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Singularities for a 2-Dimensional Semilinear Elliptic Equation with a Non-Lipschitz Nonlinearity

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We study the limit behaviour of solutions of the semilinear elliptic equation

$$\Delta u = |x|^{\sigma}\, |u|^{q-1}\, u \qquad \text{in} \quad \mathbb{R}^2, \quad q \in (0, 1), \sigma \in \mathbb{R},$$

with a non-Lipschitz nonlinearity on the right-hand side. When $|\sigma + 2| \leqslant 2$ we give a complete classification of the types of singularities as $x \to 0$ and $x \to \infty$ which in the rescaled form are essentially non-analytic and, even more, not C^{∞}. The proof is based on the asymptotic study of the corresponding evolution dynamical system and the Sturmian argument on zero set analysis. © 1999 Academic Press

Key Words: semilinear elliptic equation; asymptotics of singularities; Sturmian argument.

INTRODUCTION

The starting point of this article is to describe the asymptotic behaviour of the positive solutions of the equation

$$\Delta u = |u|^{q-1}\, u \tag{0.1}$$

in an exterior domain of the plane $\mathbb{R}^2$ where $0 < q < 1$. This kind of semilinear equation appears in several applications in mechanics and physics,

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and in particular can be treated as the equation of equilibrium states in a conductive medium with strong absorption. In order to present broader results, we shall consider a more general equation, namely

$$\Delta u = |x|^{\sigma}\, |u|^{q-1}\, u \qquad \text{in } \mathbb{R}^2, \tag{0.2}$$

with fixed constants $q \in (0, 1)$ and $\sigma \in \mathbb{R}$. Because this class of equation is invariant under the Kelvin transform in the plane, it is sufficient to study the behaviour of the solutions near the singular point $x = 0$, so we can assume that u is defined in $B' = B \backslash \{0\}$, where $B = \{x \in \mathbb{R}^2 \mid |x| \leqslant 1\}$. In particular, the asymptotic behaviour at infinity of a solution of (0.1) is changed by the Kelvin transform into the singular behaviour at 0 of a solution of (0.2) with $\sigma = -4$. Denote

$$\gamma = (\sigma + 2)/(1 - q). \tag{0.3}$$

The first striking difference of (0.2) with $q \in (0, 1)$ and with $q > 1$, studied in detail in [V1, V2, BrV, Y1, CMV], is that the Keller–Osserman estimate near 0,

$$u(x) = O(|x|^{\gamma}), \tag{0.4}$$

is not always true, even for nonnegative solutions: the linear effect can be stronger than the nonlinear one and this leads to behaviour of the type $x \mapsto |\mathrm{Ln}\, |x||$ when $\sigma > -2$, or $x \mapsto |\mathrm{Ln}\, |x||^{2/(1-q)}$ when $\sigma = -2$.

The second difference is that the solutions of (0.2) can present dead cores. Indeed the lower order nonlinearity $u \mapsto g(u) = |u|^{q-1}\, u$ is no longer regular and in fact is not locally Lipschitz continuous at $u = 0$. This implies that the strong maximum principle is no longer true in general.

The main problem is the question of convergence for the solutions of (0.2) which satisfy (0.4). Such solutions exist whenever $\sigma \neq -2$, and in fact when $|\sigma + 2|$ is large enough, there do exist an infinity of nonradial solutions, including nonnegative ones, presenting dead cores. Then the classical results of convergence of [Si] cannot be applied, because they are based on analyticity arguments. Nevertheless, we show that several ideas from [CMV] can be adapted to the essentially non-smooth case $q \in (0, 1)$.

The basic tool is to reduce the equation to a *semilinear evolution elliptic problem.* Using the idea introduced in [V1, V2, GS], we set

$$v(t, \theta) = r^{-\gamma} u(r, \theta), \qquad t = -\mathrm{Ln}\, r, \tag{0.5}$$

where (r, θ) are the polar coordinates in $\mathbb{R}^2 \backslash \{0\}$ ($r > 0$, $\theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$), to get an *autonomous* equation of the form

$$-v_{tt} + 2\gamma v_t = A(v) \equiv v_{\theta\theta} + \gamma^2 v - |v|^{q-1}\, v \tag{0.6}$$

on $[0, +\infty) \times S^1$. Thus we are reduced to studying the behaviour of the *bounded* solutions of (0.6) when $t \to +\infty$. So we define in a standard way the ω-limit set of the orbit $\{v(t, .) \mid t > 0\}$,

$$\Gamma = \{w \in C^2(S^1) \mid \exists \{t_n\} \to +\infty \text{ such that } v(t_n, .) \to w(.) \text{ in } C^2(S^1)\}, \quad (0.7)$$

on first prove that $\Gamma \subseteq \mathscr{E}$, where

$$\mathscr{E} = \{w \in C^2(S^1) \mid A(w) = 0\} \quad (0.8)$$

is the set of stationary solutions. The main purpose of this article is to prove the following convergence result.

THEOREM 0.1. *Assume* $|\sigma + 2| \leqslant 2$. *Any solution* v *of* (0.6), *bounded on* $[0, +\infty) \times S^1$, *converges to precisely one element of the set* $\mathscr{E}$ *of stationary solutions.*

In order to point out the difficulties in the convergence process, let us begin by introducing the following two explicit solutions to (0.2) in the case $\sigma \neq -2$, which describe typical asymptotic properties of the solutions under consideration and play an important role in our comparison arguments. The first one is radially symmetric,

$$u^*(x) = c^* \, |x|^\gamma, \qquad \text{with} \quad c^* = \gamma^{-2/(1-q)}, \quad (0.9)$$

for which, after rescaling, (0.6) becomes the "flat" equation $v^*(\theta) \equiv c^*$. The second solution, which exists for $\sigma \geqslant -1$ or $\sigma \leqslant -3$, has a nontrivial θ-shape,

$$U^*(x) = |x|^\gamma \, F(\theta), \quad (0.10)$$

where F is the compactly supported one-hump function

$$F(\theta) = \begin{cases} c_1[\cos(|\sigma+2| \, \theta/2)]^{2/(1-q)} & \text{if} \quad |\theta| < \theta^* = l^*/2, \\ 0, & \text{if} \quad |\theta| \geqslant \theta^*, \end{cases} \quad (0.11)$$

where

$$c_1[(1+q)\, \gamma^2/2]^{1/(q-1)}, \qquad l^* = 2\pi/|\sigma+2|. \quad (0.12)$$

Notice that F is not analytic, and in fact $F \in C^{m, \nu}(S^1)$ with $m = I(2/(1-q)) > 2$ (the integer part of $2/(1-q)$) and $\nu = 2/(1-q) - I(2/(1-q))$. Recall that due to well-known results on the interior regularity for uniformly elliptic equations with analytic coefficients, any solution u to (0.2) is smooth and analytic at any point x where $u(x) \neq 0$, so that v

can be only non-analytic on the zero set $\{v=0\} \equiv \{(t, \theta) \in [0, +\infty) \times S^1 \mid v(t, \theta)=0\}$.

Observe that function F can be continued in $\{|\theta| > \theta^*\}$ in various ways to define a multi-dimensional family of stationary solutions. For instance, the sum or the difference of functions shifted in θ, $F(\theta-\theta_1) \pm F(\theta-\theta_2)$, is a stationary solution provided that the supports (the positivity domains) of each function are not overlapping. Since the length of each support is equal to l^*, such a superposition of stationary solutions is available if $2l^* \leqslant 2\pi$, i.e. if $\sigma \geqslant 0$ or $\sigma \leqslant -4$. When $|\sigma+2| \leqslant 2$ the connected components of $\mathscr{E}$ generated by F are one-dimensional: under the form $\{\pm F(\,.+\alpha) \mid \alpha \in [0, 2\pi]\}$, or also, in the case $|\sigma+2|=2$, under the form $\{F(\,.+\alpha) \pm F(\,.+\pi+\alpha) \mid \alpha \in [0, 2\pi]\}$. In that last case the two humps have no internal freedom to move since the length of their positivity domain is π. Equation (0.2) can admit also other solutions of the form $u(x)=|x|^\gamma G(\theta)$, where G is a constant sign or changing sign C^∞ function.

The most delicate problem in the study of the asymptotic behaviour is to prove the stabilization of $v(t, \theta)$ as $t \to +\infty$ to the essentially not C^∞ and compactly supported one-hump stationary solution F or to superposition of two such elementary stationary humps with no internal freedom. On the contrary, the analysis of the stabilization to smooth C^∞-profiles is performed exactly as in [CMV], or [Si] for positive ones.

Our analysis consists in several steps. In Section 1 we give a priori estimates and make a Lyapunov-type analysis. Thus we obtain convergence for the unbounded solutions of (0.6), and prove that $\Gamma \subseteq \mathscr{E}$ for the bounded ones. In Section 2 we give the complete structure of the set of stationary solutions. In Section 3 we give the basic argument for convergence of the bounded solutions: we prove that any solution of (0.6) in a domain Q, which is small on the boundary ∂Q, presents a dead core in *int* Q as soon as Q has a suitable size. In Section 4 we assume that $|\sigma+2| \leqslant 2$ and prove Theorem 0.1.

