(t_n) = +\infty$. Then $y(s_n) \leq M/B$. But $E(t_n) \leq E(s_n)$, hence $y(t_n)$ is bounded, contradiction.

ii) *Case* $A < 0$, $B - A^2/4 > 0$. Let $d = B - A^2/4$. The function $z = e^{At/2}y$ satisfies the inequality

$$z_{tt}(t) + d\, z(t) \leq M\ e^{At/2} \leq M, \tag{5.3}$$

hence z is bounded from the first case. But $z_{tt}(t) \leq M\ e^{At/2}$, hence the function $t \mapsto z_t(t) + 2M\ e^{At/2}/\,|A|$ is decreasing to a limit which is necessarily 0. Then the function $t \mapsto z(t) - 4M\ e^{At/2}/A^2$ is increasing to a limit which is necessarily 0 from 5.3. The conclusion follows. ■

Theorem 5.1 *Let* $Y \in C^2([0, +\infty) \times S^{N-1})$ *be any nonnegative bounded solution of equation*

$$Y_{tt} + A\, Y_t + \Delta_{S^{N-1}} Y + B\, Y - Y^q = 0 \tag{5.4}$$

in $[0, +\infty) \times S^{N-1}$*, with given reals* $A, B \in \mathbb{R}$*, with* $A \neq 0$*, and* $q \in (0, 1)$.

i) Then the limit set

$$\Gamma(Y) = \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} Y(\tau, .)} \tag{5.5}$$

is a connected compact subset of the set

$$E_B^+ = \left\{ \omega \in C^2(S^{N-1}) \mid \omega(.) \geq 0, \quad \Delta_{S^{N-1}}\omega + B\, \omega - \omega^q = 0 \right\}. \tag{5.6}$$

ii) If moreover $0 \in \Gamma(Y)$*, then* $Y(t, .) \equiv 0$ *for large* t.

Proof. i) Here we adapt the proofs of [10], [3] for the superlinear case $q > 1$ to the sublinear one . Since $q < 1$, from Schauder estimates, Y is uniformly bounded in $C^{2,q}([T, T+1] \times S^{N-1})$, uniformly in T on $[1, +\infty)$. Multiplying 5.4 by Y_t and integrating over $[1, T] \times S^{N-1}$, it follows that Y_t is bounded in $L^2([1, +\infty] \times S^{N-1})$, since $A \neq 0$. In order to estimate Y_{tt} , we cannot derivate 5.4 by respect to t, but we multiply 5.4 by Y_{tt} and integrate again over $[1, T] \times S^{N-1}$:

$$\begin{aligned}\int_1^T \int_{S^{N-1}} Y_{tt}^2\, d\theta\, dt = &- \left[\int_{S^{N-1}} (\tfrac{A}{2}\, Y_t^2 + B\, Y Y_t) d\theta\right]_1^T \\ &+ \int_1^T \int_{S^{N-1}} (B\, Y_t^2 + (Y^q - \Delta_{S^{N-1}} Y)\, Y_{tt})\, d\theta\, dt.\end{aligned} \tag{5.7}$$

The last term is majorated by $\left[\int_{S^{N-1}}(Y^q - \Delta_{S^{N-1}}Y)\; Y_t)\; d\theta\right]_1^T$, by regularization and integrations by part, hence Y_{tt} is bounded in $L^2([1,+\infty]\times S^{N-1})$. Then we conclude as in [3].

ii) If $0 \in \Gamma(Y)$, we first deduce that $\Gamma(Y) = \{0\}$ as in [3] (Theorem 3.2), hence

$$\lim_{t\to+\infty} \|Y(t,.)\|_{C^0(S^{N-1})} = 0. \tag{5.8}$$

Then from Hölder inequality, since $q < 1$, there exists some $T > 0$ such that Y is a subsolution of equation

$$-Z_{tt} - A\; Z_t - \Delta_{S^{N-1}}Z + \frac{1}{2}Z^q = 0 \tag{5.9}$$

on $[T,+\infty]\times S^{N-1}$. Now the function

$$Z_K = K\;\left[(T+1-t)^+\right]^{2/(1-q)} \tag{5.10}$$

is a supersolution of this equation for $K \leq K_0 = K_0(A,q)$ small enough, with compact support. Choosing T large enough so that $\|Y(t,.)\|_{C^0(S^{N-1})} \leq K_0$, it comes $Y \leq Z_{K_0}$ in $[T,+\infty]\times S^{N-1}$, from 5.8 and the maximum principle, hence the conclusion. ■

The following theorem is used in the logarithmic cases in equation 1.1. We just give the outline of the proof, because it is quite similar to [6] (Corollary 4.2).

