Initial Blow-up for the Solutions of a Semilinear Parabolic Equation with Source Term

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Abstract

Here we study the initial blow-up of the positive solutions of the semilinear parabolic equation

$$u_t = \Delta u + u^q \qquad \text{in } \Omega \times (0,T)$$

with $q > 1$, for any domain Ω of $\mathbf{R}^N$. We show that Harnack inequality holds, and prove the following a priori estimate in any domain $\varpi \subset\subset \Omega$, (or $\varpi = \Omega = \mathbf{R}^N$):

$$u(x,t) \leq C\, t^{-1/(q-1)} \qquad \text{in } \varpi \times (0, T/2)\,,$$

with $C = C(\varpi, N, q)$, whenever $q < N(N+2)/(N-1)^2$.

1 Introduction

Let Ω be a domain of $\mathbf{R}^N$ ($N \geq 1$), $T \in (0, +\infty]$, and $u \in C^{2,1}(\Omega \times (0,T))$ be a positive solution of equation

$$u_t = \Delta u + u^q \qquad \text{in } \Omega \times (0,T)\,, \tag{1}$$

where $q > 1$ is a real number.

The classical blow-up problem concerns the behaviour of u near the point T, assuming that $(0,T)$ denotes the maximal time interval of existence, and $T < +\infty$. In [6], Giga and Kohn proved the estimate

$$u(x,t) \leq C(T-t)^{-1/(q-1)} \qquad \text{in } \Omega \times (T/2, T)\,, \tag{2}$$

when $q < (N+2)/(N-2)$ (or $N \leq 2$) and Ω is bounded, convex, or $\Omega = \mathbf{R}^N$, see [6]. Many further results concern the precise behaviour of u near the blowup point.

Here we consider the *initial blow-up problem.* Our purpose is to give an a priori estimate of the behaviour of u when t tends to 0. In case $q < (N+2)/N$, and Ω is bounded, Moutoussamy and Veron gave in [9] the precise behaviour of any solution of 1 with a punctual singularity as initial data: if $0 \in \Omega$ and $u \in C^{2,1}\left[(\overline{\Omega} \times [0,T])\backslash \{(0,0)\}\right]$ is a solution of 1 with $u(x,0) = 0$ on $\Omega\backslash \{0\}$, then there exists $\gamma \geq 0$ such that

$$\lim_{t \to 0} \max_{x \in \overline{\Omega}} |u(x,t) - \gamma\, E(x,t)| = 0,$$

where $E(x,t) = (4\pi t)^{-N/2} \exp\left(-|x|^2/4t\right)$. Furthermore, Baras and Pierre studied in [3] the removability of data measures, and showed in particular that any punctual singularity is removable if $q \geq (N+2)/N$. Assuming now $(N+2)/N < q < (N+2)/(N-2)$, Weissler showed the existence of a very singular solution of 1 on $\mathbf{R}^N \times (0,+\infty)$ under the form

$$u(x,t) = t^{-1/(q-1)}\,\omega(y), \qquad y = x/\sqrt{t},$$

where ω is a radially symmetric solution of equation

$$\Delta\omega + \frac{1}{2}\,y.\nabla\omega + \frac{1}{q-1}\,\omega + \omega^q = 0,$$

bounded in $\mathbf{R}^N$, see [11]. These results lead to conjecture an estimate in $t^{-1/(q-1)}$ when $q < (N+2)/(N-2)$ and in $t^{-N/2}$ when $q < (N+2)/N$. Our main result is a part of this conjecture:

Theorem 1 *Let Ω be any domain of $\boldsymbol{R}^N$ and $u \in C^{2,1}(\Omega \times (0,T))$ any positive solution of 1 . Assume that*

$$1 < q < N(N+2)/(N-1)^2 \;\; (\textit{or } N = 1). \tag{3}$$

Then for any domain $\varpi \subset\subset \Omega$, (or $\varpi = \Omega = \boldsymbol{R}^N$), there exists a constant $C = C(\varpi, N, q) > 0$ such that

$$u(x,t) \leq C\,t^{-1/(q-1)} \qquad \textit{in } \varpi \times (0,T/2)\,. \tag{4}$$