1. A PRIORI ESTIMATES AND LYAPUNOV-TYPE ANALYSIS

Our first results concern the nonnegative solutions of (0.2). We refer to [BG] for an analogous study in $\mathbb{R}^N$ $(N \geqslant 3)$. We denote by

$$\bar{u}(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta)\, d\theta, \qquad \bar{v} = \frac{1}{2\pi} \int_0^{2\pi} v(t, \theta)\, d\theta,$$

the mean values of u, v at points r, t. In the sequel the same letter C denotes some positive constants which may depend on u, v, but not on the variables x and t. Some estimates are available for *supersolutions*:

PROPOSITION 1.1. *Let $u \in C^2(B')$ be any nonnegative subharmonic supersolution of* (0.2); *that is,*

$$0 \leqslant \Delta u \leqslant |x|^\sigma u^q \qquad \text{in } B'. \tag{1.1}$$

(i) *If* $\sigma > -2$, *then* $u(x) \leqslant C\,|\mathrm{Ln}\,|x||$ *near* 0.

(ii) *If* $\sigma = -2$, *then* $u(x) \leqslant C\,|\mathrm{Ln}\,|x||^{2/(1-q)}$ *near* 0.

(iii) *If* $\sigma < -2$, *then* $u(x) \leqslant C\,|x|^\gamma$ *near* 0.

Proof. As pointed out in [Y2], any estimate on $\bar{u}$ implies the corresponding estimate on u. Indeed from subharmonicity u satisfies the inequality $u(x) \leqslant 8 \max_{[|x|/2,\, 3|x|/2]} \bar{u}$ whenever $|x| < 2/3$. From Jensen's inequality, $\bar{u}$ satisfies the inequality

$$0 \leqslant \Delta \bar{u} \leqslant r^\sigma \bar{u}^q \qquad \text{in } (0, 1]. \tag{1.2}$$

Thus we set

$$y(t, \theta) = u(r\,\theta) \tag{1.3}$$

hence

$$0 \leqslant \bar{y}_{tt} \leqslant e^{-(\sigma+2)\,t} \bar{y}^q \qquad \text{on } [0\ +\infty). \tag{1.4}$$

In any case we can assume that $\bar{y}$ is unbounded; hence it is nondecreasing $\bar{y}(t) \to +\infty$ as $t \to +\infty$.

(i) Assume $\sigma > -2$. From convexity, we have $t\bar{y}_t(t) \geqslant C\bar{y}(t)$ on $[1, +\infty)$, hence $(\bar{y}_t^{1-q})_t\,(t) \leqslant Ce^{-(\sigma+2)\,t/2}$ for large t. By integrating twice, it follows that $\bar{y}(t) \leqslant Ct$ on $[1, +\infty)$.

(ii) Assume $\sigma = -2$. Then, $(\bar{y}^{1-q})_{tt}\,(t) \leqslant 1 - q$; hence $\bar{y}^{1-q}(t) \leqslant Ct^2$ on $[1, +\infty)$.

(iii) Assume $\sigma < -2$. By integrating twice in (1.4) over $(1, t)$ we deduce the estimate $\bar{y}(t) \leqslant Ce^{-(\sigma+2)\,t}\bar{y}^q(t)$ on $[1, +\infty)$. ∎

Those estimates can be improved in the case of *solutions* of (0.2):

PROPOSITION 1.2. *Let $u \in C^2(B')$ be any nonnegative solution of* (0.2) *with $\sigma > -2$. Then either*

(i) $\lim_{x \to 0} |\mathrm{Ln}\,|x||^{-1}\, u(x) = \alpha > 0$,

(ii) $\lim_{x \to 0} u(x) = c > 0$, *or*

(iii) $u(x) \leqslant C\,|x|^\gamma$ *near* 0.

Proof. Let us define

$$z(t, \theta) = t^{-1} u(r, \theta). \tag{1.5}$$

Then z is a *bounded* solution of equation

$$z_{tt} + z_{\theta\theta} = \Psi = -2t^{-1} z_t + t^{q-1} e^{-(\sigma+2)t} z^q \tag{1.6}$$

on $[0, +\infty) \times S^1$. From Calderon–Zygmund and Schauder estimates, as $r \mapsto |r|^{q-1} r \in C^{0,q}(\mathbb{R})$, the orbit of z is relatively compact in $C^2(S^1)$, hence $t\Psi(t, .)$ is bounded. As in [V1] or [BR, Proposition 4.2], defining $\tilde{z} = z - \bar{z}$, $\tilde{\Psi} = \Psi - \bar{\Psi}$, we multiply (1.6) by z, integrate over S^1, and get

$$\begin{aligned} \int_{S^1} \tilde{z}\tilde{z}_{tt}(t, \theta)\, d\theta &= \int_{S^1} \tilde{z}_\theta(t, \theta)^2\, d\theta + \int_{S^1} \tilde{z}\tilde{\Psi}(t, \theta)\, d\theta \\ &\geqslant \int_{S^1} \tilde{z}(t, \theta)^2\, d\theta - \tfrac{1}{2}\left(\int_{S^1} \tilde{z}(t, \theta)^2\, d\theta + \int_{S^1} \tilde{\Psi}(t, \theta)^2\, d\theta\right) \end{aligned} \tag{1.7}$$

from Hölder and Poincaré inequalities. Then the function $g(t) = \int_{S^1} \tilde{z}^2(t, \theta)\, d\theta$ satisfies an inequality of the form $-g_{tt} + g \leqslant Ct^{-2}$ on $[1, +\infty)$. Since z is bounded, the function $t \mapsto t^2 g(t)$ is also bounded by a classical application of the maximum principle as in [V1], i.e., $\|z(t, .) - \bar{z}(t)\|_{L^2(S^1)} = O(t^{-1})$. But $e^{(\sigma/2+1)t}(t^2 \bar{z}_t)_t$ is bounded; hence $\bar{z}(t)$ has a finite limit $\alpha \geqslant 0$. Then $z(t, .)$ converges to α in $C^2(S^1)$, and $|\mathrm{Ln}\, |x||^{-1} u(x)$ tends to α. In the case $\alpha = 0$ we use a supersolution of (1.6) on $[1, +\infty) \times S^1$, $\chi(t) = t^{-1} \max_{\theta \in S^1} z(1, \theta)$, and deduce from the maximum principle that $tz(t, .)$ is bounded; i.e., $u(x)$ is bounded near 0. Now the function y defined in (1.3) is a *bounded* solution of the equation

$$y_{tt} + y_{\theta\theta} = \Phi = e^{-(\sigma+2)t} y^q. \tag{1.8}$$

Hence as above $\|y(t, .) - \bar{y}(t)\|_{L^2(S^1)} = O(t^{-1})$ (and in fact the convergence is exponential); but $\bar{y}(t)$ has a finite limit $c \geqslant 0$, hence $u(x)$ tends to c. In the case $c = 0$, as in the proof of Proposition 1.1, we integrate inequality (1.4) twice over (t, ∞) and obtain the estimate $\bar{y}(t) \leqslant Ce^{-(\sigma+2)t} \bar{y}^q(t)$; consequently $|x|^{-\gamma} u(x)$ is bounded. ∎

The critical case $\sigma = -2$ presents the usual difficulties due to the superposition of the linear and nonlinear effects (see [V1, A, BR]). Moreover, with a non-Lipschitz nonlinearity and some particularities due to the dimension $N = 2$, the first phenomena of compact support appear.

PROPOSITION 1.3. *Let* $u \in C^2(B')$ *be any nonnegative solution of* (0.2) *with* $\sigma = -2$. *Then either* $\lim_{x \to 0} |\mathrm{Ln}\, |x||^{-2/(1-q)} u(x) = b = [(1-q)^2/2(1+q)]^{1/(1-q)}$, *or* $u \equiv 0$ *near* 0.

Proof. The function

$$\xi(t,\theta)=t^{-2/(1-q)}u(r,\theta) \tag{1.9}$$

is a *bounded* solution of the equation

$$\xi_{tt}+\xi_{\theta\theta}=\Psi=-at^{-1}\xi_t+t^{-2}(\xi^q-b^{q-1}\xi), \tag{1.10}$$

where $a=4/(1-q)$. As above the orbit of ξ is relatively compact in $C^2(S^1)$; hence $t\psi(t,.)$ is bounded, and $\|\xi(t,.)-\bar{\xi}(t)\|_{L^2(S^1)}=O(t^{-1})$. We can write down the equation relative to $\bar{\xi}$ in the form

$$\bar{\xi}_{tt}+at^{-1}\bar{\xi}_t+t^{-2}(b^{q-1}\bar{\xi}-\bar{\xi}^q)=t^{-2}(\overline{\xi^q}-\bar{\xi}^q). \tag{1.11}$$