Theorem 5.2 *Let $Y \in C^2([0,+\infty)\times S^{N-1})$ be any nonnegative bounded solution of equation*

$$Y_{tt} + (A_1 + \frac{A_2}{t})\; Y_t + \Delta_{S^{N-1}}Y + \frac{1}{t^n}\left[(B_1 + \frac{B_2}{t})\; Y\; - Y^q\right] = 0 \tag{5.11}$$

in $[0,+\infty)\times S^{N-1}$, where $q \in (0,1)$, $A_1, A_2, B_1, B_2 \in \mathbb{R}$, with $B_1 > 0$, and $A_1 > 0$, $n = 1$, or $A_1 = 0$, $n = 2$. Then $Y(t,.)$ converges in $C^2(S^{N-1})$ to $B_1^{1/(q-1)}$ or to 0. In the last case, $Y(t,.) \equiv 0$ for large t.

Proof. First we prove as above that Y is bounded in $C^{2,q}([T,T+1]\times S^{N-1})$, uniformly in T on $[1,+\infty)$. Then we write 5.11 under the form

$$Y_{tt} + A_1\; Y_t + \Delta_{S^{N-1}}Y + \varphi = 0, \tag{5.12}$$

where $\|\varphi(t,.)\|_{C^0(S^{N-1})} = O(t^{-1})$. Arguying as in [6] (Proposition 4.1), it follows that

$$\left\|Y(t,.) - \overline{Y}(t)\right\|_{L^2(S^{N-1})} = O(t^{-1}), \tag{5.13}$$

see [6], but we cannot conclude directly to the convergence. From 5.12, $\overline{Y}$ satisfies

$$\overline{Y}_{tt} + (A_1 + \frac{A_2}{t})\,\overline{Y}_t + \frac{1}{t^n}(B_1\,\overline{Y}\ - \overline{Y}^q) = \psi \tag{5.14}$$

where by computation $\|\psi(t,.)\|_{C^0(S^{N-1})} = O(t^{-1-q/2})$. Then we follow the proof of [6] (Corollary 4.2), in case $A_1 > 0, n = 1$, or [5] (Proposition 1.3), in the delicate case $A_1 = 0, n = 2$, and get the convergence of $\overline{Y}$ to $B_1^{1/(q-1)}$ or to 0. From 5.13 it implies the convergence of $Y(t,.)$ in $C^2(S^{N-1})$. In case of limit zero, there exists some $T > 0$ such that Y is a subsolution of equation

$$-Z_{tt} - (A_1 + \frac{A_2}{t})\,Z_t - \Delta_{S^{N-1}}Z + \frac{1}{2t^n}Z^q = 0 \tag{5.15}$$

on $[T, +\infty] \times S^{N-1}$. If $A_1 > 0,\ n = 1$, choosing $T \geq \max(1, 2\,|A_2|\,/A_1)$, the function

$$Z_K = K\,((T^{1/2} + 1 - t^{1/2})^+)^{2/(1-q)} \tag{5.16}$$

is a supersolution of this equation on $[T, +\infty] \times S^{N-1}$, as soon as $K \leq K_1 = K_1(q)$ enough. If $A_1 = 0$, $n = 2$, a supersolution is given by

$$Z_K = K\,\left[(Ln((T+1)/t))^+\right]^{2/(1-q)}, \tag{5.17}$$

for $K \leq K_2 = K_2(q, A_2)$ small enough. We conclude to the compacity of the support as in Theorem 5.1. ■

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