If moreover $q < (N+2)/N$, then, for any domain $\varpi \subset\subset \Omega$, there exists a constant $C = C(\varpi, N, q, u) > 0$ such that

$$u(x,t) \leq C\,t^{-N/2} \qquad \textit{in } \varpi \times (0,T/2)\,. \tag{5}$$

In order to obtain such estimates, it is clear that the methods of [6] fail: they are based on the decreasingness of an energy function, so that it is impossible to reverse the time. Our proof is based on Bernstein techniques, still used in the study of the behaviour of the solutions of the elliptic problem in Ω with an isolated interior singularity x_0 :

$$\Delta w + w^q = 0 \qquad \text{in } \Omega\backslash \{x_0\}\,. \tag{6}$$

In [7], Gidas and Spruck showed the estimate

$$w(x) \leq C\, |x - x_0|^{-2/(q-1)} \qquad \text{near } x_0,$$

when $q < (N+2)/(N-2)$, by proving that w satisfies Harnack inequality. This method has been further explored in [5]. If moreover $q < N/(N-2)$, the Harnack property implies the estimate

$$w(x) \leq C\, |x - x_0|^{-(N-2)}, \qquad \text{near } x_0,$$

which can be obtained more simply, see [10] and [8]. As in [5],[7],[4], we use the Bochner-Wietzenböck formula in order to obtain suitable estimates on the total gradient of u. Up to now, our proof is limitated to the case $q < N(N+2)/(N-1)^2$, but we hope to extend it up to the case $q < (N+2)/(N-2)$.

2 Proofs of the results

We write equation 1 under the form

$$u_t = \Delta u + Hu \qquad \text{in } \Omega \times (0,T), \tag{7}$$

with $H = u^{q-1}$. As for the elliptic singularity problem, the idea is to give suitable integral estimates of the coefficient H, that means to obtain a good estimate of some power of u. In that aim, the crucial step is the estimate of the gradient term $u^{q-1}\,|\nabla u|^2$. In [4], using the Böchner-Wietzenböck formula,

$$\frac{1}{2}\Delta\left(|\nabla v|^2\right) = |Hess\, v|^2 + (\nabla \Delta v).\nabla v, \tag{8}$$

we showed the following.

Lemma 2 *Let G be any open set of $\boldsymbol{R}^N$. Then for any function $w \in C^2(G)$, any nonnegative $\xi \in \mathcal{D}(G)$, and any reals d, m such that $d \neq m+2$,*

$$\frac{\left[2(N-m)d - (N-1)(m^2+d^2)\right]}{4N}\int_G \xi w^{m-2}\,|\nabla w|^4$$
$$-\frac{N-1}{N}\int_G \xi w^m\,(\Delta w)^2 - \frac{[2(N-1)m + (N+2)d]}{2N}\int_G \xi w^{m-1}\,|\nabla w|^2\,\Delta w$$
$$\leq \frac{m+d}{2}\int_G w^{m-1}\,|\nabla w|^2\,\nabla w.\nabla\xi + \int_G w^m \Delta w \nabla w.\nabla \xi + \frac{1}{2}\int_G w^m\,|\nabla w|^2\,\Delta\xi.$$

Using this Lemma, we show the essential interior estimate.

Lemma 3 ***(Estimate of the gradients)*** *Let $u \in C^{2,1}(\Omega \times (0,T))$ be a positive solution of 1, and G be any domain of $\boldsymbol{R}^N$, such that $Q = G \times (t_1, t_2) \subset \overline{Q} \subset \Omega \times (0,T)$. Assume that 3 holds. Suppose $\zeta \in \mathcal{D}(Q)$ with value in $[0,1]$ and let $r > 4$. Then there exists a constant $C = C(N,q,r) > 0$ such that*

$$\int\int_Q \zeta^r u^{-2} |\nabla u|^4 + \int\int_Q \zeta^r u^{q-1} |\nabla u|^2 + \int\int_Q \zeta^r u_t^2$$

$$\leq C(\int\int_Q \zeta^{r-2} u^{q+1}(|\nabla \zeta|^2 + |\zeta_t|) + C(\int\int_Q \zeta^{r-4} u^2(|\Delta\zeta|^2 + |\Delta\zeta|^2 + \zeta_t^2)). \quad (9)$$