This equation is not classical, because (due to the dimension 2) the coefficient of $\bar{\xi}_t$ tends to 0 when $t\to+\infty$. In order to get rid of this term we set $\tau=\mathrm{Ln}\ t$, and (1.11) becomes

$$\bar{\xi}_{\tau\tau}+(a-1)\,\bar{\xi}_\tau+b^{q-1}\bar{\xi}-\bar{\xi}^q=\varphi=\overline{\xi^q}-\bar{\xi}^q, \tag{1.12}$$

where $a-1=(3+q)/(1-q)>0$. Moreover, $t^{q+2}\varphi=e^{(q+2)\tau}\varphi$ is bounded. Indeed, $q<1$ implies $|\xi^q(t,.)-\bar{\xi}^q(t)|\leqslant|\xi(t,.)-\bar{\xi}(t)|^q$. Integrating over S^1, we deduce from Hölder inequality that $|\overline{\xi^q}(t)-\bar{\xi}^q(t)|\leqslant(2\pi)^{-q/2}$ $\|\xi(t,.)-\bar{\xi}(t)\|^q_{L^2(S^1)}$. Then we multiply (1.12) by $\bar{\xi}_\tau$ and integrate over $[0,\tau]$ and finally find

$$(a-1)\int_0^\tau\bar{\xi}^2_\tau(\tau)\,d\tau=\left[\frac{1}{1+q}\,\bar{\xi}^q-\frac{b^{q-1}}{2}\,\bar{\xi}^2-\frac{1}{2}\,\bar{\xi}^2_\tau\right]_0^\tau+\int_0^\tau\varphi(\tau)\,\bar{\xi}_\tau(\tau)\,d\tau. \tag{1.13}$$

As in [BR] this implies $\bar{\xi}_\tau\in L^2((0,+\infty))$ from the Hölder inequality, because $a\neq 1$. Due to the non-Lipschitz nonlinearity, we cannot differentiate (1.12) in order to obtain an estimate on $\bar{\xi}_{\tau\tau}$. Thus we multiply (1.12) by $\bar{\xi}_{\tau\tau}$ and derive

$$\int_0^\tau\bar{\xi}^2_{\tau\tau}(\tau)\,d\tau+\left[\frac{a-1}{2}\,\bar{\xi}^2_\tau+b^{q-1}\bar{\xi}\bar{\xi}_\tau\right]_0^\tau-b^{q-1}\int_0^\tau\bar{\xi}^2_\tau(\tau)\,d\tau-\int_0^\tau\varphi(\tau)\,\bar{\xi}_{\tau\tau}(\tau)\,d\tau$$

$$=\int_0^\tau\bar{\xi}^q(\tau)\bar{\xi}_{\tau\tau}(\tau)\,d\tau\leqslant[\bar{\xi}^q\bar{\xi}_\tau]_0^\tau \tag{1.14}$$

by smoothing and integrating by parts in the last term. Then $\bar{\xi}_{\tau\tau}\in L^2((0,+\infty))$, hence $\lim_{\tau\to+\infty}\bar{\xi}_\tau=0$, i.e., $\lim_{t\to+\infty}t\bar{\xi}_t=0$. Now following [BR, Corollary 4.2], if $\bar{\xi}(t)$ has a limit, then it converges to b or

0; indeed the equation $x_{tt}+4/(1-q)\,x_t+\mu/ty^2=0$ has no bounded solution when $\mu\neq 0$. Assume $l=\limsup_{t\to+\infty}\bar{\xi}(t)>l'=\liminf_{t\to+\infty}\bar{\xi}(t)$. Then there exist two sequences $(s_n), (t_n)\to+\infty$ such that $s_n<t_n<s_{n+1}$, $\lim\bar{\xi}(s_n)=l$, $\lim\bar{\xi}(t_n)=l'$, and $\bar{\xi}_t(s_n)=\bar{\xi}_t(t_n)=0$, $\bar{\xi}_{tt}(s_n)\leqslant 0\leqslant\bar{\xi}_{tt}(t_n)$. This implies $l\geqslant b\geqslant l'$ from (1.12). Assume for example that $l>b$ and consider $\vartheta_n\in(t_n, s_{n+1})$ such that $\bar{\xi}(\vartheta_n)=(l+b)/2$ and $\bar{\xi}(t)>b$ on $[\vartheta_n, s_{n+1}]$. Now multiply (1.12) by t and integrate on $[\vartheta_n, s_{n+1}]$ and go to the limit. Since $\lim_{t\to+\infty}t\bar{\xi}_t=0$ and $t\varphi(t)=O(t^{-1-q})$, it implies $(a-1)(l-b)\leqslant 0$, hence a contradiction holds. The proof is similar if $b>l'$. Then $\xi(t, .)$ converges to b or 0 in $C^2(S^1)$, i.e. $|\mathrm{Ln}\,|x||^{-2/(1-q)}\,u(x)$ tends either to b or to 0. In the last case we use the function $\xi_{k,T}=k((\mathrm{Ln}((T+1)/t))^{+1})^{2/(1-q)}$, where $k, T>0$. It is a supersolution of (1.6) as soon as $k\leqslant k_0=k_0(q)$. Choosing T large enough so that $\xi(T, .)\leqslant k_0$ on S^1, we deduce that ξ vanishes in finite time. ▮

By opposition to the case $q>1$, we have no estimates for the *changing sign solutions* u of (0.2). Indeed Kato inequality implies that $|u|$ is a *subsolution* of (0.2), so that Proposition 1.1 does not apply. Anyway, if $\sigma>-2$ and we assume that $|\mathrm{Ln}\,|x||^{-1}\,u(x)$ is bounded near 0, then following the proof of Proposition 1.2 we get the following convergence estimate: $\lim_{x\to 0}|\mathrm{Ln}\,|x||^{-1}\,u(x)=\alpha>0$. Similarly if $\sigma=-2$ and we assume that $|\mathrm{Ln}\,|x||^{-2/(1-q)}\,u(x)$ is bounded, then $\lim_{x\to 0}|\mathrm{Ln}\,|x||^{-1}\,u(x)=\pm l$, or $u\equiv 0$ near 0.

Now let us go to the last case, which is the motivation of this paper.

PROPOSITION 1.4. *Assume $\sigma\neq -2$. Let $u\in C^2(B')$ be any solution of* (0.2) *such that $|x|^{-\gamma}\,u(x)$ is bounded, and let v be defined by* (0.5). *Then the ω-limit set Γ of the orbit of v is contained in the set $\mathscr{E}$ of stationary solutions of* (0.6). *Moreover, if $0\in\Gamma$, then v has a compact support, i.e., $u\equiv 0$ near* 0.

Proof. As above, the orbit of v is relatively compact in $C^2(S^1)$; therefore Γ is nonempty, compact, and connected. Multiplying (0.6) by v_t and integrating over $[1, t]\times S^1$, we get

$$E(t)-E(1)=2\gamma\int_1^t\int_{S^1}v_t^2(t,\theta)\,d\theta\,dt, \tag{1.15}$$

where

$$E(t)=\frac{1}{2}\int_{S^1}\left(v_t^2-v_\theta^2+\gamma^2 v^2-\frac{2}{1+q}\,|v|^{1+q}\right)(t,\theta)\,d\theta. \tag{1.16}$$

Thus $v_t \in L^2((1, +\infty) \times S^1)$, because $\gamma \neq 0$. Multiplying by v_{tt} we get in the same way

$$\int_1^t \int_{S^1} v_{tt}^2(t, \theta)\, d\theta\, dt - \left[\int_{S^1} (\gamma v_{tt}^2 - \gamma^2 v v_t)(s, \theta)\, d\theta \right]_{s=1}^{s=t} - \gamma^2 \int_1^t \int_{S^1} v_t^2(t, \theta)\, d\theta\, dt$$

$$= \int_1^t \int_{S^1} ((|v|^{q-1}\, v - v_{\theta\theta})\, v_{tt})(t, \theta)\, d\theta\, dt$$

$$\leqslant \left[\int_{S^1} (v_\theta v_{t\theta} + |v|^{q-1}\, v v_t)(t, \theta)\, d\theta \right]_1^t \tag{1.17}$$

by smoothing and integrating by parts in the two last terms. Then $v_{tt} \in L^2((1, +\infty) \times S^1)$. Since v_{tt} is uniformly continuous, from Schauder estimates, it follows that v_t and v_{tt} tend to 0 in $L^2(S^1)$, and $\Gamma \subseteq \mathscr{E}$.