Proof. Let us apply Lemma 2 to $w = u = u(t,.)$, with $\xi = \zeta^r = \zeta^r(t,.)$, for any $t \in [t_1, t_2]$, with $m = 0$ and any real $d \neq 2$:

$$\frac{\left[2Nd - (N-1)d^2\right]}{4N} \int_G \zeta^r u^{-2} |\nabla u|^4$$

$$-\frac{N-1}{N} \int_G \zeta^r (\Delta u)^2 - \frac{(N+2)d}{2N} \int_G \zeta^r u^{-1} |\nabla u|^2 \Delta u$$

$$\leq \frac{d}{2} \int_G u^{-1} |\nabla u|^2 \nabla u.\nabla(\zeta^r) + \int_G \Delta u \nabla u.\nabla(\zeta^r) + \frac{1}{2} \int_G |\nabla u|^2 \Delta(\zeta^r).$$

Now from 1,

$$-\int_G \zeta^r (\Delta u)^2 = -\int_G \zeta^r \Delta u (u_t - u^q)$$

$$= X + \int_G \nabla u.\nabla(\zeta^r) u_t - \int_G u^q \nabla u.\nabla(\zeta^r) - q \int_G \zeta^r u^{q-1} |\nabla u|^2,$$

with

$$X = X(t) = \int_G \zeta^r \nabla u.\nabla u_t = \frac{df}{dt} - P, \quad (10)$$

where

$$f = f(t) = \frac{1}{2} \int_G \zeta^r |\nabla u|^2, \qquad P = \frac{1}{2} \int_G (\zeta^r)_t |\nabla u|^2.$$

Hence we deduce

$$\begin{aligned} & a \int_G \zeta^r u^{-2} |\nabla u|^4 + b \int_G \zeta^r u^{q-1} |\nabla u|^2 \\ & \leq -\frac{N-1}{N} X + \frac{(N+2)d}{2N} Y + \frac{1}{N} Z - \frac{1}{N} V + \frac{d}{2} W + \frac{1}{2} R, \end{aligned} \quad (11)$$

where

$$a = \frac{\left[2Nd - (N-1)d^2\right]}{4N}, \qquad b = \frac{[(N+2)d - 2(N-1)q]}{2N},$$

$$Y = \int_G \zeta^r u^{-1} |\nabla u|^2 u_t, \qquad Z = \int_G u_t \nabla(\zeta^r).\nabla u,$$

$$V=\int_G u^q\nabla(\zeta^r).\nabla u,\quad W=\int_G u^{-1}\left|\nabla u\right|^2\nabla(\zeta^r).\nabla u,\quad R=\int_G\Delta(\zeta^r)\left|\nabla u\right|^2.$$

If $N\geq 2$, since $q<N(N+2)/(N-1)^2$, we can chose d such that

$$\frac{2(N-1)q}{N+2}<d<\frac{2N}{N-1}.$$

Then for any $N\geq 1$ we have $a>0$ and $b>0$, which allows to get an estimate of the crucial term $\int_G\zeta^r u^{q-1}\left|\nabla u\right|^2$. Now we bound some of the right-hand side terms of 11 with Hölder inequality. For any $\varepsilon\in(0,1)$, there exists $C(\varepsilon)>0$ such that

$$V\leq\varepsilon\int_G\zeta^r u^{q-1}\left|\nabla u\right|^2+C(\varepsilon)\int_G\zeta^{r-2}u^{q+1}\left|\nabla\zeta\right|^2; \tag{12}$$

$$W\leq\varepsilon\int_G\zeta^r u^{-2}\left|\nabla u\right|^4+C(\varepsilon)\int_G\zeta^{r-4}u^2\left|\nabla\zeta\right|^4; \tag{13}$$

$$\begin{aligned} R&\leq r(r-1)\int_G\zeta^{r-2}\left|\nabla\zeta\right|^2\left|\nabla u\right|^2+r\int_G\zeta^{r-1}\Delta\zeta\left|\nabla u\right|^2 \\ &\leq\varepsilon\int_G\zeta^r u^{-2}\left|\nabla u\right|^4+C(\varepsilon)\int_G\zeta^{r-4}u^2\left|\nabla\zeta\right|^4+C(\varepsilon)\int_G\zeta^{r-2}u^2\left|\Delta\zeta\right|^2.\end{aligned} \tag{14}$$

On another part, multiplying equation 1 by $\zeta^r u_t$ and integrating over G, we find from 10

$$\int_G\zeta^r u_t^2=-X+\frac{dg}{dt}-Z-S=P-Z-S+\frac{d(g-f)}{dt}, \tag{15}$$

where

$$g=g(t)=\frac{1}{q+1}\int_G\zeta^r u^{q+1},\qquad S=\frac{1}{q+1}\int_G(\zeta^r)_t u^{q+1}.$$

And we can bound P and Z by

$$P=\frac{r}{2}\int_G\zeta^{r-1}\zeta_t\left|\nabla u\right|^2\leq\varepsilon^2\int_G\zeta^r u^{-2}\left|\nabla u\right|^4+C(\varepsilon^2)\int_G\zeta^{r-2}u^2\zeta_t^2; \tag{16}$$

$$\begin{aligned} Z&=r\int_G u_t\zeta^{r-1}\nabla\zeta.\nabla u\leq\frac{1}{2}\int_G\zeta^r u_t^2+\frac{r}{2}\int_G\zeta^{r-2}\left|\nabla\zeta\right|^2\left|\nabla u\right|^2 \\ &\leq\frac{1}{2}\int_G\zeta^r u_t^2+\varepsilon^2\int_G\zeta^r u^{-2}\left|\nabla u\right|^4+C(\varepsilon^2)\int_G\zeta^{r-4}u^2\left|\nabla\zeta\right|^4.\end{aligned} \tag{17}$$

Substituting 16 and 17 into 15, we see that

$$\begin{aligned}\frac{1}{2}\int_G\zeta^r u_t^2&\leq\frac{d(g-f)}{dt}+2\varepsilon^2\int_G\zeta^r u^{-2}\left|\nabla u\right|^4 \\ &\quad+C(\varepsilon^2)\int_G\zeta^{r-2}\zeta_t^2+\frac{r}{q+1}\int_G\zeta^{r-1}u^{q+1}\left|\zeta_t\right|.\end{aligned}$$

At last

$$Y=\int_G \zeta^r u^{-1}\left|\nabla u\right|^2 u_t \leq \varepsilon\int_G \zeta^r u^{-2}\left|\nabla u\right|^4+\frac{1}{4\varepsilon}\int_G \zeta^r u_t^2 \qquad (18)$$
$$\leq \frac{1}{2\varepsilon}\frac{d(g-f)}{dt}+\varepsilon\int_G \zeta^r u^{-2}\left|\nabla u\right|^4$$
$$+\frac{C(\varepsilon^2)}{2\varepsilon}\int_G \zeta^{r-2}\zeta_t^2+\frac{2r}{(q+1)\varepsilon}\int_G \zeta^{r-1}u^{q+1}\left|\zeta_t\right|.$$

Substituting 10,16,18,17,12,13 and 14 into 11, and assuming that ζ takes its value in $[0,1]$, we find with new constants $C(\varepsilon)$,

$$(a-\varepsilon)\int_G \zeta^r u^{-2}\left|\nabla u\right|^4+(b-\varepsilon)\int_G \zeta^r u^{q-1}\left|\nabla u\right|^2$$
$$\leq -\frac{N-1}{N}\frac{dg}{dt}+C(\varepsilon)\frac{d(g-f)}{dt}+C(\varepsilon)\int_G \zeta^{r-2}u^{q+1}(\left|\nabla\zeta\right|^2+\left|\zeta_t\right|)$$
$$+C(\varepsilon)\int_G \zeta^{r-4}u^2(\left|\Delta\zeta\right|^2+\left|\nabla\zeta\right|^4+\zeta_t^2).$$

Then there exists constant $C=C(N,q,r)>0$ and two real constants $C_i=C_i(N,q,r)$ such that

$$\int_G \zeta^r u^{-2}\left|\nabla u\right|^4+\int_G \zeta^r u^{q-1}\left|\nabla u\right|^2$$
$$\leq C_1\frac{dg}{dt}+C_2\frac{df}{dt}+C(\int_G \zeta^{r-2}u^{q+1}(\left|\nabla\zeta\right|^2+\left|\zeta_t\right|)$$
$$+C(\int_G \zeta^{r-4}u^2(\left|\Delta\zeta\right|^2+\left|\Delta\zeta\right|^2+\zeta_t^2)).$$

Integrating this relation between t_1 and t_2, and observing that $f(t_i)=g(t_i)=0$ because $\zeta\in\mathcal{D}(Q)$, we deduce the estimate of the terms $\int\int_Q \zeta^r u^{-2}\left|\nabla u\right|^4$ and $\int\int_Q \zeta^r u^{q-1}\left|\nabla u\right|^2$. Then the estimate of $\int\int_Q \zeta^r u_t^2$ holds from 15, and 9 holds. ■

Now the estimate of $u^{q-1}\left|\nabla u\right|^2$ implies an interior estimate of u^{2q}.