In the case $0 \in \Gamma$, there exists a sequence $\{t_n\} \to +\infty$ such that $E(t_n) \to 0$. Then from monotonicity $E(t) \to 0$ as $t \to +\infty$, and $\int_{S^1} (w_\theta^2 - \gamma^2 w^2 + 2\,|w|^{1+q}/(1+q))(\theta)\, d\theta = 0$ for any $w \in \Gamma$. But $\int_{S^1} (w_\theta^2 - \gamma^2 w^2 + |w|^{1+q})(\theta)\, d\theta = 0$ since $w \in \mathscr{E}$, hence $\Gamma = \{0\}$ and $v(t, .) \to 0$ in $C^2(S^1)$. Now the function $v_{K,T} = K[(T+1-t)^+]^{2/(1-q)}$ with $K, T > 0$ is a supersolution of (0.6) with compact support, as soon as $K \leqslant K_0 = K_0(q, \sigma)$, and $|v|$ is a subsolution, hence the conclusion as in Proposition 1.3. ∎

Remark 1.5. The function $u_0(x) = u(x/|x|^2)$ obtained by the Kelvin transform satisfies the equation

$$\Delta u_0 = |x|^{\sigma_0}\, |u_0|^{q-1}\, u_0, \qquad \text{with} \quad \sigma_0 + 2 = -(\sigma + 2). \tag{1.18}$$

Notice also that the transformation $x \mapsto x/|x|^2$ corresponds for the function v to the transformation $t \mapsto -t$. Hence the function $v_0(t, \theta) = r^{-\gamma_0} u_0(r, \theta)$ associated to u_0 with

$$\gamma_0 = (\sigma_0 + 2)/(1-q) = -\gamma \tag{1.19}$$

is given by $v_0(t, \theta) = v(-t, \theta)$. Consequently we deduce the behaviour at infinity from Propositions 1.2–1.4. Let $u \in C^2(\mathbb{R}^2 \backslash B)$ be any nonnegative solution of (0.2). If $\sigma = -2$, then either $\lim_{x \to 0} (\mathrm{Ln}\,|x|)^{-2/(1-q)}\, u(x) = b$, or u has a compact support. If $\sigma > -2$, then $u(x) \leqslant C\,|x|^\gamma$ near infinity. If $\sigma < -2$, then either $\lim_{|x| \to +\infty} (\mathrm{Ln}\,|x|)^{-1}\, u(x) = \alpha > 0$, or $\lim_{|x| \to +\infty} u(x) = c > 0$, or $u(x) \leqslant C\,|x|^\gamma$ near infinity. Assume that $|x|^{-\gamma}\, u(x)$ is bounded, and let v be defined by (0.5) in $(-\infty, 0] \times S^1$. Then the α-limit set of the orbit of v is contained in $\mathscr{E}$. Moreover if it contains 0, then u has a compact support. One can study the possible global solutions as in [BBo] by using energy methods.

2. THE SET OF STATIONARY PROFILES

In this section we give a complete description of the set $\mathscr{E}$ of solutions of the stationary equation on S^1

$$A(w) = w_{\theta\theta} + \gamma^2 w - |w|^{q-1}\, w = 0, \tag{2.1}$$

where $\gamma \neq 0$ is given by (0.3). That is, we look for 2π-periodic solutions of (2.1) on $\mathbb{R}$. For any solution w on $\mathbb{R}$, there exists a constant K such that

$$w_\theta^2 + U(w) = K, \qquad \text{where} \quad U(w) = \gamma^2 w^2 - 2\,|w|^{1+q}/(1+q). \tag{2.2}$$

Notice that the potential $U/2$ is an even function, with $U(0) = U(c_1) = 0$, $U'(c^*) = 0$, with c^*, c_1 given by (0.9), (0.12). Then w is periodic, or constant. Any solution with $K \neq 0$ is a C^∞-function. When $K > 0$, w is an odd function changing sign, intersecting transversally the axis $w = 0$. When $K > 0$, w keeps a constant sign, positive or negative. The constant solution $w^*(\theta) \equiv c^*$ and its opposite correspond to the limit case $K = -M$, where

$$M = (1-q)\, \gamma^{-2(1+q)/(1-q)}/(1+q), \tag{2.3}$$

and the family of solutions associated to the one-hump function F (including $w \equiv 0$) corresponds to the case $K = 0$. Thus we can decompose $\mathscr{E}$ into

$$\mathscr{E} = \hat{\mathscr{E}} \cup \mathscr{E}_+ \cup (-\mathscr{E}_+) \cup \mathscr{E}_0, \tag{2.4}$$

where $\hat{\mathscr{E}}$ is the set of changing sign C^∞-functions, $\mathscr{E}_+$ the set of positive ones, and $\mathscr{E}_0$ the set of functions with at least a zero of order 2.

Our study is directly linked to the study of [BBo] about equation

$$w_{\theta\theta} - \gamma^2 w + |w|^{q-1}\, w = 0 \qquad \text{in the case } q > 1. \tag{2.5}$$

Indeed the potential V relative to (2.5) and the potential U present similar graphs; the only difference lies in the local behaviour at the origin. In the same manner we prove monotonicity results for the period functions. First let us consider the changing sign C^∞-solutions.

Lemma 2.1. *For any $\alpha > 0$, let $\hat{w}(.,\alpha)$ be the solution of (2.1) such that $\hat{w}(0,\alpha) = 0$, $\hat{w}_\theta(0,\alpha) = \alpha$, and let $\hat{P}(\alpha)$ be its least period. Then $\hat{P}$ is decreasing on $(0, +\infty)$ from $2l^*$ to $(1-q)\, l^*$.*

Proof. As in [BBo, Lemma 1.1] we obtain monotonicity. The proof differs only in the limit values of $\hat{P}$. We can write it in the form

$$\hat{P}(\alpha) = 4\left[\int_0^{c_1} (\alpha^2 - U(\varpi))^{-1/2}\, d\varpi + \int_0^1 \hat{z}'(\alpha\lambda)(1-\lambda^2)^{-1/2}\, d\lambda\right], \tag{2.6}$$

where for any $s>0$, $\hat{z}(s)$ is the positive solution of the equation $U(\hat{z}(s))=s^2$. Then $\lim_{s\to 0} \hat{z}'(s)=0$, $\lim_{s\to+\infty} \hat{z}'(s)=1/|\gamma|$, hence $\hat{z}'$ is bounded. In our case, the function $(-U)^{-1/2}$ is integrable at the origin; hence (by using the transformation $v=[(1+q)/2]^{1/2}\, \varpi^{(1-q)/2}$

$$\lim_{\alpha\to 0} \hat{P}(\alpha)=4\int_0^{c_1} (-U(\varpi))^{-1/2}\, d\varpi=\frac{8}{(1-q)\, |\gamma|}\int_0^1 (1-v)^{-1/2}\, dv=2l^*.$$

We have also $\lim_{\alpha\to+\infty} \hat{P}(\alpha)=4/|\gamma| \int_0^{c_1} (1-\varpi)^{-1/2}\, d\varpi=(1-q)\, l^*$, since the first integral in (2.6) tends to 0. ∎

The second lemma deals with the positive solutions.

LEMMA 2.2. *For any $\beta\in(0, M)$, let $w_+(.,\beta)$ be the positive solution of (2.1) such that $w_+(0,\beta)=c^*$, $(w_+)_\theta\,(0,\beta)=\beta$, and let $P_+(\beta)$ be its least period. Then P_+ is increasing on $(0, M)$ from $(1-q)^{1/2}\, l^*$ to l^*.*

Proof. Here we follow the proof of [BBo, Lemma 2.2]. We can write

$$P_+(\beta)=2\int_0^1 z'(\beta\lambda)(1-\lambda^2)^{-1/2}\, d\lambda-2\int_0^1 y'(\beta\lambda)(1-\lambda^2)^{-1/2}\, d\lambda, \quad (2.7)$$

where for any $s\in(0, M)$, $z(s)$ is the root of the equation $U(z(s))+M^2=s^2$ greater than c^*, and $y(s)$ is the smaller one. Here the difficulty comes because the two terms in (2.7) vary in the same way. Indeed, as in [BBo], we prove that

$$y''(s)\leqslant -2U^{(3)}(c^*)/3U''^2(c^*)\leqslant z''(s)\leqslant 0 \qquad \text{on } (0, M) \qquad (2.8)$$

by using the fact that $U^{(3)}(w)=2q(1-q)\, w^{q-2}\geqslant 0$ on $(0, c_1)$. Then the proof of the monotonicity is based on the fact that the function $(U'')^{-2/3}$ is convex on the set where U'' is positive. This is the case here, because $U^{(4)}(w)=-2q(1-q)(2-q)\, |w|^{q-3}\leqslant 0$, and $((U'')^{-2/3})''=\frac{2}{9}(U'')^{-5/3}\,[5(U^{(3)})^2-3U''U^{(4)}]$. Then P_+ is increasing. Now we easily obtain

$$\lim_{s\to 0} (z'(s)-y'(s))=2(2/U''(c^*))^{1/2}=2/(1-q)^{1/2}\, |\gamma|,$$

and it follows that

$$\lim_{\beta\to 0} P_+(\beta)=(1-q)^{1/2}\, l^*.$$

In order to derive the limit at the point M, we can use the Beppo–Levi theorem, because $z''-y''$ is nonnegative. This implies that

$$\lim_{\beta \to M} P_+(\beta) = 2 \int_0^1 (z' - y')(M\lambda)(1 - \lambda^2)^{-1/2} \, d\lambda$$

$$= 2 \int_0^{c_1} (-U(\varpi))^{-1/2} \, d\varpi = l^*,$$

which ends the proof. ∎

The full description of the set of stationary solutions is derived from the above lemmas.

THEOREM 2.3. *Let $\mathscr{E} = \hat{\mathscr{E}} \cup \mathscr{E}_+ \cup (-\mathscr{E}_+) \cup \mathscr{E}_0$ be the set of solutions of* (2.1), *on S^1 with $\sigma \neq -2$, where $\hat{\mathscr{E}}$, $\mathscr{E}_+$, $\mathscr{E}_0$ are as above. Then we get the following*

(i) *If $|\sigma + 2| < (1 - q)$, then $\hat{\mathscr{E}} = \varnothing$. If not, then $\hat{\mathscr{E}} = \bigcup_{k=\hat{k}_1}^{\hat{k}_2} \hat{\mathscr{C}}_k$, where $\hat{k}_1 = \min\{k \in \mathbb{N} \mid k > |\sigma + 2|/2\}$, $\hat{k}_2 = \max\{k \in \mathbb{N} \mid k < |\sigma + 2|/(1 - q)\}$, and each subset $\hat{\mathscr{C}}_k$ is one-dimensional, generated by the shifting of a solution with the least period $2\pi/k$.*

(ii) *If $|\sigma + 2| < (1 - q)^{1/2}$, then $\mathscr{E}_+ = \{c^*\}$. If not, then $\mathscr{E}_+ = \{c^*\} \cup \bigcup_{k=k_1}^{k_2} \mathscr{C}_{+k}$, where $k_1 = \min\{k \in \mathbb{N} \mid k > |\sigma + 2|\}$, $k_2 = \max\{k \in \mathbb{N} \mid k < |\sigma + 2|/(1 - q)^{1/2}\}$, and $\mathscr{C}_{+k}$ is one-dimensional, generated by the shifting of a solution with the least period $2\pi/k$.*

(iii) *If $|\sigma + 2| < 1$, then $\mathscr{E}_0 = \{0\}$.*

If $1 \leqslant |\sigma + 2| < 2$, then $\mathscr{E}_0 = \{0\} \cup (\pm \mathscr{C}_0^0)$, where $\mathscr{C}_0^0 = \{\theta \mapsto F(\theta - \theta_1) \mid \theta_1 \in S^1\}$ is one-dimensional, generated by the shifting of the one-hump solution $F(\theta)$, whose length of the support is between π and 2π.

If $|\sigma + 2| = 2$ (i.e., $\sigma = 0$ or $\sigma = -4$), then $\mathscr{E}_0 = \{0\} \cup (\pm \mathscr{C}_0^0) \cup (\pm \mathscr{C}_0^{1\pm})$, where $\mathscr{C}_0^{1\pm} = \{\theta \mapsto F(\theta - \theta_1) \pm F(\theta - \pi - \theta_1) \mid \theta_1 \in S^1\}$ are one-dimensional (here the length of the support of $F(\theta)$ is exactly π).

If $2 < |\sigma + 2| < 3$, then $\mathscr{E}_0 = \{0\} \cup (\pm \mathscr{C}_0^0) \cup (\pm \mathscr{C}_0^{1\pm}) \cup (C_0^{2\pm})$, where $\mathscr{C}_0^{2\pm} = \{\theta \mapsto F(\theta - \theta_1) \pm F(\theta - \theta_2 - \theta_1) \mid \theta_1, \theta_2 \in S^1, \theta_2 \in [\pi, 2\pi - l^*)\}$ are two-dimensional.

If $p \leqslant |\sigma + 2| < p + 1$ ($p \geqslant 3$), then the nonzero connected components of $\mathscr{E}_0$ are generated by sums or differences of m functions ($1 \leqslant m \leqslant p$), constructed by shifting from $F(\theta)$. Each component with m-humps functions is m-dimensional if $m < |\sigma + 2|$ and 1-dimensional if $m = p = |\sigma + 2|$.

Remark 2.4. A similar phenomenon has been brought to light in [GV] in a study of the set of solutions of the quasilinear equation

$$-(|w_\theta|^{p-2} w_\theta)_\theta + |w|^{q-1} w = \lambda \, |w|^{p-2} w \tag{2.9}$$

on S^1, in the case $2 < p < q + 1$.

3. THE PRINCIPLE OF THE INTERIOR DEAD CORE

In this section we give the main step for proving the convergence of the bounded solutions of (0.6) near $+\infty$ in the case $\sigma \neq -2$, i.e. $\gamma \neq 0$. The argument consists in a barrier analysis.

PROPOSITION 3.1. *Assume* $\sigma \neq -2$. *Let* $t_1, t_2 \in \mathbb{R}$ *with* $0 < t_2 - t_1 \leqslant 1$ *and consider a bounded domain* $Q \subset \mathbb{R} \times S^1$ *such that* $Q \cap \{t = t_1\} \neq \emptyset$ *and* $Q \cap \{t = t_2\} \neq \emptyset$. *For any* $\varepsilon > 0$, *set*

$$\tau_\varepsilon = k\varepsilon^{1/\gamma_0}, \tag{3.1}$$

where

$$\gamma_0 = 2/(1-q), \tag{3.2}$$

and $k \geqslant k(\gamma)$ *is a fixed and large enough real number. Consider any solution* v *of* (0.6) *in* Q *such that*

$$|v| \leqslant \varepsilon \qquad \text{on} \quad \partial Q \cap \{t \leqslant t_1\} \quad \text{and} \quad \partial Q \cap \{t \geqslant t_2\}. \tag{3.3}$$

(i) *Assume*

$$v = 0 \qquad \text{on} \quad \partial Q \cap \{t_1 \leqslant t \leqslant t_2\}. \tag{3.4}$$

If ε *is small enough such that* $\tau_\varepsilon \leqslant (t_2 - t_1)/3$, *then*

$$v = 0 \qquad \text{in} \quad Q' = Q \cap \{t_1 + \tau_\varepsilon \leqslant t \leqslant t_2 - \tau_\varepsilon\}. \tag{3.5}$$

(ii) *Assume that* $Q \supset [t_1, t_2] \times [\theta_1, \theta_2]$ *for some* $\theta_1, \theta_2 \in S^1$, $\theta_1 \neq \theta_2$, *and*

$$|v| \leqslant \varepsilon \qquad \text{on} \quad \partial Q \cap \{t_1 \leqslant t \leqslant t_2\}. \tag{3.6}$$

If ε *is small enough so that* $\tau_\varepsilon \leqslant \min\{t_2 - t_1, \theta_2 - \theta_1\}/3$, *then*

$$v = 0 \qquad \text{in} \quad Q'' = [t_1 + \tau_\varepsilon, t_2 - \tau_\varepsilon] \times [\theta_1 + \tau_\varepsilon, \theta_2 - \tau_\varepsilon]. \tag{3.7}$$

Proof. *Step* 1. *The Case* $Q \subset [-1, 0] \times S^1$ *and* $\sigma \geqslant 0$. Then the function $u(x) = u(r, \theta) = r^\gamma v(t, \theta)$ is a solution of (0.2) in a corresponding domain $\mathcal{Q}$, contained in the annulus $\mathrm{A} = \{x \in \mathbb{R}^N \mid 1 \leqslant |x| \leqslant e\}$. Hence from Kato inequality, $|u|$ is a subsolution of the equation

$$\Delta w = w^q \qquad \text{in } \mathcal{Q}, \tag{3.8}$$

and $|u(x)| \leqslant e^{\gamma}\varepsilon$ on $\partial\mathcal{Q}$, because $\gamma > 0$. Assume that $\tau_\varepsilon \leqslant (t_2 - t_1)/3$ in case (i), $\tau_\varepsilon \leqslant \min\{t_2 - t_1, \theta_2 - \theta_1\}/3$ in case (ii). Let $\mathcal{Q}'$, $\mathcal{Q}''$ be the nonempty domains corresponding to Q', Q''. Now for any $x_0 \in \mathcal{Q}$, the function

$$w(x, x_0) = c_0^* \, |x - x_0|^{\gamma_0}, \qquad \text{with} \quad c_0^* = (\gamma_0)^{-\gamma_0}, \tag{3.9}$$

is a supersolution of (3.8). If $x_0 \in \mathcal{Q}'$, then for any $x \in (\partial\mathcal{Q} \cap \{r \leqslant e^{-t_2}\}) \cup (\partial\mathcal{Q} \cap \{r \geqslant e_1^{-t_1}\})$ we have $|x - x_0| \geqslant \min\{e^{\tau_\varepsilon} - 1, 1 - e^{-\tau_\varepsilon}\} \geqslant \tau_\varepsilon/2$, hence $w(x) \geqslant \varepsilon$ provided $k \geqslant 2\gamma_0 e^{\gamma/\gamma_0}$. Choosing $\varepsilon \leqslant [k^{-1}(t_2 - t_1)/3]^{\gamma_0}$, we conclude from the maximum principle that $|u(x)| \leqslant w(x, x_0)$ in $\mathcal{Q}'$; in particular $u(x_0) = 0$, for any $x_0 \in \mathcal{Q}'$. In the same way if $x_0 \in Q''$, we get again $|x - x_0| \geqslant \tau_\varepsilon/2$ for any $x \in \partial\mathcal{Q}$, hence $w(x) \leqslant \varepsilon$ on $\partial\mathcal{Q}$. Choosing now $\varepsilon \leqslant [k^{-1} \min\{t_2 - t_1, \theta_2 - \theta_1\}/3]^{\gamma_0}$, we conclude that $u(x_0) = 0$ for any $x_0 \in \mathcal{Q}''$.