Lemma 4 ***(Estimate of the power)*** *Under the assumptions of Lemma 3 with $r>4q/(q-1)$, there exists a constant $C=C(N,q,r)>0$ such that*

$$\int\int_Q \zeta^r u^{2q}\leq C\int\int_Q \zeta^{r-4q/(q-1)}(\left|\nabla\zeta\right|^2+\left|\Delta\zeta\right|+\left|\zeta_t\right|)^{2q/(q-1)}. \qquad (19)$$

Proof. Multiplying equation 1 by $\zeta^r u^q$, we obtain

$$\int_G \zeta^r u^{2q}=\frac{dg}{dt}-S+V+q\int\int_Q \zeta^r u^{q-1}\left|\nabla u\right|^2,$$

hence from 9, with a new constant $C = C(N, q, r)$,

$$\int\int_Q \zeta^r u^{2q} \leq C \int\int_Q \zeta^{r-2} u^{q+1} (|\nabla\zeta|^2 + |\zeta_t|)$$
$$+ C \int\int_Q \zeta^{r-4} u^2 (|\Delta\zeta|^2 + |\Delta\zeta|^2 + \zeta_t^2).$$

Now 19 follows from Hölder inequality. ■

The estimate of the power in turn implies Harnack inequality.

Lemma 5 *Under the assumptions of Theorem 1, u satisfies a parabolic Harnack inequality inside* $\Omega \times (0,T)$ *: there is a constant* $C = C(N, q) > 0$ *such that for any* $t \in (0,T)$ *and any ball* $B(x, 3\rho) \subset \Omega$, *with* $\left[t - 9\rho^2,\, t + 9\rho^2\right] \subset (0,T)$,

$$\max_{\overline{B}(x,\rho)\times[t-8\rho^2,\, t-7\rho^2]} u \leq C \min_{\overline{B}(x,\rho)\times[t-\rho^2,\, t]} u. \tag{20}$$

Proof. Let $0 < t < T$ and $x \in \Omega$ and $\rho > 0$ such that $B(x, 5\rho) \subset \Omega$ and $\left(t - 10\rho^2, t + 10\rho^2\right) \subset (0,T)$. Let us apply Lemma 3 with $G = B(x, 4\rho)$, $t_1 = t - 10\rho^2$, $t_2 = t + 10\rho^2$, and chose some $r > 4q/(q-1)$. We take $\zeta(x,t) = \varphi(x)\psi(t)$, with $\varphi \in \mathcal{D}(G)$, $\psi \in \mathcal{D}\left[(t_1, t_2)\right]$ with value in $[0,1]$ and $\varphi \equiv 1$ on $B(x, 3\rho)$, $\psi \equiv 1$ on $\left[t - 9\rho^2,\, t + 9\rho^2\right]$ and $|\Delta\varphi| + |\nabla\varphi|^2 + |\psi_t| \leq C(N)\rho^{-2}$. Then from we get the estimate

$$\int\int_{B(x,3\rho)\times[t-9\rho^2,\, t+9\rho^2]} H^s \leq C\, \rho^{N+2-2s}, \tag{21}$$

where $s = 2q/(q-1) > (N+2)/2$, since $q < (N+2)/(N-2)$. From Aronson and Serrin [2], this implies the Harnack inequality 20. ■

Now we can reach the final result.

Proof of Theorem 1 Let $t < T/2$ and $\rho^2 = t/10$. From 20 and 21 we get

$$u(x,t) \leq C\, \rho^{-2s/(q-1)s} = C\, t^{-1/(q-1)},$$

for any x with $B(x, 3\rho) \subset \Omega$. This implies in particular that 4 holds for any domain $\varpi \subset\subset \Omega$ (or $\varpi = \Omega = \mathbf{R}^N$).