Step 2. *The General Case.* As we have observed it before, the equation is invariant under the change from t into $-t$ and γ into $-\gamma$. Without loss of generality, we can assume that $\gamma > 0$. By translation and dilatation, we can also reduce to the first case. Indeed consider some positive integer n and perform the change of variables

$$v(t, \theta) = n^{\gamma_0} w(t', \theta'), \qquad \textit{with} \quad t' = \frac{t - T}{n} - 1 \in [-1, 0], \quad \theta' = \frac{\theta}{n} \in \left[0, \frac{2\pi}{n}\right]. \tag{3.10}$$

Then w is a $2\pi/n$-periodic solution of the equation

$$-w_{t't'} + 2\gamma' w_{t'} = w_{\theta'\theta'} + \gamma'^2 w - |w|^{q-1} w, \tag{3.11}$$

where $\gamma' = n\gamma$. Then it is also a solution of (3.11) in an open subset of $[-1, 0] \times S^1$. Choosing $n \geqslant \gamma_0/\gamma$, then $\gamma' = (\sigma' + 2)/(1 - q)$ with $\sigma' \geqslant 0$. And $\{|v| \leqslant \varepsilon\}$ is the set $\{|w| \leqslant \varepsilon'\}$, with $\varepsilon' = n^{-\gamma_0}\varepsilon$, hence we get (3.5), (3.7) because $\tau'_\varepsilon = \tau_\varepsilon/n = k\varepsilon'^{1/\gamma_0} \leqslant (t_2/n - t_1/n)/3$, as soon as $k \geqslant 2\gamma_0 e^{\gamma'/\gamma_0}$, in particular if $k \geqslant 2\gamma_0 e^2$. ∎

Under the assumptions of Proposition 3.1, the ω-limit set Γ of the orbit of v is contained in a connected component of the set $\mathscr{E}$ of stationary solutions of (0.6). For proving the convergence to a unique element of this component, the principal difficulties come from the component containing functions with dead cores. As a consequence of Proposition 3.1, we prove the existence of *similar dead cores for function v*. For convenience and without loss of generality we formulate our result for the case when the orbit $\{v(t, .) \mid t > 0\}$ stabilizes to the component $\mathscr{C}_0^0 = \{\theta \mapsto F(\theta - \theta_1) \mid \theta_1 \in S^1\}$, generated by shifting of the one-hump function F (in case F presents a dead core, that means $1 < |\sigma + 2|$). Due to the rotational invariance of the equation, we can fix an arbitrary monotone sequence

$\{t_n\} \to +\infty$ such that $v(t_n, .)$ converges precisely to $F(.)$. In particular, this implies that

$$v(t_n, \theta) \to 0 \qquad \text{uniformly for} \quad |\theta| \geqslant \theta^* \tag{3.12}$$

as $n \to +\infty$. We now establish the following property of the support of $v(t, .)$.

PROPOSITION 3.2. *Let v be any solution of* (0.6) *on $\mathbb{R}^+ \times S^1$ with $1 < |\sigma + 2|$. Suppose that for some increasing sequence $\{t_n\} \to +\infty$,*

$$v(t_n, .) \to F(.) \qquad \text{in } C^2(S^1), \tag{3.13}$$

where F is given by (0.11). *Let $k \geqslant k(\gamma)$ be fixed, and let $\tau_\varepsilon = k\varepsilon^{1/\gamma_0}$, with $0 < \varepsilon \leqslant \varepsilon(k)$ small enough. Then there exists a $\delta(\varepsilon) > 0$, with $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$, $n(\varepsilon) \in \mathbb{N}$, and a subsequence $\{t_{\varphi_\varepsilon(n)}\}$ such that*

$$v(t, \theta) \equiv 0 \qquad \text{for} \quad |\theta| \geqslant \theta^* + \delta(\varepsilon), \quad t \in [t_{\varphi_\varepsilon(n)} + \tau_\varepsilon, t_{\varphi_\varepsilon(n)} + 2\tau_\varepsilon] \text{ and } n \geqslant n(\varepsilon). \tag{3.14}$$

Proof. After extraction of a subsequence, we can suppose that $t_{n+1} - t_n \geqslant 1$. Let $k \geqslant k(\gamma)$ and $\varepsilon > 0$ be fixed, and $\tau_\varepsilon = k\varepsilon^{1/\gamma_0}$. By the assumption,

$$v(t_n, .) \to F(.) \qquad \text{uniformly on } S^1.$$

Denote $T_n^\varepsilon = t_n + 3\tau_\varepsilon$. Then we can extract a subsequence $\{\varphi_\varepsilon(n)\}$ such that

$$v(T^\varepsilon_{\varphi_\varepsilon(n)}, .) \to F_\varepsilon(.) \qquad \text{uniformly on } S^1, \tag{3.16}$$

where F_ε is in the same connected component of $\mathscr{E}$ as F, i.e., F_ε is a translated of F. From Schauder regularity estimates,

$$C = \sup_{(t, \theta) \in [1, +\infty) \times S^1} |v_t(t, \theta)| < +\infty, \tag{3.17}$$

and this implies $|F_\varepsilon(.) - F(.)| \leqslant C\tau_\varepsilon$. Hence there exists a function $\eta(\tau) > 0$, depending on C, with $\lim_{\tau \to 0} \eta(\tau) = 0$, such that $F_\varepsilon(\theta) = 0$ for all $|\theta| \geqslant \theta^* + \eta(\tau_\varepsilon)$. Thus there exists an integer $n_1(\varepsilon)$ such that for any $n \geqslant n_1(\varepsilon)$,

$$|v(t_{\varphi_\varepsilon(n)}, \theta)| \leqslant \varepsilon \quad \text{and} \quad |v(T^\varepsilon_{\varphi_\varepsilon(n)}, \theta)| \leqslant \varepsilon \qquad \text{for all} \quad |\theta| \geqslant \theta^* + \eta(\tau_\varepsilon). \tag{3.18}$$

Consider the set

$$Q_\varepsilon = \{(t, \theta) \in [t_{\varphi_\varepsilon(n)}, T^\varepsilon_{\varphi_\varepsilon(n)}] \times S^1 \mid |\theta| \geqslant \theta^*, |v(t, \theta)| \leqslant \varepsilon\}. \tag{3.19}$$

Then there exists some $n(\varepsilon) \geqslant n_1(\varepsilon)$, such that for any $n \geqslant n(\varepsilon)$ and any $t \in [t_{\varphi_\varepsilon(n)}, T^\varepsilon_{\varphi_\varepsilon(n)}]$, $Q_\varepsilon \supset [t_{\varphi_\varepsilon(n)}, T^\varepsilon_{\varphi_\varepsilon(n)}] \times [\theta^* + \eta(\tau_\varepsilon), 2\pi - \theta^* - \eta(\tau_\varepsilon))]$. Indeed suppose it were not true. Then there exist for example a sequence $\{n_p\} \to +\infty$ and $\tilde{t}_{n_p} \in [t_{\varphi_\varepsilon(n_p)}, T^\varepsilon_{\varphi_\varepsilon(n_p)}]$ and $\tilde{\theta}_{n_p} \in (\theta^* + \eta(\tau_\varepsilon), \pi]$ such that, say, $v(\tilde{t}_{n_p}, \tilde{\theta}_{n_p}) > \varepsilon$. After extraction of subsequences, we can assume that $\tilde{\theta}_{n_p} \to \tilde{\theta}_\varepsilon \in [\theta^* + \eta(\tau_\varepsilon), \pi)$, and $v(\tilde{t}_{n_p}, .) \to \tilde{F}_\varepsilon(.)$ uniformly on S^1. Then $\tilde{F}_\varepsilon(\tilde{\theta}_\varepsilon) \geqslant \varepsilon$; but $|\tilde{F}_\varepsilon(.) - F(.)| \leqslant C\tau_\varepsilon$, hence $\tilde{F}_\varepsilon(\theta) = 0$ for all $|\theta| \geqslant \theta^* + \eta(\tau_\varepsilon)$, which is impossible. Then we can apply Proposition 3.1 to Q_ε: taking $\varepsilon \leqslant \varepsilon(k)$ small enough such that $\tau_\varepsilon \leqslant 1/3$ and $\tau_\varepsilon + \eta(\tau_\varepsilon) \leqslant 2(\pi - \theta^*)/3$, we deduce that v is identically 0 in

$$Q''_\varepsilon = [t_{\varphi_\varepsilon(n)} + \tau_\varepsilon, t_{\varphi_\varepsilon(n)} + 2\tau_\varepsilon] \times [\theta^* + \eta(\tau_\varepsilon) + \tau_\varepsilon, 2\pi - \theta^* - \eta(\tau_\varepsilon) - \tau_\varepsilon)], \tag{3.20}$$

hence (3.14) with $\delta(\varepsilon) = \eta(\tau_\varepsilon) + \tau_\varepsilon$. ∎

Remark 3.3. This last result implies in particular a phenomenon of *disappearance of a small tail.* Under assumptions of Proposition 3.2, there exists another sequence $\{t'_n\} \to +\infty$ and a positive sequence $\{\delta_n\} \to 0$ such that

$$v(t'_n, .) \to F(.) \quad \text{in} \quad C^2(S^1), \quad \text{and} \quad v(t'_n, \theta) \equiv 0 \quad \text{for} \quad |\theta| \geqslant \theta^* + \delta_n. \tag{3.21}$$

Indeed we first extract a subsequence such that $t_{n+1} - t_n \geqslant 1$. Then for any $p \in \mathbb{N}$, we choose $\varepsilon = \varepsilon_0/p$ with $\varepsilon_0 \leqslant \varepsilon(k)$ small enough, and an increasing sequence $\{n_p\} \to +\infty$ such that $n_p \geqslant n(\varepsilon_0/p)$, and let $t'_p = t_{\varphi_\varepsilon(n_p)} + \tau(\varepsilon_0)/p)$ and $\delta_p = \delta(\varepsilon_0/p)$. Then $v(t'_p, \theta) \equiv 0$ for $|\theta| \geqslant \theta^* + \delta_p$, and $|v(t'_p, .) - F(.)| \leqslant 2C\tau_{\varepsilon_0/p}$ and therefore (3.21) holds.

4. CONVERGENCE RESULTS

In this section we assume essentially that $|\sigma + 2| \leqslant 2$ and we prove Theorem 0.1. Let $u \in C^2(B')$ be any solution of (0.2) such that $|x|^{-\gamma}$- is bounded, and let v be defined by (0.6) in $[0, +\infty) \times S^1$. Remember that the ω-limit set Γ of the orbit of v is contained in a connected component $\mathscr{C}$ of the set $\mathscr{E}$.

The case $\Gamma \subset \mathscr{E}_+$ can be treated separately. Indeed, let G be any fixed function in Γ. The case $G \equiv c^*$ is obvious because the set Γ is connected, hence $\Gamma = \{c^*\}$. In the general case we can use the methods of [BV]: the connected component of G, generated by shifting of G, is bounded far away from 0, since G is positive and continuous. Then there exists some $T > 0$ such that $\liminf_{[T, +\infty] \times S^1} v = \ell > 0$. Now the function $g(u) = |u|^{q-1} u$ is uniformly analytic on any compact subset of $[\ell/2, +\infty)$ containing the range of v after time T. Then we apply the Simon theorem [Si] and conclude that the convergence to a single element holds.

In the case $\Gamma \subset \hat{\mathscr{E}} \cup \mathscr{E}_0$ we can no more use this result. Therefore we carry on the ideas of [CMV] which are specific to dimension 2. It turns out that for the elliptic autonomous equation (0.6), a powerful technique is the reflection ideas which come from the Sturmian zero-set argument for second order ordinary differential equations and parabolic equations, the Jordan curve theorem, and Aleksandrov's reflection principle. These ideas were used in [CMV] for Eq. (0.6) in the regular case $q > 1$. The principle of the interior dead core makes it possible to apply a similar technique in the non-smooth case $0 < q < 1$. The following proof is also available in the case $\Gamma \subset \mathscr{E}_+$ and allows to avoid the difficult result of Simon. Our argument is by contradiction.

Proof of Theorem 0.1. Let G be any fixed nonconstant function in Γ. Assume that the connected component of G is one-dimensional. Then without loss of generality we can assume that there exists some $v > 0$ such that $\theta \mapsto G(\theta - v)$ and $\theta \mapsto G(\theta + v)$ lie in Γ, and $G(\theta) = G(-\theta)$ on S^1, and $G(\theta) > 0$ for small $\theta > 0$. We can suppose that $v > 0$ is small enough, according to the connectedness of Γ. Observe that the reflected function $v(t, -\theta)$ satisfies (0.6) again. Following the idea of [CMV], we introduce the function $\tilde{v}$ defined by

$$\tilde{v}(t, \theta) = v(t, \theta) - v(t, -\theta). \tag{4.1}$$

It satisfies the equation

$$-\tilde{v}_{tt} + 2\gamma \tilde{v}_t = \tilde{v}_{\theta\theta} + \gamma^2 \tilde{v} - h\tilde{v}, \tag{4.2}$$

where

$$h(t, \theta) = q \int_0^1 |\eta v(t, \theta) + (1-\eta)\, v(t, -\theta)|^{q-1}\, d\eta \geqslant 0, \tag{4.3}$$

and

$$\tilde{v}(t, 0) = \tilde{v}(t, \pi) = 0 \qquad \text{for all} \quad t \geqslant 0. \tag{4.4}$$

From our assumptions, there exists a monotone sequence $\{t_n\} \to +\infty$ such that

$$\begin{cases} v(t_{2p}, \theta) \to G(\theta - v), \\ v(t_{2p+1}, \theta) \to G(\theta + v). \end{cases} \tag{4.5}$$

Therefore

$$\begin{cases} \tilde{v}(t_{2p}, \theta) \to \Phi(\theta) = G(\theta - v) - G(-\theta - v), \\ \tilde{v}(t_{2p+1}, \theta) \to -\Phi(\theta) = G(\theta + v) - G(-\theta + v), \end{cases} \tag{4.6}$$

in $C^2(S^1)$. Notice that all the zeros of Φ coincide with the points of extremum of G, since v is small enough. The set Z of zeros of Φ in $[0, \pi)$ is composed of isolated points and (or) intervals. The set $\{\Phi \neq 0\}$ has a finite number of connected components. Then from (4.6), for any $\varepsilon > 0$ there exists $d'(\varepsilon) > 0$, with $\lim_{\varepsilon \to 0} d'(\varepsilon) = 0$, and $n'(\varepsilon) \in \mathbb{N}$ such that

$$\operatorname{dist}(\theta, Z) \leqslant d'(\varepsilon) \quad \text{for any } \theta \in [0, \pi) \text{ and } n \geqslant n'(\varepsilon) \quad \text{such that } \tilde{v}(t_n, \theta) = 0. \tag{4.7}$$

Moreover, 0 is a simple zero. Then, from (4.6), $\tilde{v}(t_{2p}, \theta)$ is positive and $\tilde{v}(t_{2p+1}, \theta)$ is negative on $(0, \theta^* - v]$ for large p, since the convergence holds in $C^1(S^1)$. In the half-disk $O = [0, +\infty) \times (0, \pi)$, we consider the three sets

$$\begin{cases} O^+ = \{(t, \theta) \in O \mid \tilde{v}(t, \theta) > 0\}, \\ O^- = \{(t, \theta) \in O \mid \tilde{v}(t, \theta) < 0\}, \\ O_o = \{(t, \theta) \in O \mid \tilde{v}(t, \theta) = 0\}. \end{cases} \tag{4.8}$$

From (4.2), (4.3), the continuous function $\tilde{u}(x) = |x|^\gamma \, \tilde{v}(t, \theta)$ satisfies the elliptic equation

$$-\Delta \tilde{u} + H\tilde{u} = 0, \qquad \text{with some} \quad H \geqslant 0, \tag{4.9}$$

hence it satisfies the maximum principle. Consequently for any connected component C of O^+, O^-, either $C \cap [t = 0] \neq \emptyset$, or $C \cap [t = T] \neq \emptyset$ for any large T. Let C_{2p} be the connected component of O^+ such that $(t_{2p}, 0) \in \partial C_{2p}$, and C_{2p+1} be the connected component of O^- such that $(t_{2p+1}, 0) \in \partial C_{2p+1}$. As in [CMV], from the Jordan curve theorem, either there exists an integer n_0 such that

$$\text{for any } n \geqslant n_0, \, C \cap [t = T] \neq \emptyset \qquad \text{for any large } T, \tag{4.10}$$

or

$$\text{for any } n \in \mathbb{N}, \, C \cap [t = 0] \neq \emptyset \tag{4.11}$$

Then for any $n > n_0$ (resp. any $n \in \mathbb{N}$) there exist at least $n - n_0$ components $C \subset \bigcup_{m \in \mathbb{N}} C_m$ such that $C \cap [t = T] \neq \emptyset$ for any $T \geqslant t_n$. Now we divide the proof according to the different subcases.

Case 1. *Stabilization to a Function with Isolated Zeros.* Here we suppose that the zeros of the function G are isolated. It is the case when $\Gamma \subset \mathscr{E}^+$, or $\Gamma \subset \hat{\mathscr{E}}$. It also happens when $\Gamma \subset \mathscr{E}_0$ and $|\sigma+2|=2$, and G is composed of two humps, obtained by shifting $\pm F$, which period is exactly π.

In that case the set Z of zeros of Φ in $[0, \pi)$ is finite, and each of them is simple. For any $n > n_0$ there exists $C \subset \bigcup_{m \in \mathbb{N}} C_m$ such that $\operatorname{meas}(C \cap [t = t_n]) \leqslant 2\pi/(n - n_0)$. As in [CMV] this implies that Φ admits at least one multiple zero, which is impossible.

Case 2. *Stabilization to a One-Hump Function with Dead Core.* Here we assume that $G = F$, where F is defined in (0.11), with $|\sigma+2| > 1$, that implies $\theta^* < \pi$. Let $k \geqslant 3k(\gamma)$ and $\tau_\varepsilon = k\varepsilon^{1/\gamma_0}$, with $0 < \varepsilon \leqslant \varepsilon(k)$ small enough. From Proposition 3.2, there exists $\delta(\varepsilon) > 0$ with $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$, $n(\varepsilon) \in \mathbb{N}$ and a subsequence $\{t_{\varphi_\varepsilon(n)}\}$ such that for any even integer $n \geqslant n(\varepsilon)$

$$v(t, \theta) \equiv 0 \qquad \text{for} \quad |\theta| \geqslant \theta^* + v + \delta(\varepsilon),\, t \in [t_{\varphi_\varepsilon(n)} + \tau_\varepsilon, t_{\varphi_\varepsilon(n)} + 2\tau_\varepsilon]. \tag{4.12}$$

We can also assume, after extraction of a subsequence, that

$$v(t_{\varphi_\varepsilon(n)} + \tau_\varepsilon, .) \xrightarrow[n \to \infty]{} F_\varepsilon^1 \qquad \text{and} \qquad v(t_{\varphi_\varepsilon(n)} + 2\tau_\varepsilon, .) \xrightarrow[n \to \infty]{} F_\varepsilon^2, \tag{4.13}$$

uniformly on S^1, where F_ε^1 and F_ε^2 are translations of F.