On another part $u \in L^\infty\left((0, T/2)\,; L^1_{loc}(\Omega)\right)$. Indeed consider any domains $\varpi \subset\subset \varpi' \subset\subset \Omega$. Following the proof of [9], let $\lambda_{\varpi'}$ be the first eigenvalue and $\Phi_{\varpi'}$ be the first positive eigenfunction of $-\Delta$ with Dirichlet conditions on $\partial\varpi'$, with $\int_{\varpi'} \Phi_{\varpi'} = 1$, and set $X(t) = \int_{\varpi'} u(.,t)\Phi_{\varpi'}$. Since $u_t - \Delta u \geq 0$, we find that $d(e^{-\lambda_{\varpi'} t} X(t))/dt \geq 0$, as $\partial\Phi_{\varpi'} < 0$. Hence $X(t)$ is bounded on $(0, T/2)$, and

$$\int_\varpi u(.,t) \leq C \int_{\varpi'} u(., T/2), \tag{22}$$

where now the constant C depends also on u, hence $u \in L^\infty\left((0, T/2)\,; L^1(\varpi)\right)$. From [2], this implies 5 . Let us recall the simple proof: under the assumption 3, the Harnack inequality also implies that

$$u(x,t) \leq C\rho^{-N} \int_{B(x,2\rho)} u(.,t). \tag{23}$$

Hence from 23, for any $x \in \varpi$ and t small enough,

$$u(x,t) \leq C\rho^{-N} \leq Ct^{-N/2}$$

and 5 holds . If $q < (N+2)/N$, the estimate 5 is better than 4, unless $\Omega = \mathbf{R}^N$. $\blacksquare$

Remark Recall that the Harnack inequalities proved by Aronson and Serrin are obtained by the well-known Moser technique. As for the interior singularity problem 6, one can give a shorter proof of estimate 5 in case $q < (N+2)/N$, which does not uses Lemmas 1 to 3. As above, for any domains $\varpi \subset\subset \varpi' \subset\subset \Omega$, we have $u \in L^\infty\left((0,T/2)\,;L^1(\varpi)\right)$. Moreover $u \in L^q\left((0,T/2)\,;L^q(\varpi)\right)$, since for any $t < T/2$,

$$\int\int_{\varpi\times[t,T/2]} u^q \leq C\int\int_{\varpi'\times[t,T/2]} u^q\Phi_{\varpi'} \leq C\left(X(T/2)+\lambda_{\varpi'}\int_0^{T/2} X\right).$$

Now we write equation 1 under the form

$$u_t = \Delta u + f. \tag{24}$$

Taking $t < T/2$ and $\rho^2 = t$, we apply an L^∞ estimate due to Andreucci, Herrero and Velazquez, see [1], which is also based on Moser technique: for any solution of equation 24, and any $\sigma > (N+2)/2$, there is a constant $C = C(N,q,r) > 0$ such that for any x with $B(x,2\rho) \subset \Omega$,

$$\sup_{B(x,\rho)\times[t/2,t]} u(x,t) \leq C\,\frac{1}{|B(x,3\rho/2)\times[t/4,t]|}\int\int_{B(x,3\rho/2)\times[t/4,t]} u$$

$$+C\left(\int\int_{B(x,3\rho/2)\times[t/4,t]} f^\sigma\right)^{\nu/\sigma}\times\left(\int\int_{B(x,3\rho/2)\times[t/4,t]} u^q\right)^{(1-\nu)/q},$$

where $\nu = (N+2)\sigma/\left[(N+2)\sigma + q(2\sigma-(N+2)\right]$. Now taking $f = u^q$ with $q < (N+2)/N$, one can choose $\sigma \in \left[(N+2)/2, q/(q-1)\right]$. By an iterative argument it follows that

$$\sup_{B(x,\rho)\times[t/2,t]} u(x,t) \leq C\,\frac{1}{|B(x,3\rho/2)\times[t/4,t]|}\int\int_{B(x,3\rho/2)\times[t/4,t]} u$$

$$+\,C\left(\int\int_{B(x,3\rho/2)\times[t/4,t]} u^q\right)^{2/(N+2-Nq)};$$

see [1]. In particular, it implies that for any $x \in \varpi$, and t small enough,

$$u(x,t) \leq C\,\left(\rho^{-N}+1\right) \leq C\,t^{-N/2},$$

which again proves 5.

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