Now we extend (4.7). For any $t \in [t_{\varphi_\varepsilon(n)} + \tau_\varepsilon, t_{\varphi_\varepsilon(n)} + 2\tau_\varepsilon]$ and any $\theta \in [0, \pi)$ such that $\tilde{v}(t, \theta) = 0$, we have $|\tilde{v}(t_{\varphi_\varepsilon(n)}, \theta)| \leqslant C\tau_\varepsilon$ from (3.17). Hence from (4.6) we deduce that there exists some $n'(\varepsilon) \geqslant n(\varepsilon)$ such that $|\Phi(\theta)| \leqslant \varepsilon + C\tau_\varepsilon$ whenever $n \geqslant n'(\varepsilon)$. Then there exists $\delta'(\varepsilon) > 0$, with $\lim_{\varepsilon \to 0} \delta'(\varepsilon) = 0$ such that for any $n \geqslant n'(\varepsilon)$, there holds

$$\begin{aligned} \operatorname{dist}(\theta, Z) \leqslant \delta'(\varepsilon) \qquad &\text{for any} \quad t \in [t_{\varphi_\varepsilon(n)} + \tau_\varepsilon, t_{\varphi_\varepsilon(n)} + 2\tau_\varepsilon] \\ &\text{and any } \theta \in [0, \pi) \text{ such that } \tilde{v}(t, \theta) = 0. \end{aligned} \tag{4.14}$$

Then there exist at least $n - n_0$ components $C \subset \bigcup_{m \in \mathbb{N}} C_m$ (resp. an infinity) such that $C \cap [t = t_{\varphi_\varepsilon(n)} + 2\tau_\varepsilon] \neq \emptyset$ and $C \cap [t = t_{\varphi_\varepsilon(n)}] \neq \emptyset$. From (4.13), such components satisfy

$$\partial C \cap \{t = \tilde{t}\} \subset [0, \delta'(\varepsilon)] \cup [\theta^* + v - \delta'(\varepsilon), \pi], \tag{4.15}$$

for any $\tilde{t} \in [t_{\varphi_\varepsilon(n)} + \tau_\varepsilon, t_{\varphi_\varepsilon(n)} + 2\tau_\varepsilon]$.

Now we claim that there exist an ε small enough (namely $\varepsilon \leqslant \varepsilon(k)$, with $\delta(\varepsilon) + 2\delta'(\varepsilon) < v$), an integer $n \geqslant n'(\varepsilon)$, and a component C such that

$$\begin{aligned} C \cap \{t_{\varphi_\varepsilon(n)} + \tau_\varepsilon \leqslant t \leqslant t_{\varphi_\varepsilon(n)} + 2\tau_\varepsilon\} \subset \{&t_{\varphi_\varepsilon(n)} + \tau_\varepsilon, t_{\varphi_\varepsilon(n)} + 2\tau_\varepsilon] \\ &\times [\theta^* + v - \delta'(\varepsilon), \pi]. \end{aligned} \tag{4.16}$$

First, notice that (4.16) is satisfied as soon as $C \cap \{t = t_{\varphi_\varepsilon(n)} + \tau_\varepsilon\} \subset [\theta^* + \nu - \delta''(\varepsilon), \pi]$, by connexity. Suppose it is not true. Then for any small ε and any $n \geqslant n'(\varepsilon)$, any such component C would intersect $[0, \delta'(\varepsilon)]$ at the point $t = t_{\varphi_\varepsilon(n)} + \tau_\varepsilon$. But taking $n > 2\pi/\theta^*$, there exists a component C^n such that $\text{meas}(C^n \cap \{t = < t_{\varphi_\varepsilon(n)}\}) < \theta^*$ and since $2\delta'(\varepsilon) < \varepsilon$, we deduce that $C^n \cap \{t = t_{\varphi_\varepsilon(n)} + \tau_\varepsilon\} \subset [0, \delta'(\varepsilon)]$. Then taking $\varepsilon = \varepsilon_0/p$, there exists some increasing sequences $\{n_p\} \to +\infty$ and $\{\tilde{t}_{n_p}\} \to +\infty$ such that for any $p \geqslant 1$ there exists some $C \subset \bigcup_{m \in \mathbb{N}} C_m$ such that $C \cap \{t = \tilde{t}_{n_p}\} \subset [0, \delta'(\varepsilon_0/p)]$. This implies again that 0 is a multiple zero of Φ, which is false; therefore (4.16) holds. But from Proposition 3.2, the inequality $\delta(\varepsilon) + \delta'(\varepsilon) \leqslant \nu$ implies that

$$v(t, -\theta) = 0 \qquad \text{in} \quad [t_n + \tau_\varepsilon, t_n + 2\tau_\varepsilon] \times [\theta^* - \nu + \delta(\varepsilon), \pi]. \tag{4.17}$$

We deduce that

$$v = \tilde{v} \qquad \text{on} \quad [t_n + \tau_\varepsilon, t_n + 2\tau_\varepsilon] \times [\theta^* + \nu - \delta'(\varepsilon), \pi]. \tag{4.18}$$

Then we can apply Proposition 3.1 in $C \cap \{t_{\varphi_\varepsilon(n)} + \tau_\varepsilon \leqslant t \leqslant t_{\varphi_\varepsilon(n)} + 2\tau_\varepsilon\}$ for n large enough, with τ_ε replaced by $\tau_\varepsilon/3$, since we have chosen $k \geqslant 3k(\gamma)$. Indeed we only need to prove that $v \leqslant \varepsilon$ on the sets $I_{1,n} = C \cap \{t = t_{\varphi_\varepsilon(n)} + \tau_\varepsilon\}$ and $I_{2,n} = C \cap \{t = t_{\varphi_\varepsilon(n)} + 2\tau_\varepsilon\}$. We denote $a_{j,n} = \min I_{j,n}$ and $b_{j,n} = \max I_{j,n}$ ($j = 1, 2$). From (4.13) there exists $N(\varepsilon)$ such that for any $n \geqslant N(\varepsilon)$ there holds $|v(t_{\varphi_\varepsilon(n)} + \tau_\varepsilon, .) - F^1_\varepsilon(.)| \leqslant \varepsilon/2$, which implies $F^1_\varepsilon(a_{1,n}) \leqslant \varepsilon/2$ and also $F^1_\varepsilon(\theta) \leqslant \varepsilon/2$ for any $\theta \in I_{1,n}$ since F^1_ε is nonincreasing and therefore $|v(t_{\varphi_\varepsilon(n)} + \tau_\varepsilon, \theta)| \leqslant F^1_\varepsilon(\theta) + F^1_\varepsilon(a_{1,n}) \leqslant \varepsilon$ on $I_{1,n}$. Similarly $|v(t_{\varphi_\varepsilon(n)} + 2\tau_\varepsilon, \theta)| \leqslant F^2_\varepsilon(\theta) + F^2_\varepsilon(a_{1,n}) \leqslant \varepsilon$ on $I_{2,n}$. We deduce that $v \equiv 0$ and therefore $\tilde{v} \equiv 0$ in $C \cap \{t_{\varphi_\varepsilon(n)} + (4/3)\tau_\varepsilon \leqslant t \leqslant t_{\varphi_\varepsilon(n)} + (5/3)\tau_\varepsilon\}$, which contradicts the fact that $C \subset O^+ \cup O^-$. Thus we get a contradiction.

Remark 4.1. By using the above method we can obtain some convergence results in the case $|\sigma + 2| > 2$. There is convergence if the limit set Γ is included into a zero- or one-dimensional connected component of $\mathscr{E}$: this is the case if $\Gamma \subset \mathscr{E}_+$, or if $\Gamma \subset \hat{\mathscr{E}}$. This is also the case if $\Gamma \subset \mathscr{E}_0$ is composed of k-jointed humps, in which case the one-hump analysis can be performed. Finally, if $\Gamma \subset \mathscr{E}_0$ is composed of k totally separated humps, which means humps such that the distance between their supports is minorized by a positive real number α. In that case, it follows from the interior dead core principle that for $t \geqslant t_0$ large enough, the support of $\theta \mapsto v(t, \theta)$ is the union of k totally separated intervals, and each of them is essentially concentrated in the support of a single hump. Therefore we can write $v(t, \theta) = \sum_{j=1}^k v_j(t, \theta)$, where each of the v_j satisfies (0.6) in $[t_0, +\infty) \times S^1$ and has its limit set included into a connected component of $\mathscr{E}_0$ generated by shifting by a one-hump stationary solution. The convergence of v_j follows from case 2 above.

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